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Hyperbolic Conservation Laws in Continuum Physics

Fourth Edition

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For Mihalis and Thalia

Preface to the Fourth Edition

The purpose of this work is to present a broad overview of the theory of hyperbolic conservation laws, with emphasis on its genetic relation to classical continuum physics. The background, scope and plan are outlined in the Introduction, following this preface. The book was originally published fifteen years ago, and a third, revised edition appeared in 2010. Nevertheless, in order to keep pace with recent developments in the area, it has become necessary to prepare this substantially expanded and updated new edition.

In the face of the explosive growth of research, in volume, diversity and technical complexity, the encyclopedic ambitions of the project had to be moderated. Thus, a number of significant recent theoretical developments are barely touched upon here, or are merely sketched. For the same reason, it is not feasible to present the multitude of diverse applications that have mushroomed over the past few years. Still, the updated bibliography, now comprising close to two thousand entries, provides a panoramic view of the entire area.

The underlying objective of the work to promote synergy between the analysis of hyperbolic systems of conservation laws and continuum physics is particularly relevant at the present time, as the analytical theory is finally preparing the ground for taking up the challenge posed by systems in several spatial dimensions. The Euler equations of gas dynamics currently serve as the port of entry into that area of research. The new edition provides a brief account of recent developments is that direction and also strives to bring to the fore the noteworthy, albeit undeservedly neglected, paradigm of the system of elastodynamics.

The present edition places increased emphasis on the theory of hyperbolic systems of balance laws with dissipative source, modeling relaxation phenomena. The part of the theory pertaining to classical solutions in several spatial dimensions is expounded in the heavily revised and expanded Chapter V, while weak *BV* solutions in one spatial dimension are discussed in a newly added chapter (XVI).

A substantial portion of the original text has been reorganized so as to streamline the exposition, update the information, and enrich the collection of examples. In particular, several chapters of the latest edition have been expanded by the addition of new sections, elaborating on previously raised issues or introducing new topics for discussion.

Acknowledgments

My mentors, Jerry Ericksen and Clifford Truesdell, initiated me to continuum physics, as living scientific subject and as formal mathematical structure with fascinating history. I trust that both views are somehow reflected in this work.

I am grateful to many scientists—teachers, colleagues, and students alike—who have helped me, over the past fifty years, to learn continuum physics and the theory of hyperbolic conservation laws. Since it would be impossible to list them all here by name, let me single out Stu Antman, John Ball, Alberto Bressan, Gui-Qiang Chen, Cleopatra Christoforou, Bernie Coleman, Ron DiPerna, Hermano Frid, Jim Glimm, Jim Greenberg, Mort Gurtin, Ling Hsiao, Barbara Keyfitz, Peter Lax, Philippe LeFloch, Tai-Ping Liu, Andy Majda, Piero Marcati, Ingo Müller, Walter Noll, Jim Serrin, Denis Serre, Marshall Slemrod, Joel Smoller, Luc Tartar, Konstantina Trivisa, Thanos Tzavaras, Dehua Wang and Zhouping Xin, who have also honored me with their friendship. In particular, Denis Serre's persistent encouragement helped me to carry this arduous project to completion.

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Introduction

The seeds of continuum physics were planted with the works of the natural philosophers of the eighteenth century, most notably Euler; by the mid-nineteenth century, the trees were fully grown and ready to yield fruit. It was in this environment that the study of gas dynamics gave birth to the theory of quasilinear hyperbolic systems in divergence form, commonly called *hyperbolic conservation laws*; and these two subjects have been traveling hand in hand over the past two hundred years. This book aims at presenting the theory of hyperbolic conservation laws from the standpoint of its genetic relation to continuum physics. A sketch of the early history of this relation follows the Introduction. Even though research is still marching at a brisk pace, both fields have attained by now the degree of maturity that would warrant the writing of such an exposition.

In the realm of continuum physics, material bodies are realized as continuous media, and so-called "extensive quantities," such as mass, momentum and energy, are monitored through the fields of their densities, which are related by balance laws and constitutive equations. A self-contained, though skeletal, introduction to this branch of classical physics is presented in Chapter II. The reader may flesh it out with the help of a specialized text on the subject.

In its primal formulation, the typical balance law stipulates that the time rate of change in the amount of an extensive quantity stored inside any subdomain of the body is balanced by the rate of flux of this quantity through the boundary of the subdomain together with the rate of its production inside the subdomain. In the absence of production, a balanced extensive quantity is conserved. The special feature that renders continuum physics amenable to analytical treatment is that, under quite natural assumptions, statements of gross balance, as above, reduce to field equations, i.e., partial differential equations in divergence form.

The collection of balance laws in force demarcates and identifies particular continuum theories, such as mechanics, thermomechanics, electrodynamics, and so on. In the context of a continuum theory, constitutive equations encode the material properties of the medium, for example, heat-conducting viscous fluid, elastic solid, elastic dielectric, etc. The coupling of these constitutive relations with the field equations gives birth to closed systems of partial differential equations, dubbed "balance laws" or "conservation laws," from which the equilibrium state or motion of the continuous medium is to be determined. Historically, the vast majority of noteworthy partial differential equations were generated through that process. This is eminently the case for hyperbolic systems of conservation laws, as may be seen from the historical account. The central thesis of the book is that the umbilical cord joining continuum physics with the theory of partial differential equations should not be severed, as it is still carrying nourishment in both directions.

Systems of balance laws may be elliptic, typically in statics; hyperbolic, in dynamics, for media with "elastic" response; mixed elliptic-hyperbolic, in statics or dynamics, when the medium undergoes phase transitions; parabolic or mixed parabolic-hyperbolic, in the presence of viscosity, heat conductivity or other diffusive mechanisms. Accordingly, the basic notions shall be introduced, in Chapter I, at a level of generality that would encompass all of the above possibilities. Nevertheless, since the subject of this work is hyperbolic conservation laws, the discussion will eventually focus on such systems, beginning with Chapter III.

The term "homogeneous hyperbolic conservation law" refers to first-order systems of partial differential equations in divergence form,

$$(HCL) \qquad \partial_t H(U) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0,$$

that are of hyperbolic type. The state vector U, with values in $\mathbb{R}^n$, is to be determined as a function of the spatial variables $(x_1, \dots, x_m)$ and time t. The given functions H and $G_1, \dots, G_m$ are smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The symbol ∂_t stands for $\partial/\partial t$ and ∂_α denotes $\partial/\partial x_\alpha$. The notion of hyperbolicity will be specified in Section 3.1.

Solutions to hyperbolic conservation laws may be visualized as propagating waves. When the system is nonlinear, the profiles of compression waves get progressively steeper and eventually break, generating jump discontinuities which propagate on as shocks. Hence, inevitably, the theory has to deal with weak solutions. This difficulty is compounded further by the fact that, in the context of weak solutions, uniqueness is lost. It thus becomes necessary to devise proper criteria for singling out admissible weak solutions. Continuum physics naturally induces such admissibility criteria through the Second Law of thermodynamics. These may be incorporated in the analytical theory, either directly, by stipulating outright that admissible solutions should satisfy "entropy" inequalities, or indirectly, by equipping the system with a minute amount of diffusion, which has negligible effect on smooth solutions but reacts stiffly in the presence of shocks, weeding out those that are not thermodynamically admissible. The notions of "entropy" and "vanishing diffusion," which will play a central role throughout the book, will be introduced in Chapters III and IV.

Chapter V discusses the Cauchy problem and the initial-boundary value problem for hyperbolic systems of balance laws, in the context of classical solutions. It is shown that these problems are locally well-posed and the resulting smooth solutions are stable, even within the broader class of admissible weak solutions, but their life span is finite, unless there is a dissipative source that thwarts the breaking of waves.

The analysis underscores the stabilizing effect of the Second Law of thermodynamics and the role of dissipation modeling relaxation.

The Cauchy problem in the large may be considered only in the context of weak solutions. This is still terra incognita for systems of more than one equation in several space dimensions, as the analysis is at present facing seemingly insurmountable obstacles. It may turn out that the Cauchy problem is not generally well-posed, either because of catastrophic failure of uniqueness (see Section 4.8), or because distributional solutions fail to exist. In the latter case one would have to resort to the class of weaker, measure-valued solutions (see Section 17.3). It is even conceivable that hyperbolic systems should be perceived as mere shadows, in the Platonic sense, of diffusive systems with minute viscosity or dispersion. Nevertheless, this book will focus on success stories, namely problems admitting standard distributional weak solutions. These encompass scalar conservation laws in one or several space dimensions, systems of hyperbolic conservation laws in a single space dimension, as well as systems in several space dimensions whenever invariance (radial symmetry, stationarity, self-similarity, etc.) reduces the number of independent variables to two.

Chapter VI provides a detailed presentation of the rich and definitive theory of L^∞ and BV solutions to the Cauchy problem and the initial-boundary value problem for scalar conservation laws in several space dimensions.

Beginning with Chapter VII, the focus of the investigation is fixed on systems of conservation laws in one space dimension. In that setting, the theory has a number of special features that are of great help to the analyst, so major progress has been achieved.

Chapter VIII provides a systematic exposition of the properties of shocks. In particular, various shock admissibility criteria are introduced, compared and contrasted. Admissible shocks are then combined, in Chapter IX, with another class of particular solutions, called centered rarefaction waves, to synthesize wave fans that solve the classical Riemann problem. Solutions of the Riemann problem may in turn be employed as building blocks for constructing solutions to the Cauchy problem, in the class BV of functions of bounded variation. Two construction methods based on this approach will be presented here: the random choice scheme, in Chapter XIII, and a front tracking algorithm, in Chapter XIV. Uniqueness and stability of these solutions will also be established.

Chapter XV outlines an alternative construction of BV solutions to the Cauchy problem, for general strictly hyperbolic systems of conservation laws, by the method of vanishing viscosity.

Chapter XVI discusses the construction of BV solutions by the random choice method for strictly hyperbolic systems of balance laws with a dissipative source, governing relaxation phenomena.

The above construction methods generally apply when the initial data have sufficiently small total variation. This restriction seems to be generally necessary because, in certain systems, when the initial data are "large" even weak solutions to the Cauchy problem may blow up in finite time. Whether such catastrophes may occur to solutions of the field equations of continuum physics is at present a major open problem. For a limited class of systems, which however contains several important representatives, solutions with large initial data can be constructed by means of the functional analytic method of compensated compactness. This approach, which rests on the notions of measure-valued solution and the Young measure, will be outlined in Chapter XVII.

There are other interesting properties of weak solutions, beyond existence and uniqueness. In Chapter X, the notion of characteristic is extended from classical to weak solutions; and it is employed for obtaining a very precise description of regularity and long-time behavior of solutions to scalar conservation laws, in Chapter XI, as well as to systems of two conservation laws, in Chapter XII.

The final Chapter XVIII discusses problems in two spatial dimensions, and time, in which geometry and invariance reduce the number of variables to two, namely the Riemann problem for scalar conservation laws, flows past obstacles and shock collisions in gas dynamics, cavitation in elastodynamics and isometric immersion of surfaces in differential geometry.

The bibliography, comprising close to two thousand entries, is quite extensive but far from comprehensive. Next to recent developments, it also provides reference to earlier work that may have been superseded, so as to enable the reader to trace the evolution of the field.

In order to highlight the fundamental ideas, the discussion proceeds from the general to the particular, notwithstanding the clear pedagogical merits of the reverse course. Even so, under proper guidance, the book may also serve as a text. With that in mind, the pace of the proofs is purposely uneven: slow for the basic, elementary propositions that may provide material for an introductory course; faster for the more advanced technical results that are addressed to the experienced analyst. Even though the various parts of this work fit together to form an integral entity, readers may select a number of independent itineraries through the book. Thus, those principally interested in the conceptual foundations of the theory of hyperbolic conservation laws, in connection to continuum physics, need go through Chapters I-V only. Chapter VI, on the scalar conservation law, may be read virtually independently of the rest. Students intending to study solutions as compositions of interacting elementary waves may begin with Chapters VII-IX and then either continue on to Chapters X-XII or else pass directly to Chapters XIII, XIV and XVI. Similarly, Chapter XV relies solely on Chapters VII and VIII, while Chapter XVIII depends on Chapters III, VII, VIII and IX. Finally, only Chapter VII is needed as a prerequisite for the functional analytic approach expounded in Chapter XVII.

Certain topics are perhaps discussed in excessive detail, as they are of special interest to the author; and a number of results are published here for the first time. On the other hand, several important aspects of the theory and its applications are barely touched upon, or are only sketched very briefly. They include the stability theory of multi-space-dimensional shocks and boundary conditions, the newly emerging theory of hyperbolic conservation laws with random initial data, the derivation of the balance laws of continuum physics from the kinetic theory of gases, the study of phase transitions and a host of diverse applications. Each one of these areas would warrant the writing of a specialized monograph. The most conspicuous absence is a discussion of numerics, which, beyond its practical applications, also provides valuable insight to the theory. Fortunately, a number of texts on the numerical analysis of hyperbolic conservation laws are currently available and may fill this gap.

Geometric measure theory, functional analysis and dynamical systems provide the necessary tools in the theory of hyperbolic conservation laws, but to a great extent the analysis employs custom-made techniques, with strong geometric flavor, underscoring wave propagation and wave interaction. This may leave the impression that the area is insular, detached from the mainland of partial differential equations. However, the reader will soon realize that the field of hyperbolic conservation laws is far-reaching and highly diversified, as it is connected by bridges with the realms of elliptic equations, parabolic equations, dispersive equations and the equations of the kinetic theory.

A Sketch of the Early History of Hyperbolic Conservation Laws

The general theory, and even the name itself, of hyperbolic conservation laws emerged just fifty years ago, and yet the special features of this class of systems of partial differential equations had been identified long before, in the context of particular examples arising in mathematical physics. The aim here is to trace the early seminal works that launched the field and set it on its present course. A number of relevant classic papers have been collected in Johnson and Chéret [1].

The ensuing exposition will describe how the subject emerged out of fluid dynamics, how its early steps were frustrated by the confused state of thermodynamics, how it was set on a firm footing, and how it finally evolved into a special branch of the theory of partial differential equations.

This section may be read independently of the rest of the book, as it is essentially self-contained, but the student will draw extra benefit by revisiting it after getting acquainted with the current state of the art expounded in the main body of the text. Accordingly, in order to highlight the connection between past and present, the history is presented here with the benefit of hindsight: current terminology is freely used, and symbols and equations drawn from the original sources have been transliterated to modern notation.

Since the early history of hyperbolic conservation laws is inextricably intertwined with gas dynamics, we begin with a brief review of the theory of ideal gases, as it stood at the turn of the nineteenth century. Details on this topic are found in the historical tract by Truesdell [1].

The state of the ideal gas is determined by its density ρ, pressure p and (absolute) temperature θ, which are interrelated by the law associated with the names of Boyle, Gay-Lussac and Mariotte:

$$p = R\rho\theta, \tag{1}$$

where R is the universal gas constant. In the place of ρ, one may equally employ its inverse $u = 1/\rho$, namely the specific volume.

The specific heat at constant pressure or at constant volume, c_p or c_u, is the rate of change in the amount of heat stored in the gas as the temperature varies, while the pressure or the specific volume is held fixed. The ratio $\gamma = c_p/c_u$ is a constant bigger than one, called the *adiabatic exponent*.

Barotropic thermodynamic processes, in which $p = p(\rho)$, may be treated in the realm of mechanics, with no regard to temperature. The simplest example is an isothermal process, in which the temperature is held constant, so that, by (1),

$$p = a^2\rho. \tag{2}$$

Subtler is the case of an isentropic or adiabatic process,[1] in which the temperature and the specific volume vary simultaneously in such a proportion that the amount of heat stored in any part of the gas remains fixed. As shown by Laplace and by Poisson, this assumption leads to

$$p = a^2\rho^\gamma. \tag{3}$$

The oldest, and still most prominent, paradigm of a hyperbolic system of conservation laws is provided by the Euler equations for barotropic gas flow, which express the conservation of mass and momentum, relating the velocity field v with the density field and the pressure field. The pertinent publications by Euler, culminating in his definitive formulation of hydrodynamics, are collected in Euler [1], which also contains informative commentary by Truesdell. In addition to the equations that bear his name, Euler derived what is now called the Bernoulli equation for irrotational flow, so named because in steady flow it reduces to the celebrated law discovered by Daniel Bernoulli. We will encounter the aforementioned equations on several occasions in the main body of this book, beginning with Section 3.3.6.

Internal forces in an elastic fluid are transmitted by the hydrostatic pressure, which is a scalar field. As a result, the Euler equations form a system of conservation laws with distinctive geometric structure. Conservation laws of more generic type, manifesting the tensorial nature of the flux field, as discussed here in Chapter I, emerged in the 1820s from the pioneering work of Cauchy [1,2,3,4] on the theory of elasticity. Nevertheless, as we shall see below, the early work on hyperbolic conservation laws dealt almost exclusively with the one-space-dimensional setting, for which the Euler equations constitute a fully representative example.

In an important memoir on the theory of sound, published in 1808, Poisson [1] considers the Euler equations and the Bernoulli equation for rectilinear isothermal flow of an ideal gas, namely

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2) + a^2\partial_x\rho = 0, \end{cases} \tag{4}$$

[1] The term "adiabatic" was coined in 1859 by Rankine, who also originated the use of the symbol γ for the adiabatic exponent. However, in the sequel we will employ the newer terminology "isentropic," while reserving "adiabatic" for a related but different use; see Section 2.5.

$$\partial_t \phi + \frac{1}{2}(\partial_x \phi)^2 + a^2 \log \rho = 0, \tag{5}$$

where ϕ is the velocity potential, $v = \partial_x \phi$. By eliminating ρ between (5) and $(4)_1$, he derives a second order equation for ϕ alone:

$$\partial_t^2 \phi + 2(\partial_x \phi)(\partial_x \partial_t \phi) + (\partial_x \phi)^2 \partial_x^2 \phi - a^2 \partial_x^2 \phi = 0. \tag{6}$$

Employing a method of solving differential equations devised by Laplace and by Legendre, he concludes that any ϕ that satisfies the functional equation

$$\partial_x \phi = f(x + (a - \partial_x \phi)t), \tag{7}$$

for some arbitrary smooth function f, is a particular solution of (6).

In current terminology, one recognizes Poisson's solution as a simple wave (see Section 7.6) on which the Riemann invariant (see Section 7.3) $v + a \log \rho$ is constant, and thus v satisfies the equation

$$\partial_t v + v \partial_x v - a \partial_x v = 0, \tag{8}$$

admitting solutions

$$v = f(x + (a - v)t), \tag{9}$$

with f an arbitrary smooth function.

Forty years after the publication of Poisson's paper, the British astronomer Challis [1] made the observation that (9), with $f(x) = -\sin(\frac{1}{2}\pi x)$, yields $v = 0$ along the straight line $x = -at$ and $v = 1$ along the straight line $x = -1 - (a-1)t$, which raises the paradox that v must be simultaneously equal to 0 and 1 at the point $(-a, 1)$ of intersection of these straight lines. This is the earliest reference to the breakdown of classical solutions, which pervades the entire theory of hyperbolic conservation laws.

The issue raised by Challis was addressed almost immediately by Stokes [1], his colleague at the University of Cambridge. Stokes notes that, according to Poisson's solution (9), along each straight line $x = \bar{x} - (a - f(\bar{x}))t$, we have $v(x,t) = f(\bar{x})$ and

$$\partial_x v(x,t) = \frac{f'(\bar{x})}{1 + t f'(\bar{x})}. \tag{10}$$

Thus, unless f is nondecreasing, the wave will break at $t = -1/f'(\bar{x})$, where $f'(\bar{x})$ is the minimum of f'. He then ponders what would happen after singularities develop and comes up with an original and bold conjecture. In his own words: "Perhaps the most natural supposition to make for trial is that a surface of discontinuity is formed, in passing across which there is an abrupt change of density and velocity." He seems highly conscious that this is a far-reaching idea, going well beyond the particular setting of Poisson's solution, as he writes: "Although I was led to the subject by considering the interpretation of the integral (9), the consideration of a discontinuous motion is not here introduced in connection with that interpretation, but simply for its own sake; and I wish the two subjects to be considered as quite distinct."

Stokes then proceeds to characterize the jump discontinuities that conform to the governing physical laws of conservation of mass and momentum, which underlie the Euler equations in the realm of smooth flows. Assuming that density and velocity jump from (ρ_-, v_-) to (ρ_+, v_+) across a line of discontinuity propagating with speed σ (i.e., having slope σ), he shows that

$$\tag{11} \begin{cases} \rho_+ v_+ - \rho_- v_- = \sigma(\rho_+ - \rho_-) \\ \rho_+ v_+^2 + a^2 \rho_+ - \rho_- v_-^2 - a^2 \rho_- = \sigma(\rho_+ v_+ - \rho_- v_-). \end{cases}$$

By eliminating σ between the above two equations, he gets

$$\tag{12} \rho_- \rho_+ (v_+ - v_-)^2 = a^2 (\rho_+ - \rho_-)^2.$$

Thus, Stokes [1] introduces, in the context of the Euler equations (4) for isothermal flow, the notion of a shock wave and derives what are now known as the Rankine-Hugoniot jump conditions (see Section 8.1), which characterize distributional weak solutions of (4). This paper is one of the cornerstones of the theory of hyperbolic conservation laws. However, the development of the subject was soon to hit a roadblock.

Stokes's idea of contemplating flows with jump discontinuities was criticized, apparently in private, by Sir William Thomson (Lord Kelvin), and later by Lord Rayleigh, in private correspondence (Rayleigh [1]) as well as in print (Rayleigh [2, §253]), on the following grounds: they argued that jump discontinuities should not produce or consume mechanical energy. A calculation shows that this would require

$$\tag{13} 2\rho_- \rho_+ \log\left(\frac{\rho_-}{\rho_+}\right) = \rho_-^2 - \rho_+^2,$$

which is incompatible with $\rho_- \neq \rho_+$.

In order to place the above argument in the present context of the theory of conservation laws, one should notice that any smooth solution of the Euler equations (4) automatically satisfies the conservation law of mechanical energy

$$\tag{14} \partial_t \left(\tfrac{1}{2}\rho v^2 + a^2 \rho \log \rho\right) + \partial_x \left(\tfrac{1}{2}\rho v^3 + a^2 \rho v \log \rho + a^2 \rho v\right) = 0.$$

In current terminology, $\frac{1}{2}\rho v^2 + a^2 \rho \log \rho$ is an entropy for the system (4), with entropy flux $\frac{1}{2}\rho v^3 + a^2 \rho v \log \rho + a^2 \rho v$; see Section 7.4. Assuming that mechanical energy should be conserved, even in the presence of shocks, induces the jump condition

$$\tag{15} \begin{aligned} &\tfrac{1}{2}\rho_+ v_+^3 + a^2 \rho_+ v_+ \log \rho_+ + a^2 \rho_+ v_+ - \tfrac{1}{2}\rho_- v_-^3 - a^2 \rho_- v_- \log \rho_- - a^2 \rho_- v_- \\ &\qquad = \sigma\left[\tfrac{1}{2}\rho_+ v_+^2 + a^2 \rho_+ \log \rho_+ - \tfrac{1}{2}\rho_- v_-^2 - a^2 \rho_- \log \rho_-\right]. \end{aligned}$$

Eliminating σ between $(11)_1$ and (15), and making use of (12), one arrives at (13).

Stokes was convinced, and perhaps also intimidated, by the above criticism. On June 5, 1877 he answered Lord Rayleigh in an apologetic tone, renouncing his theory: “Thank you for pointing out the objection to the queer kind of motion in the paper you refer to. Sir W. Thomson pointed the same out to me many years ago, and I should have mentioned it if I had the occasion to write anything bearing on the subject, or if, without that, my paper had attracted attention. It seemed, however, hardly worthwhile to write a criticism on a passage in a paper which was buried among other scientific antiquities” (Stokes [2]). Moreover, when his collected works were published in 1880, he deleted the part of the 1848 paper that discussed discontinuous flows and inserted an annotation to the effect that shocks are impossible because they are incompatible with energy conservation, expressed by (13) (Stokes [3]). A more detailed account of the correspondence between Stokes, Rayleigh and Thomson, with references to the original sources, is found in Salas [1].

Stokes’s vindication had to wait for the development of thermodynamics. Still, it is puzzling that he was prepared to abandon his theory so readily, even though, as we shall see below, by 1880 substantial progress had already been made in the subject.

To follow these developments in chronological order, let us return to the late 1840s. Airy [1], the Astronomer Royal, had demonstrated in 1845 that the propagation of longitudinal water waves in a shallow channel is governed, in Lagrangian coordinates, by the second-order equation

$$\partial_t^2 w = a^2(1+\partial_x w)^{-3}\partial_x^2 w. \tag{16}$$

One may recast (16) as a hyperbolic system of conservation laws, by setting $u = 1+\partial_x w$ and $v = \partial_t w$:

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x\left(\frac{1}{2}a^2 u^{-2}\right) = 0. \end{cases} \tag{17}$$

This system, written in Eulerian coordinates, is known as the shallow-water equations, and is identical to the Euler equations for isentropic flow, with adiabatic exponent $\gamma = 2$. The derivation is found in Section 7.1.

Reacting to Challis’s paper of 1848, Airy [2] observes that, in a similar fashion, System (4) of the Euler equations for isothermal flow may be recast, in Lagrangian coordinates, as the scalar second-order equation

$$\partial_t^2 w = a^2(1+\partial_x w)^{-2}\partial_x^2 w, \tag{18}$$

where w is the displacement, so that $u = 1+\partial_x w$ is specific volume and $v = \partial_t w$ is velocity.

Airy then notes that (16) and (18), respectively, admit solutions

$$\partial_t w - 2a(1+\partial_x w)^{-\frac{1}{2}} = \text{constant}, \tag{19}$$

$$\partial_t w - a\log(1+\partial_x w) = \text{constant}, \tag{20}$$

whose derivation he attributes to De Morgan. This is the first instance that Riemann invariants appear explicitly in print, in connection to simple waves.

With reference to Stokes's work, Airy conjectures that tidal bores in long rivers and surf on extensive flat sand may turn out to be represented by solutions of his shallow-water equations, with jump discontinuities.

The next notable contribution came from Earnshaw [1]. The starting point in his analysis is the observation that if $y(x,t)$ satisfies

$$\partial_t y - f(\partial_x y) = \text{constant}\,, \tag{21}$$

for some function f, then

$$\partial_t^2 y = [f'(\partial_x y)]^2 \partial_x^2 y. \tag{22}$$

He compares (22) with the equation of rectilinear barotropic motion of a fluid in Lagrangian coordinates,

$$\rho_0 \partial_t^2 y + \partial_x p(\partial_x y) = 0, \tag{23}$$

where $y(x,t)$ is the position of the fluid particle x at time t, ρ_0 is the reference density and p is the pressure. Thus one may obtain particular solutions of (23) in the form (21), with f computed from $[f'(u)]^2 = -\rho_0^{-1} p'(u)$.

Assuming $\rho_0 = 1$, in current practice one introduces the specific volume $u = \partial_x y$, the velocity $v = \partial_t y$, and recasts (23) as the equivalent first-order system

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x p(u) = 0\,. \end{cases} \tag{24}$$

Then $v - f(u)$ are Riemann invariants of (24) and the solutions $v - f(u) = \text{constant}$ represent simple waves.

Earnshaw computes $f(u)$, and thereby finds solutions (21), for isothermal flow, in which $p = a^2\rho = a^2 u^{-1}$, and for isentropic flow, where $p = a^2\rho^\gamma = a^2 u^{-\gamma}$, with $\gamma > 1$. In current notation,

$$v \pm a \log u = \text{constant}\,, \tag{25}$$

$$v \pm \frac{2a\gamma^{\frac{1}{2}}}{\gamma - 1} u^{-\frac{\gamma-1}{2}} = \text{constant}\,. \tag{26}$$

Depending on the sign, he envisages these solutions as waves of "condensation" or waves of "rarefaction". The term "compression" is currently used in the place of "condensation".

Another interesting observation due to Earnshaw is that the Euler equations admit traveling wave solutions ("waves propagating without undergoing a change of type," in his terminology) if and only if $p = a - b/\rho$, which identifies what is now known as the Chaplygin gas; see Section 2.5. This is the earliest encounter with a nonlinear system that is linearly degenerate; see Section 7.5. Earnshaw further notes that for this special p, the equation of motion (23), in Lagrangian coordinates, is linear.

We now come to a landmark in the development of the theory of hyperbolic conservation laws, namely the celebrated paper by Riemann [1]. This is the earliest work in which the Euler equations are treated from the perspective of analysis. It should be noted, however, that the author takes great pains to motivate his work from physics. He expresses the hope that, beyond their purely mathematical interest, his results will find applications in acoustics.

Riemann considers the rectilinear barotropic flow of a gas with general equation of state $p = p(\rho)$, subject only to the condition $p'(\rho) > 0$, and writes the Euler equations in the form

$$\begin{cases} \partial_t \log\rho + v\partial_x \log\rho = -\partial_x v \\ \\ \partial_t v + v\partial_x v = -p'(\rho)\partial_x \log\rho. \end{cases} \tag{27}$$

As we saw above, what are now known as Riemann invariants were already present in the works of Poisson, Airy, and Earnshaw, albeit exclusively in connection with simple waves. By contrast, Riemann defines his invariants

$$r = \frac{1}{2}\left[v + \int\sqrt{p'(\rho)}d\rho\right], \qquad s = \frac{1}{2}\left[-v + \int\sqrt{p'(\rho)}d\rho\right], \tag{28}$$

in the context of any smooth solution of (27), and shows that they satisfy

$$\partial_t r = -(v + \sqrt{p'(\rho)})\partial_x r, \qquad \partial_t s = -(v - \sqrt{p'(\rho)})\partial_x s, \tag{29}$$

so that r and s, respectively, stay constant along what are now deemed forward and backward characteristics.

Riemann then devises what is now known as a hodograph transformation (see Section 12.2), which recasts the nonlinear system (29) as a linear equation by reversing the roles of (x,t) and (r,s) as independent and dependent variables. To that end, upon observing that

$$\partial_r\left[x - (v - \sqrt{p'(\rho)})t\right] = -\partial_s\left[x - (v + \sqrt{p'(\rho)})t\right], \tag{30}$$

he introduces the potential $w(r,s)$,

$$\partial_r w = x - (v + \sqrt{p'(\rho)})t, \qquad -\partial_s w = x - (v - \sqrt{p'(\rho)})t, \tag{31}$$

and shows that it satisfies a linear equation

$$\partial_r\partial_s w = m(\partial_r w + \partial_s w), \tag{32}$$

where m is some function of ρ, induced by $p(\rho)$, and thus depends solely on $r+s$. In particular, $m = -(2a)^{-1} =$ constant, in the isothermal case $p = a^2\rho$. One has to determine $w(r,s)$ by solving (32), then get $x(r,s)$ and $t(r,s)$ from (31), and finally invert these functions to obtain $r(x,t)$ and $s(x,t)$.

Next, Riemann considers solutions with jump discontinuities, under the additional assumption $p''(\rho) \geq 0$. By balancing mass and momentum across a shock $x = \xi(t)$, with left state (ρ_-, v_-), right state (ρ_+, v_+), and speed $\dot{\xi}$, he derives the jump conditions in the form

$$(33) \qquad \dot{\xi} = v_- \pm \left[\frac{\rho_+}{\rho_-} \, \frac{p(\rho_+) - p(\rho_-)}{\rho_+ - \rho_-} \right]^{\frac{1}{2}} = v_+ \pm \left[\frac{\rho_-}{\rho_+} \, \frac{p(\rho_+) - p(\rho_-)}{\rho_+ - \rho_-} \right]^{\frac{1}{2}} .$$

He does not address the issue of energy conservation, as he was probably unaware of the objections to Stokes's work raised by Kelvin and by Rayleigh. Nevertheless, he emphasizes that, on grounds of stability, only compressive shocks are physically meaningful. Actually, he postulates shock admissibility in the guise of what is now called the Lax E-condition (see Section 8.3), namely

$$(34) \qquad v_+ + \sqrt{p'(\rho_+)} < \dot{\xi} < v_- + \sqrt{p'(\rho_-)}$$

for forward shocks, and similarly for backward shocks. He also constructs what is now known as the shock curve or the Hugoniot locus (see Section 8.2) and describes its shape.

Finally, Riemann introduces and solves the celebrated problem that now bears his name (see Chapter IX), namely, he demonstrates that a jump in the state variables (ρ, v) is generally resolved into an outgoing wave fan consisting of a backward and a forward wave, each of which may be either a compressive shock or a centered rarefaction simple wave.

Riemann's remarkable paper provides the foundation for the general theory of hyperbolic systems of conservation laws in one space dimension. As we shall see, shocks cannot be isothermal or isentropic in gas flow that conserves energy, together with mass and momentum. This has prompted the complaint, widely circulating in the literature, that Riemann's treatment of shocks is deficient. We have seen already that it is the same argument that forced Stokes to abandon his theory of isothermal shocks. Nevertheless, such criticism is off the mark. Riemann's equation (33) on jump condition is mathematically accurate within the framework of isentropic gas dynamics, which is a physically legitimate simplification of the more complete, thermodynamic theory.

After 1860, the notion of shocks was progressively gaining acceptance. By 1870, judging by a footnote in Rankine [1], Kelvin, one of the early critics of Stokes's ideas, was prepared to admit compressive shocks, while rejecting rarefaction shocks as unstable.

In 1877, Christoffel [1] considered shocks of barotropic gas flow in three-dimensional space, and derived the corresponding jump conditions. In fact, the common practice of bracketing the symbol of a field to denote its jump across a shock was introduced by him. See Hölder [1].

Between 1875 and 1889, Ernst Mach, with his students' assistance, performed the earliest experiments on shock waves, at the German University of Prague. He employed electric sparks triggered by the discharge of a Leiden jar to generate shock waves in the layer of air confined between parallel glass plates, one of which was covered with soot. The path of the shock was then determined from the marks it left on the soot. By means of this device, he managed to measure the speed of the shocks, showing that they are supersonic. He also observed that the oblique incidence of a shock on a rigid boundary may induce regular or irregular (now called Mach) reflection. The phenomenon of Mach reflection was also detected in surface waves, in liquids (mercury, milk, syrup). The shocks were generated by dropping a V-shaped iron wire on the liquid surface. Later on, by using the more powerful optical schlieren method, Mach and his students demonstrated that projectiles moving through the air at supersonic speed are surrounded by bow shocks. A historical account of Mach's experiments, with references to the original publications, which appeared in Vols. 72-92 of the Vienna Academy Sitzungsherichte, is found in Krehl and van der Geest [1].

The mid-nineteenth century was a period of rapid strides in mastering the principles of thermodynamics. The central role of the internal energy ε was recognized, and the law of energy conservation was established. The notion of entropy was introduced, and named, by Clausius [1], and the Second Law of thermodynamics was formulated. Soon the specific entropy s took its place on the list of thermodynamic variables, next to ε, θ, p, u and ρ. General thermal equations of state $\varepsilon = \varepsilon(\rho,\theta)$ were considered by Kirchhoff [1], and equivalent caloric equations of state $\varepsilon = \varepsilon(u,s)$ were introduced by Gibbs [1], who postulated the rule that now bears his name:

$$d\varepsilon = \theta ds - pdu. \tag{35}$$

Nevertheless, as we shall see below, most authors adhered to the special case of the ideal gas, with equations of state

$$\varepsilon = \frac{R}{\gamma - 1}\theta, \qquad p = R\rho\theta, \qquad s = \frac{R}{\gamma-1}\log\frac{\theta}{\rho^{\gamma-1}}. \tag{36}$$

As regards fluid dynamics, it became clear that the system of the Euler equations had to be supplemented with an additional, independent field equation, expressing the conservation of (combined mechanical and thermal) energy. This equation was derived by Kirchhoff [1,2] for thermoviscoelastic, heat-conducting gases. In the absence of viscosity and heat conductivity, attaching the conservation of energy equation to the Euler equations yields another important paradigm of a hyperbolic system of conservation laws, which will be encountered on several occasions in the main body of this book, beginning with Section 3.3.5.

The task of determining the jump conditions that express energy conservation in the presence of shocks was undertaken by Rankine [1] and by Hugoniot [2].

Rankine derives his jump conditions for a shock wave moving into an undisturbed medium, by balancing the loss in mechanical energy at the shock against the heat flux, so that the total energy production is nil. It is not easy to follow his discussion,[2] which is based on physical arguments, but eventually he arrives at the correct equations.

[2] For a reconstruction of Rankine's argument, see Rayleigh [4].

In contrast to Rankine's approach, the treatment by Hugoniot [2] is in a clear mathematical style. He considers the rectilinear motion of general nonlinear elastic media, in Lagrangian coordinates. First, he discusses at length barotropic motion, governed by the scalar second-order equation (23), or equivalently by the first-order system (24). He reproduces (reinvents?) the results of Earnshaw and Riemann, without citing either of these authors. In particular, he determines the two families of Riemann invariants, by the same procedure as Earnshaw, and then shows, as Riemann already had, that they are constant along characteristics of the same family. He also points out the connection between Riemann invariants and simple waves.

The pathbreaking contribution of Hugoniot is found in Chapter V of his memoir, where he discusses shocks, and especially in §§ 149-158, where he derives the jump condition dictated by energy conservation. It is there that one encounters for the first time the full set of jump conditions in Lagrangian coordinates:

$$\begin{cases} \sigma(u_+ - u_-) = v_- - v_+ \\ \sigma(v_+ - v_-) = p_+ - p_- \\ \sigma(\varepsilon_+ + \frac{1}{2}v_+^2 - \varepsilon_- - \frac{1}{2}v_-^2) = p_+v_+ - p_-v_- \,, \end{cases} \tag{37}$$

written for an ideal gas, with internal energy $\varepsilon = pu/(\gamma - 1)$; see (36). Combining the three equations in (37) yields the famous jump condition

$$\varepsilon_+ - \varepsilon_- + \frac{1}{2}(p_+ + p_-)(u_+ - u_-) = 0, \tag{38}$$

which does not involve v or σ. Hugoniot derives this equation for the case of the ideal gas, in the form

$$\frac{p_+}{p_-} = \frac{2u_- + (\gamma - 1)(u_- - u_+)}{2u_+ + (\gamma - 1)(u_+ - u_-)} \,. \tag{39}$$

He seems unaware that Rankine [1] had already obtained a similar result.

The above equations have had a great impact in the theory of gas dynamics and its applications. Consequently, all jump conditions associated with shocks are now collectively known as "*Rankine-Hugoniot jump conditions*," even though the name "Stokes jump conditions" would represent a more accurate reflection of the historical record.

Another important contribution by Hugoniot [1] is the introduction and study of weak waves (also known as acceleration waves), namely propagating characteristic surfaces across which the state variables themselves are continuous but their derivatives experience jump discontinuities. Weak waves had appeared earlier in the acoustic research of Euler.

By the turn of the twentieth century, the field of hyperbolic conservation laws was branching out, finding applications in the science and technology of combustion, detonation, and aerodynamics. We shall not pursue the history of developments in those new directions, because the aim here is to trace the evolution of the core ideas that have led to the present state of affairs in the mathematical theory. An account of the subject of detonations, as it stood in the early 1900s, is found in the book by Jouguet [3]. A good starting point for getting acquainted with the enormous literature on aerodynamics is the text by von Mises [1], which contains historical references and an extensive bibliography.

The state of the art in the basic theory of hyperbolic conservation laws around 1900 is exemplified by the books of Duhem [1], Hadamard [1], and Weber [1]. Weber mainly elaborates upon the aforementioned paper by Riemann. For his part, Hadamard makes a presentation of the work of Riemann in conjunction with the results of Hugoniot. Both Duhem and Hadamard provide detailed expositions on the propagation of shock and weak waves, in several space dimensions. In particular, Hadamard makes the important observation that when an irrotational flow crosses a weak wave it remains irrotational, while after crossing a non-planar shock it acquires vorticity. Duhem emphasizes the implications of thermodynamics. He postulates the Second Law in the form of the celebrated field inequality that now bears his name (see Section 2.3):

$$\rho_0 \dot{s} + \operatorname{Div}\left(\frac{1}{\theta} Q\right) \geq 0, \tag{40}$$

where Q is the heat flux vector. Furthermore, he shows that in the absence of viscosity and heat conductivity, the system of conservation laws for mass, momentum and energy, in conjunction with the Gibbs rule (35), implies that smooth thermodynamic processes are necessarily isentropic: $\dot{s} = 0$.

With the dawn of the new century, the theory was confronted by the issue of physical admissibility of shocks. The reader may recall that in the previous century several authors, beginning with Riemann, had subscribed to the view that compressive shocks are stable, and thereby admissible, in contrast to rarefaction shocks, which are patently unstable, as they are apt to disintegrate into rarefaction simple waves. However, the connection between shock stability and the Second Law of thermodynamics was still elusive. A related point of contention was the physical status of shocks in barotropic flows, in view of Rayleigh's argument that they fail to conserve mechanical energy.

The first step in addressing these questions was taken by Jouguet [1,2], who demonstrated that in an ideal gas only compressive shocks are compatible with the Second Law of thermodynamics. Indeed, in the absence of heat flux, $Q = 0$, (40) reduces to $\dot{s} \geq 0$, and $\dot{s}$ generated by a shock propagating with speed σ has the same sign as $\sigma(s_- - s_+)$. By virtue of (36),

$$s_+ - s_- = \frac{R}{\gamma - 1} \log\left(\frac{p_+ u_+^\gamma}{p_- u_-^\gamma}\right). \tag{41}$$

Furthermore, p_+/p_- is related to u_+/u_- by Hugoniot's equation (39). One then easily sees that $s_+ - s_-$ has the same sign as $u_+ - u_-$ and hence the opposite sign from $\rho_+ - \rho_-$.

Following upon the above work, Duhem [2] shows that weak compressive shocks in fact satisfy the Second Law of thermodynamics in any gas with equation of state $p = p(u,s)$, subject only to the constraint $p_{uu} > 0$. In the process, he makes the important observation that the entropy jump across a weak shock is of cubic order in the strength of the shock. He further proves that weak compressive shocks are subsonic relative to the denser gas and supersonic relative to the more rarefied gas. Thus, in current terminology, Duhem demonstrates that when the system of conservation laws for gas dynamics is genuinely nonlinear, the Lax E-condition manifests the Second Law of thermodynamics for weak shocks; see Section 8.5.

In parallel with Jouguet's work, Zemplén [1] investigates the balance between mechanical and thermal energy produced by shocks in ideal gases, and shows that mechanical energy is converted into heat at compressive shocks, while at rarefaction shocks this process would be reversed. Thus the Second Law of thermodynamics allows only for shocks that convert mechanical energy into heat.

With regard to barotropic flow, $p = p(\rho)$, Burtton [1], Weber [1], and Rayleigh [3] are ultimately prepared to grant physical status to shocks so long as they comply with the Second Law of thermodynamics, which they interpret as a requirement that the production of mechanical energy be nonpositive. Weber demonstrates that only compressive shocks meet this requirement, when $p''(\rho) \geq 0$. He operates in Eulerian coordinates, but in order to avoid writing too many new equations, here we transcribe his calculation to Lagrangian coordinates: The rate of mechanical energy production by a shock propagating with speed σ is

$$\dot{E} = -\sigma\left\{\frac{1}{2}v_+^2 - \frac{1}{2}v_-^2 - \int_{u_-}^{u_+} p(u)du\right\} + p_+v_+ - p_-v_- \,. \tag{42}$$

Since mass and momentum are conserved, $u_\pm$, $p_\pm$ and $v_\pm$ are related by the first two jump conditions in (37), with the help of which (42) may be written as

$$\dot{E} = \sigma\left\{\int_{u_-}^{u_+} p(u)du - \frac{1}{2}(p_+ + p_-)(u_+ - u_-)\right\}. \tag{43}$$

It is now clear that, assuming $p''(u) > 0$, $\dot{E} < 0$ if and only if $\sigma(u_+ - u_-) > 0$, i.e., $\sigma(\rho_+ - \rho_-) < 0$.

In current terminology, Weber is employing mechanical energy $\frac{1}{2}v^2 - \int p(u)du$ as a convex entropy for the system (24), with associated entropy flux $vp(u)$, and is showing that when the system is genuinely nonlinear, then the entropy admissibility condition is equivalent to the Lax E-condition; see Section 8.5.

An alternative, albeit related, way of identifying physically admissible shocks is through the "vanishing viscosity" approach. Stokes [1] and Hugoniot [2], among others, were aware that viscosity and/or heat conductivity would smear shocks. This was formalized by Duhem [1]. Thus the loss of mechanical energy incurred at shocks could be attributed to the workings of "internal friction" induced by viscosity and heat conductivity. Passing to the zero viscosity limit may be justified by showing that physically admissible shocks can be paired with viscous traveling waves having the same end-states and the same speed. By changing coordinates, one may consider just stationary shocks and steady-state viscous waves.

The conservation laws of mass, momentum and energy for rectilinear flow of heat-conducting viscous fluids read

$$
(44) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\[2ex] \partial_t(\rho v) + \partial_x(\rho v^2 + p - \mu \partial_x v) = 0 \\[2ex] \partial_t(\rho \varepsilon + \frac{1}{2}\rho v^2) + \partial_x(\rho v \varepsilon + \frac{1}{2}\rho v^3 + p v - \mu v \partial_x v - k \partial_x \theta) = 0, \end{cases}
$$

where μ is the viscosity and k is the conductivity. In the absence of viscosity and heat conductivity, stationary shocks satisfy the jump conditions

$$
(45) \qquad \begin{cases} \rho_+ v_+ = \rho_- v_- = m \\[2ex] m v_+ + p_+ = m v_- + p_- = a \\[2ex] m \varepsilon_+ + \frac{1}{2} m v_+^2 + p_+ v_+ = m \varepsilon_- + \frac{1}{2} m v_-^2 + p_- v_- = b. \end{cases}
$$

The objective is to determine the shock layer (also called shock profile or shock structure), namely a steady-state solution

$$
(46) \qquad \begin{cases} \rho v = m \\[2ex] \rho v^2 + p - \mu v' = a \\[2ex] \rho v \varepsilon + \frac{1}{2}\rho v^3 + p v - \mu v v' - k \theta' = b \end{cases}
$$

of (44), on $(-\infty, \infty)$, for the assigned parameters m, a and b. Here and below, the prime denotes differentiation with respect to x.

The above program was initiated almost simultaneously, and apparently independently, by Rayleigh[3] [4] and by G.I. Taylor [1]. Their approaches are surprisingly similar. Assuming the gas is ideal (36), they eliminate p, ρ, ε and θ between the equations in (46), ending up with a second-order equation for v alone:

$$
(47) \qquad \frac{k\mu}{Rm}(v v')' = \left(\frac{2k}{R} + \frac{\mu}{\gamma - 1} \right) v v' - \frac{ka}{Rm} v' - \frac{\gamma + 1}{2(\gamma - 1)} m v^2 + \frac{a\gamma}{\gamma - 1} v - b.
$$

For $k = 0$ or $\mu = 0$, Taylor solves (47) in closed form. He also derives the asymptotic form of the solution when both k and μ are positive but $|v_+ - v_-|$ is small. In particular, he points out that only compressive shocks may support viscous profiles.

Becker [1] treated the same problem, still for ideal gases, by observing that, in consequence of (46), the temperature as a function of the specific volume satisfies a first-order differential equation that may be integrated in closed form in the special situation where k/μ equals the specific heat c_p at constant pressure.

[3] In addition to dealing with the issue at hand, Rayleigh's memoir provides an interesting review of the development of the theory of shock waves in the nineteenth century.

During the Second World War, prominent physicists and mathematicians, coming from various areas of expertise, were attracted to gas dynamics. An important issue at the time was the behavior of real gases at high temperature and at high pressure, beyond the range of validity of the polytropic model. In that connection, Bethe [1] demonstrates that in gases with equation of state $p = p(u,s)$, the condition $p_{uu} > 0$ alone does not guarantee that compressive shocks of arbitrary strength are compatible with the Second Law of thermodynamics and stable. For that purpose, one must make additional assumptions, such as $up_s + 2\theta > 0$ and $\theta p_u + pp_u < 0$, which may fail when the gas undergoes phase transitions.

Another important contribution from the same era is the work of Weyl [1], which extends the investigation on shock layers, initiated by Rayleigh [4], G.I. Taylor [1], and Becker [1], to gases with general equations of state. By combining the equations in (46), Weyl derives the first-order system

$$\begin{cases} \mu m u' = m^2 u + p - a \\ k\theta' = m\varepsilon - \frac{1}{2}m^3u^2 + amu - b, \end{cases} \tag{48}$$

in the variables (u,θ), where u is the specific volume ρ^{-1}. Noting that, by virtue of (45), $(u_\pm, \theta_\pm)$ are equilibrium points of (48), he realizes the shock profile as the orbit joining (u_-, θ_-), which is an unstable node, with (u_+, θ_+), which is a saddle. The definitive treatment of this problem was provided two years later by Gilbarg [1], establishing the existence of the shock layer, for arbitrary positive μ, k, and showing that it converges to (a) a shock when both μ and k tend to zero; (b) a continuous shock layer when μ tends to zero, while k is held fixed; and (c) to a generally discontinuous shock layer when k tends to zero, while μ is held fixed. As we shall see in Section 8.6, Weyl's approach has now become standard practice in the general theory of hyperbolic conservation laws.

The roster of prominent scientists who contributed to the field as part of the war effort includes von Neumann. He prepared a number of expository reports [1,2,3] on the theory of shock waves in gas dynamics, with many insightful observations. In particular, he elaborated on the problem of oblique shock reflection, reviving and popularizing Mach's contributions from the nineteenth century. He also championed the idea of obtaining solutions by means of scientific computation; see von Neumann [4]. The proceedings of a panel discussion, held on August 17, 1949, chaired by von Neumann [5] and involving Burgers, Heisenberg, von Karman and other experts, provide a glimpse of what were perceived as major open problems at that time. Remarkably, many of the issues raised there are still unresolved.

By the late 1940s, a large amount of information on hyperbolic conservation laws had been amassed, mainly in the guise of gas dynamics. It had been derived by mathematicians, physicists, chemists and engineers over a period of 150 years, and had been presented in a wide variety of styles and levels of rigor. The task of consolidating this material was undertaken by Courant and Friedrichs [1], who provide a magisterial synthesis of the subject, in mathematical language. Their book has played – and continues to play – an important role in disseminating the physical underpinnings of the theory to the mathematical community.

After 1950, following the contemporaneous trends in the general area of partial differential equations, research in hyperbolic conservation laws focuses on the qualitative theory of the Cauchy problem. The seeds for establishing the local existence of classical solutions had already been planted in the 1930s by the work of Schauder [1] on quasilinear hyperbolic equations of second order. Schauder's strategy, which employs a hierarchy of L^2 "energy" estimates on the derivatives of solutions to linearized equations and then passes to the quasilinear case via a fixed point argument, has been exploited widely over the past decades, culminating in the definitive theory of the Cauchy problem for symmetrizable systems of conservation laws, expounded here in Chapter V.

Since classical solutions to the Cauchy problem generally break down in finite time, one may at best hope to establish the existence of weak solutions in the large. The first successful attempt in that direction was made in the seminal paper by Hopf [1], which treats the Cauchy problem for the simplest nonlinear scalar conservation law

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0. \tag{49}$$

This equation was originally proposed by Bateman [1], in an obscure publication, as a simple model for the system of conservation laws of gas dynamics. It reappeared independently in the work of Burgers [1] on turbulence, and is now universally known as the *Burgers equation*.

Adopting Burgers's viewpoint, Hopf treats (49) as the $\mu \downarrow 0$ limit of the *Burgers equation with viscosity*

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \mu \partial_x^2 u. \tag{50}$$

By employing the celebrated *Hopf-Cole transformation*

$$u = -2\mu \partial_x \log \phi \,, \tag{51}$$

so named because it was also discovered independently by Cole [1], he reduces (50) to the classical heat equation

$$\partial_t \phi = \mu \partial_x^2 \phi. \tag{52}$$

This enables him to solve the Cauchy problem for the equation (50), with initial data u_0, in the explicit form

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left\{ -\frac{1}{2\mu} F(x,y,t) \right\} dy}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2\mu} F(x,y,t) \right\} dy} \,, \tag{53}$$

where

$$F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(z) dz \,. \tag{54}$$

Letting $\mu \downarrow 0$ in (53), he arrives at a weak solution $u(x,t)$ to the Cauchy problem for the Burgers equation (49), which is determined explicitly by the initial data $u_0(x)$ through

$$(55) \qquad u(x,t) = \frac{x - y(x,t)}{t}, \qquad \text{a.e. on } (-\infty,\infty)\times(0,\infty),$$

where $y(x,t)$ is the minimizer of the function $F(x,y,t)$, with respect to y on $(-\infty,\infty)$. He also investigates the geometric structure and the large-time behavior of this solution.

Hopf's paper stimulated intensive research on the scalar conservation law, initially in one and eventually in several space dimensions, which generated the rich theory presented in Chapters VI and XI.

The next major milestone, marking the conclusion of this historical introduction, is the landmark paper by Lax [2], which coins the term "hyperbolic conservation law" and launches the field as a new principal branch in the theory of partial differential equations. This was accomplished by distilling, generalizing and formalizing the raw material that had accumulated over the years in the context of special systems, as reported above.

The first part of Lax's paper extends the aforementioned work of Hopf, and in particular devises a generalization of (55) that solves the Cauchy problem, for general convex scalar conservation laws. The reader may find an account of this theory in Section 11.4.

The second part of the paper lays the foundations for the general theory of systems of hyperbolic conservation laws in one space dimension, by introducing the notions of strict hyperbolicity, genuine nonlinearity, Riemann invariants, simple waves and the Lax E-condition, which all come together in the construction of shock and rarefaction wave curves and the solution of the Riemann problem. To a great extent, the present state of the art in the theory of hyperbolic systems of conservation laws in one space dimension, as presented here in Chapters VII, VIII and IX, is an elaboration of the above themes. It is fair to say that Lax's paper set the direction for the development of the field of hyperbolic conservation laws over the past fifty years.

Contents

I

Balance Laws

The general, mathematical theory of balance laws expounded in this chapter has been designed to provide a unifying framework for the multitude of balance laws of classical continuum physics, obeyed by so-called "extensive quantities" such as mass, momentum, energy, etc. The ambient space for the balance law will be $\mathbb{R}^k$, with typical point X. In the applications to continuum physics, $\mathbb{R}^k$ will stand for physical space, of dimension one, two or three, in the context of statics; and for space-time, of dimension two, three or four, in the context of dynamics.

The generic balance law will be introduced through its primal formulation, as a postulate that the production of an extensive quantity in any domain is balanced by a flux through the boundary; it will then be reduced to a field equation. It is this reduction that renders continuum physics mathematically tractable. It will be shown that the divergence form of the field equation is preserved under change of coordinates, and that the balance law, in its original form, may be retrieved from the field equation. The properties discussed in this chapter derive solely from the divergence form of the field equations and thus apply equally to balance laws governing equilibrium and evolution.

The field equations for a system of balance laws will be combined with constitutive equations, relating the flux and production density with a state vector, to obtain a closed quasilinear first order system of partial differential equations in divergence form.

It will be shown that symmetrizable systems of balance laws are endowed with companion balance laws which are automatically satisfied by smooth solutions, though not necessarily by weak solutions. The issue of admissibility of weak solutions will be raised.

Solutions will be considered with shock fronts or weak fronts, in which the state vector field or its derivatives experience jump discontinuities across a manifold of codimension one.

The theory of *BV* functions, which provide the natural setting for solutions with shock fronts, will be surveyed and the geometric structure of *BV* solutions will be described.

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Highly oscillatory weak solutions will be constructed, and a first indication of the stabilizing role of admissibility conditions will be presented.

The setting being Euclidean space, it will be expedient to employ matrix notation at the expense of obscuring the tensorial nature of the fields. The symbol $\mathbb{M}^{r\times s}$ will denote throughout the vector space of $r \times s$ matrices and $\mathbb{R}^r$ shall be identified with $\mathbb{M}^{r\times 1}$. The $(r-1)$-dimensional Hausdorff measure in $\mathbb{R}^r$ will be denoted by $\mathscr{H}^{r-1}$. Other standard notation to be used here includes $\mathbb{S}^{r-1}$ for the unit sphere in $\mathbb{R}^r$ and $\mathscr{B}_\rho(X)$ for the ball of radius ρ centered at X. In particular, $\mathscr{B}_\rho$ will stand for $\mathscr{B}_\rho(0)$.

1.1 Formulation of the Balance Law

Let $\mathscr{X}$ be an open subset of $\mathbb{R}^k$. A *proper domain* in $\mathscr{X}$ is any open bounded subset of $\mathscr{X}$, with Lipschitz boundary. A balance law on $\mathscr{X}$ postulates that the *production* of a certain "extensive" quantity in any proper domain $\mathscr{D}$ is balanced by the *flux* of this quantity through the boundary $\partial\mathscr{D}$ of $\mathscr{D}$.

The salient feature of an extensive quantity is that both its production and its flux are additive over disjoint subsets. Thus, the production in the proper domain $\mathscr{D}$ is given by the value $\mathscr{P}(\mathscr{D})$ of a (signed) Radon measure $\mathscr{P}$ on $\mathscr{X}$. Similarly, with every proper domain $\mathscr{D}$ is associated a countably additive set function $\mathscr{Q}_\mathscr{D}$, defined on Borel subsets of $\partial\mathscr{D}$, such that the flux in or out of $\mathscr{D}$ through any Borel subset $\mathscr{C}$ of $\partial\mathscr{D}$ is given by $\mathscr{Q}_\mathscr{D}(\mathscr{C})$. Hence, the balance law simply states

$$\mathscr{Q}_\mathscr{D}(\partial\mathscr{D}) = \mathscr{P}(\mathscr{D}), \tag{1.1.1}$$

for every proper domain $\mathscr{D}$ in $\mathscr{X}$.

For the purposes of this book, it will suffice to consider flux set functions $\mathscr{Q}_\mathscr{D}$ that are absolutely continuous with respect to the Hausdorff measure $\mathscr{H}^{k-1}$. Hence with each proper domain $\mathscr{D}$ in $\mathscr{X}$ is associated a *density flux function* $q_\mathscr{D} \in L^1(\partial\mathscr{D})$ such that

$$\mathscr{Q}_\mathscr{D}(\mathscr{C}) = \int_\mathscr{C} q_\mathscr{D}(X)d\mathscr{H}^{k-1}(X), \tag{1.1.2}$$

for any Borel subset $\mathscr{C}$ of $\partial\mathscr{D}$.

Borel subsets $\mathscr{C}$ of $\partial\mathscr{D}$ are oriented by means of the outward unit normal N to $\mathscr{D}$, at points of $\mathscr{C}$. The fundamental postulate in the theory of balance laws is that the flux depends solely on the surface and its orientation, i.e., if $\mathscr{C}$ is concurrently a Borel subset of the boundaries of two distinct proper domains $\mathscr{D}_1$ and $\mathscr{D}_2$, sharing the same outward normal on $\mathscr{C}$, then $\mathscr{Q}_{\mathscr{D}_1}(\mathscr{C}) = \mathscr{Q}_{\mathscr{D}_2}(\mathscr{C})$, and thereby $q_{\mathscr{D}_1}(X) = q_{\mathscr{D}_2}(X)$, for almost all (with respect to $\mathscr{H}^{k-1}$) $X \in \mathscr{C}$.

In analogy to the flux measure, one might be tempted to limit consideration to production measures $\mathscr{P}$ that are absolutely continuous with respect to Lebesgue measure, and are thus represented by a *production density function* $p \in L^1_{loc}(\mathscr{X})$:

$$\mathscr{P}(\mathscr{D}) = \int_\mathscr{D} p(X)dX. \tag{1.1.3}$$

Though adequate for many applications, such a simplification would be too restrictive for our needs, as we must deal extensively with balance laws for entropy-like quantities, which incur nonzero production on shock fronts of zero Lebesgue measure.

1.2 Reduction to Field Equations

At first glance, the notion of a balance law, as introduced in Section 1.1, appears too general to be of any use. It turns out, however, that the balancing requirement (1.1.1) induces severe restrictions on density flux functions. Namely, the value of $q_{\mathscr{D}}$ at $X \in \partial\mathscr{D}$ may depend on $\mathscr{D}$ solely through the outward normal N at X, and the dependence is "linear". This renders the balance law quite concrete, reducing it to a field equation.

1.2.1 Theorem. *Consider the balance law* (1.1.1) *on $\mathscr{X}$ where $\mathscr{P}$ is a signed Radon measure and the $\mathscr{Q}_{\mathscr{D}}$ are induced, through* (1.1.2), *by density flux functions $q_{\mathscr{D}}$ that are bounded, $|q_{\mathscr{D}}(X)| \le C$, for all proper domains $\mathscr{D}$ and any $X \in \partial\mathscr{D}$. Then,*
(i) With each $N \in \mathbb{S}^{k-1}$ is associated a bounded measurable function a_N on $\mathscr{X}$, with the following property: Let $\mathscr{D}$ be any proper domain in $\mathscr{X}$ and suppose X is some point on $\partial\mathscr{D}$ where the outward unit normal to $\mathscr{D}$ exists and is N. Assume further that X is a Lebesgue point of $q_{\mathscr{D}}$, relative to $\mathscr{H}^{k-1}$, and that the upper derivate of $|\mathscr{P}|$ at X, with respect to Lebesgue measure, is finite. Then

$$q_{\mathscr{D}}(X) = a_N(X). \tag{1.2.1}$$

(ii) There exists a vector field $A \in L^\infty\left(\mathscr{X};\mathbb{M}^{1\times k}\right)$ such that, for any fixed $N \in \mathbb{S}^{k-1}$,

$$a_N(X) = A(X)N, \quad a.e. \text{ on } \mathscr{X}. \tag{1.2.2}$$

(iii) The function A satisfies the field equation

$$\operatorname{div} A = \mathscr{P}, \tag{1.2.3}$$

in the sense of distributions on $\mathscr{X}$.

Proof. Fix any $N \in \mathbb{S}^{k-1}$ and then take any hyperplane $\mathscr{C}$, of codimension one, with normal N and nonempty intersection with $\mathscr{X}$. For $X \in \mathscr{C} \cap \mathscr{X}$, let $\mathscr{B}_r^-(X)$ denote the semiball $\{Y \in \mathscr{B}_r(X) : (Y - X)\cdot N < 0\}$. The limit

$$a_N(X) = \lim_{r\downarrow 0} \frac{1}{\mathscr{H}^{k-1}\left(\mathscr{C} \cap \mathscr{B}_r(X)\right)} \int_{\mathscr{C}\cap\mathscr{B}_r(X)} q_{\mathscr{B}_r^-(X)}(Y)\, d\mathscr{H}^{k-1}(Y), \tag{1.2.4}$$

exists for almost all (with respect to $\mathscr{H}^{k-1}$) $X \in \mathscr{C} \cap \mathscr{X}$ and defines a bounded, $\mathscr{H}^{k-1}$-measurable function. By repeating the above construction for every hyperplane with normal N, we define a_N on all of $\mathscr{X}$.

In order to study the properties of a_N, we fix $N \in \mathbb{S}^{k-1}$, together with a hyperplane $\mathscr{C}$ with normal N, and a ball $\mathscr{B}$ in $\mathscr{X}$, centered at some point on $\mathscr{C} \cap \mathscr{X}$. We then apply the balance law to cylindrical domains

$$(1.2.5) \qquad \mathscr{D} = \bigcup_{-\delta<\tau<\varepsilon} \mathscr{A}_\tau, \qquad \mathscr{A}_\tau = \{X : X - \tau N \in \mathscr{C} \cap \mathscr{B}\},$$

where δ and ε are small nonnegative numbers. This yields

$$(1.2.6) \quad \int_{\mathscr{A}_\varepsilon} a_N(X) d\mathscr{H}^{k-1}(X) + \int_{\mathscr{A}_{-\delta}} a_{-N}(X) d\mathscr{H}^{k-1}(X) = \mathscr{P}(\mathscr{D}) + O(\delta) + O(\varepsilon),$$

where the terms $O(\delta)$ and $O(\varepsilon)$ account for the contribution of the flux through the lateral boundary of the cylindrical domain. Setting $\delta = 0$ and letting $\varepsilon \downarrow 0$, we derive from (1.2.6) an estimate which, applied to all balls $\mathscr{B}$, implies that, as $\tau \downarrow 0$, $a_N(X + \tau N) \to -a_{-N}(X)$, in $L^\infty(\mathscr{C} \cap \mathscr{X})$ weak*. Similarly, setting $\varepsilon = 0$ and letting $\delta \downarrow 0$, we deduce that, as $\tau \uparrow 0$, $a_{-N}(X + \tau N) \to -a_N(X)$, again in $L^\infty(\mathscr{C} \cap \mathscr{X})$ weak*. In particular, this implies that a_N is Lebesgue measurable on $\mathscr{X}$.

Returning to (1.2.6), and now letting both $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we conclude that $a_{-N}(X) = -a_N(X)$, for almost all (with respect to $\mathscr{H}^{k-1}$) $X \in \mathscr{C} \cap \mathscr{X}$, unless $\mathscr{C}$ belongs to the (at most) countable family of hyperplanes with normal N for which $|\mathscr{P}|(\mathscr{C} \cap \mathscr{X}) > 0$. Henceforth, we will refer to these exceptional hyperplanes as *singular*.

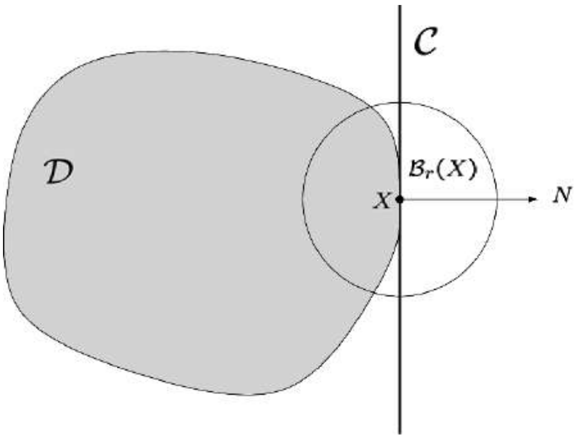

Fig. 1.2.1

To show (1.2.1), consider any proper domain $\mathscr{D}$ in $\mathscr{X}$ and fix any $X \in \partial\mathscr{D}$ where the outward unit normal is N and the tangential hyperplane is $\mathscr{C}$. Assume, further, that X is a Lebesgue point of $q_\mathscr{D}$ and that the upper derivate of $|\mathscr{P}|$ at X, with respect to Lebesgue measure, is finite. For r positive and small, write the balance law, first for the domain $\mathscr{D} \cap \mathscr{B}_r(X)$, then for the semiball $\{Y \in \mathscr{B}_r(X) : (Y - X) \cdot N < 0\}$; see Fig. 1.2.1.

Combining the resulting two equations yields

$$\int_{\partial\mathscr{D}\cap\mathscr{B}_r(X)} q_{\mathscr{D}}(Y)d\mathscr{H}^{k-1}(Y) - \int_{\mathscr{C}\cap\mathscr{B}_r(X)} a_N(Y)d\mathscr{H}^{k-1}(Y) = o\big(r^{k-1}\big). \tag{1.2.7}$$

Dividing (1.2.7) by r^{k-1}, letting $r \downarrow 0$, and recalling (1.2.4), we arrive at (1.2.1), thus establishing assertion (i) of the theorem.

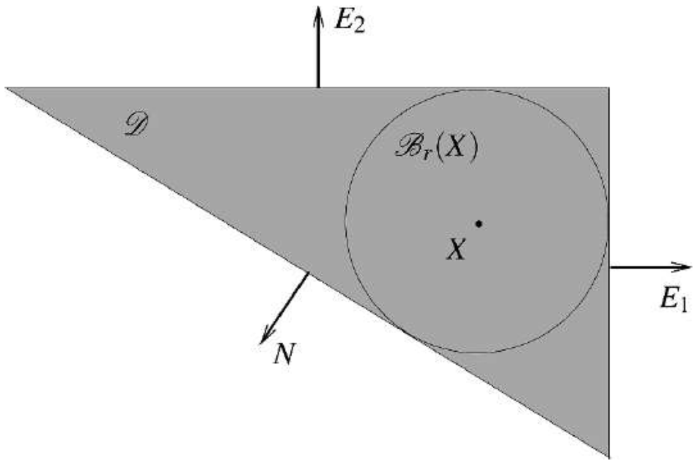

Fig. 1.2.2

We will verify (1.2.2) by employing the celebrated *Cauchy tetrahedron argument.* We introduce the standard orthonormal basis $\{E_\alpha : \alpha = 1, \cdots, k\}$ in $\mathbb{R}^k$ and assemble the m-row vector field $A \in L^\infty\big(\mathscr{X}; \mathbb{M}^{1\times k}\big)$ with components a_{E_α}:

$$A(X) = \big[a_{E_1}(X), \cdots, a_{E_k}(X)\big]. \tag{1.2.8}$$

Fix any $N \in \mathbb{S}^{k-1}$ with nonzero components N_α (the argument has to be slightly modified when some of the N_α vanish), and take any $X \in \mathscr{X}$ with the following properties: X is a Lebesgue point of the $k+1$ functions $a_{E_1}, \cdots, a_{E_k}$ and a_N; the upper derivate of $|\mathscr{P}|$ at X, with respect to Lebesgue measure, is finite. For r positive and small, consider the simplex[1]

$$\mathscr{D} = \{Y : (Y_\alpha - X_\alpha)\operatorname{sgn} N_\alpha > -r,\ \alpha = 1, \cdots, k;\ (Y - X) \cdot N < r\}. \tag{1.2.9}$$

Notice that $\partial\mathscr{D}$ consists of one face $\mathscr{C}$ with outward normal N and k faces $\mathscr{C}_\alpha$, for $\alpha = 1, \cdots, k$, with respective outward normals $(-\operatorname{sgn} N_\alpha)E_\alpha$. Furthermore, we have $\mathscr{H}^{k-1}(\mathscr{C}_\alpha) = |N_\alpha|\mathscr{H}^{k-1}(\mathscr{C})$. We select r so that none of the faces of $\mathscr{D}$ lies on a singular hyperplane. The balance law for $\mathscr{D}$ then reads

[1] The Cauchy tetrahedron argument derives its name from the special case $k = 3$. Figure 1.2.2 depicts the setting when $k = 2$ and both N_1 and N_2 are negative.

$$\int_{\mathscr{C}} a_N d\mathscr{H}^{k-1} - \sum_{\alpha=1}^{k} (\operatorname{sgn} N_\alpha) \int_{\mathscr{C}_\alpha} a_{E_\alpha} d\mathscr{H}^{k-1} = \mathscr{P}(\mathscr{D}). \tag{1.2.10}$$

Upon dividing (1.2.10) by $\mathscr{H}^{k-1}(\mathscr{C})$ and then passing to the limit along any sequence of r that tends to zero, while avoiding the (at most countable) set of values for which some face of $\mathscr{D}$ lies on a singular hyperplane, one arrives at

$$a_N(X) = \sum_{\alpha=1}^{k} a_{E_\alpha}(X) N_\alpha = A(X)N, \tag{1.2.11}$$

which establishes (1.2.2).

It remains to show (1.2.3). For Lipschitz continuous A, one may derive (1.2.3) by applying the divergence theorem to the balance law. In the general case where A is merely in L^∞, we resort to mollification. We fix any test function $\psi \in C_0^\infty(\mathbb{R}^k)$ with total mass one, supported in the unit ball, we rescale it by ε,

$$\psi_\varepsilon(X) = \varepsilon^{-k}\psi\big(\varepsilon^{-1}X\big), \tag{1.2.12}$$

and employ it to mollify, in the customary fashion, $\mathscr{P}$ and A on the set $\mathscr{X}_\varepsilon \subset \mathscr{X}$ of points whose distance from $\mathscr{X}^c$ exceeds ε:

$$p_\varepsilon = \psi_\varepsilon * \mathscr{P}, \qquad A_\varepsilon = \psi_\varepsilon * A. \tag{1.2.13}$$

For any hypercube $\mathscr{D} \subset \mathscr{X}_\varepsilon$, we apply the divergence theorem to the smooth field A_ε and use Fubini's theorem to get

$$\begin{aligned} \int_{\mathscr{D}} \operatorname{div} A_\varepsilon(X) dX &= \int_{\partial\mathscr{D}} A_\varepsilon(X) N(X) d\mathscr{H}^{k-1}(X) \\ &= \int_{\partial\mathscr{D}} \int_{\mathbb{R}^k} \psi_\varepsilon(Y) A(X-Y) N(X) dY d\mathscr{H}^{k-1}(X) \\ &= \int_{\mathbb{R}^k} \psi_\varepsilon(Y) \int_{\partial\mathscr{D}_Y} A(Z)N(Z) d\mathscr{H}^{k-1}(Z) dY, \end{aligned} \tag{1.2.14}$$

where $\mathscr{D}_Y$ denotes the Y-translate of $\mathscr{D}$, that is $\mathscr{D}_Y = \{Z : Z+Y \in \mathscr{D}\}$. By virtue of the balance law,

$$\int_{\partial\mathscr{D}_Y} A(Z)N(Z) d\mathscr{H}^{k-1}(Z) = \int_{\partial\mathscr{D}_Y} a_N(Z) d\mathscr{H}^{k-1}(Z) = \mathscr{P}(\mathscr{D}_Y), \tag{1.2.15}$$

for almost all Y in the ball $\{Y : |Y| < \varepsilon\}$. Hence (1.2.14) gives

$$\int_{\mathscr{D}} \operatorname{div} A_\varepsilon(X) dX = \int_{\mathbb{R}^k} \psi_\varepsilon(Y) \mathscr{P}(\mathscr{D}_Y) dY = \int_{\mathscr{D}} p_\varepsilon(X) dX, \tag{1.2.16}$$

whence we infer

$$\operatorname{div} A_\varepsilon(X) = p_\varepsilon(X), \qquad X \in \mathscr{X}_\varepsilon\,. \tag{1.2.17}$$

Letting $\varepsilon \downarrow 0$ yields (1.2.3), in the sense of distributions on $\mathscr{X}$. This completes the proof.

In the following section we shall see that the course followed in the proof of the above theorem can be reversed: departing from the field equation (1.2.3), one may retrieve the flux density functions $q_{\mathscr{D}}$ and thereby restore the balance law in its original form (1.1.1).

1.3 Change of Coordinates and a Trace Theorem

The divergence form of the field equations of balance laws is preserved under coordinate changes, so long as the fields transform according to appropriate rules. In fact, this holds even when the flux fields are merely locally integrable.

1.3.1 Theorem. *Let $\mathscr{X}$ be an open subset of $\mathbb{R}^k$ and let $A \in L^1_{loc}\left(\mathscr{X};\mathbb{M}^{1\times k}\right)$ and $\mathscr{P} \in \mathscr{M}(\mathscr{X})$ satisfy the field equation*

$$\operatorname{div} A = \mathscr{P}, \tag{1.3.1}$$

in the sense of distributions on $\mathscr{X}$. Consider any bilipschitz homeomorphism X^* of $\mathscr{X}$ to a subset $\mathscr{X}^*$ of $\mathbb{R}^k$, with Jacobian matrix

$$J = \frac{\partial X^*}{\partial X} \tag{1.3.2}$$

such that

$$\det J \geq a > 0, \;\; \textit{a.e. on } \mathscr{X}. \tag{1.3.3}$$

Then, $A^* \in L^1_{loc}\left(\mathscr{X}^*;\mathbb{M}^{1\times k}\right)$ and $\mathscr{P}^* \in \mathscr{M}\left(\mathscr{X}^*\right)$, defined by

$$A^* \circ X^* = (\det J)^{-1} A J^\top, \tag{1.3.4}$$

$$\langle \mathscr{P}^*, \varphi^* \rangle = \langle \mathscr{P}, \varphi \rangle, \quad \textit{where } \varphi = \varphi^* \circ X^*, \tag{1.3.5}$$

satisfy the field equation

$$\operatorname{div} A^* = \mathscr{P}^*, \tag{1.3.6}$$

in the sense of distributions on $\mathscr{X}^*$.

Proof. It follows from (1.3.1) that

$$\int_{\mathscr{X}} A \operatorname{grad} \varphi \, dX + \langle \mathscr{P}, \varphi \rangle = 0, \tag{1.3.7}$$

for any Lipschitz function φ with compact support in $\mathscr{X}$, since one can always construct a sequence $\{\varphi_m\}$ of test functions in $C_0^\infty(\mathscr{X})$, supported in a compact subset of $\mathscr{X}$, such that, as $m \to \infty$, $\varphi_m \to \varphi$, uniformly, and $\operatorname{grad}\varphi_m \to \operatorname{grad}\varphi$, boundedly almost everywhere on $\mathscr{X}$.

Given any test function $\varphi^* \in C_0^\infty(\mathscr{X}^*)$, consider the function $\varphi = \varphi^* \circ X^*$, Lipschitz with compact support in $\mathscr{X}$. Notice that $\operatorname{grad}\varphi = J^\top \operatorname{grad}\varphi^*$. Furthermore, $dX^* = (\det J)dX$. By virtue of these and (1.3.4), (1.3.5), we can write (1.3.7) as

$$\int_{\mathscr{X}^*} A^* \operatorname{grad}\varphi^* dX^* + \langle \mathscr{P}^*, \varphi^* \rangle = 0, \tag{1.3.8}$$

which establishes (1.3.6). The proof is complete.

1.3.2 Remark. In the special, yet common, situation (1.1.3) where the measure $\mathscr{P}$ is induced by a production density field $p \in L^1_{loc}(\mathscr{X})$, (1.3.5) implies that $\mathscr{P}^*$ is also induced by a production density field $p^* \in L^1_{loc}(\mathscr{X}^*)$, given by

$$p^* \circ X^* = (\det J)^{-1} p. \tag{1.3.9}$$

Even though in general the field A is only defined almost everywhere on an open subset of $\mathbb{R}^k$, it turns out that the field equation induces a modicum of regularity, manifesting itself in trace theorems, which will allow us to identify the flux through surfaces of codimension one and thus retrieve the balance law in its original form. We begin with planar surfaces.

1.3.3 Lemma. *Assume $A \in L^\infty(\mathscr{K};\mathbb{M}^{1\times k})$ and $\mathscr{P} \in \mathscr{M}(\mathscr{K})$ satisfy* (1.3.1)*, in the sense of distributions, on a cylindrical domain $\mathscr{K} = \mathscr{B} \times (\alpha,\beta)$, where $\mathscr{B}$ is a ball in $\mathbb{R}^{k-1}$. Let E_k denote the k-base vector in $\mathbb{R}^k$ and set $X = (x,t)$, with x in $\mathscr{B}$ and t in (α,β). Then, after one modifies, if necessary, A on a set of measure zero, the function $a(x,t) = A(x,t)E_k$ acquires the following properties: One-sided limits $a(\cdot,\tau\pm)$ in $L^\infty(\mathscr{B})$ weak* exist, for any $\tau \in (\alpha,\beta)$, and can be determined by*

$$\begin{cases} a(x,\tau-) = \operatorname*{ess\,lim}_{t\uparrow\tau} A(x,t)E_k = \lim_{\varepsilon\downarrow 0} \dfrac{1}{\varepsilon} \displaystyle\int_{\tau-\varepsilon}^{\tau} A(x,t)E_k dt, \\[2ex] a(x,\tau+) = \operatorname*{ess\,lim}_{t\downarrow\tau} A(x,t)E_k = \lim_{\varepsilon\downarrow 0} \dfrac{1}{\varepsilon} \displaystyle\int_{\tau}^{\tau+\varepsilon} A(x,t)E_k dt, \end{cases} \tag{1.3.10}$$

where the limits are taken in $L^\infty(\mathscr{B})$ weak. Furthermore, for any $\tau \in (\alpha,\beta)$ and any Lipschitz continuous function φ with compact support in $\mathscr{K}$,*

$$
(1.3.11) \quad \begin{cases} \displaystyle\int_{\mathscr{B}} a(x,\tau-)\varphi(x,\tau)dx = \int_{\mathscr{B}\times(\alpha,\tau)} A(X)\,\mathrm{grad}\,\varphi(X)\,dX + \langle \mathscr{P},\varphi\rangle_{\mathscr{B}\times(\alpha,\tau)}\,, \\ \displaystyle -\int_{\mathscr{B}} a(x,\tau+)\varphi(x,\tau)dx = \int_{\mathscr{B}\times(\tau,\beta)} A(X)\,\mathrm{grad}\,\varphi(X)\,dX + \langle \mathscr{P},\varphi\rangle_{\mathscr{B}\times(\tau,\beta)}\,. \end{cases}
$$

Thus, $a(\cdot,\tau-) = a(\cdot,\tau+) = a(\cdot,\tau)$, unless τ belongs to the (at most) countable set of points with $|\mathscr{P}|(\mathscr{B}\times\{\tau\}) > 0$. In particular, when $\mathscr{P}$ is absolutely continuous with respect to Lebesgue measure, the function $\tau \mapsto a(\cdot,\tau)$ is continuous on (α,β), in the weak* topology of $L^\infty(\mathscr{B})$.

Proof. Fix ε positive and small. If r is the radius of $\mathscr{B}$, let $\mathscr{B}_\varepsilon$ denote the ball in $\mathbb{R}^{k-1}$ with the same center as $\mathscr{B}$ and radius $r-\varepsilon$. As in the proof of Theorem 1.2.1, we mollify A and $\mathscr{P}$ on $\mathscr{B}_\varepsilon \times (\alpha+\varepsilon, \beta-\varepsilon)$ through (1.2.13). The resulting smooth fields A_ε and p_ε satisfy (1.2.17). We also set $a_\varepsilon(x,t) = A_\varepsilon(x,t)E_k$.

We multiply (1.2.17) by any Lipschitz function φ on $\mathbb{R}^{k-1}$, with compact support in $\mathscr{B}_\varepsilon$, and integrate the resulting equation over $\mathscr{B}_\varepsilon\times(r,s)$, $\alpha+\varepsilon<r<s<\beta-\varepsilon$. After an integration by parts, this yields

$$
(1.3.12) \qquad \int_{\mathscr{B}_\varepsilon} a_\varepsilon(x,s)\varphi(x)dx - \int_{\mathscr{B}_\varepsilon} a_\varepsilon(x,r)\varphi(x)dx
$$

$$
= \int_r^s \int_{\mathscr{B}_\varepsilon} \{A_\varepsilon(x,t)\,\Pi_k\,\mathrm{grad}\,\varphi(x) + p_\varepsilon(x,t)\varphi(x)\}\,dx\,dt,
$$

where Π_k denotes the projection of $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$. It follows that the total variation of the function $t \mapsto \int_{\mathscr{B}_\varepsilon} a_\varepsilon(x,t)\varphi(x)dx$, over the interval $(\alpha+\varepsilon,\beta-\varepsilon)$, is bounded, uniformly in $\varepsilon > 0$. Therefore, starting out from some countable family $\{\varphi_\ell\}$ of test functions, with compact support in $\mathscr{B}$, that is dense in $L^1(\mathscr{B})$, we may invoke Helly's theorem in conjunction with a diagonal argument to extract a sequence $\{\varepsilon_m\}$, with $\varepsilon_m \to 0$ as $m\to\infty$, and identify a countable subset G of (α,β), such that, for any $\ell = 1,2\cdots$, the sequence $\{\int_{\mathscr{B}} a_{\varepsilon_m}(x,t)\varphi_\ell(x)dx\}$ converges, as $m\to\infty$, for every $t\in(\alpha,\beta)\backslash G$, and the limit function has bounded variation over (α,β). The resulting limit functions, for all ℓ, may be collectively represented as $\int_{\mathscr{B}} a(x,t)\varphi_\ell(x)dx$, for some function $t\mapsto a(\cdot,t)$ taking values in $L^\infty(\mathscr{B})$. Clearly, $a(x,t) = A(x,t)E_k$, a.e. in $\mathscr{K}$. Thus, a does not depend on the particular sequence $\{\varepsilon_m\}$ employed for its construction, and (1.3.10) holds for any $\tau\in(\alpha,\beta)$.

Given any $\tau\in(\alpha,\beta)$ and any Lipschitz function φ with compact support in $\mathscr{K}$, we multiply (1.2.17) by φ and integrate the resulting equation over $\mathscr{B}_\varepsilon\times(\alpha+\varepsilon,s)$, where $s\in(\alpha+\varepsilon,\tau)\backslash G$. After an integration by parts, this yields

$$
(1.3.13) \qquad \int_{\mathscr{B}_\varepsilon} a_\varepsilon(x,s)\varphi(x,s)dx = \int_{\mathscr{B}_\varepsilon\times(a+\varepsilon,s)} [A_\varepsilon(X)\,\mathrm{grad}\,\varphi(X) + p_\varepsilon(X)\varphi(X)]\,dX.
$$

In (1.3.13) we first let $\varepsilon \downarrow 0$ and then $s \uparrow \tau$ thus arriving at $(1.3.11)_1$. The proof of $(1.3.11)_2$ is similar.

When $\mathscr{P}$ is absolutely continuous with respect to Lebesgue measure, (1.3.12) implies that the family of functions $t \mapsto \int_{\mathscr{B}_\varepsilon} a_\varepsilon(x,t)\varphi_\ell(x)dx$, parametrized by ε, is actually equicontinuous, and hence $\int_{\mathscr{B}} a(x,t)\varphi_\ell(x)dx$ is continuous on (α,β), for any $\ell = 1,2,\cdots$. Thus, $t \mapsto a(\cdot;t)$ is continuous on (α,β), in $L^\infty(\mathscr{B})$ weak*. This completes the proof.

The k-coordinate direction was singled out, in the above proposition, just for convenience. Analogous continuity properties are clearly enjoyed by AE_α, in the direction of any base vector E_α, and indeed by AN, in the direction of any $N \in \mathbb{S}^{k-1}$. Thus, departing from the field equation (1.2.3), one may retrieve the flux density functions a_N, for planar surfaces, encountered in Theorem 1.2.1. The following proposition demonstrates that even the flux density functions $q_{\mathscr{D}}$, for general proper domains $\mathscr{D}$, may be retrieved by the same procedure.

1.3.4 Theorem. *Assume that $A \in L^\infty\left(\mathscr{X};\mathbb{M}^{1\times k}\right)$ and $\mathscr{P} \in \mathscr{M}(\mathscr{X})$ satisfy* (1.3.1)*, in the sense of distributions, on an open subset $\mathscr{X}$ of $\mathbb{R}^k$. Then, with any proper domain $\mathscr{D}$ in $\mathscr{X}$ is associated a bounded $\mathscr{H}^{k-1}$-measurable function $q_{\mathscr{D}}$ on $\partial\mathscr{D}$ such that*

$$\int_{\partial\mathscr{D}} q_{\mathscr{D}}(X)\varphi(X)d\mathscr{H}^{k-1}(X) = \int_{\mathscr{D}} A(X)\,\mathrm{grad}\,\varphi(X)\,dX + \langle\mathscr{P},\varphi\rangle_{\mathscr{D}}\,, \tag{1.3.14}$$

for any Lipschitz continuous function φ on $\mathbb{R}^k$, with compact support in $\mathscr{X}$.

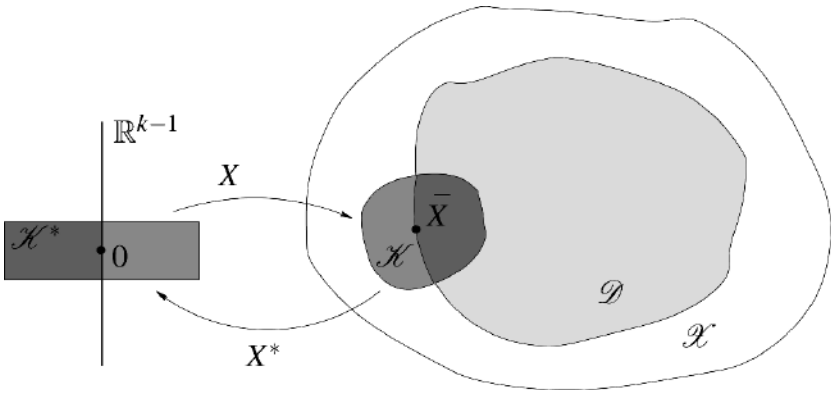

Fig. 1.3.1

Proof. Consider the cylindrical domain $\mathscr{K}^* = \{X^* = (x,t) : x \in \mathscr{B},\ t \in (-1,1)\}$, where $\mathscr{B}$ is the unit ball in $\mathbb{R}^{k-1}$. Fix any proper domain $\mathscr{D}$ in $\mathscr{X}$.

Since $\mathscr{D}$ is a Lipschitz domain, with any point $\bar{X} \in \partial\mathscr{D}$ is associated a bilipschitz homeomorphism X from $\mathscr{K}^*$ to some open subset $\mathscr{K}$ of $\mathscr{X}$ such that $X(0) = \bar{X}$, $X(\mathscr{B}\times(-1,0)) = \mathscr{D}\cap\mathscr{K}$ and $X(\mathscr{B}\times\{0\}) = \partial\mathscr{D}\cap\mathscr{K}$; see Fig. 1.3.1.

Consider the inverse map X^* of X, with Jacobian matrix J, given by (1.3.2) and satisfying (1.3.3). Construct $A^* \in L^\infty(\mathscr{K}^*;\mathbb{M}^{1\times k})$, by (1.3.4), and $\mathscr{P}^* \in \mathscr{M}(\mathscr{K}^*)$, by (1.3.5), which will satisfy (1.3.6) on $\mathscr{K}^*$, in the sense of distributions.

We now apply Lemma 1.3.3 to identify the function $a^*(x,t)$ on $\mathscr{K}^*$, which is equal to $A^*(x,t)E_k$, a.e on $\mathscr{K}^*$, and by (1.3.10) satisfies

$$a^*(x,0-) = \operatorname*{ess\,lim}_{t\to 0} A^*(x,t)E_k = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} A^*(x,t)E_k dt. \tag{1.3.15}$$

We fix any Lipschitz continuous function φ on $\mathbb{R}^k$, with compact support in $\mathscr{K}$, and let $\varphi^* = \varphi \circ X$, for $X^* \in \mathscr{K}^*$. By virtue of (1.3.11),

$$\int_{\mathscr{B}} a^*(x,0-)\varphi^*(x,0)dx = \int_{\mathscr{B}\times(-1,0)} A^*(X^*)\operatorname{grad}\varphi^*(X^*)\,dX^* + \langle \mathscr{P}^*,\varphi^*\rangle_{\mathscr{B}\times(-1,0)}\,. \tag{1.3.16}$$

We employ the homeomorphism X^* in order to transform (1.3.16) into an equation on $\mathscr{X}$. Using that $\operatorname{grad}\varphi = J^\top \operatorname{grad}\varphi^*$ and recalling (1.3.4) and (1.3.5), we may rewrite (1.3.16) as

$$\int_{\partial\mathscr{D}\cap\mathscr{K}} q_{\mathscr{D}}(X)\varphi(X)d\mathscr{H}^{k-1}(X) = \int_{\mathscr{D}} A(X)\operatorname{grad}\varphi(X)\,dX + \langle \mathscr{P},\varphi\rangle_{\mathscr{D}}\,, \tag{1.3.17}$$

where we have set

$$q_{\mathscr{D}} = \frac{dx}{d\mathscr{H}^{k-1}}\,a^*\circ X^* = \frac{\det J}{E_k^\top JN}\,a^*\circ X^*, \tag{1.3.18}$$

with N denoting the outward unit normal to $\mathscr{D}$.

Equation (1.3.17) establishes (1.3.14) albeit only for φ with compact support in $\mathscr{K}$. It should be noted, however, that the right-hand side of (1.3.17) does not depend on the homeomorphism X^* and thus the values of $q_{\mathscr{D}}$ on $\partial\mathscr{D}\cap\mathscr{K}$ are intrinsically defined, independently of the particular construction employed above. Hence, one may easily pass from (1.3.17) to (1.3.14), for arbitrary Lipschitz continuous functions φ with compact support in $\mathscr{X}$, by a straightforward partition of unity argument. This completes the proof.

The reader can find, in the literature cited in Section 1.10, more refined versions of the above proposition, in which A is assumed to be merely locally integrable or even just a measure, as well as alternative methods of proof. For instance, in a more abstract approach, one establishes the existence of $q_{\mathscr{D}}$ by showing that the right-hand side of (1.3.14) can be realized as a bounded linear functional on $L^1(\partial\mathscr{D})$.

1.3.5 Remark. In the applications of the theory, one often needs an explicit construction of $q_{\mathscr{D}}$ from A. This is easily obtained for domains $\mathscr{D}$ with simple geometric structure. To begin with, when $\mathscr{D}$ is the half-space $\{X \in \mathbb{R}^k : X \cdot N < 0\}$ for some $N \in \mathbb{S}^{k-1}$, Lemma 1.3.3 implies

$$q_{\mathscr{D}}(X) = \operatorname*{ess\,lim}_{t \to 0} A(X - tN)N = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} A(x - tN)N dt, \tag{1.3.19}$$

with limits taken in $L^\infty(\mathbb{R}^{k-1})$ weak*. Consider next any $\mathscr{D}$ with the following property. With any $X_0 \in \partial\mathscr{D}$ are associated $r > 0$, $\varepsilon_0 > 0$ and $N \in \mathbb{S}^{k-1}$ such that for all X in the set $\mathscr{C} = \partial\mathscr{D} \bigcap \mathscr{B}_r(X_0)$ and $t \in (0, \varepsilon_0)$, the point $X - tN$ lies in $\mathscr{D}$. Then, applying (1.3.14) with test function φ foliated by the translates of $\mathscr{C}$ in the direction – N, we conclude that (1.3.19) holds for X in $\mathscr{C}$, with the limits now taken in $L^\infty(\mathscr{C})$ weak*.

1.4 Systems of Balance Laws

We consider the situation where n distinct balance laws, with production measures induced by production density fields, act simultaneously in $\mathscr{X}$, and collect their field equations (1.2.3) into the system

$$\operatorname{div} A(X) = P(X), \tag{1.4.1}$$

where now A is a $n \times k$ matrix field and P is a n-column vector field. The divergence operator acts on the row vectors of A, yielding as $\operatorname{div} A$ a n-column vector field.

We assume that the state of the medium is described by a *state vector* field U, taking values in an open subset $\mathscr{O}$ of $\mathbb{R}^n$, which determines the flux density field A and the production density field P at the point $X \in \mathscr{X}$ by *constitutive equations*

$$A(X) = G(U(X), X), \qquad P(X) = \Pi(U(X), X), \tag{1.4.2}$$

where G and Π are given smooth functions defined on $\mathscr{O} \times \mathscr{X}$ and taking values in $\mathbb{M}^{n \times k}$ and $\mathbb{R}^n$, respectively.

Combining (1.4.1) with (1.4.2) yields

$$\operatorname{div} G(U(X), X) = \Pi(U(X), X), \tag{1.4.3}$$

namely a (formally) closed quasilinear first order system of partial differential equations from which the state vector field is to be determined. Any equation of the form (1.4.3) will henceforth be called a *system of balance laws*, if $n \geq 2$, or a *scalar balance law* when $n = 1$. In the special case where there is no production, $\Pi \equiv 0$, (1.4.3) will be called a *system of conservation laws*, if $n \geq 2$, or a *scalar conservation law* when $n = 1$. This terminology is not quite standard: in lieu of "system of balance laws" certain authors favor the term "system of conservation laws with source." When G and Π do not depend explicitly on X, the system of balance laws is called *homogeneous*.

Notice that when coordinates are stretched in the vicinity of some fixed point $\bar{X} \in \mathscr{X}$, i.e., $X = \bar{X} + \varepsilon Y$, then, as $\varepsilon \downarrow 0$, the system of balance laws (1.4.3) reduces to a homogeneous system of conservation laws with respect to the Y variable. This is why local properties of solutions of general systems of balance laws may be investigated, without loss of generality, in the simpler setting of homogeneous systems of conservation laws.

A Lipschitz continuous field U that satisfies (1.4.3) almost everywhere on $\mathscr{X}$ will be called a *classical solution*. A measurable field U that satisfies (1.4.3) in the sense of distributions, i.e., $G(U(X),X)$ and $\Pi(U(X),X)$ are locally integrable and

$$\int_{\mathscr{X}}[G(U(X),X)\operatorname{grad}\varphi(X)+\varphi(X)\Pi(U(X),X)]dX=0, \tag{1.4.4}$$

for any test function $\varphi\in C_0^\infty(\mathscr{X})$, is a *weak solution*. Any weak solution which is Lipschitz continuous is necessarily a classical solution.

1.4.1 Notation. For $\alpha=1,\cdots,k$, $G_\alpha(U,X)$ will denote the α-th column vector of the matrix $G(U,X)$.

1.4.2 Notation. Henceforth, D will denote the differential with respect to the U variable. When used in conjunction with matrix notations, D shall be regarded as a row operation: $\mathrm{D}=\left[\partial/\partial U^1,\cdots,\partial/\partial U^n\right]$.

1.5 Companion Balance Laws

Consider a system (1.4.3) of balance laws on an open subset $\mathscr{X}$ of $\mathbb{R}^k$, resulting from combining the field equation (1.4.1) with constitutive relations (1.4.2). A smooth function Q, defined on $\mathscr{O}\times\mathscr{X}$ and taking values in $\mathbb{M}^{1\times k}$, is called a *companion* of G if there is a smooth function B, defined on $\mathscr{O}\times\mathscr{X}$ and taking values in $\mathbb{R}^n$, such that, for all $U\in\mathscr{O}$ and $X\in\mathscr{X}$,

$$\mathrm{D}Q_\alpha(U,X)=B(U,X)^\top\mathrm{D}G_\alpha(U,X),\qquad \alpha=1,\cdots,k. \tag{1.5.1}$$

The relevance of (1.5.1) stems from the observation that any classical solution U of the system of balance laws (1.4.3) is automatically also a (classical) solution of the *companion balance law*

$$\operatorname{div}Q(U(X),X)=h(U(X),X), \tag{1.5.2}$$

with

$$h(U,X)=B(U,X)^\top\Pi(U,X)+\nabla\cdot Q(U,X)-B(U,X)^\top\nabla\cdot G(U,X). \tag{1.5.3}$$

In (1.5.3) $\nabla\cdot$ denotes divergence with respect to X, holding U fixed – as opposed to div, which treats U as a function of X.

One determines the companion balance laws (1.5.2) of a given system of balance laws (1.4.3) by identifying the integrating factors B that render the right-hand side of (1.5.1) a gradient of a function of U. The relevant integrability condition is

$$\mathrm{D}B(U,X)^\top \mathrm{D}G_\alpha(U,X) = \mathrm{D}G_\alpha(U,X)^\top \mathrm{D}B(U,X), \qquad \alpha = 1,\cdots,k, \tag{1.5.4}$$

for all $U \in \mathscr{O}$ and $X \in \mathscr{X}$. Clearly, one can satisfy (1.5.4) by employing any B that does not depend on U; in that case, however, the resulting companion balance law (1.5.2) is just a trivial linear combination of the equations of the original system (1.4.3). For nontrivial B, which vary with U, (1.5.4) imposes $\frac{1}{2}n(n-1)k$ conditions on the n unknown components of B. Thus, when $n = 1$ and k is arbitrary one may use any (scalar-valued) function B. When $n = 2$ and $k = 2$, (1.5.4) reduces to a system of two equations in two unknowns from which a family of B may presumably be determined. In all other cases, however, (1.5.4) is formally overdetermined and the existence of nontrivial companion balance laws should not be generally expected. Nevertheless, as we shall see in Chapter III, the systems of balance laws of continuum physics are endowed with natural companion balance laws.

The system of balance laws (1.4.3) is called *symmetric* when the $n \times n$ matrices $\mathrm{D}G_\alpha(U,X), \alpha = 1,\cdots,k$, are symmetric, for any $U \in \mathscr{O}$ and $X \in \mathscr{X}$; say $\mathscr{O}$ is simply connected and

$$G(U,X)^\top = \mathrm{D}\Gamma(U,X)^\top, \tag{1.5.5}$$

for some smooth function Γ, defined on $\mathscr{O} \times \mathscr{X}$ and taking values in $\mathbb{M}^{1\times k}$. In that case one may satisfy (1.5.4) by taking $B(U,X) \equiv U$, which induces the companion

$$Q(U,X) = U^\top G(U,X) - \Gamma(U,X). \tag{1.5.6}$$

Conversely, if (1.5.1) holds for some B with the property that, for every fixed $X \in \mathscr{X}$, $B(\cdot,X)$ maps diffeomorphically $\mathscr{O}$ to some open subset $\mathscr{O}^*$ of $\mathbb{R}^n$, then the change $U^* = B(U,X)$ of state vector reduces (1.4.3) to the equivalent system of balance laws

$$\operatorname{div} G^*(U^*(X),X) = \Pi^*(U^*(X),X), \tag{1.5.7}$$

with

$$G^*(U^*,X) = G(B^{-1}(U^*,X),X), \qquad \Pi^*(U^*,X) = \Pi(B^{-1}(U^*,X),X), \tag{1.5.8}$$

which is symmetric. Indeed, upon setting

$$Q^*(U^*,X) = Q(B^{-1}(U^*,X),X), \tag{1.5.9}$$

$$\Gamma^*(U^*,X) = U^{*\top} G^*(U^*,X) - Q^*(U^*,X), \tag{1.5.10}$$

one easily obtains from (1.5.1) that

$$G^*(U^*,X)^\top = \mathrm{D}\Gamma^*(U^*,X)^\top. \tag{1.5.11}$$

We have thus demonstrated that a system of balance laws is endowed with nontrivial companion balance laws if and only if it is *symmetrizable*.

We shall see that the presence of companion balance laws has major implications for the theory of systems of balance laws arising in physics. Quite often, in order to simplify the analysis, it becomes necessary to make simplifying physical assumptions that truncate the system of balance laws while simultaneously trimming proportionately the size of the state vector. Such truncations cannot be performed arbitrarily without destroying the mathematical structure of the system, which goes hand in hand with its relevance to physics. For a canonical truncation, it is necessary to operate on (or at least think in terms of) the symmetric form (1.5.7) of the system, and adhere to the rule that dropping the i-th balance law should be paired with "freezing" (i.e., assigning fixed values to) the i-th component U^{*i} of the special state vector U^*. Then, the resulting truncated system will still be symmetric and will inherit the companion

$$\hat{Q} = Q^* - \sum U^{*i} G^{*i}, \tag{1.5.12}$$

where the summation runs over all i for which the i-balance law has been eliminated and U^{*i} has been frozen. G^{*i} denotes the i-th row vector of G^*.

Despite (1.5.1), and in contrast to the behavior of classical solutions, weak solutions of (1.4.3) need not satisfy (1.5.2). Nevertheless, one of the tenets of the theory of systems of balance laws is that *admissible weak solutions* should at least satisfy the inequality

$$\operatorname{div} Q(U(X),X) \leq h(U(X),X), \tag{1.5.13}$$

in the sense of distributions, for a designated family of companions. Relating this postulate to the Second Law of thermodynamics and investigating its implications for stability of weak solutions are among the principal objectives of this book.

Notice that an inequality (1.5.13), holding in the sense of distributions, can always be turned into an equality by subtracting from the right-hand side some nonnegative measure $\mathscr{M}$,

$$\operatorname{div} Q(U(X),X) = h(U(X),X) - \mathscr{M}, \tag{1.5.14}$$

and may thus be realized, by virtue of Theorem 1.3.4, as the field equation of a balance law.

1.6 Weak and Shock Fronts

The regularity of solutions of a system of balance laws will depend on the nature of the constitutive functions. The focus will be on solutions with "fronts", that is singularities assembled on manifolds of codimension one. To get acquainted with this sort of solutions, we consider here two kinds of fronts in a particularly simple setting.

In what follows, $\mathscr{F}$ will be a smooth $(k-1)$-dimensional manifold, embedded in the open subset $\mathscr{X}$ of $\mathbb{R}^k$, with orientation induced by the unit normal field N.

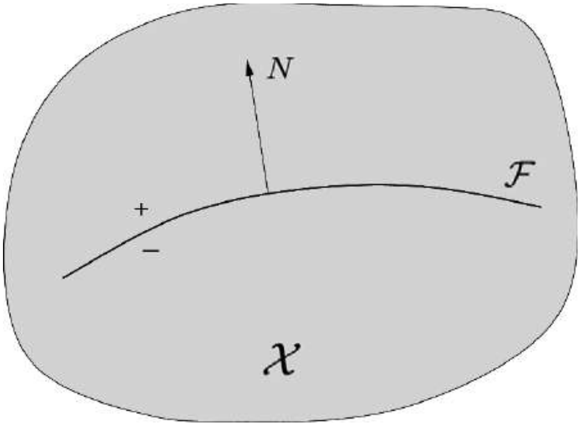

Fig. 1.6.1

U will be a (generally weak) solution of the system of balance laws (1.4.3) on $\mathscr{X}$ which is continuously differentiable on $\mathscr{X} \setminus \overline{\mathscr{F}}$, but is allowed to be singular on $\mathscr{F}$. In particular, (1.4.3) holds for any $X \in \mathscr{X} \setminus \overline{\mathscr{F}}$. See Fig. 1.6.1.

First we consider the case where $\mathscr{F}$ is a *weak front*, that is, U is Lipschitz continuous on $\mathscr{X}$ and as one approaches $\mathscr{F}$ from either side the gradient of U attains distinct limits $\mathrm{grad}^- U$, $\mathrm{grad}^+ U$. Thus $[\![\mathrm{grad}\, U]\!] = \mathrm{grad}^+ U - \mathrm{grad}^- U$ is the jump experienced by $\mathrm{grad}\, U$ across $\mathscr{F}$. Since U is continuous, tangential derivatives of U cannot jump across $\mathscr{F}$ and hence $[\![\mathrm{grad}\, U]\!] = [\![\partial U/\partial N]\!] N^\top$, where $[\![\partial U/\partial N]\!]$ denotes the jump of the normal derivative $\partial U/\partial N$ across $\mathscr{F}$. Therefore, taking the jump of (1.4.3) across $\mathscr{F}$ at any point $X \in \mathscr{F}$ yields the following condition on $[\![\partial U/\partial N]\!]$:

(1.6.1) $$\mathrm{D}[G(U(X),X)N] \left[\!\!\left[\frac{\partial U}{\partial N} \right]\!\!\right] = 0.$$

Next we assume $\mathscr{F}$ is a *shock front*, that is, as one approaches $\mathscr{F}$ from either side, U attains distinct limits U_-, U_+ and thus experiences a jump $[\![U]\!] = U_+ - U_-$ across $\mathscr{F}$. Both U_- and U_+ are continuous functions on $\mathscr{F}$. Since U is a (weak) solution of (1.4.3), we may write (1.4.4) for any $\varphi \in C_0^\infty(\mathscr{X})$. In (1.4.4) integration over $\mathscr{X}$ may be replaced with integration over $\mathscr{X} \setminus \overline{\mathscr{F}}$. Since U is C^1 on $\mathscr{X} \setminus \overline{\mathscr{F}}$, we may integrate by parts in (1.4.4). Using that φ has compact support in $\mathscr{X}$ and that (1.4.3) holds for any $X \in \mathscr{X} \setminus \overline{\mathscr{F}}$, we get

(1.6.2) $$\int_{\mathscr{F}} \varphi(X)[G(U_+,X) - G(U_-,X)]N \, d\mathscr{H}^{k-1}(X) = 0,$$

whence we deduce that the following *jump condition* must be satisfied at every point X of the shock front $\mathscr{F}$:

(1.6.3) $$[G(U_+,X) - G(U_-,X)]N = 0.$$

Notice that (1.6.3) may be rewritten in the form

(1.6.4)
$$\left\{ \int_0^1 \mathrm{D}[G(\tau U_+ + (1-\tau)U_-, X)N]d\tau \right\} [\![U]\!] = 0.$$

Comparing (1.6.4) with (1.6.1) we conclude that weak fronts may be regarded as shock fronts with "infinitesimal" strength: $|[\![U]\!]|$ vanishingly small.

With each $U \in \mathscr{O}$ and $X \in \mathscr{X}$ we associate the variety

(1.6.5)
$$\mathscr{V}(U,X) = \left\{ (N,V) \in \mathbb{S}^{k-1} \times \mathbb{R}^n : \mathrm{D}[G(U,X)N]V = 0 \right\}.$$

The number of weak fronts and shock fronts of small strength that may be sustained by solutions of (1.4.3) will depend on the size of $\mathscr{V}$. In the extreme case where, for all (U,X), the projection of $\mathscr{V}(U,X)$ onto $\mathbb{R}^n$ contains only the vector $V = 0$, (1.4.3) is called *elliptic*. Thus a system of balance laws is elliptic if and only if it cannot sustain any weak fronts or shock fronts of small strength. The opposite extreme to ellipticity, where $\mathscr{V}$ attains the maximal possible size, is hyperbolicity, which will be introduced in Chapter III.

1.7 Survey of the Theory of *BV* Functions

The space of *BV* functions provides a natural setting for solutions of systems of balance laws with shock fronts. Indeed, a prominent feature of these functions is that their points of discontinuity assemble on manifolds of codimension one. Comprehensive treatment of the theory of *BV* functions can be found in the references cited in Section 1.10, so only properties relevant to our purposes will be listed here, without proofs.

1.7.1 Definition. A scalar function v is of *locally bounded variation* on an open subset $\mathscr{X}$ of $\mathbb{R}^k$ if $v \in L^1_{loc}(\mathscr{X})$ and $\mathrm{grad}\, v$ is a ($\mathbb{R}^k$-valued) Radon measure $\mathscr{M}$ on $\mathscr{X}$, i.e.,

(1.7.1)
$$-\int_{\mathscr{X}} v \,\mathrm{div}\, \Psi(X) dX = \int_{\mathscr{X}} \Psi(X) d\mathscr{M}(X),$$

for any test function $\Psi \in C_0^\infty(\mathscr{X}; \mathbb{M}^{1\times k})$. When $v \in L^1(\mathscr{X})$ and $\mathscr{M}$ is finite, v is a function of *bounded variation* on $\mathscr{X}$, with *total variation*

(1.7.2)
$$TV_{\mathscr{X}} v = |\mathscr{M}|(\mathscr{X}) = \sup_{|\Psi(X)|=1} \int_{\mathscr{X}} v(X) \,\mathrm{div}\, \Psi(X) \, dX.$$

The set of functions of bounded variation and locally bounded variation on $\mathscr{X}$ will be denoted by $BV(\mathscr{X})$ and $BV_{\mathrm{loc}}(\mathscr{X})$, respectively.

Clearly, the Sobolev space $W^{1,1}(\mathscr{X})$, of $L^1(\mathscr{X})$ functions with derivatives in $L^1(\mathscr{X})$, is contained in $BV(\mathscr{X})$; and $W^{1,1}_{\mathrm{loc}}(\mathscr{X})$ is contained in $BV_{\mathrm{loc}}(\mathscr{X})$.

The following proposition provides a useful criterion for testing whether a given function has bounded variation:

1.7.2 Theorem. *Let $\{E_\alpha, \alpha = 1, \cdots, k\}$ denote the standard orthonormal basis of $\mathbb{R}^k$. If $v \in BV_{\mathrm{loc}}(\mathscr{X})$, then*

$$\limsup_{h \downarrow 0} \frac{1}{h} \int_{\mathscr{Y}} |v(X + hE_\alpha) - v(X)| dX = |\mathscr{M}_\alpha|(\mathscr{Y}), \qquad \alpha = 1, \cdots, k, \tag{1.7.3}$$

for any open bounded set $\mathscr{Y}$ with $\overline{\mathscr{Y}} \subset \mathscr{X}$. Conversely, if $v \in L^1_{\mathrm{loc}}(\mathscr{X})$ and the left-hand side of (1.7.3) *is finite for every $\mathscr{Y}$ as above, then $v \in BV_{\mathrm{loc}}(\mathscr{X})$.*

As a corollary, the above proposition yields the following result on compactness:

1.7.3 Theorem. *Any sequence $\{v_\ell\}$ in $BV_{loc}(\mathscr{X})$, such that $\|v_\ell\|_{L^1(\mathscr{Y})}$ and $TV_{\mathscr{Y}} v_\ell$ are uniformly bounded on every open bounded $\mathscr{Y} \subset \mathscr{X}$, contains a subsequence which converges in $L^1_{loc}(\mathscr{X})$, as well as almost everywhere on $\mathscr{X}$, to some function v in $BV_{loc}(\mathscr{X})$, with $TV_{\mathscr{Y}} v \leq \liminf_{\ell \to \infty} TV_{\mathscr{Y}} v_\ell$.*

Functions of bounded variation are endowed with fine geometric structure, as described in

1.7.4 Theorem. *The domain $\mathscr{X}$ of any $v \in BV_{\mathrm{loc}}(\mathscr{X})$ is the union of three, pairwise disjoint, subsets $\mathscr{C}$, $\mathscr{J}$, and $\mathscr{I}$ with the following properties*:

(a) *$\mathscr{C}$ is the set of points of approximate continuity of v, i.e., with each $\bar{X} \in \mathscr{C}$ is associated $v_0 \in \mathbb{R}$ such that*

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_{\mathscr{B}_r(\bar{X})} |v(X) - v_0| dX = 0. \tag{1.7.4}$$

(b) *$\mathscr{J}$ is the set of points of approximate jump discontinuity of v, i.e., with each $\bar{X} \in \mathscr{J}$ are associated N in $\mathbb{S}^{k-1}$ and distinct v_-, v_+ in $\mathbb{R}$ such that*

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_{\mathscr{B}_r^{\pm}(\bar{X})} |v(X) - v_\pm| dX = 0, \tag{1.7.5}$$

where $\mathscr{B}_r^{\pm}(\bar{X})$ denote the semiballs $\mathscr{B}_r(\bar{X}) \cap \{X : (X - \bar{X}) \cdot N \gtrless 0\}$. Moreover, $\mathscr{J}$ is countably rectifiable, i.e., it is essentially covered by the countable union of C^1 $(k-1)$-dimensional manifolds $\{\mathscr{F}_i\}$ embedded in $\mathbb{R}^k$: $\mathscr{H}^{k-1}(\mathscr{J} \backslash \bigcup \mathscr{F}_i) = 0$. Furthermore, when $\bar{X} \in \mathscr{J} \cap \mathscr{F}_i$ then N is normal on $\mathscr{F}_i$ at $\bar{X}$.
(c) $\mathscr{I}$ is the set of irregular points of v; its $(k-1)$-dimensional Hausdorff measure is zero: $\mathscr{H}^{k-1}(\mathscr{I}) = 0$.

Up to this point, the identity of a *BV* function is unaffected by modifying its values on any set of (k-dimensional Lebesgue) measure zero, i.e., $BV_{\mathrm{loc}}(\mathscr{X})$ is actually a space of equivalence classes of functions, specified only up to a set of measure zero. However, when dealing with the finer behavior of these functions, it is expedient to designate a canonical representative of each equivalence class, with values specified up to a set of $(k-1)$-dimensional Hausdorff measure zero. This will be effected in the following way.

Suppose g is a continuous function on $\mathbb{R}$ and let $v \in BV_{\mathrm{loc}}(\mathscr{X})$. With reference to the notation of Theorem 1.7.4, the *normalized composition* $\widetilde{g \circ v}$ of g and v is defined by

$$(1.7.6) \qquad \widetilde{g \circ v}(X) = \begin{cases} g(v_0), & \text{if } X \in \mathscr{C} \\ \displaystyle\int_0^1 g(\tau v_- + (1-\tau)v_+)\,d\tau, & \text{if } X \in \mathscr{J} \end{cases}$$

and arbitrarily on the set $\mathscr{I}$ of irregular points, whose $(k-1)$-dimensional Hausdorff measure is zero. In particular, one may normalize v itself:

$$(1.7.7) \qquad \tilde{v}(X) = \begin{cases} v_0\,, & \text{if } X \in \mathscr{C} \\ \frac{1}{2}(v_- + v_+), & \text{if } X \in \mathscr{J}\,. \end{cases}$$

Thus every point of $\mathscr{C}$ becomes a Lebesgue point.

The appropriateness of the above normalization is indicated by the following generalization of the classical chain rule:

1.7.5 Theorem. *Assume g is continuously differentiable on $\mathbb{R}$, with derivative $\mathrm{D}g$, and let $v \in BV_{\mathrm{loc}}(\mathscr{X}) \cap L^\infty(\mathscr{X})$. Then $g \circ v \in BV_{loc}(\mathscr{X}) \cap L^\infty(\mathscr{X})$. The normalized function $\widetilde{\mathrm{D}g \circ v}$ is locally integrable with respect to the measure $\mathscr{M} = \mathrm{grad}\, v$ and*

$$(1.7.8) \qquad \mathrm{grad}\,(g \circ v) = (\widetilde{\mathrm{D}g \circ v})\,\mathrm{grad}\, v$$

in the sense

$$(1.7.9) \qquad -\int_{\mathscr{X}} g(v(X))\,\mathrm{div}\,\Psi(X)\,dX = \int_{\mathscr{X}} (\widetilde{\mathrm{D}g \circ v})(X)\Psi(X)\,d\mathscr{M}(X),$$

for any test function $\Psi \in C_0^\infty(\mathscr{X}; \mathbb{M}^{1\times k})$.

Next we review certain important geometric properties of a class of sets in $\mathbb{R}^k$ that are intimately related to the theory of BV functions.

1.7.6 Definition. A subset $\mathscr{D}$ of $\mathbb{R}^k$ has (*locally*) *finite perimeter* when its indicator function $\chi_{\mathscr{D}}$ has (locally) bounded variation on $\mathbb{R}^k$.

Let us apply Theorem 1.7.4 to the indicator function $\chi_{\mathscr{D}}$ of a set $\mathscr{D}$ with locally finite perimeter. Clearly, the set $\mathscr{C}$ of points of approximate continuity of $\chi_{\mathscr{D}}$ is the union of the sets of density points of $\mathscr{D}$ and $\mathbb{R}^k \backslash \mathscr{D}$. The complement of $\mathscr{C}$, i.e., the set of X in $\mathbb{R}^k$ that are not points of density of either $\mathscr{D}$ or $\mathbb{R}^k \backslash \mathscr{D}$, constitutes the *measure theoretic boundary* $\partial\mathscr{D}$ of $\mathscr{D}$. It can be shown that $\mathscr{D}$ has finite perimeter if and only if $\mathscr{H}^{k-1}(\partial\mathscr{D}) < \infty$, and its perimeter may be measured by $TV_{\mathbb{R}^k}\chi_{\mathscr{D}}$ or by $\mathscr{H}^{k-1}(\partial\mathscr{D})$. The set of points of approximate jump discontinuity of $\chi_{\mathscr{D}}$ is called the *reduced boundary* of $\mathscr{D}$ and is denoted by $\partial^*\mathscr{D}$. By Theorem 1.7.4, $\partial^*\mathscr{D} \subset \partial\mathscr{D}$, $\mathscr{H}^{k-1}(\partial\mathscr{D}\backslash\partial^*\mathscr{D}) = 0$ and $\partial^*\mathscr{D}$ is covered by the countable union of C^1 $(k-1)$-dimensional manifolds. Moreover, the vector $N \in \mathbb{S}^{k-1}$ associated with each point X of $\partial^*\mathscr{D}$ may naturally be interpreted as the *measure theoretic outward normal* to $\mathscr{D}$ at X. Sets with Lipschitz boundary have finite perimeter. In fact, one can reformulate the entire theory of balance laws by considering as proper domains sets that are not necessarily Lipschitz, as postulated in Section 1.1, but merely have finite perimeter.

1.7.7 Definition. Assume $\mathscr{D}$ has finite perimeter and let $v \in BV_{\mathrm{loc}}(\mathbb{R}^k)$. v has *inward* and *outward traces* v_- and v_+ at the point $\bar{X}$ of the reduced boundary $\partial^*\mathscr{D}$ of $\mathscr{D}$, where the outward normal is N, if

$$\lim_{r\downarrow 0} \frac{1}{r^k} \int\limits_{\mathscr{B}_r^{\pm}} |v(X) - v_{\pm}| dX = 0. \tag{1.7.10}$$

It can be shown that the traces $v_{\pm}$ are defined for almost all (with respect to $\mathscr{H}^{k-1}$) points of $\partial^*\mathscr{D}$ and are locally integrable on $\partial^*\mathscr{D}$. Furthermore, the following version of the Gauss-Green theorem holds:

1.7.8 Theorem. *Assume $v \in BV(\mathbb{R}^k)$ so $\mathscr{M} = \operatorname{grad} v$ is a finite measure. Consider any bounded set $\mathscr{D}$ of finite perimeter, with set of density points $\mathscr{D}^*$ and reduced boundary $\partial^*\mathscr{D}$. Then*

$$\mathscr{M}(\mathscr{D}^*) = \int_{\partial^*\mathscr{D}} v_+ N d\mathscr{H}^{k-1}. \tag{1.7.11}$$

Furthermore, for any Borel subset $\mathscr{F}$ of $\partial\mathscr{D}$,

$$\mathscr{M}(\mathscr{F}) = \int_{\mathscr{F}} (v_- - v_+) N d\mathscr{H}^{k-1}. \tag{1.7.12}$$

In particular, the set $\mathscr{J}$ of points of approximate jump discontinuity of any v in $BV_{\mathrm{loc}}(\mathbb{R}^k)$ may be covered by the countable union of oriented surfaces and so (1.7.12) will hold for any measurable subset $\mathscr{F}$ of $\mathscr{J}$.

For $v \in BV(\mathscr{X})$, the measure $\mathscr{M} = \operatorname{grad} v$ may be decomposed into the sum of three mutually singular measures: its *continuous part,* which is absolutely continuous with respect to k-dimensional Lebesgue measure; its *jump part,* which is concentrated on the set $\mathscr{J}$ of points of approximate jump discontinuity of v; and its *Cantor part.* In particular, the Cantor part of the measure of any Borel subset of $\mathscr{X}$ with finite $(k-1)$-dimensional Hausdorff measure vanishes.

1.7.9 Definition. $v \in BV(\mathscr{X})$ is a *special function of bounded variation,* namely $v \in SBV(\mathscr{X})$, if the Cantor part of the measure $\operatorname{grad} v$ vanishes.

It turns out that $SBV(\mathscr{X})$ is a proper subspace of $BV(\mathscr{X})$ and it properly contains $W^{1,1}(\mathscr{X})$.

For $k = 1$, the theory of BV functions is intimately related with the classical theory of functions of bounded variation. Assume v is a BV function on a (bounded or unbounded) interval $(a,b) \subset (-\infty,\infty)$. Let $\tilde{v}$ be the normalized form of v. Then

$$TV_{(a,b)}v = \sup \sum_{j=1}^{\ell-1} |\tilde{v}(x_{j+1}) - \tilde{v}(x_j)|, \tag{1.7.13}$$

where the supremum is taken over all (finite) meshes $a < x_1 < x_2 < \cdots < x_\ell < b$. Furthermore, (classical) one-sided limits $\tilde{v}(x\pm)$ exist at every $x \in (a,b)$ and are both equal to $\tilde{v}(x)$, except possibly on a countable set of points. When $k = 1$, the compactness Theorem 1.7.3 reduces to the classical *Helly theorem*.

Any $v \in SBV(a,b)$ is the sum of an absolutely continuous function and a saltus function. Accordingly, the measure $\operatorname{grad} v$ is the sum of the pointwise derivative v' of v, which exists almost everywhere on (a,b), and the (at most) countable sum of weighted Dirac masses, located at the points of jump discontinuity of v and weighted by the jump.

A vector-valued function U is of (locally) bounded variation on $\mathscr{X}$ when each one of its components has (locally) bounded variation on $\mathscr{X}$; and its total variation $TV_{\mathscr{X}}U$ is the sum of the total variations of its components. All of the discussions, above, for scalar-valued functions, and in particular the assertions of Theorems 1.7.2, 1.7.3, 1.7.4, 1.7.5 and 1.7.8, generalize immediately to (and will be used below for) vector-valued functions of bounded variation.

1.8 *BV* Solutions of Systems of Balance Laws

We consider here weak solutions $U \in L^\infty(\mathscr{X})$ of the system (1.4.3) of balance laws, which are in $BV_{\text{loc}}(\mathscr{X})$. In that case, by virtue of Theorem 1.7.5, the function $G \circ U$ is also in $BV_{\text{loc}}(\mathscr{X}) \cap L^\infty(\mathscr{X})$ and (1.4.3) is satisfied as an equality of measures. The first task is to examine the local form of (1.4.3), in the light of Theorems 1.7.4, 1.7.5, and 1.7.8.

1.8.1 Theorem. *A function $U \in BV_{\text{loc}}(\mathscr{X}) \cap L^\infty(\mathscr{X})$ is a weak solution of the system* (1.4.3) *of balance laws if and only if* (a) *the measure equality*

$$[\mathrm{D}G(\tilde{U}(X),X), \operatorname{grad} U(X)] + \nabla \cdot G(\tilde{U}(X),X) = \Pi(\tilde{U}(X),X) \tag{1.8.1}$$

holds on the set $\mathscr{C}$ of points of approximate continuity of U; and (b) *the jump condition*

$$[G(U_+,X) - G(U_-,X)]N = 0 \tag{1.8.2}$$

is satisfied for almost all (with respect to $\mathscr{H}^{k-1}$) X on the set $\mathscr{J}$ of points of approximate jump discontinuity of U, with normal vector N and one-sided limits U_-, U_+.

Proof. In (1.8.1) and in (1.8.6), (1.8.7), below, the symbol $\nabla\cdot$ denotes divergence with respect to X, holding U fixed – as opposed to div which treats U as a function of X. Let $\mathscr{M}$ denote the measure defined by the left-hand side of (1.4.3). On $\mathscr{C}$, $\mathscr{M}$ reduces to the measure on the left-hand side of (1.8.1), by virtue of Theorem 1.7.5. Recalling the Definition 1.7.7 of trace and the characterization of one-sided limits in Theorem 1.7.4, we deduce $(G\circ U)_\pm = G\circ U_\pm$ at every point of $\mathscr{J}$. Thus, if $\mathscr{F}$ is any Borel subset of $\mathscr{J}$, then on account of the remark following the proof of Theorem 1.7.8,

$$\mathscr{M}(\mathscr{F}) = \int_{\mathscr{F}} [G(U_-,X) - G(U_+,X)]N d\mathscr{H}^{k-1}. \tag{1.8.3}$$

Therefore, $\mathscr{M} = \Pi$ in the sense of measures if and only if (1.8.1) and (1.8.2) hold. This completes the proof.

Consequently, the set of points of approximate jump discontinuity of a BV solution is the countable union of *shock fronts*.

As we saw in Section 1.5, when G has a companion Q, the companion balance law (1.5.2) is automatically satisfied by any classical solution of (1.4.3). The following proposition describes the situation in the context of BV weak solutions.

1.8.2 Theorem. *Assume the system of balance laws* (1.4.3) *is endowed with a companion balance law* (1.5.2). *Let $U \in BV_{\mathrm{loc}}(\mathscr{X}) \cap L^\infty(\mathscr{X})$ be a weak solution of* (1.4.3). *Then the measure*

$$\mathscr{N} = \operatorname{div} Q(U(X),X) - h(U(X),X) \tag{1.8.4}$$

is concentrated on the set $\mathscr{J}$ of points of approximate jump discontinuity of U and the inequality (1.5.13) *will be satisfied in the sense of measures if and only if*

$$[Q(U_+,X) - Q(U_-,X)]N \geq 0 \tag{1.8.5}$$

holds for almost all (with respect to $\mathscr{H}^{k-1}$) $X \in \mathscr{J}$.

Proof. By virtue of Theorem 1.7.5, we may write (1.4.3) and (1.8.4) as

$$[\widetilde{\mathrm{D}G\circ U}, \operatorname{grad} U] + \nabla\cdot G - \Pi = 0, \tag{1.8.6}$$

$$\mathscr{N} = [\widetilde{\mathrm{D}Q\circ U}, \operatorname{grad} U] + \nabla\cdot Q - h. \tag{1.8.7}$$

On account of (1.7.6), if X is in the set $\mathscr{C}$ of points of approximate continuity of U,

$$\widetilde{\mathrm{D}G\circ U}(X) = \mathrm{D}G(\tilde{U}(X),X), \qquad \widetilde{\mathrm{D}Q\circ U}(X) = \mathrm{D}Q(\tilde{U}(X),X). \tag{1.8.8}$$

Combining (1.8.6), (1.8.7), (1.8.8) and using (1.5.1), (1.5.3), we deduce that $\mathscr{N}$ vanishes on $\mathscr{C}$.

From the Definition 1.7.7 of trace and the characterization of one-sided limits in Theorem 1.7.4, we infer $(Q \circ U)_{\pm} = Q \circ U_{\pm}$. If $\mathscr{F}$ is a bounded Borel subset of $\mathscr{J}$, we apply (1.7.12), keeping in mind the remark following the proof of Theorem 1.7.8. This yields

$$\mathscr{N}(\mathscr{F}) = \int_{\mathscr{F}} [Q(U_-, X) - Q(U_+, X)] N d\mathscr{H}^{k-1}. \tag{1.8.9}$$

Therefore, $\mathscr{N} \leq 0$ if and only if (1.8.5) holds. This completes the proof.

1.9 Rapid Oscillations and the Stabilizing Effect of Companion Balance Laws

Consider a homogeneous system of conservation laws

$$\operatorname{div} G(U(X)) = 0 \tag{1.9.1}$$

and assume that

$$[G(W) - G(V)] N = 0 \tag{1.9.2}$$

holds for some states V, W in $\mathscr{O}$ and $N \in \mathbb{S}^{k-1}$. Then one may construct highly oscillatory weak solutions of (1.9.1) on $\mathbb{R}^k$ by the following procedure: start with any finite family of parallel $(k-1)$-dimensional hyperplanes, all of them orthogonal to N, and define a function U on $\mathbb{R}^k$ which is constant between adjacent hyperplanes, taking the values V and W in alternating order. It is clear that U is a weak solution of (1.9.1), by virtue of (1.9.2) and Theorem 1.8.1.

One may thus construct a sequence of solutions that converges in L^∞ weak* to some U of the form $U(X) = \rho(X \cdot N)V + [1 - \rho(X \cdot N)]W$, where ρ is any measurable function from $\mathbb{R}$ to $[0,1]$. It is clear that, in general, such U will not be solutions of (1.9.1), unless $G(\cdot)N$ happens to be affine along the straight line segment in $\mathbb{R}^n$ that connects V to W. This type of instability distinguishes systems that may support shock fronts from elliptic systems that cannot.

Assume now G is equipped with a companion Q and $[Q(W) - Q(V)]N \neq 0$. Notice that imposing the admissibility condition $\operatorname{div} Q(U) \leq 0$ would disqualify the oscillating solutions constructed above, because, by virtue of Theorem 1.8.2, it would not allow jumps both from V to W and from W to V, in the direction N. Consequently, inequalities (1.5.13) seem to play a stabilizing role. To what extent this stabilizing is effective will be a major issue for discussion in the book.

1.10 Notes

The principles of the theory of balance laws were conceived in the process of laying down the foundations of elasticity, in the 1820's. Theorem 1.2.1 has a long and celebrated history. The crucial discovery that the flux density is necessarily a linear function of the outward normal was made by Cauchy [1,2]. The argument that the flux density through a surface may depend on the surface solely through its outward normal is attributed to Hamel and to Noll [2]. For recent developments of these ideas in the context of continuum physics, see Gurtin and Martins [1], Degiovanni, Marzocchi and Musesti [1], Marzocchi and Musesti [1,2] and Šilhavý [2,3,4]. The proof here borrows ideas from Ziemer [1]. With regard to the issue of retrieving the balance law from its field equation, which is addressed by Theorem 1.3.4, Chen and Frid [1,5,6] have developed a comprehensive theory of divergence measure fields which yields a more explicit construction of the trace than by the method presented here, under the additional mild technical assumption that the surface may be foliated (akin to Remark 1.3.5). For further developments of that approach, see Chen [9,10], Chen and Frid [8,9], Frid [7], Chen and Torres [1,2], and Chen, Torres and Ziemer [1,2]. An alternative, less explicit, functional analytic approach is found in Anzellotti [1]. An important question, currently under investigation, is whether the conclusion of Theorem 1.3.1 still holds in the more general situation where the change of coordinates belongs to some Sobolev space $W^{1,p}$ or even to the space BV.

The observation that systems of balance laws are endowed with nontrivial companions if and only if they are symmetrizable is due to Godunov [1,2,3], and to Friedrichs and Lax [1]; see also Boillat [1] and Ruggeri and Strumia [1]. For a discussion of proper truncations of systems of balance laws arising in physics, see Boillat and Ruggeri [1].

As already noted in the historical introduction, in one space dimension, weak fronts are first encountered in the acoustic research of Euler while shock fronts were introduced by Stokes [1]. Fronts in several space dimensions were first studied by Christoffel [1]. The connection between shock fronts and phase transitions will not be pursued here. For references to this active area of research see Section 8.7.

Comprehensive expositions of the theory of BV functions can be found in the treatise of Federer [1], the monographs of Giusti , and Ambrosio, Fusco and Pallara [1], and the texts of Evans and Gariepy [1] and Ziemer [2]. Theorems 1.7.5 and 1.7.8 are taken from Volpert [1]. The theory of special functions of bounded variation is elaborated in Ambrosio, Fusco and Pallara [1].

An insightful discussion of the issues raised in Section 1.9 is found in DiPerna [10]. These questions will be elucidated by the presentation of the method of compensated compactness, in Chapter XVI.

II

Introduction to Continuum Physics

In continuum physics, material bodies are modeled as continuous media whose motion and equilibrium is governed by balance laws and constitutive relations.

The list of balance laws identifies the theory, for example mechanics, thermomechanics, electrodynamics, etc. The referential (Lagrangian) and the spatial (Eulerian) formulation of the typical balance law will be presented. The balance laws of mass, momentum, energy, and the Clausius-Duhem inequality, which demarcate continuum thermomechanics, will be recorded.

The type of constitutive relation encodes the nature of material response. The constitutive equations of thermoelasticity and thermoviscoelasticity will be introduced. Restrictions imposed by the Second Law of thermodynamics, the principle of material frame indifference, and material symmetry will be discussed.

2.1 Kinematics

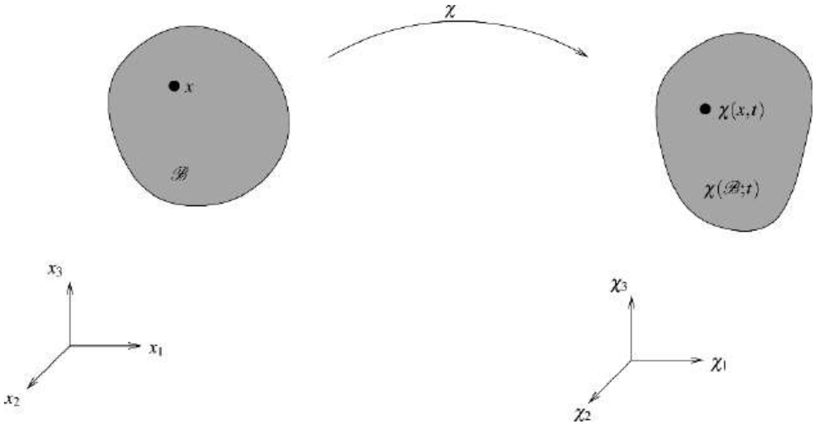

Fig. 2.1.1

The ambient space is $\mathbb{R}^m$, of dimension one, two or three. Two copies of $\mathbb{R}^m$ shall be employed, one for the *reference space*, the other for the *physical space.* A *body*

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is identified by a *reference configuration*, namely an open subset $\mathscr{B}$ of the reference space. Points of $\mathscr{B}$ will be called *particles*. The typical particle will be denoted by x and time will be denoted by t.

A *placement* of the body is a bilipschitz homeomorphism of its reference configuration $\mathscr{B}$ to some open subset of the physical space. A *motion* of the body over the time interval (t_1,t_2) is a Lipschitz map χ of $\mathscr{B} \times (t_1,t_2)$ to $\mathbb{R}^m$ whose restriction to each fixed t in (t_1,t_2) is a placement. Thus, for fixed $x \in \mathscr{B}$ and $t \in (t_1,t_2)$, $\chi(x,t)$ specifies the position in physical space of the particle x at time t; for fixed $t \in (t_1,t_2)$, the map $\chi(\cdot,t) : \mathscr{B} \to \mathbb{R}^m$ yields the placement of the body at time t; finally, for fixed $x \in \mathscr{B}$, the curve $\chi(x,\cdot) : (t_1,t_2) \to \mathbb{R}^m$ describes the trajectory of the particle x in physical space. See Fig. 2.1.1.

The reference configuration generally renders an abstract representation of the body. In practice, however, one often identifies the reference space with the physical space and employs as reference configuration an actual placement of the body, by identifying material particles with the point in physical space that they happen to occupy in that particular placement.

The aim of continuum physics is to monitor the evolution of various fields associated with the body, such as density, stress, temperature, etc. In the *referential* or *Lagrangian* description, one follows the evolution of fields along particle trajectories, while in the *spatial* or *Eulerian* description one monitors the evolution of fields at fixed position in space. The motion allows us to pass from one formulation to the other. For example, considering some illustrative field w, we write $w = f(x,t)$ for its referential description and $w = \phi(\chi,t)$ for its spatial description. The motion relates f and ϕ by $\phi(\chi(x,t),t) = f(x,t)$, for $x \in \mathscr{B}$, $t \in (t_1,t_2)$.

Either formulation has its relative merits, so both will be used here. Thus, in order to keep proper accounting, three symbols would be needed for each field, one to identify it, one for its referential description, and one for its spatial description (w, f, and ϕ in the example, above). However, in order to control the proliferation of symbols and make the physical interpretation of the equations transparent, the standard notational convention is to employ the single identifying symbol of the field for all three purposes. To prevent ambiguity in the notation of derivatives, the following rules will apply: Partial differentiation with respect to t will be denoted by an overdot in the referential description and by a t-subscript in the spatial description. Gradient, differential and divergence[1] will be denoted by Grad, ∇ and Div, with respect to the material variable x, and by grad, d and div, with respect to the spatial variable χ. Thus, referring again to the typical field w with referential description $w = f(x,t)$ and spatial description $w = \phi(\chi,t)$, $\dot{w}$ will denote $\partial f/\partial t$, w_t will denote $\partial \phi/\partial t$, $\operatorname{Grad} w$ will denote $\operatorname{grad}_x f$, and $\operatorname{grad} w$ will denote $\operatorname{grad}_\chi \phi$. This notation may appear confusing at first but the student of the subject soon learns to use it efficiently and correctly.

The motion χ induces two important kinematical fields, namely the *velocity*

[1] For consistency with matrix notations, gradients will be realized as m-column vectors and differentials will be m-row vectors, namely the transpose of gradients. As in Chapter I, the divergence operator will be acting on row vectors.

$$v = \dot{\chi}, \tag{2.1.1}$$

in $L^\infty(\mathscr{B} \times (t_1, t_2); \mathbb{R}^m)$, and the *deformation gradient,* which, its name notwithstanding, is the differential of the motion:

$$F = \nabla\chi, \tag{2.1.2}$$

in $L^\infty(\mathscr{B} \times (t_1, t_2); \mathbb{M}^{m\times m})$. In accordance with the definition of placement, we shall be assuming

$$\det F \geq a > 0 \qquad \text{a.e.} \tag{2.1.3}$$

These fields allow us to pass from spatial to material derivatives; for example, assuming w is a Lipschitz field,

$$\dot{w} = w_t + (\mathrm{d}w)v, \tag{2.1.4}$$

$$\operatorname{Grad} w = F^\top \operatorname{grad} w, \qquad \nabla w = (\mathrm{d}w)F. \tag{2.1.5}$$

By virtue of the polar decomposition theorem, the local deformation of the medium, expressed by the deformation gradient F, may be realized as the composition of a pure stretching and a rotation:

$$F = RU, \tag{2.1.6}$$

where the symmetric, positive definite matrix

$$U = (F^\top F)^{1/2} \tag{2.1.7}$$

is called the *right stretch tensor* and the proper orthogonal matrix R is called the *rotation tensor*.

Turning to the rate of change of deformation, we introduce the referential and spatial *velocity gradients* (which are actually differentials):

$$\dot{F} = \nabla v, \qquad L = \mathrm{d}v. \tag{2.1.8}$$

L is decomposed into the sum of the symmetric *stretching tensor* D and the skew-symmetric *spin tensor* W:

$$L = D + W, \qquad D = \tfrac{1}{2}(L + L^\top), \qquad W = \tfrac{1}{2}(L - L^\top). \tag{2.1.9}$$

The spin tensor is just a representation of the *vorticity* vector $\omega = \operatorname{curl} v$ as a skew symmetric matrix.

The class of Lipschitz continuous motions allows for shocks but is not sufficiently broad to also encompass motions involving cavitation in elasticity, vortices in hydrodynamics, vacuum in gas dynamics, etc. Even so, we shall continue to develop the theory under the assumption that motions are Lipschitz continuous, deferring considerations of generalization until such need arises.

2.2 Balance Laws in Continuum Physics

Consider a motion χ of a body with reference configuration $\mathscr{B} \subset \mathbb{R}^m$, over a time interval (t_1, t_2). The typical balance law of continuum physics postulates that the change over any time interval in the amount of a certain extensive quantity stored in any part of the body is balanced by a flux through the boundary and a production in the interior during that time interval. With space and time fused into space-time, the above statement yields a balance law of the type considered in Chapter I, ultimately reducing to a field equation of the form (1.2.3).

To adapt to the present setting the notation of Chapter I, we take space-time $\mathbb{R}^{m+1}$ as the ambient space $\mathbb{R}^k$, and set $\mathscr{X} = \mathscr{B} \times (t_1, t_2)$, $X = (x,t)$. With reference to (1.4.1), we partition the flux density field A into a $n \times m$ matrix-valued spatial part Ψ and a $\mathbb{R}^n$-valued temporal part Θ, namely $A = [-\Psi \,|\, \Theta]$. In the notation introduced in the previous section, (1.4.1) now takes the form

$$\dot{\Theta} = \operatorname{Div} \Psi + P. \tag{2.2.1}$$

This is the referential field equation for the typical balance law of continuum physics. The field Θ is the density of the balanced quantity; Ψ is the flux density field through material surfaces; and P is the production density.

The corresponding spatial field equation may be derived by appealing to Theorem 1.3.1. The map X^* that carries (x,t) to $(\chi(x,t),t)$ is a bilipschitz homeomorphism of $\mathscr{X}$ to some subset $\mathscr{X}^*$ of $\mathbb{R}^{m+1}$, with Jacobian matrix (cf. (1.3.2), (2.1.1), and (2.1.2)):

$$J = \left[\begin{array}{c|c} F & v \\ \hline 0 & 1 \end{array}\right]. \tag{2.2.2}$$

Notice that (1.3.3) is satisfied by virtue of (2.1.3). Theorem 1.3.1 and Remark 1.3.2 now imply that if $\Theta \in L^1_{loc}(\mathscr{X}; \mathbb{R}^n)$, $\Psi \in L^1_{loc}(\mathscr{X}; \mathbb{M}^{n\times k})$ and $P \in L^1_{loc}(\mathscr{X}; \mathbb{R}^n)$, then (2.2.1) holds in the sense of distributions on $\mathscr{X}$ if and only if

$$\Theta^*_t + \operatorname{div}(\Theta^* v^\top) = \operatorname{div} \Psi^* + P^* \tag{2.2.3}$$

holds in the sense of distributions on $\mathscr{X}^*$, where the fields $\Theta^* \in L^1_{loc}(\mathscr{X}^*; \mathbb{R}^n)$, $\Psi^* \in L^1_{loc}(\mathscr{X}^*; \mathbb{M}^{n\times m})$ and $P^* \in L^1_{loc}(\mathscr{X}^*, \mathbb{R}^n)$ are defined by

$$\Theta^* = (\det F)^{-1}\Theta, \qquad \Psi^* = (\det F)^{-1}\Psi F^\top, \qquad P^* = (\det F)^{-1} P. \tag{2.2.4}$$

It has thus been established that the referential (Lagrangian) field equations (2.2.1) and the spatial (Eulerian) field equations (2.2.3) of the balance laws of continuum physics are related by (2.2.4) and are equivalent within the function class of fields considered here.

As we have seen, in order to pass from Lagrangian to Eulerian coordinates, and vice versa, one has to apply Theorem 1.3.1 for a bilipschitz homeomorphism that is not given in advance, but is generated by the motion itself, which also affects the balanced fields. This coupling, which has no bearing on whether the referential and the spatial formulations of balance laws are equivalent, has nonetheless fostered an unwarranted aura of mystery about the issue.

In anticipation of the forthcoming discussion of material symmetry, it is useful to investigate how the fields Θ, Ψ, P and Θ^*, Ψ^*, P^* transform under *isochoric* changes of the reference configuration of the body, induced by a bilipschitz homeomorphism $\bar{x}$ of $\mathscr{B}$ to some subset $\bar{\mathscr{B}}$ of another reference space $\mathbb{R}^m$, with Jacobian matrix

(2.2.5) $$H = \frac{\partial \bar{x}}{\partial x}, \qquad \det H = 1,$$

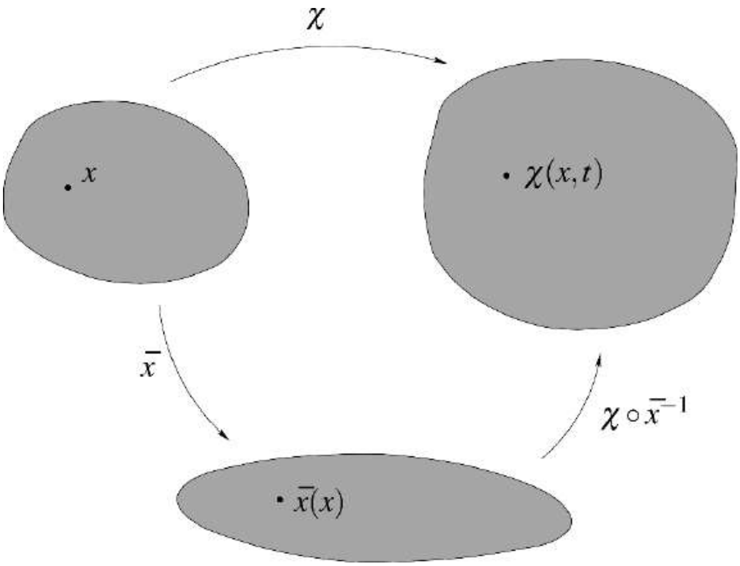

Fig. 2.2.1

see Figure 2.2.1. By virtue of Theorem 1.3.1, the Lagrangian field equation (2.2.1) on $\mathscr{B}$ will transform into an equation of exactly the same form on $\bar{\mathscr{B}}$, with fields $\bar{\Theta}$, $\bar{\Psi}$ and $\bar{P}$ related to Θ, Ψ and P by

(2.2.6) $$\bar{\Theta} = \Theta, \qquad \bar{\Psi} = \Psi H^\top, \qquad \bar{P} = P.$$

In the corresponding Eulerian field equations, the fields $\bar{\Theta}^*, \bar{\Psi}^*$ and $\bar{P}^*$ are obtained through (2.2.4): $\bar{\Theta}^* = (\det \bar{F})^{-1}\bar{\Theta}$, $\bar{\Psi}^* = (\det \bar{F})^{-1}\bar{\Psi}\bar{F}^\top$ and $\bar{P}^* = (\det \bar{F})^{-1}\bar{P}$, where $\bar{F}$ denotes the deformation gradient relative to the new reference configuration $\bar{\mathscr{B}}$. By the chain rule, $\bar{F} = FH^{-1}$ and so

(2.2.7) $$\bar{\Theta}^* = \Theta^*, \qquad \bar{\Psi}^* = \Psi^*, \qquad \bar{P}^* = P^*,$$

i.e., as was to be expected, the spatial fields are not affected by changing the reference configuration of the body.

In continuum physics, theories are identified by means of the list of balance laws that apply in their context. The illustrative example of thermomechanics will be presented in the next section. It should be noted, however, that in addition to balance laws with physical content there are others that simply express useful, purely kinematic properties. Equation (2.1.8), $\dot{F} = \nabla v$, which expresses the compatibility between the fields F and v, provides an example in that direction.

At first reading, one may skip the remainder of this section, which deals with a special topic for future use, and pass directly to the next Section 2.3.

In what follows, we derive, for $m = 3$, a set of kinematic balance laws whose referential form is quite complicated and yet whose spatial form is very simple or even trivial. This will demonstrate the usefulness of switching from the Lagrangian to the Eulerian formulation and vice versa.

A smooth function φ on the set of $F \in \mathbb{M}^{3\times 3}$ with $\det F > 0$ is called a *null Lagrangian* if the Euler-Lagrange equation

$$\operatorname{Div}\left[\partial_F \varphi(F)\right] = 0, \tag{2.2.8}$$

associated with the functional $\int \varphi(F)dx$, holds for every smooth deformation gradient field F. Any null Lagrangian φ admits a representation as an affine function

$$\varphi(F) = \operatorname{tr}(AF) + \operatorname{tr}(BF^*) + \alpha \det F + \beta \tag{2.2.9}$$

of F, its determinant $\det F$, and its adjugate matrix $F^* = (\det F)F^{-1} = (\partial_F \det F)^\top$.

By combining (2.2.8) with $\dot{F} = \nabla v$, one deduces that if φ is any null Lagrangian (2.2.9), then the conservation law

$$\dot{\varphi}(F) = \operatorname{Div}\left[v^\top \partial_F \varphi(F)\right] \tag{2.2.10}$$

holds for any smooth motion with deformation gradient F and velocity v.

The aim here is to show that, for any null Lagrangian (2.2.9), the "quasi-static" conservation law (2.2.8) as well as the "kinematic" conservation law (2.2.10) actually hold even for motions that are merely Lipschitz continuous, i.e.,

$$\operatorname{Div}\left(\partial_F F\right) = 0, \tag{2.2.11}$$

$$\operatorname{Div}\left(\partial_F F^*\right) = 0, \tag{2.2.12}$$

$$\operatorname{Div}\left(\partial_F \det F\right) = 0, \tag{2.2.13}$$

$$\dot{F} = \operatorname{Div}\left(v^\top \partial_F F\right), \tag{2.2.14}$$

$$\dot{F}^* = \operatorname{Div}\left(v^\top \partial_F F^*\right), \tag{2.2.15}$$

$$\overline{\det F}^{\boldsymbol{\cdot}} = \operatorname{Div}\left(v^\top \partial_F \det F\right), \tag{2.2.16}$$

for any bounded measurable deformation gradient field F and velocity field v.

Clearly, (2.2.11) is obvious and (2.2.14) is just an alternative way of writing the familiar $\dot{F} = \nabla v$. Furthermore, since

$$\frac{\partial F^*_{\alpha i}}{\partial F_{j\beta}} = \sum_{k=1}^{3}\sum_{\gamma=1}^{3} \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}F_{k\gamma}\,, \tag{2.2.17}$$

where ε_{ijk} and $\varepsilon_{\alpha\beta\gamma}$ are the standard permutation symbols, (2.2.12) follows from the observation that $\partial F_{k\gamma}/\partial x_\beta = \partial^2\chi_k/\partial x_\beta \partial x_\gamma$ is symmetric in (β,γ) while $\varepsilon_{\alpha\beta\gamma}$ is skew-symmetric in (β,γ).

To see (2.2.13), consider the trivial balance law (2.2.3), with $\Theta^* = 0$, $\Psi^* = I$, $P^* = 0$, and write its Lagrangian form (2.2.1), where on account of (2.2.4), $\Theta = 0$, $\Psi = (\det F)(F^\top)^{-1} = (F^*)^\top = \partial \det F/\partial F$, $P = 0$. Similarly, (2.2.16) is the Lagrangian form (2.2.1) of the trivial balance law (2.2.3), with $\Theta^* = 1, \Psi^* = v^\top$, and $P^* = 0$. Indeed, in that case, by virtue of (2.2.4) we deduce that $\Theta = \det F$, $\Psi = (\det F)(F^{-1}v)^\top = (F^*v)^\top = v^\top(\partial \det F/\partial F)$, and $P = 0$.

It remains to verify (2.2.15). We begin with the simple conservation law

$$(F^{-1})_t = (\mathrm{d}x)_t = \mathrm{d}x_t = -\mathrm{d}(F^{-1}v), \tag{2.2.18}$$

in Eulerian coordinates, and derive its Lagrangian form (2.2.1), through (2.2.4). Thus $\Theta = (\det F)F^{-1} = F^*$, while the flux Ψ, in components form, reads

$$\Psi_{\alpha i\beta} = \sum_{j=1}^{3}(\det F)\left[F^{-1}_{\beta j}F^{-1}_{\alpha i} - F^{-1}_{\alpha j}F^{-1}_{\beta i}\right]v_j\,. \tag{2.2.19}$$

The quantity in brackets vanishes when $\alpha = \beta$ and/or $i = j$; otherwise, it represents a minor of the matrix F^{-1} and thus is equal to $\det F^{-1}$ multiplied by the corresponding entry of the matrix $(F^{-1})^{-1} = F$. Hence, recalling (2.2.17),

$$(\det F)\left[F^{-1}_{\beta j}F^{-1}_{\alpha i} - F^{-1}_{\alpha j}F^{-1}_{\beta i}\right] = \sum_{k=1}^{3}\sum_{\gamma=1}^{3}\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}F_{k\gamma} = \frac{\partial F^*_{\alpha i}}{\partial F_{j\beta}}\,, \tag{2.2.20}$$

and this establishes (2.2.15).

2.3 The Balance Laws of Continuum Thermomechanics

Continuum thermomechanics, which will serve as a representative model throughout this work, is demarcated by the balance laws of mass, linear momentum, angular momentum, energy, and entropy whose referential and spatial field equations will now be introduced.

In the *balance law of mass,* there is neither flux nor production so the referential and spatial field equations read

$$\dot{\rho}_0 = 0, \tag{2.3.1}$$

$$\rho_t + \operatorname{div}(\rho v^\top) = 0, \tag{2.3.2}$$

where ρ_0 is the *reference density* and ρ is the *density* associated with the motion, related through

$$\rho = \rho_0 (\det F)^{-1}. \tag{2.3.3}$$

Note that (2.3.1) implies that the value of the reference density associated with a particle does not vary with time: $\rho_0 = \rho_0(x)$. (2.3.2) is also referred to as the *equation of continuity*.

In the *balance law of linear momentum*, the production is induced by the *body force* (per unit mass) vector b, with values in $\mathbb{R}^m$, while the flux is represented by a *stress tensor* taking values in $\mathbb{M}^{m\times m}$. The referential and spatial field equations read

$$(\rho_0 v)^{\cdot} = \operatorname{Div} S + \rho_0 b, \tag{2.3.4}$$

$$(\rho v)_t + \operatorname{div}(\rho v v^\top) = \operatorname{div} T + \rho b, \tag{2.3.5}$$

where S denotes the *Piola-Kirchhoff stress* and T denotes the *Cauchy stress*, related by

$$T = (\det F)^{-1} S F^\top. \tag{2.3.6}$$

For any unit vector ν, the value of $S\nu$ at (x,t) yields the stress (force per unit area) vector transmitted at the particle x and time t across a material surface with normal ν; while the value of $T\nu$ at (χ,t) gives the stress vector transmitted at the point χ in space and time t across a spatial surface with normal ν.

In the *balance law of angular momentum*, production and flux are the moments about the origin of the production and flux involved in the balance of linear momentum. Consequently, the referential field equation is

$$(\chi \wedge \rho_0 v)^{\cdot} = \operatorname{Div}(\chi \wedge S) + \chi \wedge \rho_0 b, \tag{2.3.7}$$

where $\wedge$ denotes cross product. Under the assumption that $\rho_0 v$, S and $\rho_0 b$ are in L^1_{loc} while the motion χ is Lipschitz continuous, we may use (2.3.4), (2.1.1) and (2.1.2) to reduce (2.3.7) into

$$S F^\top = F S^\top. \tag{2.3.8}$$

Similarly, the spatial field equation of the balance of angular momentum reduces, by virtue of (2.3.5), to the statement that the Cauchy stress tensor is symmetric:

$$T^\top = T. \tag{2.3.9}$$

There is no need to perform that calculation since (2.3.9) also follows directly from (2.3.6) and (2.3.8).

In the *balance law of energy*, the energy density is the sum of the (specific) *internal energy* (per unit mass) ε and kinetic energy. The production is the sum of the rate of work of the body force and the *heat supply* (per unit mass) r. Finally, the flux is the sum of the rate of work of the stress tensor and the *heat flux*. The referential and spatial field equations thus read

$$(\rho_0\varepsilon + \tfrac{1}{2}\rho_0|v|^2)^{\cdot} = \mathrm{Div}\,(v^\top S + Q^\top) + \rho_0 v^\top b + \rho_0 r, \tag{2.3.10}$$

$$(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)_t + \mathrm{div}\,[(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)v^\top] = \mathrm{div}\,(v^\top T + q^\top) + \rho v^\top b + \rho r, \tag{2.3.11}$$

where the referential and spatial heat flux vectors Q and q, with values in $\mathbb{R}^m$, are related by

$$q = (\det F)^{-1} F Q. \tag{2.3.12}$$

Finally, the *balance law of entropy* is expressed by the *Clausius-Duhem inequality*

$$(\rho_0 s)^{\cdot} \geq \mathrm{Div}\left(\frac{1}{\theta}Q^\top\right) + \rho_0 \frac{r}{\theta}, \tag{2.3.13}$$

$$(\rho s)_t + \mathrm{div}\,(\rho s v^\top) \geq \mathrm{div}\left(\frac{1}{\theta}q^\top\right) + \rho\frac{r}{\theta}, \tag{2.3.14}$$

in its referential and spatial form, respectively. The symbol s stands for (specific) *entropy* and θ denotes the (absolute) *temperature*. Thus, the *entropy flux* is just the heat flux divided by temperature. The term $\frac{r}{\theta}$ represents the external entropy supply (per unit mass), induced by the heat supply r. However, the fact that (2.3.13) and (2.3.14) are mere inequalities rather than equalities signifies that there may be additional *internal entropy production,* which is not specified a priori in the context of this theory, apart from being constrained to be nonnegative. This last condition is dictated by (and in fact expresses) the *Second Law of thermodynamics*. As a nonnegative distribution, the internal entropy production is necessarily a measure $\mathscr{N}$. Adding $\mathscr{N}$ to the right-hand side turns the Clausius-Duhem inequality into an equality which, by virtue of Theorem 1.3.3, is the field equation of a balance law. In particular, this demonstrates that the referential form (2.3.13) and the spatial form (2.3.14) are equivalent even when the fields are merely locally integrable.

The motion and the entropy (or temperature) field together constitute a *thermodynamic process.* The fields of internal energy, stress, heat flux, and temperature (or entropy) are determined from the thermodynamic process by means of constitutive relations that characterize the material response of the body. In particular, the constitutive equation for the stress is required to satisfy identically the balance law of angular momentum as expressed by (2.3.8) or (2.3.9). Representative material classes will be introduced in the following Sections, 2.5 and 2.6.

The field equations of the balance laws of mass, linear momentum and energy, coupled with the constitutive relations, render a closed system of evolution equations that should determine the thermodynamic process from assigned body force field b, heat supply field r, boundary conditions, and initial conditions.

The remaining balance law of entropy plays a markedly different role. The Clausius-Duhem inequality (2.3.13) or (2.3.14) is regarded as a criterion of *thermodynamic admissibility* for thermodynamic processes that already comply with the balance laws of mass, momentum and energy. In this regard, smooth thermodynamic processes are treated differently from thermodynamic processes with discontinuities.

It is a tenet of continuum thermodynamics that the constitutive relations should be constrained by the requirement that any smooth thermodynamic process that balances mass, momentum and energy must be automatically thermodynamically admissible. To implement this requisite, the first step is to derive from the Clausius-Duhem inequality the *dissipation inequality*

$$\rho_0\dot{\varepsilon} - \rho_0\theta\dot{s} - \operatorname{tr}(S\dot{F}^\top) - \frac{1}{\theta}Q\cdot G \le 0, \tag{2.3.15}$$

$$\rho\dot{\varepsilon} - \rho\theta\dot{s} - \operatorname{tr}(TD) - \frac{1}{\theta}q\cdot g \le 0, \tag{2.3.16}$$

in Lagrangian or Eulerian form, respectively, which does not involve the extraneously assigned body force and heat supply. The new symbols G and g appearing in (2.3.15) and (2.3.16) denote the *temperature gradient:*

$$G = \operatorname{Grad}\theta, \qquad g = \operatorname{grad}\theta, \qquad G = F^\top g. \tag{2.3.17}$$

To establish (2.3.15), one first eliminates the body force b between the field equations (2.3.1), (2.3.4) and (2.3.10) of the balance laws of mass, linear momentum and energy to get

$$\rho_0\dot{\varepsilon} = \operatorname{tr}(S\dot{F}^\top) + \operatorname{Div}Q^\top + \rho_0 r, \tag{2.3.18}$$

and then eliminates the heat supply r between the above equation and the Clausius-Duhem inequality (2.3.13). Similarly, (2.3.16) is obtained by combining (2.3.2), (2.3.5) and (2.3.11) with (2.3.14) in order to eliminate b and r. Of course, (2.3.15) and (2.3.16) are equivalent: either one implies the other by virtue of (2.3.3), (2.3.6), (2.3.17), (2.1.9) and (2.3.9). In the above calculations it is crucial that the underlying thermodynamic process is assumed smooth, because this allows us to apply the classical product rule of differentiation on terms like $|v|^2$, $v^\top S$, $\theta^{-1}Q$ etc., which induces substantial cancellation. It should be emphasized that the dissipation inequalities (2.3.15) and (2.3.16) are generally meaningless for thermodynamic processes with discontinuities.

The constitutive equations are required to satisfy identically the dissipation inequality (2.3.15) or (2.3.16), which will guarantee that any smooth thermodynamic process that balances mass, momentum and energy is automatically thermodynamically admissible. The implementation of this requisite for specific material classes will be demonstrated in the following Sections 2.5 and 2.6.

Beyond taking care of smooth thermodynamic processes, as above, the Clausius-Duhem inequality is charged with the additional responsibility of certifying the thermodynamic admissibility of discontinuous processes. This is a central issue, with many facets, which will surface repeatedly in the remainder of the book.

When dealing with continuous media with complex structure, e.g., mixtures of different materials, it becomes necessary to replace the Clausius-Duhem inequality with a more general entropy inequality in which the entropy flux is no longer taken a priori as heat flux divided by temperature but is instead specified by an individual constitutive relation. It turns out, however, that in the context of thermoelastic or thermoviscoelastic media, which are the main concern of this work, the requirement that such an inequality must hold identically for any smooth thermodynamic process that balances mass, momentum and energy implies in particular that entropy flux is necessarily heat flux divided by temperature, so that we fall back to the classical Clausius-Duhem inequality.

To prepare the ground for the forthcoming investigation of material symmetry, it is necessary to discuss the law of transformation of the fields involved in the balance laws when the reference configuration undergoes a change induced by an isochoric bilipshitz homeomorphism $\bar{x}$, with unimodular Jacobian matrix H (2.2.5); see Fig. 2.2.1. The deformation gradient F and the stretching tensor D (cf. (2.1.9)) will transform into new fields $\bar{F}$ and $\bar{D}$:

$$\bar{F} = FH^{-1}, \qquad \bar{D} = D. \tag{2.3.19}$$

The reference density ρ_0, internal energy ε, Piola-Kirchhoff stress S, entropy s, temperature θ, referential heat flux vector Q, density ρ, Cauchy stress T, and spatial heat flux vector q, involved in the balance laws, will also transform into new fields $\bar{\rho}_0, \bar{\varepsilon}, \bar{S}, \bar{s}, \bar{\theta}, \bar{Q}, \bar{\rho}, \bar{T}$, and $\bar{q}$ according to the rule (2.2.6) or (2.2.7), namely,

$$\bar{\rho}_0 = \rho_0, \quad \bar{\varepsilon} = \varepsilon, \quad \bar{S} = SH^{\top}, \quad \bar{s} = s, \quad \bar{\theta} = \theta, \quad \bar{Q} = HQ, \tag{2.3.20}$$

$$\bar{\rho} = \rho, \quad \bar{T} = T, \quad \bar{q} = q. \tag{2.3.21}$$

Also the referential and spatial temperature gradients G and g will transform into $\bar{G}$ and $\bar{g}$ with

$$\bar{G} = (H^{-1})^{\top} G, \qquad \bar{g} = g. \tag{2.3.22}$$

2.4 Material Frame Indifference

The body force and heat supply are usually induced by external factors and are assigned in advance, while the fields of internal energy, stress, entropy and heat flux are determined by the thermodynamic process. Motions may influence these fields inasmuch as they deform the body: rigid motions, which do not change the distance between particles, should have no effect on internal energy, temperature or referential heat flux and should affect the stress tensor in such a manner that the resulting stress vector, observed from a frame attached to the moving body, looks fixed. This requirement is postulated by the fundamental *principle of material frame indifference* which will now be stated with precision

Consider any two thermodynamic processes (χ,s) and $(\chi^{\#},s^{\#})$ of the body such that the entropy fields coincide, $s^{\#}=s$, while the motions differ by a rigid (time dependent) rotation[2]:

(2.4.1) $$\chi^{\#}(x,t)=O(t)\chi(x,t), \qquad x\in\mathscr{B},\ t\in(t_1,t_2),$$

(2.4.2) $$O^{\top}(t)O(t)=O(t)O^{\top}(t)=I, \qquad \det O(t)=1, \quad t\in(t_1,t_2).$$

Note that the fields of deformation gradient F, $F^{\#}$, spatial velocity gradient L, $L^{\#}$ and stretching tensor D, $D^{\#}$ (cf. (2.1.8), (2.1.9)) of the two processes (χ,s), $(\chi^{\#},s^{\#})$ are related by

(2.4.3) $$F^{\#}=OF, \qquad L^{\#}=OLO^{\top}+\dot{O}O^{\top}, \quad D^{\#}=ODO^{\top}.$$

Let (ε,S,θ,Q) and $(\varepsilon^{\#},S^{\#},\theta^{\#},Q^{\#})$ denote the fields for internal energy, Piola-Kirchhoff stress, temperature and referential heat flux associated with the processes (χ,s) and $(\chi^{\#},s^{\#})$. The principle of material frame indifference postulates:

(2.4.4) $$\varepsilon^{\#}=\varepsilon, \qquad S^{\#}=OS, \qquad \theta^{\#}=\theta, \qquad Q^{\#}=Q.$$

From (2.4.4), (2.3.17) and (2.4.3) it follows that the referential and spatial temperature gradients G, $G^{\#}$ and g, $g^{\#}$ of the two processes are related by

(2.4.5) $$G^{\#}=G, \qquad g^{\#}=Og.$$

Furthermore, from (2.3.6), (2.3.12) and (2.4.3) we deduce the following relations between the Cauchy stress tensors T, $T^{\#}$ and the spatial heat flux vectors q, $q^{\#}$ of the two processes:

(2.4.6) $$T^{\#}=OTO^{\top}, \qquad q^{\#}=Oq.$$

The principle of material frame indifference should be reflected in the constitutive relations of continuous media, irrespectively of the nature of material response. Illustrative examples will be considered in the following two sections.

2.5 Thermoelasticity

In the framework of continuum thermomechanics, a *thermoelastic* medium is identified by the constitutive assumption that, for any fixed particle x and any motion, the

[2] An alternative, albeit equivalent, realization of this setting is to visualize a single thermodynamic process monitored by two observers attached to individual coordinate frames that rotate relative to each other. When adopting that approach, certain authors are allowing for reflections, in addition to proper rotations.

value of the internal energy ε, the Piola-Kirchhoff stress S, the temperature θ, and the referential heat flux vector Q, at x and time t, is determined solely by the value at (x,t) of the deformation gradient F, the entropy s, and the temperature gradient G, through constitutive equations

$$(2.5.1)\qquad \begin{cases} \varepsilon = \hat{\varepsilon}(F,s,G), \\ S = \hat{S}(F,s,G), \\ \theta = \hat{\theta}(F,s,G), \\ Q = \hat{Q}(F,s,G), \end{cases}$$

where $\hat{\varepsilon}$, $\hat{S}$, $\hat{\theta}$ and $\hat{Q}$ are smooth functions defined on the subset of $\mathbb{M}^{m\times m}\times\mathbb{R}\times\mathbb{R}^m$ with $\det F > 0$. Moreover, $\hat{\theta}(F,s,G) > 0$. When the thermoelastic medium is *homogeneous*, the same functions $\hat{\varepsilon}$, $\hat{S}$, $\hat{\theta}$ and $\hat{Q}$ and the same value ρ_0 of the reference density apply to all particles $x \in \mathscr{B}$.

The Cauchy stress T and the spatial heat flux q are also determined by constitutive equations of the same form, which may be derived from (2.5.1) and (2.3.6), (2.3.12). When employing the spatial description of the motion, it is natural to substitute on the list (2.5.1) the constitutive equations of T and q for the constitutive equations of S and Q; also on the list (F,s,G) of the state variables to replace the referential temperature gradient G with the spatial temperature gradient g (cf. (2.3.17)).

The above constitutive equations will have to comply with the conditions stipulated earlier. To begin with, as postulated in Section 2.3, every smooth thermodynamic process that balances mass, momentum and energy must satisfy identically the Clausius-Duhem inequality (2.3.13) or, equivalently, the dissipation inequality (2.3.15). Substituting from (2.5.1) into (2.3.15) yields

$$(2.5.2)\qquad \operatorname{tr}[(\rho_0\partial_F\hat{\varepsilon} - \hat{S})\dot{F}^\top] + \rho_0(\partial_s\hat{\varepsilon} - \hat{\theta})\dot{s} + \rho_0\partial_G\hat{\varepsilon}\,\dot{G} - \hat{\theta}^{-1}\hat{Q}\cdot G \le 0.$$

It is clear that by suitably controlling the body force b and the heat supply r one may construct smooth processes that balance mass, momentum and energy and attain at some point (x,t) arbitrarily prescribed values for F, s, G, $\dot{F}$, $\dot{s}$ and $\dot{G}$, subject only to the constraint $\det F > 0$. Hence (2.5.2) cannot hold identically unless the constitutive relations (2.5.1) are of the following special form:

$$(2.5.3)\qquad \begin{cases} \varepsilon = \hat{\varepsilon}(F,s), \\ S = \rho_0\partial_F\hat{\varepsilon}(F,s), \\ \theta = \partial_s\hat{\varepsilon}(F,s), \\ Q = \hat{Q}(F,s,G), \end{cases}$$

$$(2.5.4)\qquad \hat{Q}(F,s,G)\cdot G \ge 0.$$

Thus the internal energy may depend on the deformation gradient and on the entropy but not on the temperature gradient. The constitutive equations for stress and temperature are induced by the constitutive equation of internal energy, through caloric relations, and are likewise independent of the temperature gradient. Only the heat flux may depend on the temperature gradient, subject to the condition (2.5.4) which implies that heat always flows from the hotter to the colder part of the body.

Another requirement on constitutive relations is that they observe the principle of material frame indifference, formulated in Section 2.4. By combining (2.4.4) and $(2.4.3)_1$ with (2.5.3), we deduce that the functions $\hat{\varepsilon}$ and $\hat{Q}$ must satisfy the conditions

$$\text{(2.5.5)} \qquad \hat{\varepsilon}(OF,s) = \hat{\varepsilon}(F,s), \qquad \hat{Q}(OF,s,G) = \hat{Q}(F,s,G),$$

for all proper orthogonal matrices O. A simple calculation verifies that when (2.5.5) hold, then the remaining conditions in (2.4.4) will be automatically satisfied, by virtue of $(2.5.3)_2$ and $(2.5.3)_3$.

To see the implications of (2.5.5), we apply it with $O = R^\top$, where R is the rotation tensor in (2.1.6), to deduce

$$\text{(2.5.6)} \qquad \hat{\varepsilon}(F,s) = \hat{\varepsilon}(U,s), \qquad \hat{Q}(F,s,G) = \hat{Q}(U,s,G).$$

It is clear that, conversely, if (2.5.6) hold then (2.5.5) will be satisfied for any proper orthogonal matrix O. Consequently, the principle of material frame indifference is completely encoded in the statement (2.5.6) that the internal energy and the referential heat flux vector may depend on the deformation gradient F solely through the right stretch tensor U.

When the spatial description of motion is to be employed, the constitutive equation for the Cauchy stress

$$\text{(2.5.7)} \qquad T = \rho \partial_F \hat{\varepsilon}(F,s) F^\top,$$

which follows from (2.3.6), (2.3.3) and $(2.5.3)_2$, will satisfy the principle of material frame indifference $(2.4.6)_1$ so long as (2.5.6) hold. For the constitutive equation of the spatial heat flux vector

$$\text{(2.5.8)} \qquad q = \hat{q}(F,s,g),$$

the principle of material frame indifference requires (recall $(2.4.6)_2$, $(2.4.3)_1$ and $(2.4.5)_2$):

$$\text{(2.5.9)} \qquad \hat{q}(OF,s,Og) = O\hat{q}(F,s,g),$$

for all proper orthogonal matrices O.

The final general requirement for constitutive relations is that the Piola-Kirchhoff stress satisfy (2.3.8), for the balance of angular momentum. This imposes no additional restrictions, however, because a simple calculation reveals that once $(2.5.5)_1$ holds, S computed through $(2.5.3)_2$ will automatically satisfy (2.3.8). Thus in thermoelasticity, material frame indifference implies balance of angular momentum.

The constitutive equations undergo further reduction when the medium is endowed with *material symmetry*. Recall from Section 2.3 that when the reference configuration of the body is changed by means of an isochoric bilipschitz homeomorphism $\bar{x}$ with unimodular Jacobian matrix H (2.2.5), then the fields transform according to the rules (2.3.19), (2.3.20), (2.3.21) and (2.3.22). It follows, in particular, that any medium that is thermoelastic relative to the original reference configuration will stay so relative to the new one, as well, even though the constitutive functions will generally change. Any isochoric transformation of the reference configuration that leaves invariant the constitutive functions for ε, T and θ manifests material symmetry of the medium. Consider any such transformation and let H be its Jacobian matrix. By virtue of $(2.3.19)_1$, $(2.3.20)_2$ and $(2.5.3)_1$, the constitutive function $\hat{\varepsilon}$ of the internal energy will remain invariant, provided

$$\hat{\varepsilon}(FH^{-1}, s) = \hat{\varepsilon}(F, s). \tag{2.5.10}$$

A simple calculation verifies that when (2.5.10) holds, the constitutive functions for T and θ, determined through (2.5.7) and (2.5.3), are automatically invariant under that H. On account of $(2.3.19)_1$ and $(2.3.22)_2$, the constitutive function $\hat{q}$ of the heat flux will be invariant under H if

$$\hat{q}(FH^{-1}, s, g) = \hat{q}(F, s, g). \tag{2.5.11}$$

It is clear that the set of matrices H with determinant one for which (2.5.10) and (2.5.11) hold forms a subgroup $\mathscr{G}$ of the special linear group $\mathrm{SL}(m)$, called the *symmetry group* of the medium. In certain media, $\mathscr{G}$ may contain only the identity matrix I in which case material symmetry is minimal. When $\mathscr{G}$ is nontrivial, it dictates through (2.5.10) and (2.5.11) conditions on the constitutive functions of the medium.

Maximal material symmetry is attained when $\mathscr{G} \equiv \mathrm{SL}(m)$. In that case the medium is a *thermoelastic fluid*. Applying (2.5.10) and (2.5.11) with selected matrix $H = (\det F)^{-1/m} F \in \mathrm{SL}(m)$, we deduce that $\hat{\varepsilon}$ and $\hat{q}$ may depend on F solely through its determinant or, equivalently by virtue of (2.3.3), through the density ρ :

$$\varepsilon = \tilde{\varepsilon}(\rho, s), \qquad q = \tilde{q}(\rho, s, g). \tag{2.5.12}$$

The Cauchy stress may then be obtained from (2.5.7) and the temperature from $(2.5.3)_3$. The calculation gives

$$T = -pI, \tag{2.5.13}$$

$$p = \rho^2 \partial_\rho \tilde{\varepsilon}(\rho, s), \qquad \theta = \partial_s \tilde{\varepsilon}(\rho, s). \tag{2.5.14}$$

In the standard texts on thermodynamics, (2.5.14) are usually presented in the guise of the *Gibbs relation:*

$$\theta ds = d\varepsilon + pd\left(\frac{1}{\rho}\right). \tag{2.5.15}$$

The constitutive function $\tilde{q}$ in (2.5.12) must also satisfy the requirement (2.5.9) of material frame indifference which now assumes the simple form

$$\tilde{q}(\rho, s, Og) = O\tilde{q}(\rho, s, g), \tag{2.5.16}$$

for all proper orthogonal matrices O. The final reduction of $\tilde{q}$ that satisfies (2.5.16) is

$$q = \kappa(\rho, s, |g|)g, \tag{2.5.17}$$

where κ is a scalar-valued function. We have thus shown that in a thermoelastic fluid the internal energy depends solely on density and entropy. The Cauchy stress is a *hydrostatic pressure*, likewise depending only on density and entropy. The heat flux obeys Fourier's law with *thermal conductivity* κ which may vary with density, entropy and the magnitude of the heat flux.

The simplest classical example of a thermoelastic fluid is the *ideal gas*, which is identified by Boyle's law

$$p = R\rho\theta, \tag{2.5.18}$$

combined with the constitutive assumption that internal energy is proportional to temperature:

$$\varepsilon = c\theta. \tag{2.5.19}$$

In (2.5.18), R is the *universal gas constant* divided by the molecular weight of the gas, and c in (2.5.19) is the *specific heat*. The constant $\gamma = 1 + R/c$ is the *adiabatic exponent.* The classical kinetic theory predicts $\gamma = 1 + 2/n$, where n is the number of degrees of freedom of the gas molecule. The maximum value $\gamma = 5/3$ is attained when the gas is monatomic.

Combining (2.5.18) and (2.5.19) with (2.5.13) and (2.5.14), one easily deduces that the constitutive relations for the ideal gas, in normalized units, read

$$\varepsilon = c\rho^{\gamma-1}e^{\frac{s}{c}}\,, \qquad p = R\rho^{\gamma}e^{\frac{s}{c}}\,, \qquad \theta = \rho^{\gamma-1}e^{\frac{s}{c}}\,. \tag{2.5.20}$$

The ideal gas model provides a satisfactory description of the behavior of ordinary gases, over a wide range of density and temperature, but it becomes less reliable at extreme values of the state variables, especially near the point of transition to the liquid phase. Accordingly, a large number of equations have been proposed, with theoretical or empirical provenances, that would apply to "real gases". The most classical example is the *van der Waals gas*, in which (2.5.18) is replaced by

$$(p + a\rho^2)(1 - b\rho) = R\rho\theta\,, \tag{2.5.21}$$

where a and b are positive parameters. It corresponds to constitutive relations

(2.5.22)

$$\varepsilon = c\left(\frac{\rho}{1-b\rho}\right)^{\gamma-1} e^{\frac{s}{c}} + a\rho,\ p = R\left(\frac{\rho}{1-b\rho}\right)^{\gamma} e^{\frac{s}{c}},\ \theta = \left(\frac{\rho}{1-b\rho}\right)^{\gamma-1} e^{\frac{s}{c}}.$$

A more exotic model is the *Chaplygin gas*, with equations of state in the form

(2.5.23)

$$\varepsilon = \frac{1}{2\rho^2}f(s) - \frac{1}{\rho}g(s) + h(s),\ p = g(s) - \frac{1}{\rho}f(s),\ \theta = \frac{1}{2\rho^2}f'(s) - \frac{1}{\rho}g'(s) + h'(s).$$

Notice that at low density the pressure becomes negative, which runs counter to conventional wisdom. However, it is this feature that renders the Chaplygin gas attractive to cosmologists, as they are relating it to "dark matter".

An *isotropic thermoelastic solid* is a thermoelastic material with symmetry group $\mathscr{G}$ the proper orthogonal group $\mathrm{SO}(m)$. In that case, to obtain the reduced form of the internal energy function $\hat{\varepsilon}$ we combine (2.5.10) with $(2.5.6)_1$. Recalling (2.1.7) we conclude that

$$\hat{\varepsilon}(OUO^\top, s) = \hat{\varepsilon}(U, s), \tag{2.5.24}$$

for any proper orthogonal matrix O. In particular, we apply (2.5.24) for the proper orthogonal matrices O that diagonalize the symmetric matrix $U : OUO^\top = \Lambda$. This establishes that, in consequence of material frame indifference and material symmetry, the internal energy of an isotropic thermoelastic solid may depend on F solely as a symmetric function of the eigenvalues of the right stretch tensor U. Equivalently,

$$\varepsilon = \tilde{\varepsilon}(J_1, \cdots, J_m, s), \tag{2.5.25}$$

where $(J_1, \cdots, J_m)$ are invariants of U. In particular, when $m = 3$, one may employ $J_1 = |F|^2$, $J_2 = |F^*|^2$ and $J_3 = \det F$, where F^* is the adjugate matrix of F. The reduced form of the Cauchy stress for the isotropic thermoelastic solid, computed from (2.5.25) and (2.5.7), is recorded in the references cited in Section 2.9. The reader may also find there explicit examples of constitutive functions for specific compressible or incompressible isotropic elastic solids.

In an alternative, albeit equivalent, formulation of thermoelasticity, one regards the temperature θ, rather than the entropy s, as a state variable and writes a constitutive equation for s rather than for θ. In that case it is also expedient to monitor the *Helmholtz free energy*

$$\psi = \varepsilon - \theta s \tag{2.5.26}$$

in the place of the internal energy ε. One thus starts out with constitutive equations

$$\begin{cases} \psi = \bar{\psi}(F, \theta, G), \\ S = \bar{S}(F, \theta, G), \\ s = \bar{s}(F, \theta, G), \\ Q = \bar{Q}(F, \theta, G), \end{cases} \tag{2.5.27}$$

in the place of (2.5.1). The requirement that all smooth thermodynamic processes that balance mass, momentum and energy must satisfy identically the dissipation inequality (2.3.15) reduces (2.5.27) to

$$\begin{cases} \psi = \bar{\psi}(F,\theta), \\ S = \rho_0 \partial_F \bar{\psi}(F,\theta), \\ s = -\partial_\theta \bar{\psi}(F,\theta), \\ Q = \bar{Q}(F,\theta,G), \end{cases} \tag{2.5.28}$$

$$\bar{Q}(F,\theta,G)\cdot G \geq 0, \tag{2.5.29}$$

which are the analogs [3] of (2.5.3), (2.5.4). The principle of material frame indifference and the presence of material symmetry further reduce the above constitutive equations. In particular, $\bar{\psi}$ satisfies the same conditions as $\hat{\varepsilon}$, above.

We conclude the discussion of thermoelasticity with remarks on special thermodynamic processes. A process is called *adiabatic* if the heat flux Q vanishes identically; it is called *isothermal* when the temperature field θ is constant; and it is called *isentropic* if the entropy field s is constant. Note that (2.5.29) implies $\bar{Q}(F,\theta,0) = 0$ so, in particular, all isothermal processes are adiabatic. Materials that are poor conductors of heat are commonly modeled as *nonconductors of heat*, characterized by the constitutive assumption $\hat{Q} \equiv 0$. Thus every thermodynamic process of a nonconductor is adiabatic.

In an isentropic process, the entropy is set equal to a constant, $s \equiv \bar{s}$; the constitutive relations for the temperature and the heat flux are discarded and those for the internal energy and the stress are restricted to $s = \bar{s}$:

$$\begin{cases} \varepsilon = \hat{\varepsilon}(F,\bar{s}), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F,\bar{s}). \end{cases} \tag{2.5.30}$$

In particular, for an ideal gas, on account of (2.5.20),

$$\varepsilon = \frac{\kappa}{\gamma-1}\rho^{\gamma-1}, \qquad p = \kappa\rho^{\gamma}, \tag{2.5.31}$$

where $\kappa = R\exp(\bar{s}/c)$.

In an isentropic process, the motion is determined solely by the balance laws of mass and momentum, in conjunction with the constitutive relations (2.5.30). This may create the impression that isentropic thermoelasticity is isomorphic to the purely mechanical theory of *hyperelasticity*. However, this is not entirely accurate, because

[3] The constitutive equations in the form (2.5.3) are called *caloric* and in the form (2.5.28) are called *thermal*.

isentropic thermoelasticity inherits from thermodynamics the Second Law in the following guise: To sustain an isentropic process, one must control the heat supply r in such a manner that the ensuing motion, under the constant entropy field, satisfies the energy balance law (2.3.10). When the process is also adiabatic, $Q = 0$, the Clausius-Duhem inequality (2.3.13) reduces to $r \leq 0$, in which case (2.3.10) implies

$$(\rho_0\varepsilon + \tfrac{1}{2}\rho_0|v|^2)^{\cdot} \leq \mathrm{Div}\,(v^\top S) + \rho_0 v^\top b. \tag{2.5.32}$$

The Eulerian form of this inequality is

$$(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)_t + \mathrm{div}\,[(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2)v^\top] \leq \mathrm{div}\,(v^\top T) + \rho v^\top b. \tag{2.5.33}$$

The above inequalities play in isentropic thermoelasticity the role played by the Clausius-Duhem inequality (2.3.13), (2.3.14) in general thermoelasticity: for smooth motions, they hold identically, as equalities, by virtue of (2.3.4) and (2.5.30). By contrast, in the context of motions that are merely Lipschitz continuous, they are extra conditions serving as the test of *thermodynamic admissibility* of the motion.

In practice, the isentropic theory is employed when it is judged that the effect of entropy fluctuation is negligible. This is not an uncommon situation, for the following reason. In smooth adiabatic processes, $Q = 0$, and if in addition $r = 0$, (2.3.18) in conjunction with (2.5.3) yields $\dot{s} = 0$. Thus, in the absence of heat supply, adiabatic processes starting out isentropically stay isentropic for as long as they are smooth. The smoothness requirements are met when F, v and s are merely Lipschitz continuous, which allows for processes with weak fronts, though not with shocks. As we shall see later, even after shocks develop, so long as they remain weak, entropy fluctuation is small (of third order) in comparison to the fluctuation of density and velocity, and may thus be neglected.

In isothermal thermoelasticity, θ is set equal to a constant $\hat{\theta}$, the heat supply r is regulated to balance the energy equation, and the motion is determined solely by the balance laws of mass and momentum. The only constitutive equations needed are

$$\begin{cases} \psi = \bar{\psi}(F,\hat{\theta}), \\ \\ S = \rho_0 \partial_F \bar{\psi}(F,\hat{\theta}), \end{cases} \tag{2.5.34}$$

namely the analogs of (2.5.30). The implications of the Second Law of thermodynamics are seen, as before, by combining (2.3.10) with (2.3.13), assuming now $\theta = \hat{\theta}$ =constant. This yields

$$(\rho_0\psi + \tfrac{1}{2}\rho_0|v|^2)^{\cdot} \leq \mathrm{Div}\,(v^\top S) + \rho_0 v^\top b, \tag{2.5.35}$$

with Eulerian form

$$(\rho\psi + \frac{1}{2}\rho|v|^2)_t + \mathrm{div}\,[(\rho\psi + \frac{1}{2}\rho|v|^2)v^\top] \leq \mathrm{div}\,(v^\top T) + \rho v^\top b, \tag{2.5.36}$$

which should be compared to (2.5.32) and (2.5.33). We conclude that isothermal and isentropic thermoelasticity are essentially isomorphic, with the Helmholtz free energy, at constant temperature, in the former, playing the role of internal energy, at constant entropy, in the latter.

In an isothermal process $\theta = \hat{\theta}$ for an ideal gas,

$$\psi = k \log \rho , \qquad p = k\rho , \tag{2.5.37}$$

where $k = R\hat{\theta}$.

2.6 Thermoviscoelasticity

We now consider an extension of thermoelasticity that encompasses materials with *internal dissipation* induced by *viscosity of the rate type*. The internal energy ε, the Piola-Kirchhoff stress S, the temperature θ, and the referential heat flux vector Q may now depend not only on the deformation gradient F, the entropy s and the temperature gradient G, as in (2.5.1), but also on the time rate $\dot{F}$ of the deformation gradient:

$$\begin{cases} \varepsilon = \hat{\varepsilon}(F, \dot{F}, s, G), \\ S = \hat{S}(F, \dot{F}, s, G), \\ \theta = \hat{\theta}(F, \dot{F}, s, G), \\ Q = \hat{Q}(F, \dot{F}, s, G). \end{cases} \tag{2.6.1}$$

As stipulated in Section 2.3, every smooth thermodynamic process that balances mass, momentum and energy must satisfy identically the dissipation inequality (2.3.15). Substituting from (2.6.1) into (2.3.15) yields

$$\operatorname{tr}[(\rho_0 \partial_F \hat{\varepsilon} - \hat{S})\dot{F}^\top] + \operatorname{tr}(\rho_0 \partial_{\dot{F}} \hat{\varepsilon}\, \ddot{F}^\top) + \rho_0(\partial_s \hat{\varepsilon} - \hat{\theta})\dot{s} + \rho_0 \partial_G \hat{\varepsilon} \dot{G} - \hat{\theta}^{-1} \hat{Q} \cdot G \leq 0. \tag{2.6.2}$$

By suitably controlling the body force b and heat supply r, one may construct smooth processes that balance mass, momentum and energy and attain at some point (x,t) arbitrarily prescribed values for F, $\dot{F}$, s, G, $\ddot{F}$, $\dot{s}$ and $\dot{G}$, subject only to the constraint $\det F > 0$. Consequently, the inequality (2.6.2) cannot hold identically unless the constitutive functions in (2.6.1) have the following special form:

$$\begin{cases} \varepsilon = \hat{\varepsilon}(F, s), \\ S = \rho_0 \partial_F \hat{\varepsilon}(F, s) + Z(F, \dot{F}, s, G), \\ \theta = \partial_s \hat{\varepsilon}(F, s), \\ Q = \hat{Q}(F, \dot{F}, s, G), \end{cases} \tag{2.6.3}$$

$$\text{tr}\,[Z(F,\dot{F},s,G)\dot{F}^\top] + \frac{1}{\hat{\theta}(F,s)}\hat{Q}(F,\dot{F},s,G)\cdot G \geq 0. \tag{2.6.4}$$

Comparing (2.6.3) with (2.5.3) we observe that, again, the internal energy, which may depend solely on the deformation gradient and the entropy, determines the constitutive equation for the temperature by the same caloric equation of state. On the other hand, the constitutive equation for the stress now includes the additional term Z which contributes the viscous effect and induces internal dissipation manifested in (2.6.4).

The constitutive functions must be reduced further to comply with the principle of material frame indifference, postulated in Section 3.4. In particular, frame indifference imposes on internal energy the same condition $(2.5.5)_1$ as in thermoelasticity, and the resulting reduction is, of course, the same:

$$\hat{\varepsilon}(F,s) = \hat{\varepsilon}(U,s), \tag{2.6.5}$$

where U denotes the right stretch tensor (2.1.7). Furthermore, when (2.6.5) holds, the constitutive equation for the temperature, derived through $(2.6.3)_3$, and the term $\rho_0\partial_F\hat{\varepsilon}(F,s)$, in the constitutive equation for the stress, will be automatically frame indifferent. It remains to investigate the implications of frame indifference on Z and on the heat flux. Since the analysis will focus eventually on thermoviscoelastic fluids, it will be expedient to switch at this point from S and Q to T and q; also to replace, on the list $(F,\dot{F},s,G)$ of state variables, $\dot{F}$ with L (cf. (2.1.8)) and G with g (cf. (2.3.17)). We thus write

$$T = \rho\partial_F\hat{\varepsilon}(F,s)F^\top + \hat{Z}(F,L,s,g), \tag{2.6.6}$$

$$q = \hat{q}(F,L,s,g). \tag{2.6.7}$$

Recalling (2.4.3) and (2.4.5), we deduce that the principle of material frame indifference requires

$$\begin{cases} \hat{Z}(OF, OLO^\top + \dot{O}O^\top, s, Og) = O\hat{Z}(F,L,s,g)O^\top \\ \hat{q}(OF, OLO^\top + \dot{O}O^\top, s, Og) = O\hat{q}(F,L,s,g), \end{cases} \tag{2.6.8}$$

for any proper orthogonal matrix O. In particular, for any fixed state (F,L,s,g) with spin W (cf. (2.1.9)), we may pick $O(t) = \exp(-tW)$, in which case $O(0) = I$, $\dot{O}(0) = -W$. It then follows from (2.6.8) that $\hat{Z}$ and $\hat{q}$ may depend on L solely through its symmetric part D and hence (2.6.6) and (2.6.7) may be written as

$$T = \rho\partial_F\hat{\varepsilon}(F,s)F^\top + \hat{Z}(F,D,s,g), \tag{2.6.9}$$

$$q = \hat{q}(F,D,s,g), \tag{2.6.10}$$

with $\hat{Z}$ and $\hat{q}$ such that

$$\left\{\begin{aligned}&\hat{Z}(OF,ODO^\top,s,Og)=O\hat{Z}(F,D,s,g)O^\top\\&\hat{q}(OF,ODO^\top,s,Og)=O\hat{q}(F,D,s,g),\end{aligned}\right. \tag{2.6.11}$$

for all proper orthogonal matrices O.

For the balance law of angular momentum (2.3.9) to be satisfied, $\hat{Z}$ must also be symmetric: $\hat{Z}^\top=\hat{Z}$. Notice that in that case the dissipation inequality (2.6.4) may be rewritten in the form

$$\operatorname{tr}[\hat{Z}(F,D,s,g)D]+\frac{1}{\hat{\theta}(F,s)}\hat{q}(F,D,s,g)\cdot g\geq 0. \tag{2.6.12}$$

Further reduction of the constitutive functions results when the medium is endowed with material symmetry. The rules of transformation of the fields under isochoric change of the reference configuration are recorded in (2.3.19), (2.3.20), (2.3.21) and (2.3.22). As in Section 2.5, we introduce here the *symmetry group* $\mathscr{G}$ of the material, namely the subgroup of $\mathrm{SL}(m)$ formed by the Jocobian matrices H of those isochoric transformations $\bar{x}$ of the reference configuration that leave the constitutive functions for ε, T, θ and q invariant. Thus, $\mathscr{G}$ is the set of all $H\in\mathrm{SL}(m)$ with the property

$$\left\{\begin{aligned}&\hat{\varepsilon}(FH^{-1},s)=\hat{\varepsilon}(F,s),\\&\hat{Z}(FH^{-1},D,s,g)=\hat{Z}(F,D,s,g),\\&\hat{q}(FH^{-1},D,s,g)=\hat{q}(F,D,s,g).\end{aligned}\right. \tag{2.6.13}$$

The material will be called a *thermoviscoelastic fluid* when $\mathscr{G}\equiv\mathrm{SL}(m)$. In that case, applying (2.6.13) with $H=(\det F)^{-1/m}F\in\mathrm{SL}(m)$, we conclude that $\hat{\varepsilon}$, $\hat{Z}$ and $\hat{q}$ may depend on F solely through its determinant or, equivalently, through the density ρ. Therefore, the constitutive equations of the thermoviscoelastic fluid reduce to

$$\left\{\begin{aligned}&\varepsilon=\tilde{\varepsilon}(\rho,s),\\&T=-pI+\tilde{Z}(\rho,D,s,g),\\&p=\rho^2\partial_\rho\tilde{\varepsilon}(\rho,s),\qquad \theta=\partial_s\tilde{\varepsilon}(\rho,s),\\&q=\tilde{q}(\rho,D,s,g).\end{aligned}\right. \tag{2.6.14}$$

For frame indifference, $\tilde{Z}$ and $\tilde{q}$ should still satisfy, for any proper orthogonal matrix O, the conditions

$$\left\{\begin{aligned}&\tilde{Z}(\rho,ODO^\top,s,Og)=O\tilde{Z}(\rho,D,s,g)O^\top,\\&\tilde{q}(\rho,ODO^\top,s,Og)=O\tilde{q}(\rho,D,s,g),\end{aligned}\right. \tag{2.6.15}$$

which follow from (2.6.11). It is possible to write down explicitly the form of the most general functions $\tilde{Z}$ and $\tilde{q}$ that conform with (2.6.15). Here, it will suffice to record the most general constitutive relations, for m=3, that are compatible with (2.6.15) and are linear in (D,g), namely

(2.6.16) $$T = -p(\rho,s)I + \lambda(\rho,s)(\mathrm{tr}D)I + 2\mu(\rho,s)D,$$

(2.6.17) $$q = \kappa(\rho,s)g,$$

which identify the compressible, heat-conducting *Newtonian fluid*.

The *bulk viscosity* $\lambda + \frac{2}{3}\mu$, *shear viscosity* μ and *thermal conductivity* κ of a Newtonian fluid are constrained by the inequality (2.6.12), which here reduces to

(2.6.18) $$\lambda(\rho,s)(\mathrm{tr}D)^2 + 2\mu(\rho,s)\mathrm{tr}D^2 + \frac{\kappa(\rho,s)}{\tilde{\theta}(\rho,s)}|g|^2 \geq 0.$$

This inequality will hold for arbitrary D and g if and only if

(2.6.19) $$\mu(\rho,s) \geq 0, \qquad 3\lambda(\rho,s) + 2\mu(\rho,s) \geq 0, \qquad \kappa(\rho,s) \geq 0.$$

For actual dissipation, at least one of μ, $3\lambda + 2\mu$ and κ should be strictly positive.

2.7 Incompressibility

Many fluids, and even certain solids, such as rubber, may be stretched or sheared with relative ease, while exhibiting disproportionately high stiffness when subjected to deformations that would change their volume. Continuum physics treats such materials as incapable of sustaining any volume change, so that the density ρ stays constant along particle trajectories. The incompressibility condition

(2.7.1) $$\det F = 1, \qquad \mathrm{tr}D = \mathrm{div}\, v^\top = 0,$$

in Lagrangian or Eulerian coordinates, is then appended to the system of balance laws, as a kinematic constraint. In return, the stress tensor is decomposed into two parts:

(2.7.2) $$S = -p(F^{-1})^\top + \hat{S}, \qquad T = -pI + \hat{T},$$

where $\hat{S}$ or $\hat{T}$, called the *extra stress*, is determined, as before, by the thermodynamic process, through constitutive equations, while the other term, which represents a *hydrostatic pressure,* is not specified by a constitutive relation but is to be determined, together with the thermodynamic process, by solving the system of balance laws of mass, momentum and energy, subject to the kinematic constraint (2.7.1).

The salient property of the hydrostatic pressure is that it produces no work under isochoric deformations. To motivate (2.7.2) by means of the Second Law of thermodynamics, let us consider an incompressible thermoelastic material with constitutive equations for ε, θ and Q as in (2.5.1), but only defined for F with $\det F = 1$, and S unspecified. The dissipation inequality again implies (2.5.2) with $\hat{S}$ replaced by

S, $\partial_F \hat{\varepsilon}$ replaced by the tangential derivative $\partial_F^\tau \hat{\varepsilon}$ on the manifold $\det F = 1$, and $\dot{F}$ constrained to lie on the subspace

$$\text{(2.7.3)} \qquad \operatorname{tr}[(F^{-1})^\top \dot{F}^\top] = \operatorname{tr}[(F^*)^\top \dot{F}^\top] = \operatorname{tr}[(\partial_F \det F)\dot{F}^\top] = \overline{\det F}^{\,\cdot} = 0.$$

Therefore, $\operatorname{tr}[(\rho_0 \partial_F^\tau \hat{\varepsilon} - S)\dot{F}^\top] \le 0$ for all $\dot{F}$ satisfying (2.7.3) if and only if

$$\text{(2.7.4)} \qquad S = -p(F^{-1})^\top + \rho_0 \partial_F^\tau \hat{\varepsilon}(F,s),$$

for some scalar p.

In incompressible Newtonian fluids, the stress is still given by (2.6.16), where, however, ρ is constant and $p(\rho,s)$ is replaced by the undetermined hydrostatic pressure p. When the incompressible fluid is inviscid, the entire stress tensor is subsumed by the undetermined hydrostatic pressure.

2.8 Relaxation

The state variables of continuum physics, introduced in the previous sections, represent statistical averages of certain physical quantities, such as velocity, translational kinetic energy, rotational kinetic energy, chemical energy etc., associated with the molecules of the material. These quantities evolve and eventually settle, or “relax”, to states in local equilibrium, characterized by equipartition of energy and other conditions dictated by the laws of statistical physics. The constitutive relations of thermoelasticity, considered in earlier sections, are relevant so long as local equilibrium is attained in a time scale much shorter than the time scale of the gross motion of the material body. In the opposite case, where the relaxation time is of the same order of magnitude as the time scale of the motion, relaxation mechanisms must be accounted for even within the framework of continuum physics. This is done by introducing additional, *internal state variables,* measuring the deviation from local equilibrium. The states in local equilibrium span a manifold embedded in the extended state space. The internal state variables satisfy special constitutive relations, in the form of balance laws with dissipative source terms that act to drive the state vector towards local equilibrium.

An enormous variety of relaxation theories are discussed in the literature; the reader may catch a glimpse of their common underlying structure through the following example.

We consider a continuous medium that does not conduct heat and whose isentropic response is governed by constitutive relations

$$\text{(2.8.1)} \qquad \varepsilon = \hat{\varepsilon}(F,\Sigma),$$

$$\text{(2.8.2)} \qquad S = P(F) + \rho_0 \Sigma,$$

for the internal energy and the Piola-Kirchhoff stress, where Σ is an internal variable taking values in $\mathbb{M}^{m\times m}$ and satisfying a balance law of the form

$$\rho_0 \dot{\Sigma} = \frac{1}{\tau}[\Pi(\Sigma) - F]. \tag{2.8.3}$$

Thus, the material exhibits instantaneous elastic response, embodied in the term $P(F)$, combined with viscous response induced by relaxation of Σ. The positive constant τ is called the *relaxation time.*

The postulate that any smooth motion of the medium that balances linear momentum (2.3.4) must satisfy identically the entropy inequality (2.5.32) yields

$$S = \rho_0 \partial_F \hat{\varepsilon}(F, \Sigma), \tag{2.8.4}$$

$$\operatorname{tr}\left[\partial_\Sigma \hat{\varepsilon}(F, \Sigma)\dot{\Sigma}^\top\right] \leq 0. \tag{2.8.5}$$

Upon combining (2.8.4) and (2.8.5) with (2.8.2) and (2.8.3), we deduce

$$\varepsilon = \sigma(F) + \operatorname{tr}(\Sigma F^\top) + h(\Sigma), \tag{2.8.6}$$

$$P(F) = \rho_0 \partial_F \sigma(F), \qquad \Pi(\Sigma) = -\partial_\Sigma h(\Sigma). \tag{2.8.7}$$

When h is strictly convex, the source term in (2.8.3) is dissipative and acts to drive Σ towards local equilibrium $\Sigma = H(F)$, where H is the inverse function of Π. Π^{-1} exists since $-\Pi$ is strictly monotone, namely,

$$\operatorname{tr}\{[\Pi(\Sigma) - \Pi(\bar{\Sigma})][\Sigma - \bar{\Sigma}]^\top\} < 0, \qquad \text{for any } \Sigma \neq \bar{\Sigma}. \tag{2.8.8}$$

In local equilibrium the medium responds like an elastic material with internal energy

$$\varepsilon = \tilde{\varepsilon}(F) = \sigma(F) + \operatorname{tr}[H(F)F^\top] + h(H(F)) \tag{2.8.9}$$

and Piola-Kirchhoff stress

$$S = P(F) + \rho_0 H(F) = \rho_0 \partial_F \tilde{\varepsilon}(F). \tag{2.8.10}$$

2.9 Notes

The venerable field of continuum physics has been enjoying a resurgence, concomitant with the rise of interest in the behavior of materials with nonlinear response. The encyclopedic works of Truesdell and Toupin [1] and Truesdell and Noll [1] contain reliable historical information as well as massive bibliographies and may serve as excellent guides for following the development of the subject from its inception, in the 18th century, to the mid 1960's. The text by Gurtin [1] provides a clear, elementary introduction to the area. A more advanced treatment, with copious references, is found in the book of Silhavy [1]. The text by Müller [2] is an excellent presentation of thermodynamics from the perspective of modern continuum physics. Other good sources, emphasizing elasticity theory, are the books of Ciarlet [1], Hanyga [1], Marsden and Hughes [1] and Wang and Truesdell [1]. The monograph by Antman [3] contains a wealth of material on the theory of elastic strings, rods, shells and three-dimensional bodies, with emphasis on the qualitative analysis of the governing balance laws.

The referential description of motion was conceived by Euler, and was eventually named Lagrangian so as to highlight the analogy with the formulation of Analytical Dynamics by Lagrange. On the other hand, the spatial, or Eulerian, description, which was effectively employed by Euler, was introduced by Daniel Benoulli and by D'Alembert.

On the equivalence of the referential (Lagrangian) and spatial (Eulerian) description of the field equations for the balance laws of continuum physics, see Dafermos [17] and Wagner [2,3]. It would be useful to know whether this holds under more general assumptions on the motion than Lipschitz continuity. For instance, when the medium is a thermoelastic gas, it is natural to allow regions of vacuum in the placement of the body. In such a region the density vanishes and the specific volume (determinant of the deformation gradient) becomes infinitely large. For other examples in which the equations get simpler as one passes from Eulerian to Lagrangian coordinates, see Peng [2].

The kinematic balance laws (2.2.15) and (2.2.16) were first derived by Qin[1], in the context of smooth motions, by direct calculation. It is interesting that, as we see here, they are valid when the motions are merely Lipschitz continuous and in fact, as shown by Demoulini, Stuart and Tzavaras [2], even under slightly weaker hypotheses. The connection to null Lagrangians was first pointed out in this last reference. For a detailed treatment of null Lagrangians, see Ball, Currie and Olver [1]. For the differential geometric interpretation of the kinematic balance laws, see Wagner [3,4].

The field equations for the balance laws considered here were originally derived by Euler [1,2], for mass, Cauchy [3,4], for linear and angular momentum, and Kirchhoff [1], for energy. The Clausius-Duhem inequality was postulated by Clausius [1], for the adiabatic case; the entropy flux term was introduced by Duhem [1] and the entropy production term was added by Truesdell and Toupin [1]. More general entropy inequalities were first considered by Müller [1].

The use of frame indifference and material symmetry to reduce constitutive equations originated in the works of Cauchy [4] and Poisson [2]. In the ensuing century, this program was implemented (mostly correctly but occasionally incorrectly) by many authors, for a host of special constitutive equations. In particular, the work of the Cosserats [1], Rivlin and Ericksen [1], and others in the 1940's and 1950's contributed to the clarification of the concepts. The principle of material frame indifference and the definition of the symmetry group were ultimately postulated with generality and mathematical precision by Noll [1].

The postulate that constitutive equations should be reduced so that the Clausius-Duhem inequality will be satisfied automatically by smooth thermodynamic processes that balance mass, momentum and energy was first stated as a general principle by Coleman and Noll [1]. The examples presented here were adapted from Coleman and Noll [1], for thermoelasticity, and Coleman and Mizel [1], for thermoviscoelasticity.

In his doctoral dissertation (1873), van der Waals introduced the equation of state that now bears his name, in order to account for the volume of gas molecules and for intermolecular forces. Gallavotti [1] discusses its interpretation from the standpoint of statistical physics. The van der Waals gas has served over the years as a simple model for phase transitions.

As we saw in the historical introduction, the special features of the Chaplygin gas were first noticed by Earnshaw [1]. The classical contributions of Chaplygin [1] are expounded in the text by von Mises [1]. The Chaplygin gas is currently finding new applications in cosmology. For a surprising application of the equation of state of this gas to differential geometry, see Section 18.7.

Coleman and Gurtin [1] have developed a general theory of thermoviscoelastic materials with internal state variables, of which the example presented in Section 2.8 is a special case. Constitutive relations of this type were first considered by Maxwell [1]. A detailed discussion of relaxation phenomena in gas dynamics is found in the book by Vincenti and Kruger [1].

III

Hyperbolic Systems of Balance Laws

The ambient space for the system of balance laws, introduced in Chapter I, will be visualized here as space-time, and the central notion of hyperbolicity in the time direction will be motivated and defined. Companions to the flux, considered in Section 1.5, will now be realized as entropy-entropy flux pairs.

Numerous examples will be presented of hyperbolic systems of balance laws arising in continuum physics.

3.1 Hyperbolicity

Returning to the setting of Chapter I, we visualize $\mathbb{R}^k$ as $\mathbb{R}^m \times \mathbb{R}$, where $\mathbb{R}^m$, with $m = k-1$, is "space" with typical point x, and $\mathbb{R}$ is "time" with typical value t, so $X = (x,t)$. We write ∂_t for $\partial / \partial X_k$ and ∂_α for $\partial / \partial X_\alpha$, $\alpha = 1, \dots, m$. We retain the symbol div to denote divergence with respect to the x-variable in $\mathbb{R}^m$. As in earlier chapters, in matrix operations div will be acting on row vectors. We also recall the Notation 1.4.2, which will remain in force throughout this work: D denotes the differential $[\partial / \partial U^1, \dots, \partial / \partial U^n]$, regarded as a row operation.

We denote G_k by H, reassign the symbol G to denote the $n \times m$ matrix with column vectors $(G_1, \dots, G_m)$, and rewrite the system of balance laws (1.4.3) in the form

$$\partial_t H(U(x,t),x,t) + \operatorname{div} G(U(x,t),x,t) = \Pi(U(x,t),x,t). \tag{3.1.1}$$

3.1.1 Definition. The system of balance laws (3.1.1) is called *hyperbolic* in the t-direction if, for any fixed $U \in \mathscr{O}$, $(x,t) \in \mathscr{X}$ and $\nu \in \mathbb{S}^{m-1}$, the $n \times n$ matrix $\mathrm{D}H(U,x,t)$ is nonsingular and the eigenvalue problem

$$\left[\sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D}G_\alpha(U,x,t) - \lambda \mathrm{D}H(U,x,t) \right] R = 0 \tag{3.1.2}$$

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has real eigenvalues $\lambda_1(\nu;U,x,t),\cdots,\lambda_n(\nu;U,x,t)$, called *characteristic speeds*, and n linearly independent eigenvectors $R_1(\nu;U,x,t),\cdots,R_n(\nu;U,x,t)$.

A class of great importance are the *symmetric hyperbolic* systems of balance laws (3.1.1), in which, for any $U\in\mathscr{O}$ and $(x,t)\in\mathscr{X}$, the $n\times n$ matrices $\mathrm{D}G_\alpha(U,x,t)$, for $\alpha=1,\cdots,m$, are symmetric and $\mathrm{D}H(U,x,t)$ is symmetric positive definite.

The definition of hyperbolicity may be naturally interpreted in terms of the notion of fronts, introduced in Section 1.6. A front $\mathscr{F}$ of the system of balance laws (3.1.1) may be visualized as a one-parameter family of $m-1$ dimensional manifolds in $\mathbb{R}^m$, parametrized by t, i.e., as a surface propagating in space. In that context, if we renormalize the normal N on $\mathscr{F}$ so that $N=(\nu,-s)$ with $\nu\in\mathbb{S}^{m-1}$, then the wave will be propagating in the direction ν with speed s. Therefore, comparing (3.1.2) with (1.6.1) we conclude that a system of n balance laws is hyperbolic if and only if n distinct weak waves can propagate in any spatial direction. The eigenvalues of (3.1.2) will determine the speed of propagation of these waves while the corresponding eigenvectors will specify the direction of their amplitude.

When $\mathscr{F}$ is a shock front, (1.6.3) may be written in the current notation as

$$-s[H(U_+,x,t)-H(U_-,x,t)]+[G(U_+,x,t)-G(U_-,x,t)]\nu=0, \tag{3.1.3}$$

which is called the *Rankine-Hugoniot jump condition*. By virtue of Theorem 1.8.1, this condition should hold at every point of approximate jump discontinuity of any function U of class BV_{loc} that satisfies the system (3.1.1) in the sense of measures.

It is clear that hyperbolicity is preserved under any change $U^*=U^*(U,x,t)$ of state vector with $U^*(\cdot,x,t)$ a diffeomorphism for every fixed $(x,t)\in\mathscr{X}$. In particular, since $\mathrm{D}H(U,x,t)$ is nonsingular, we may employ, locally at least, H as the new state vector. Thus, without essential loss of generality, one may limit the investigation to hyperbolic systems of balance laws that have the special form

$$\partial_t U(x,t)+\mathrm{div}\, G(U(x,t),x,t)=\Pi(U(x,t),x,t). \tag{3.1.4}$$

For simplicity and convenience, we shall henceforth regard the special form (3.1.4) as *canonical*. The reader should keep in mind, however, that when dealing with systems of balance laws arising in continuum physics it may be advantageous to keep the state vector naturally provided, even at the expense of having to face the more complicated form (3.1.1) rather than the canonical form (3.1.4).

3.2 Entropy-Entropy Flux Pairs

Assume that the system of balance laws (1.4.3), which we now write in the form (3.1.1), is endowed with a companion balance law (1.5.2). We set $Q_k\equiv\eta$, reassign Q to denote the m-row vector $(Q_1,\dots,Q_m)$ and recast (1.5.2) in the new notation:

$$\partial_t\eta(U(x,t),x,t)+\mathrm{div}\, Q(U(x,t),x,t)=h(U(x,t),x,t). \tag{3.2.1}$$

As we shall see in Section 3.3, in the applications to continuum physics, companion balance laws of the form (3.2.1) are intimately related with the Second Law of thermodynamics. For that reason, η is called an *entropy* for the system (3.1.1) of balance laws and Q is called the *entropy flux* associated with η.

Equation (1.5.1), for $\alpha = k$, should now be written as

$$\mathrm{D}\eta(U,x,t) = B(U,x,t)^\top \mathrm{D}H(U,x,t). \tag{3.2.2}$$

Assume the system is in canonical form (3.1.4) so that (3.2.2) reduces to $\mathrm{D}\eta = B^\top$. Then (1.5.1) and the integrability condition (1.5.4) become

$$\mathrm{D}Q_\alpha(U,x,t) = \mathrm{D}\eta(U,x,t)\mathrm{D}G_\alpha(U,x,t), \qquad \alpha = 1,\cdots,m, \tag{3.2.3}$$

$$\mathrm{D}^2\eta(U,x,t)\mathrm{D}G_\alpha(U,x,t) = \mathrm{D}G_\alpha(U,x,t)^\top \mathrm{D}^2\eta(U,x,t), \quad \alpha = 1,\cdots,m. \tag{3.2.4}$$

Notice that (3.2.4) imposes $\frac{1}{2}n(n-1)m$ conditions on the single unknown function η. Therefore, as already noted in Section 1.5, the problem of determining a nontrivial entropy-entropy flux pair for (3.1.1) is formally overdetermined, unless either $n = 1$ and m is arbitrary, or $n = 2$ and $m = 1$. However, when the system is symmetric, we may satisfy (3.2.4) with $\eta = \frac{1}{2}|U|^2$. Conversely, if (3.2.4) holds and $\eta(U,x,t)$ is uniformly convex in U, then the change $U^* = \mathrm{D}\eta(U,x,t)^\top$ of state vector renders the system symmetric. Thus, systems of balance laws in canonical form (3.1.4) that are endowed with a convex entropy are necessarily hyperbolic.

An interesting, alternative form of the integrability condition is obtained by projecting (3.2.4) in the direction of an arbitrary $\nu \in \mathbb{S}^{m-1}$ and then multiplying the resulting equation from the left by $R_j(\nu;U,x,t)^\top$ and from the right by $R_k(\nu;U,x,t)$, with $j \neq k$. So long as $\lambda_j(\nu;U,x,t) \neq \lambda_k(\nu;U,x,t)$, this calculation yields

$$R_j(\nu;U,x,t)^\top \mathrm{D}^2\eta(U,x,t)R_k(\nu;U,x,t) = 0, \qquad j \neq k. \tag{3.2.5}$$

Moreover, (3.2.5) holds even when $\lambda_j(\nu;U,x,t) = \lambda_k(\nu;U,x,t)$, provided that one selects the eigenvectors $R_j(\nu;U,x,t)$ and $R_k(\nu;U,x,t)$ judiciously in the eigenspace of this multiple eigenvalue.

Notice that (3.2.5) imposes on η $\frac{1}{2}n(n-1)$ conditions for each fixed ν, and hence a total of $\frac{1}{2}n(n-1)m$ conditions for m linearly independent – and thereby all – ν in $\mathbb{S}^{m-1}$. A notable exception occurs for systems in which the Jacobian matrices of the components of their fluxes commute:

$$\mathrm{D}G_\alpha(U,x,t)\mathrm{D}G_\beta(U,x,t) = \mathrm{D}G_\beta(U,x,t)\mathrm{D}G_\alpha(U,x,t), \quad \alpha,\beta = 1,\ldots,m. \tag{3.2.6}$$

Indeed, in that case the $R_i(\nu;U,x,t)$ do not vary with ν and hence (3.2.5) represents just $\frac{1}{2}n(n-1)$ conditions on η. We will revisit this very special class of systems in Section 6.10.

The issue of the overdeterminacy of (3.2.5), in one spatial dimension, will be examined in depth in Section 7.4.

3.3 Examples of Hyperbolic Systems of Balance Laws

Out of a host of hyperbolic systems of balance laws in continuum physics, only a small sample will be presented here. They will serve as beacons for guiding the development of the general theory.

3.3.1 The Scalar Balance Law:
The single balance law $(n = 1)$

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t),x,t) = \varpi(u(x,t),x,t) \tag{3.3.1}$$

is always hyperbolic. Any function $\eta(u,x,t)$ may serve as entropy, with associated entropy flux and entropy production computed by

$$Q = \int^u \frac{\partial \eta}{\partial u} \frac{\partial G}{\partial u} du, \tag{3.3.2}$$

$$h = \sum_{\alpha=1}^{m} \left[\frac{\partial \eta}{\partial u} \frac{\partial G_\alpha}{\partial x_\alpha} - \frac{\partial Q_\alpha}{\partial x_\alpha} \right] + \varpi \frac{\partial \eta}{\partial u} + \frac{\partial \eta}{\partial t}. \tag{3.3.3}$$

Equation (3.3.1), the corresponding homogeneous scalar conservation law, and especially their one space dimensional $(m = 1)$ versions will serve extensively as models for developing the theory of general systems.

3.3.2 Thermoelastic Nonconductors of Heat:
The theory of thermoelastic media was discussed in Chapter II. Here we shall employ the referential (Lagrangian) description so the fields will be functions of (x,t). For consistency with the notation of the present chapter, we shall use ∂_t to denote material time derivative (in lieu of the overdot employed in Chapter II) and ∂_α to denote partial derivative with respect to the α-component x_α of x. For definiteness, we assume the physical space has dimension $m = 3$. We also adopt the standard summation convention: repeated indices are summed over the range 1,2,3.

The constitutive equations are recorded in Section 2.5. Since there is no longer danger of confusion, we may simplify the notation by dropping the "hat" from the symbols of the constitutive functions. Also for simplicity we assume that the medium is homogeneous, with reference density $\rho_0 = 1$.

As explained in Chapter II, a thermodynamic process is determined by a motion χ and an entropy field s. In order to cast the field equations of the balance laws as a first order system of the form (3.1.1), we monitor χ through its derivatives (2.1.1), (2.1.2) and thus work with the state vector $U = (F, v, s)$, taking values in $\mathbb{R}^{13}$. In that case we must append to the balance laws of linear momentum (2.3.4) and energy (2.3.10) the compatibility condition $(2.1.8)_1$. Consequently, our system of balance laws reads

$$(3.3.4) \qquad \begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i,\alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F,s) = b_i\,, & i = 1,2,3 \\ \partial_t \left[\varepsilon(F,s) + \frac{1}{2}|v|^2\right] - \partial_\alpha \left[v_i S_{i\alpha}(F,s)\right] = b_i v_i + r, & \end{cases}$$

with (cf. (2.5.3))

$$(3.3.5) \qquad S_{i\alpha}(F,s) = \frac{\partial\varepsilon(F,s)}{\partial F_{i\alpha}}\,, \qquad \theta(F,s) = \frac{\partial\varepsilon(F,s)}{\partial s}\,.$$

A lengthy calculation verifies that the system (3.3.4) is hyperbolic on a certain region of the state space if for every (F,s) lying in that region

$$(3.3.6) \qquad \frac{\partial\varepsilon(F,s)}{\partial s} > 0,$$

$$(3.3.7) \qquad \frac{\partial^2\varepsilon(F,s)}{\partial F_{i\alpha}\partial F_{j\beta}} \nu_\alpha \nu_\beta \xi_i \xi_j > 0, \quad \text{for all } \nu \text{ and } \xi \text{ in } \mathbb{S}^2\,.$$

On account of $(3.3.5)_2$, condition (3.3.6) simply states that the absolute temperature must be positive. (3.3.7), called the *Legendre-Hadamard condition*, means that ε is *rank-one convex* in F, i.e., it is convex along any direction $\xi^\top\nu$ with rank one. An alternative way of expressing (3.3.7) is to state that for any unit vector ν the *acoustic tensor* $N(\nu;F,s)$, defined by

$$(3.3.8) \qquad N_{ij}(\nu;F,s) = \frac{\partial^2\varepsilon(F,s)}{\partial F_{i\alpha}\partial F_{j\beta}} \nu_\alpha \nu_\beta\,, \qquad i,j = 1,2,3$$

is positive definite. In fact, for the system (3.3.4), the characteristic speeds in the direction ν are the six square roots of the three eigenvalues of the acoustic tensor, and zero with multiplicity seven.

Recall from Chapter II that, in addition to the system of balance laws (3.3.4), thermodynamically admissible processes should also satisfy the Clausius-Duhem inequality (2.3.13), which here takes the form

$$(3.3.9) \qquad -\partial_t s \le -\frac{r}{\theta(F,s)}\,.$$

By virtue of (3.3.5), every classical solution of (3.3.4) will satisfy (3.3.9) identically as an equality.[1] Hence, in the terminology of Section 3.2, $-s$ is an entropy for the system (3.3.4) with associated entropy flux zero.[2] Weak solutions of (3.3.4) will not

[1] Thus, for classical solutions it is convenient to substitute the equality (3.3.9) for the third equation in (3.3.4). In particular, if $r \equiv 0$, the entropy s stays constant along particle trajectories and one may determine F and v just by solving the first two equations of (3.3.4).

[2] Identifying $-s$ as the "entropy", rather than s itself which is the physical entropy, may look strange. This convention is adopted because it is more convenient to deal with functionals of the solution that are nonincreasing with time.

necessarily satisfy (3.3.9). Therefore, the role of (3.3.9) is to weed out undesirable weak solutions. The extension of a companion balance law from an identity for classical solutions into an inequality for weak solutions will play a crucial role in the general theory of hyperbolic systems of balance laws.

It should be noted that a solution (F,v,s) of (3.3.4) is not relevant to elastodynamics, unless F is a deformation gradient (2.1.2), or equivalently

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0, \qquad i=1,2,3, \quad \alpha,\beta=1,2,3. \tag{3.3.10}$$

In that case, as shown in Section 2.3, (F,v) will also satisfy the kinematic conservation laws (2.2.15) and (2.2.16), namely

$$\partial_t F^*_{\gamma k} - \partial_\alpha(\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}v_iF_{j\beta}) = 0, \qquad k=1,2,3, \quad \gamma=1,2,3 \tag{3.3.11}$$

$$\partial_t(\det F) - \partial_\alpha(v_iF^*_{\alpha i}) = 0. \tag{3.3.12}$$

Recall that (3.3.11) and (3.3.12) hold even when F and v are merely in L^∞.

The Rankine-Hugoniot jump conditions (3.1.3) for a shock front propagating in the direction $\nu \in \mathbb{S}^2$ with speed σ here take the form

$$\begin{cases} -\sigma[\![F_{i\alpha}]\!] = [\![v_i]\!]\nu_\alpha & i,\alpha=1,2,3 \\ -\sigma[\![v_i]\!] = [\![S_{i\alpha}(F,s)]\!]\nu_\alpha & i=1,2,3 \\ -\sigma[\![\varepsilon(F,s)+\tfrac{1}{2}|v|^2]\!] = [\![v_iS_{i\alpha}(F,s)]\!]\nu_\alpha\,, \end{cases} \tag{3.3.13}$$

where the double bracket denotes the jump of the enclosed quantity across the shock. By combining the three equations in (3.3.13), we can eliminate the velocity:

$$-\sigma\left\{[\![\varepsilon]\!] - \mathrm{tr}\left(\tfrac{1}{2}(S_+ + S_-)^\top[\![F]\!]\right)\right\} = 0. \tag{3.3.14}$$

Any shock associated with the physically relevant solution must also satisfy the jump condition

$$[\![F_{i\alpha}]\!]\nu_\beta = [\![F_{i\beta}]\!]\nu_\alpha\,, \qquad i=1,2,3, \qquad \alpha,\beta=1,2,3 \tag{3.3.15}$$

induced by (3.3.10), or equivalently,

$$[\![F_{i\alpha}]\!] = w_i\nu_\alpha\,, \qquad i=1,2,3, \qquad \alpha=1,2,3 \tag{3.3.16}$$

for some vector $w \in \mathbb{R}^3$. By virtue of $(3.3.13)_1$, this condition holds automatically, with $w = \sigma^{-1}[\![v]\!]$, for any shock with speed $\sigma \neq 0$. However, (3.3.16) disqualifies any isentropic shock with speed $\sigma = 0$. Indeed, for any such shock joining (F_-,s,v_-) and (F_+,s,v_+), $(3.3.13)_2$ together with (3.3.5), (3.3.8) and (3.3.16) would imply $\bar{N}w = 0$, where

$$\bar{N} = \int_0^1 N(\nu,(1-\tau)F_- + \tau F_+,s)d\tau. \tag{3.3.17}$$

Since $\bar{N}$ is positive definite, this yields $w = 0$ so no such shock is possible. On the other hand, there exist stationary nonisentropic shocks compatible with (3.3.16).

The inequality (1.8.5) induced by (3.3.9) takes the form

$$\sigma[\![s]\!] \leq 0, \tag{3.3.18}$$

which implies that whenever a material particle crosses a nonstationary shock, its physical entropy increases.

3.3.3 Isentropic Motion of Thermoelastic Nonconductors of Heat:

The physical background of isentropic processes was discussed in Section 2.5. In particular, as noted earlier, in the absence of heat supply, $r = 0$, any thermoelastic process that starts out isentropically remains isentropic for as long as it stays smooth. Moreover, the assumption of constant entropy is often a satisfactory approximation even for weak solutions. The entropy is fixed at a constant value $\bar{s}$ and, for simplicity, is dropped from the notation. The state vector reduces to $U = (F, v)$ with values in $\mathbb{R}^{12}$. The system of balance laws results from (3.3.4) by discarding the balance of energy:

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i, \alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = b_i\,, & i = 1,2,3 \end{cases} \tag{3.3.19}$$

and we still have

$$S_{i\alpha}(F) = \frac{\partial \varepsilon(F)}{\partial F_{i\alpha}}\,, \qquad i, \alpha = 1,2,3. \tag{3.3.20}$$

The system (3.3.19) is hyperbolic if ε is rank-one convex, i.e., (3.3.7) holds at $s = \bar{s}$. The characteristic speeds in the direction $\nu \in \mathbb{S}^2$ are the six square roots of the three eigenvalues of the acoustic tensor (3.3.8), at $s = \bar{s}$, and zero with multiplicity six.

As explained in Section 2.5, in addition to (3.3.19) thermodynamically admissible isentropic motions must also satisfy the inequality (2.5.32), which in the current notation reads

$$\partial_t[\varepsilon(F) + \tfrac{1}{2}|v|^2] - \partial_\alpha[v_i S_{i\alpha}(F)] \leq b_i v_i\,. \tag{3.3.21}$$

By virtue of (3.3.20), any classical solution of (3.3.19) satisfies identically (3.3.21) as an equality. Thus, in the terminology of Section 3.2, $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ is an entropy for the system (3.3.19). Note that (3.3.19) is in canonical form (3.1.4) and that $\mathrm{D}\eta = (S(F)\,, v)$. Therefore, as shown in Section 3.2, when the internal energy $\varepsilon(F)$ is uniformly convex, then changing the state vector from $U = (F, v)$ to $U^* = (S, v)$ will render the system (3.3.19) symmetric hyperbolic. It should be noted, however, that even though $\varepsilon(F)$ may be convex on a portion of the state space (especially near its minimum point), it cannot be globally convex, unless it is quadratic, in which case (3.3.19) is linear. This is a consequence of the principle of material frame indifference, which requires that $\varepsilon(F)$ be invariant under rigid rotations.

Weak solutions of (3.3.19) will not necessarily satisfy (3.3.21). We thus encounter again the situation in which a companion balance law is extended from an identity for classical solutions into an inequality serving as admissibility condition on weak solutions.

Classical or weak solutions of (3.3.19) that are relevant to elastodynamics must also satisfy (3.3.10), and thereby the kinematic conservation laws (3.3.11) and (3.3.12).

The Rankine-Hugoniot jump conditions (3.1.3) for a shock front propagating in the direction $\nu \in \mathbb{S}^2$ with speed σ take the form

$$(3.3.22)\qquad \begin{cases} -\sigma\, [\![F_{i\alpha}]\!] = [\![v_i]\!]\, \nu_\alpha & i,\alpha = 1,2,3 \\ \\ -\sigma\, [\![v_i]\!] = [\![S_{i\alpha}(F)]\!]\, \nu_\alpha & i = 1,2,3. \end{cases}$$

Shocks associated with physically relevant solutions should also satisfy (3.3.14) and (3.3.15), which, as we saw above, disqualifies all stationary shocks.

The inequality (1.8.5) induced by (3.3.21) is

$$(3.3.23)\qquad -\sigma\, \left[\!\!\left[\varepsilon(F) + \tfrac{1}{2}|v|^2\right]\!\!\right] - [\![v_i S_{i\alpha}(F)]\!]\, \nu_\alpha \leq 0\,,$$

which, in conjunction with (3.3.22), reduces to

$$(3.3.24)\qquad -\sigma \left\{ [\![\varepsilon]\!] - \mathrm{tr}\left(\tfrac{1}{2}(S_+ + S_-)^\top\, [\![F]\!]\right)\right\} \leq 0\,,$$

namely the analog of (3.3.14) that does not involve the velocity.

The passage from (3.3.4) to (3.3.19) provides an example of the truncation process that is commonly employed in continuum physics for simplifying systems of balance laws by dropping a number of the equations while simultaneously reducing proportionately the size of the state vector, according to the rules laid down in Section 1.5. In fact, one may derive the companion balance law (3.3.21) for the truncated system (3.3.19) from the companion balance law (3.3.9) of the original system (3.3.4) by using the recipe (1.5.12). Recall that in a canonical truncation, the elimination of any equation should be paired with freezing the corresponding component of the special state vector that symmetrizes the system. Thus, for instance, one may canonically truncate the system (3.3.19) by dropping the i-th of the last three equations while freezing the i-th component v_i of velocity, or else by dropping the (i,α)-th of the first nine equations while freezing the (i,α)-th component $S_{i\alpha}(F)$ of the Piola-Kirchhoff stress.

As explained in Section 2.5, the balance laws for isothermal processes of thermoelastic materials are obtained by replacing in (3.3.19), (3.3.20) and (3.3.21) the internal energy $\varepsilon(F)$, at constant entropy, with the Helmholtz free energy $\psi(F)$, at constant temperature.

3.3.4 Isentropic Motion with Relaxation:

We consider isentropic motions of the material considered in Section 2.8, assuming for simplicity that the reference density $\rho_0 = 1$ and the body force $b = 0$. The state vector is $U = (F, v, \Sigma)$, with values in $\mathbb{R}^{21}$. The system of balance laws is composed of the compatibility equation $(2.1.8)_1$, the balance of linear momentum (2.3.4) and the balance law (2.8.3) for the internal variable Σ:

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i,\alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha [P_{i\alpha}(F) + \Sigma_{i\alpha}] = 0, & i = 1,2,3 \\ \partial_t \Sigma_{i\alpha} = \dfrac{1}{\tau}[\Pi_{i\alpha}(\Sigma) - F_{i\alpha}], & i,\alpha = 1,2,3. \end{cases} \tag{3.3.25}$$

Furthermore,

$$P_{i\alpha}(F) = \frac{\partial \sigma(F)}{\partial F_{i\alpha}}, \qquad \Pi_{i\alpha}(\Sigma) = -\frac{\partial h(\Sigma)}{\partial \Sigma_{i\alpha}}. \tag{3.3.26}$$

In addition to (3.3.25), thermodynamically admissible (isentropic) motions should satisfy the entropy inequality (2.5.32), which here takes the form

$$\partial_t \left[\sigma(F) + \mathrm{tr}(\Sigma F^\top) + h(\Sigma) + \tfrac{1}{2}|v|^2\right] - \partial_\alpha [v_i P_{i\alpha}(F) + v_i \Sigma_{i\alpha}] \leq 0, \tag{3.3.27}$$

so that, in the terminology of Section 3.2, $\sigma(F) + \mathrm{tr}(\Sigma F^\top) + h(\Sigma) + \frac{1}{2}|v|^2$ is an entropy for (3.3.25).

The system (3.3.25) is hyperbolic when

$$\frac{\partial^2 \sigma(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta \xi_i \xi_j + \nu_\alpha \nu_\alpha \xi_i \zeta_i + \frac{\partial^2 h(\Sigma)}{\partial \Sigma_{i\alpha} \partial \Sigma_{j\beta}} \nu_\alpha \nu_\beta \zeta_i \zeta_j > 0, \tag{3.3.28}$$

for all $\nu \in \mathbb{S}^2$ and $(\xi, \zeta)^\top \in \mathbb{S}^5$.

3.3.5 Thermoelastic Fluid Nonconductors of Heat:

The system of balance laws (3.3.4) governs the adiabatic thermodynamic processes of all thermoelastic media, including, in particular, thermoelastic fluids. In the latter case, however, it is advantageous to employ spatial (Eulerian) description. The reason is that, as shown in Section 2.5, the internal energy, the temperature, and the Cauchy stress in a thermoelastic fluid depend on the deformation gradient F solely through the density ρ. We may thus dispense with F and describe the state of the medium through the state vector $U = (\rho, v, s)$ which takes values in the (much smaller) space $\mathbb{R}^5$.

The fields will now be functions of (χ, t). However, for consistency with the notational conventions of this chapter, we will replace the symbol χ by x. Also we will be using ∂_t (rather than a t-subscript as in Chapter II) to denote partial derivatives with respect to t.

The balance laws in force are for mass (2.3.2), linear momentum (2.3.5) and energy (2.3.11). The constitutive relations are (2.5.12), with $\tilde{q} \equiv 0$, (2.5.13) and (2.5.14). To simplify the notation, we drop the "tilde" and write $\varepsilon(\rho, s)$ in place of $\tilde{\varepsilon}(\rho, s)$. Therefore, the system of balance laws takes the form

$$(3.3.29)\qquad \begin{cases} \partial_t\rho + \mathrm{div}\,(\rho v^\top) = 0 \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v v^\top) + \mathrm{grad}\, p(\rho,s) = \rho b, \\ \partial_t[\rho\varepsilon(\rho,s) + \frac{1}{2}\rho|v|^2] + \mathrm{div}\,[(\rho\varepsilon(\rho,s) + \frac{1}{2}\rho|v|^2 + p(\rho,s))v^\top] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \rho b\cdot v + \rho r, \end{cases}$$

with

$$(3.3.30)\qquad p(\rho,s) = \rho^2\varepsilon_\rho(\rho,s), \qquad \theta(\rho,s) = \varepsilon_s(\rho,s).$$

The system (3.3.29) will be hyperbolic if

$$(3.3.31)\qquad \varepsilon_s(\rho,s) > 0, \qquad p_\rho(\rho,s) > 0.$$

The characteristic speeds in the direction $\nu \in \mathbb{S}^2$ are $v\cdot\nu$, with multiplicity three, and $v\cdot\nu \pm \sqrt{p_\rho(\rho,s)}$, with multiplicity one. The quantity $c(\rho,s) = \sqrt{p_\rho(\rho,s)}$ expresses the speed of propagation of a weak front as perceived by an observer carried by the fluid flow, and is called the *sonic speed*.

In addition to (3.3.29), thermodynamically admissible processes must also satisfy the Clausius-Duhem inequality (2.3.14), which here reduces to

$$(3.3.32)\qquad \partial_t(-\rho s) + \mathrm{div}\,(-\rho s v^\top) \leq -\rho\frac{r}{\theta(\rho,s)}\,.$$

When the process is smooth, it follows from (3.3.29) and (3.3.30) that (3.3.32) holds identically, as an equality.[3] Consequently, $\eta = -\rho s$ is an entropy for the system (3.3.29) with associated entropy flux $-\rho s v^\top$. Once again we see that a companion balance law is extended from an identity for classical solutions into an inequality serving as a test for the physical admissibility of weak solutions.

Changing the state variables from (ρ, v, s) to (ρ, m, E), where $m = \rho v$ is the momentum density and $E = \rho\varepsilon + \frac{1}{2}\rho|v|^2$ is the energy density, reduces (3.3.29) to its canonical form. A long, routine calculation shows that, by virtue of (3.3.31), the entropy $\eta = -\rho s$ is a convex function of (ρ, m, E).

The Rankine-Hugoniot jump conditions for a shock front propagating in the direction $\nu \in \mathbb{S}^2$ with speed σ read as follows:

$$(3.3.33)\qquad \begin{cases} [\![\rho(v\cdot\nu - \sigma)]\!] = 0 \\ [\![\rho(v\cdot\nu - \sigma)v + p(\rho,s)\nu]\!] = 0 \\ [\![\rho(v\cdot\nu - \sigma)\left(\varepsilon(\rho,s) + \frac{1}{2}|v|^2\right) + p(\rho,s)v\cdot\nu]\!] = 0. \end{cases}$$

[3] Thus for smooth solutions it is often convenient to substitute the simpler equality (3.3.32) for the third equation of (3.3.29). Notice that the smoothness requirement is met when ρ, v and s are merely Lipschitz continuous, which allows for flows with weak fronts.

By combining these equations, one derives the analog of (3.3.14):

$$\rho_\pm(v_\pm \cdot \nu - \sigma)\left\{[\![\varepsilon]\!] + \tfrac{1}{2}(p_+ + p_-)[\![\rho^{-1}]\!]\right\} = 0. \tag{3.3.34}$$

Notice that (3.3.33) admits shocks with $v_\pm \cdot \nu = \sigma$ and $[\![p]\!] = 0$. These fronts propagate with characteristic speed and are called contact discontinuities or *vortex sheets*. Fluid particles may slide at different speeds on either side of a vortex sheet, but they cannot cross it. In addition, (3.3.33) support shocks with $v_\pm \cdot \nu \neq \sigma$, which are traversed by the orbits of fluid particles. The quantity in braces, in (3.3.34), vanishes along these shocks.

The inequality (1.8.5) induced by (3.3.32) here takes the form

$$\rho_\pm(v_\pm \cdot \nu - \sigma)[\![s]\!] \geq 0\,, \tag{3.3.35}$$

which implies that when fluid particles cross a shock the physical entropy increases.

3.3.6 Isentropic Flow of Thermoelastic Fluids:
As shown above, in the absence of heat supply, $r = 0$, thermoelastic flows starting out isentropically remain isentropic for as long as they stay smooth. Furthermore, the assumption of constant entropy is satisfactory even after shocks develop, provided their amplitude is small. In an isentropic flow, the entropy is fixed at a constant value and is dropped from the notation. The state vector reduces to $U = (\rho, v)$, with values in $\mathbb{R}^4$. The system of balance laws results from (3.3.29) by discarding the balance of energy:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \\ \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) = \rho b, \end{cases} \tag{3.3.36}$$

with

$$p(\rho) = \rho^2 \varepsilon'(\rho). \tag{3.3.37}$$

The system (3.3.36) is hyperbolic if

$$p'(\rho) > 0. \tag{3.3.38}$$

The characteristic speeds in the direction $\nu \in \mathbb{S}^2$ are $v \cdot \nu$, with multiplicity two, and $v \cdot \nu \pm \sqrt{p'(\rho)}$, with multiplicity one. The quantity $c(\rho) = \sqrt{p'(\rho)}$ is the *sonic speed*. In particular, (3.3.38) is satisfied in the case of the ideal gas (2.5.31), as long as $\rho > 0$.

Thermodynamically admissible isentropic motions must satisfy the inequality (2.5.33), which here reduces to

$$\partial_t[\rho\varepsilon(\rho) + \tfrac{1}{2}\rho|v|^2] + \operatorname{div}[(\rho\varepsilon(\rho) + \tfrac{1}{2}\rho|v|^2 + p(\rho))v^\top] \leq \rho b \cdot v. \tag{3.3.39}$$

It should be noted that the system (3.3.36) results from the system (3.3.29) by canonical truncation, as described in Section 1.5, and in particular the companion balance law (3.3.39) can be derived from the companion balance law (3.3.32) by means of (1.5.12).

The pattern has by now become familiar: By virtue of (3.3.37), any classical solution of (3.3.36) satisfies identically (3.3.39), as an equality, so that the function $\eta = \rho\varepsilon(\rho) + \frac{1}{2}\rho|v|^2$ is an entropy for the system (3.3.36). At the same time, the inequality (3.3.39) is employed to weed out physically inadmissible weak solutions.

The system (3.3.36) attains its canonical form by changing the state variables from (ρ, v) to (ρ, m), where $m = \rho v$ is the momentum density. It is easily seen that, on account of (3.3.37) and (3.3.38), the above entropy η is a convex function of (ρ, m).

The Rankine-Hugoniot jump conditions for a shock front propagating in the direction $\nu \in \mathbb{S}^2$ with speed σ take the form

$$\begin{cases} [\![\rho(v\cdot\nu - \sigma)]\!] = 0 \\ [\![\rho(v\cdot\nu - \sigma)v + p(\rho)\nu]\!] = 0. \end{cases} \tag{3.3.40}$$

As in the nonisentropic case, we have contact discontinuities or vortex sheets, with $v_\pm \cdot \nu = \sigma$, and $[\![p]\!] = 0$, as well as shocks with $v_\pm \cdot \nu \neq \sigma$.

The inequality (1.8.5) induced by (3.3.39) is

$$[\![\rho(v\cdot\nu - \sigma)(\varepsilon(\rho) + \tfrac{1}{2}|v|^2) + p(\rho)v\cdot\nu]\!] \leq 0. \tag{3.3.41}$$

By virtue of (3.3.40), the inequality (3.3.41) may be written as

$$\rho_\pm(v_\pm\cdot\nu - \sigma)\left\{[\![\varepsilon]\!] + \tfrac{1}{2}(p_+ + p_-)[\![\rho^{-1}]\!]\right\} \leq 0. \tag{3.3.42}$$

For a broad class of equations of state, which includes the polytropic gas, (3.3.42) implies that when fluid particles cross a shock their density increases and their normal speed decreases.

We now assume that we have a smooth isentropic flow, with body force derived from a potential

$$b = -\operatorname{grad} g, \tag{3.3.43}$$

and monitor the evolution of the spin tensor W introduced in Section 2.1. Upon combining the two equations in (3.3.36), we get

$$\partial_t v + Lv + \operatorname{grad}[h(\rho) + g] = 0, \tag{3.3.44}$$

where L is the velocity gradient and $h = \varepsilon + p/\rho$ is the *enthalpy*, with derivative $h'(\rho) = p'(\rho)/\rho$. From (2.1.9), $Lv = 2Wv + L^\top v$ and $L^\top v = \operatorname{grad}(\frac{1}{2}|v|^2)$, so that

$$\partial_t v + 2Wv + \operatorname{grad}\left[h(\rho) + \tfrac{1}{2}|v|^2 + g\right] = 0. \tag{3.3.45}$$

We differentiate (3.3.45) and take the skew-symmetric part, which gives

$$\partial_t W + (\mathrm{d}W)v + WL - L^\top W^\top = 0, \tag{3.3.46}$$

or, upon using (2.1.4) and (2.1.9),

$$\dot{W} + WD + DW = 0. \tag{3.3.47}$$

Thus, when W, and thereby the vorticity $\omega = \operatorname{curl} v$, vanish at some point in space-time, they must vanish all along the trajectory of the particle that happens to occupy that position. In particular, if the vorticity vanishes for all x at $t = 0$, then it must vanish everywhere, for as long as the flow stays smooth. Such a flow is called *irrotational* or *potential*, as the velocity field derives from a potential ϕ:

$$v = \operatorname{grad} \phi. \tag{3.3.48}$$

Substituting v from (3.3.48) into (3.3.45), setting $W = 0$ and integrating the resulting equation, we deduce that irrotational flows satisfy the *Bernoulli equation*

$$\partial_t \phi + \tfrac{1}{2} |\operatorname{grad} \phi|^2 + h(\rho) + g = 0. \tag{3.3.49}$$

The constants of integration have been absorbed into the term $\partial_t \phi$.

Thus irrotational flows may be determined by solving the Bernoulli equation (3.3.49) together with the continuity equation $(3.3.36)_1$, which now takes the form

$$\partial_t \rho + \operatorname{div} [\rho (\operatorname{grad} \phi)^\top] = 0. \tag{3.3.50}$$

The above analysis applies even when ρ and v are merely Lipschitz continuous, i.e., irrotationality is preserved even in the presence of weak fronts, but it generally breaks down when jump discontinuities develop, because shocks generate vorticity. However, it has been common practice to employ the system (3.3.49), (3.3.50) even in the regime of weak solutions, notwithstanding that the resulting flows may fail to satisfy the balance of momentum equation $(3.3.36)_2$. This is in the same spirit as considering discontinuous isentropic flows, $s =$constant, even though they may fail to satisfy the energy balance equation $(3.3.29)_3$. Similar to the isentropic approximation, the error is small when the discontinuities (shocks) are weak. A major limitation is that irrotational flow cannot support vortex sheets. Indeed, since ϕ is Lipschitz, its tangential derivatives must be continuous across jump discontinuities. The condition $v_\pm \cdot \nu = \sigma$, characterizing vortex sheets, would imply that the normal derivative of ϕ would also be continuous, so no such jump discontinuities may exist.

The equations governing isothermal flow are obtained by replacing in (3.3.36), (3.3.37) and (3.3.39) the internal energy $\varepsilon(\rho)$, at constant entropy, with the Helmholtz free energy $\psi(\rho)$, at constant temperature.

Isentropic and isothermal flows of thermoelastic fluids are examples of flows in which the pressure depends solely on the density, which are known as *barotropic*.

3.3.7 The Boltzmann Equation and Extended Thermodynamics:

In contrast to continuum physics, kinetic theories realize matter as an aggregate of interacting molecules, and characterize the state by means of the molecular density function $f(\xi, x, t)$ of the velocity $\xi \in \mathbb{R}^3$ of molecules occupying the position $x \in \mathbb{R}^3$

at time t. In the classical kinetic theory, which applies to monatomic gases, $f(\xi,x,t)$ satisfies the *Boltzmann equation*

$$\partial_t f + \xi \cdot \mathrm{grad}_x f = \mathscr{Q}[f], \tag{3.3.51}$$

where $\mathscr{Q}$ stands for a complicated integral operator that accounts for changes in f incurred by collisions between molecules.

A formal connection between the continuum and the kinetic approach can be established by monitoring the family of moments

$$F_{i_1\dots i_N} = \int_{\mathbb{R}^3} \xi_{i_1}\cdots\xi_{i_N} f d\xi, \qquad i_1,\cdots,i_N = 1,2,3 \tag{3.3.52}$$

of the density f. Indeed, these moments satisfy an infinite system of evolution equations

$$\begin{cases} \partial_t F + \partial_j F_j = 0 & \\ \partial_t F_i + \partial_j F_{ij} = 0\,, & i = 1,2,3 \\ \partial_t F_{ij} + \partial_k F_{ijk} = P_{ij}\,, & i,j = 1,2,3 \\ \partial_t F_{ijk} + \partial_\ell F_{ijk\ell} = P_{ijk}\,, & i,j,k = 1,2,3 \\ \cdots\cdots\cdots\cdots\cdots & \\ \partial_t F_{i_1\dots i_N} + \partial_m F_{i_1\dots i_N m} = P_{i_1\dots i_N}\,, & i_1,\dots,i_N = 1,2,3. \end{cases} \tag{3.3.53}$$

In the above equations, and throughout this section, ∂_i denotes $\partial/\partial x_i$ and we employ the summation convention: repeated indices are summed over the range 1,2,3. The term $P_{i_1\dots i_N}$ denotes the integral of $\xi_{i_1}\cdots\xi_{i_N}\mathscr{Q}[f]$ over $\mathbb{R}^3$. Because of the special structure of $\mathscr{Q}$, the trace P_{ii} of P_{ij} vanishes.

We notice that each equation of (3.3.53) may be regarded as a balance law, in the spirit of continuum physics. In that interpretation, the moments of f are playing the role of both density and flux of balanced extensive quantities. In fact, the flux in each equation becomes the density in the following one. Under the identification

$$F = \rho\,, \tag{3.3.54}$$

$$F_i = \rho v_i\,, \qquad i = 1,2,3, \tag{3.3.55}$$

$$F_{ij} = \rho v_i v_j - T_{ij}\,, \qquad i,j = 1,2,3, \tag{3.3.56}$$

$$\tfrac{1}{2}F_{ii} = \rho\varepsilon + \tfrac{1}{2}\rho|v|^2, \tag{3.3.57}$$

$$\tag{3.3.58} \tfrac{1}{2}F_{iik} = \left(\rho\varepsilon + \tfrac{1}{2}\rho|v|^2\right)v_k - T_{ki}v_i + q_k, \qquad k = 1,2,3,$$

the first equation in (3.3.53) renders conservation of mass, the second equation renders conservation of linear momentum, and one half the trace of the third equation renders conservation of energy, for a heat-conducting viscous gas with density ρ, velocity v, internal energy ε, Cauchy stress T and heat flux q. We regard T as the superposition, $T = -pI + \sigma$, of a pressure $p = -\frac{1}{3}T_{ii}$ and a shearing stress σ that is traceless, $\sigma_{ii} = 0$. By virtue of (3.3.56) and (3.3.57),

$$\tag{3.3.59} \rho\varepsilon = \tfrac{3}{2}p,$$

which is compatible with the constitutive equations (2.5.20) of the ideal gas, for $\gamma = 5/3$.

Motivated by the above observations, one may construct a full hierarchy of continuum theories by truncating the infinite system (3.3.53), retaining only a finite number of equations. The resulting systems, however, will not be closed, because the highest order moments, appearing as flux(es) in the last equation(s), and also the production terms on the right-hand side remain undetermined. In the spirit of continuum physics, *extended thermodynamics* closes these systems by postulating that the highest order moments and the production terms are related to the lower order moments by constitutive equations that are determined by requiring that all smooth solutions of the system satisfy identically a certain inequality, akin to the Clausius-Duhem inequality. This induces a companion balance law which renders the system symmetrizable and thereby hyperbolic. The principle of material frame indifference should also be observed by the constitutive relations.

To see how the program works in practice, let us construct a truncation of (3.3.53) with state vector $U = (\rho, v, p, \sigma, q)$, which has dimension 13, as σ is symmetric and traceless. For that purpose, we retain the first three of the equations of (3.3.53), for a total of 10 independent scalar equations, and also extract 3 equations from the fourth equation of (3.3.53) by contracting two of the indices. By virtue of (3.3.54), (3.3.55), (3.3.56), (3.3.57), (3.3.58) and since P, P_i and P_{ii} vanish, we end up with the system

$$\tag{3.3.60} \begin{cases} \partial_t\rho + \partial_j\left(\rho v_j\right) = 0 \\[2ex] \partial_t\left(\rho v_i\right) + \partial_j\left(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}\right) = 0 \\[2ex] \partial_t\left(\rho\varepsilon + \frac{1}{2}\rho|v|^2\right) + \partial_k\left\{\left(\rho\varepsilon + \frac{1}{2}\rho|v|^2 + p\right)v_k - \sigma_{kj}v_j + q_k\right\} = 0 \\[2ex] \partial_t\left(\rho v_i v_j - \frac{1}{3}\rho|v|^2\delta_{i_j} - \sigma_{ij}\right) + \partial_k\left(F_{ijk} - \frac{1}{3}F_{\ell\ell k}\delta_{ij}\right) = P_{ij} \\[2ex] \partial_t\left\{\left(\rho\varepsilon + \frac{1}{2}\rho|v|^2 + p\right)v_k - \sigma_{kj}v_j + q_k\right\} + \frac{1}{2}\partial_i F_{jjik} = \frac{1}{2}P_{iik}\,. \end{cases}$$

This system can be closed by postulating that F_{ijk}, F_{jjik}, P_{ij} and P_{iik} are functions of the state vector $U = (\rho, v, p, \sigma, q)$, which are determined by requiring that all smooth solutions satisfy identically an inequality

$$\tag{3.3.61} \partial_t\varphi + \partial_i\psi_i \leq 0,$$

where φ and ψ_i are (unspecified) functions of U, and $\varphi(U)$ is convex. After a lengthy calculation (see the references cited in Section 3.4), one obtains complicated albeit quite explicit constitutive relations:

$$F_{ijk} = \rho v_i v_j v_k + \big(p v_k + \tfrac{2}{5} q_k\big)\delta_{ij} + \big(p v_i + \tfrac{2}{5} q_i\big)\delta_{jk} + \big(p v_j + \tfrac{2}{5} q_j\big)\delta_{ik}\,, \tag{3.3.62}$$

$$\begin{aligned} F_{jjik} = {} & \big(\rho|v|^2 + 7p\big) v_i v_k + \big(p\delta_{ik} - \sigma_{ik}\big)|v|^2 - \sigma_{ij} v_j v_k \\ & - \sigma_{kj} v_j v_i + \frac{14}{5}\,(q_i v_k + q_k v_i) + \frac{4}{5} q_j v_j \delta_{ik} + \frac{p}{\rho}\,(5p\delta_{ik} - 7\sigma_{ik})\,, \end{aligned} \tag{3.3.63}$$

$$P_{ij} = \tau_0 \sigma_{ij}\,, \qquad P_{iik} = 2\tau_0 \sigma_{ki} v_i - \tau_1 q_k\,. \tag{3.3.64}$$

To complete the picture, p, τ_0 and τ_1 must be specified as functions of (ρ, θ).

The special vector $U^* = B(U)$, in the notation of Section 1.5, that symmetrizes the system has components

$$U^* = \frac{1}{\theta} \begin{pmatrix} \dfrac{5p}{2\rho} - \theta s - \dfrac{1}{2}|v|^2 + \dfrac{1}{2p}\sigma_{ij} v_i v_j - \dfrac{\rho}{5p^2} q_i v_i |v|^2 \\[2ex] v_i - \dfrac{1}{p}\sigma_{ij} v_j + \dfrac{\rho}{5p^2}\left(|v|^2 q_i + 2 q_j v_j v_i\right) \\[2ex] -1 + \dfrac{2\rho}{3p^2} q_k v_k \\[2ex] -\dfrac{1}{2p}\sigma_{ij} - \dfrac{\rho}{5p^2}\left(v_i q_j + v_j q_i - \tfrac{2}{3} v_k q_k \delta_{ij}\right) \\[2ex] \dfrac{\rho}{5p^2} q_i \end{pmatrix} \tag{3.3.65}$$

In particular, as explained in Section 1.5, truncating the system (3.3.60) by dropping the last two equations should be paired with "freezing" the last two components of U^*, i.e., by setting $q = 0$ and $\sigma = 0$. In that case, the system of the first three equations of (3.3.60) reduces to the system (3.3.29), in the particular situation where $b = 0$, $r = 0$ and $\rho\varepsilon$ and p are related by (3.3.59). If one interprets (ρ, v, p) as the basic state variables and (σ, q) as internal state variables, as explained in Section 2.8, then (3.3.29) becomes the relaxed form of the system (3.3.60).

3.3.8 Nonlinear Electrodynamics:

Another rich source of interesting systems of hyperbolic balance laws is electromagnetism. The underlying system consists of Maxwell's equations

$$\begin{cases} \partial_t B = -\operatorname{curl} E \\[1ex] \partial_t D = \operatorname{curl} H - J \end{cases} \tag{3.3.66}$$

$$\operatorname{div} B^\top = 0, \qquad \operatorname{div} D^\top = \rho \tag{3.3.67}$$

on $\mathbb{R}^3$, relating the *electric field* E, the *magnetic field* H, the *electric displacement* D, the *magnetic induction* B, the *current* J, and the *charge* ρ. In turn, the current and charge are interrelated by the *continuity equation*

$$\partial_t \rho + \operatorname{div} J^\top = 0. \tag{3.3.68}$$

Constitutive relations determine E and H from the state vector $U = (B, D)$. For example, when the medium is a homogeneous electric conductor, with linear dielectric response, at rest relative to the inertial frame, then $D = \varepsilon E$ and $B = \mu H$, where ε is the *dielectric constant* and μ is the *magnetic permeability*. In order to account for nonlinear dielectric response and cross-coupling of electromagnetic fields, one postulates general constitutive equations

$$E = \frac{\partial \eta(B,D)}{\partial D}, \qquad H = \frac{\partial \eta(B,D)}{\partial B}, \tag{3.3.69}$$

or

$$D = \frac{\partial h(B,E)}{\partial E}, \qquad H = -\frac{\partial h(B,E)}{\partial B}, \tag{3.3.70}$$

where η is the *electromagnetic field energy* and h is the *Lagrangian*. Notice that η is the Legendre transform of h with respect to E.

Physically admissible fields must also satisty the dissipation inequality

$$\partial_t \eta(B,D) + \operatorname{div} Q^\top(B,D) \leq -J \cdot E, \tag{3.3.71}$$

where

$$Q = E \wedge H \tag{3.3.72}$$

is the *Poynting vector*. A straightforward calculation shows that smooth solutions of (3.3.66), (3.3.69) satisfy (3.3.71) identically, as an equality. Therefore, (η, Q) constitutes an entropy-entropy flux pair for the system of balance laws (3.3.66), (3.3.69). Since $\mathrm{D}\eta = (H, E)$, it follows from the discussion in Section 3.2 that when the electromagnetic field energy function is uniformly convex, then the change of state vector from $U = (B, D)$ to $U^* = (H, E)$ renders the system symmetric hyperbolic. It should be noted, however, that even though η may be convex on a portion of the state space (especially near its minimum point) it cannot be globally convex, unless it is quadratic, in which case (3.3.66) is linear. This is a consequence of the requirement that the Lagrangian $h(B, E)$ be invariant under Lorentz transformations, and hence may depend on (B, E) solely through the scalar quantities $|B|^2 - |E|^2$ and $B \cdot E$. This in turn implies that η must be invariant under rigid rotations and thus may depend on (B, D) solely through the scalar quantities $|B|^2, |D|^2$ and $B \cdot D$. No such function of (B, D) may be globally convex, unless it is quadratic. In this respect, there is remarkable similarity between electomagnetism and elastodynamics. Another important consequence of the Lorentz invariance of the Lagrangian is that $E \wedge H = B \wedge D$. An illustration is provided by the *Born-Infeld constitutive relations*[4]

[4] These were designed so that, contrary to the classical linear theory, the electromagnetic energy generated by a point charge at rest is finite.

$$(3.3.73) \qquad \begin{cases} E = \dfrac{\partial \eta}{\partial D} = \dfrac{1}{\eta}[D + B \wedge Q], & H = \dfrac{\partial \eta}{\partial B} = \dfrac{1}{\eta}[B - D \wedge Q] \\ \eta = \sqrt{1 + |B|^2 + |D|^2 + |Q|^2}, & Q = D \wedge B = E \wedge H. \end{cases}$$

Here η is not globally convex, but the resulting system (3.3.66) is still hyperbolic.

Under constitutive relations (3.3.69), any smooth solution of both (3.3.66) and (3.3.67) satisfies the additional balance law

$$(3.3.74) \qquad \partial_t P(B,D) + \operatorname{div} L(B,D) = -\rho E + B \wedge J,$$

with

$$(3.3.75) \qquad P = B \wedge D, \qquad L = ED^\top + HB^\top + (\eta - E \cdot D - H \cdot B)I.$$

In particular, in the Born-Infeld case,

$$(3.3.76) \qquad P = Q, \qquad L = \eta^{-1}(I + BB^\top + DD^\top - QQ^\top).$$

The reader should note the difference between (3.3.71) and (3.3.74): The former is contingent solely on (3.3.66), while the latter requires both (3.3.66) and (3.3.67) to hold.

The Rankine-Hugoniot jump conditions for a shock front propagating in the direction $\nu \in \mathbb{S}^2$ with speed σ take the form

$$(3.3.77) \qquad \begin{cases} -\sigma [\![B]\!] = [\![E]\!] \wedge \nu \\ -\sigma [\![D]\!] = -[\![H]\!] \wedge \nu. \end{cases}$$

For compatibility with (3.3.67), shocks should also satisfy the jump conditions

$$(3.3.78) \qquad [\![B]\!] \cdot \nu = 0, \qquad [\![D]\!] \cdot \nu = 0.$$

Notice that (3.3.78) follow from (3.3.77), when $\sigma \neq 0$. On the other hand, (3.3.78) in conjunction with hyperbolicity rule out the possibility of shocks with $[\![E]\!]$ and $[\![H]\!]$ collinear to ν, that would satisfy (3.3.77) for $\sigma = 0$.

The inequality (1.8.5) induced by (3.3.71) reads

$$(3.3.79) \qquad -\sigma [\![\eta(B,D)]\!] + [\![E \wedge H]\!] \cdot \nu \leq 0.$$

It is also noteworthy that under the Born-Infeld constitutive relations (3.3.73), the jump conditions (3.3.77), (3.3.78) imply

$$(3.3.80) \qquad -\sigma [\![P]\!] + [\![L]\!] \nu = 0\,,$$

with P and L given by (3.3.76). This means, in particular, that the extra balance law (3.3.74) is automatically satisfied not only by smooth, but even by BV weak solutions of (3.3.66) and (3.3.67).

3.3.9 Magnetohydrodynamics:
Interesting systems of hyperbolic balance laws arise in the context of electromechanical phenomena, where the balance laws of mass, momentum and energy of continuum thermomechanics are coupled with Maxwell's equations. As an illustrative example, we consider here the theory of *magnetohydrodynamics*, which describes the interaction of a magnetic field with an electrically conducting thermoelastic fluid. The equations follow from a number of simplifying assumptions, which will now be outlined.

Beginning with Maxwell's equations, the electric displacement D is considered negligible so (3.3.66) yields $J = \operatorname{curl} H$. The magnetic induction B is related to the magnetic field H by the classical relation $B = \mu H$. The electric field is generated by the motion of the fluid in the magnetic field and so is given by $E = B \wedge v = \mu H \wedge v$.

The fluid is a thermoelastic nonconductor of heat whose thermomechanical properties are still described by the constitutive relations (3.3.30). The balance of mass $(3.3.29)_1$ remains unaffected by the presence of the electromagnetic field. On the other hand, the electromagnetic field exerts a force on the fluid which should be accounted as body force in the balance of momentum $(3.3.29)_2$. The contribution of the electric field E to this force is assumed negligible while the contribution of the magnetic field is $J \wedge B = -\mu H \wedge \operatorname{curl} H$. On account of the identity

$$-H \wedge \operatorname{curl} H = \operatorname{div}[HH^\top - \tfrac{1}{2}(H \cdot H)I], \tag{3.3.81}$$

this body force may be realized as the divergence of the *Maxwell stress tensor*. We assume there is no external body force. To account for the electromagnetic effects on the energy equation $(3.3.29)_3$, the internal energy should be augmented by the electromagnetic field energy $\frac{1}{2}\mu|H|^2$, and $\mu(H \wedge v) \wedge H = \mu|H|^2 v - \mu(H \cdot v)H$, namely the Poynting vector, should be added to the flux. The electromagnetic energy production $-J \cdot E = -\mu(H \wedge v) \cdot \operatorname{curl} H$ and the rate of work $(J \wedge B) \cdot v = -\mu(H \wedge \operatorname{curl} H) \cdot v$ of the electromagnetic body force cancel each other out.

We thus arrive at

(3.3.82)
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \operatorname{div}[\rho v v^\top - \mu H H^\top] + \operatorname{grad}\left[p(\rho,s) + \frac{1}{2}\mu|H|^2\right] = 0 \\ \partial_t\left[\rho\varepsilon(\rho,s) + \frac{1}{2}\rho|v|^2 + \frac{1}{2}\mu|H|^2\right] \\ \quad + \operatorname{div}\left[(\rho\varepsilon(\rho,s) + \frac{1}{2}\rho|v|^2 + p(\rho,s) + \mu|H|^2)v^\top - \mu(H\cdot v)H^\top\right] = \rho r \\ \partial_t H - \operatorname{curl}(v \wedge H) = 0. \end{cases}$$

The above system of balance laws, with state vector $U = (\rho, v, s, H)$, will be hyperbolic if (3.3.31) hold. Thermodynamically admissible solutions of (3.3.82) should also satisfy the Clausius-Duhem inequality (3.3.32), with $r = 0$. By virtue of (3.3.30), it is easily seen that any classical solution of (3.3.82) satisfies identically (3.3.32) as an equality. Thus $-\rho s$ is an entropy for the system (3.3.82), with associated entropy flux $-\rho s v$.

The system (3.3.82) is supplemented by (3.3.67), namely

$$\operatorname{div} H = 0. \tag{3.3.83}$$

Magnetohydrodynamics supports a richer family of shocks than plain fluid dynamics. The Rankine-Hugoniot jump conditions associated with the system (3.3.82), for a shock propagating in the direction $\nu \in \mathbb{S}^2$ with speed σ, read

$$\begin{cases} [\![\rho(v\cdot\nu-\sigma)]\!]=0 \\ [\![\rho(v\cdot\nu-\sigma)v+(p+\frac{1}{2}\mu|H|^2)\nu-\mu(H\cdot\nu)H]\!]=0 \\ [\![(v\cdot\nu-\sigma)(\rho\varepsilon+\frac{1}{2}\rho|v|^2+\frac{1}{2}\mu|H|^2)+(v\cdot\nu)(p+\frac{1}{2}\mu|H|^2)-\mu(H\cdot\nu)(H\cdot v)]\!]=0 \\ [\![(v\cdot\nu-\sigma)H-(H\cdot\nu)v]\!]=0. \end{cases} \tag{3.3.84}$$

They are supplemented by the jump condition

$$[\![H\cdot\nu]\!]=0, \tag{3.3.85}$$

for (3.3.83), which asserts that the jump of the magnetic field must be tangential to the shock.

Upon combining (3.3.84) with (3.3.85), and after a lengthy calculation, one arrives at the analog of (3.3.34):

$$\rho_\pm(v_\pm\cdot\nu-\sigma)\left\{[\![\varepsilon]\!]+\frac{1}{2}(p_++p_-)[\![\rho^{-1}]\!]-\frac{1}{4}[\![\rho^{-1}]\!]\big|[\![H]\!]\big|^2\right\}=0. \tag{3.3.86}$$

The jump conditions (3.3.84), (3.3.85) support three types of shocks, namely:

(a) *Contact discontinuities*, akin to vortex sheets in fluid dynamics, with speed $\sigma = v_\pm\cdot\nu$ and $[\![p]\!]=0$, $[\![v]\!]=0$, $[\![H]\!]=0$, but $[\![\rho]\!]\neq 0$. No entropy is produced by such jumps.

(b)*Transverse shocks*, with $[\![\rho]\!]=0$, $[\![p]\!]=0$ and $[\![s]\!]=0$. In that case, $[\![v]\!]$ and $[\![H]\!]$ must be collinear,

$$[\![v]\!]=\pm\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}[\![H]\!], \tag{3.3.87}$$

so in particular the velocity jump is tangential to the shock, $[\![v\cdot\nu]\!]=0$. Furthermore, the strength of the magnetic field is continuous across these shocks, $[\![\,|H|\,]\!]=0$.

(c) Shocks across which all fields, ρ, p, η, v and H sustain nonzero jumps. When the Clausius-Duhem inequality (3.3.32) is satisfied, these shocks are termed *compressive,* as density, pressure and entropy increase across them. Compressive shocks form two subfamilies, distinguished by their speed and accordingly dubbed *fast* and *slow*. The strength of the magnetic field increases across fast compressive shocks and decreases across slow compressive shocks.

There are corresponding types of weak magnetohydrodynamic waves, forming intricate geometric patterns.

3.4 Notes

As pointed out in the historical introduction, the theory of nonlinear hyperbolic systems of balance laws traces its origins to the mid 19th century and has developed over the years conjointly with gas dynamics. The classic monograph by Courant and Friedrichs [1] amply surveys, in mathematical language, the state of the subject at the end of the Second World War. It is the distillation of this material that has laid the foundations of the formalized mathematical theory in its present form.

The great number of books on the theoretical and the numerical analysis of hyperbolic systems of conservation laws published in recent years is a testament to the vitality of the field. The fact that these books complement each other, as they differ in scope, style and even content, is indicative of the breadth of the area.

Students who prefer to make their first acquaintance with the subject through a bird's-eye view may begin with the outlines in the treatise by M.E. Taylor [2], the textbooks by Evans [2], Hörmander [2], and Lax [7], or the lecture notes of Lax [5], Liu [28], Dafermos [6,10], and Bressan [15,16]. To a certain extent, some of the above references are dated, but the last two are current. In fact, Bressan [15] lists many of the important open problems, while Bressan [16] is an introduction to the field with text adorned with numerous informative figures.

On the theoretical side, Jeffrey [2], Rozdestvenski and Janenko [1], and Smoller [3] are early comprehensive texts at an introductory level. The more recent books by Serre [11], Bressan [9], Holden and Risebro [2], and LeFloch [5] combine a general introduction to the subject with advanced, deeper investigations in selected directions. The encyclopedic article by Chen and Wang [1] uses the Euler equations of gas dynamics as a springboard for surveying broadly the theory of strictly hyperbolic systems of conservation laws in one space dimension. Finally, Majda [4], Chang and Hsiao [3], Li, Zhang and Yang [1], Yuxi Zheng [1], Lu [2], Perthame [2], Benzoni-Gavage and Serre [2], Ben-Artzi and Falcovitz [1], and Tartar [4] are specialized monographs, more narrowly focussed. The above books will be cited again, in later chapters, as their content becomes relevant to the discussion, and thus the reader will get some idea of their respective offerings.

Turning to numerical analysis, LeVeque [1] is an introductory text, while the books by Godlewski and Raviart [1,2], and LeVeque [2] provide a more comprehensive and technical coverage together with a voluminous list of references.

Other useful sources are the books by Kröner [1], Sod [1], Toro [1], Kulikovski, Pogorelov and Semenov [1], and Holden and Risebro [2], and the lecture notes of Tadmor [2].

Another rich resource is the text by Whitham [2] which presents a panorama of connections of the theory with a host of diverse applications as well as a survey of ideas and techniques devised over the years by applied mathematicians studying wave propagation, of which many are ready for more rigorous analytical development. Zeldovich and Raizer [1,2], and Vincenti and Kruger [1] are excellent introductions to gas dynamics from the perspective of physicists and may be consulted for building intuition.

The student may get a sense of the evolution of research activity in the field over the past twenty years by consulting the Proceedings of the International Conferences on Hyperbolic Problems which are held biennially. Those that have already appeared at the time of this writing, listed in chronological order and under the names of their editors, are: Carasso, Raviart and Serre [1], Ballmann and Jeltsch [1], Engquist and Gustafsson [1], Donato and Oliveri [1], Glimm, Grove, Graham and Plohr [1], Fey and Jeltsch [1], Freistühler and Warnecke [1], Hou and Tadmor [1], Benzoni-Gavage and Serre [3], Tadmor, Liu and Tzavaras [1], Tatsien Li and Song Jiang [1], and Ancona, Bressan, Marcati and Marson [1].

An insightful perspective on the state of the subject at the turn of the century is provided by Serre [16].

The term ”entropy” in the sense employed here was introduced by Lax [4]. A collection of informative essays on various notions of “entropy” in physics and mathematics is found in the book edited by Greven, Keller and Warnecke [1]. The question whether entropies may exist for systems endowed with symmetry groups, such as invariance under rotations and Galilean transformations, is addressed in Sever [15,16].

The systems (3.3.29) and (3.3.36) are commonly called *Euler's equations*. There is voluminous literature on various aspects of their theory, some of which will be presented in subsequent chapters. For rudimentary aspects, the reader may consult any text on fluid mechanics, for example Chorin and Marsden [1]. A classification of convex entropies is found in Harten [1] and Harten, Lax, Levermore and Morokoff [1].

The literature on nonlinear elastodynamics is less extensive. Good references, with copious bibliography, are the books by Truesdell and Noll [1], and Antman [3].

The book by Cercignani [1] is an excellent introduction to the Boltzmann equation. For recent developments in the program of bridging the kinetic with the continuum theory of gases, see the informative survey articles by Villani [1], and Vasseur [7]. See also Berthelin and Vasseur [1].

For a thorough treatment of extended thermodynamics and its relation to the kinetic theory, the reader should consult the monograph by Müller and Ruggeri [1]. The issue of generating simpler systems by truncating more complex ones is addressed in detail by Boillat and Ruggeri [1].

For a systematic development of electrothermomechanics, along the lines of the development of continuum thermomechanics in Chapter II, see Coleman and Dill [1], and Grot [1]. Numerous examples of electrodynamical problems involving hyperbolic systems of balance laws are presented in Bloom [1]. Particularly relevant to the presentation here is the article by Serre [25]. See also Boillat [3,5]. The constitutive equations (3.3.73) were proposed by Born and Infeld [1]. The reader may find some of their remarkable properties, together with relevant references, in Chapter V. For magnetohydrodynamics see for example the texts of Cabannes [1], Jeffrey [1], Landau and Lifshitz [1], and Kulikovskiy and Lyubimov [1].

The theory of relativity is a rich source of interesting hyperbolic systems of balance laws, which will not be tapped in this work. When the fluid velocity is comparable to the speed of light, the Euler equations should be modified to account for special relativistic effects; cf. Taub [1], Friedrichs [3], and the book by Christodoulou [1]. The study of these equations from the perspective of the theory of hyperbolic balance laws has already produced a substantial body of literature. For orientation and extensive bibliography, the reader may consult the recent monograph by Groah, Smoller and Temple [2] or the survey articles by Smoller and Temple [3], and Groah, Smoller and Temple [1]. See also Ruggeri [1,2], Smoller and Temple [1,2], Pant [1], and Jing Chen [1].

Interesting hyperbolic systems of conservation laws also arise in differential geometry, in connection to the isometric immersion and evolution of surfaces, with shocks manifesting themselves as “kinks”; see Section 18.7 and Arun and Prasad [1].

Numerous additional examples of hyperbolic conservation laws in one space dimension will be presented in Chapter VII.

IV

The Cauchy Problem

The theory of the Cauchy problem for hyperbolic conservation laws is confronted with two major challenges. First, classical solutions, starting out from smooth initial values, spontaneously develop discontinuities; hence, in general, only weak solutions may exist in the large. Next, weak solutions to the Cauchy problem fail to be unique. One does not have to dig too deep in order to encounter these difficulties. As shown in Sections 4.2, 4.4 and 4.8, they arise even at the level of the simplest nonlinear hyperbolic conservation laws, in one or several space dimensions.

The Cauchy problem for weak solutions will be formulated in Section 4.3. To overcome the obstacle of nonuniqueness, restrictions need to be imposed that will weed out unstable, physically irrelevant, or otherwise undesirable solutions, in hope of singling out a unique admissible solution. Two admissibility criteria will be introduced in this chapter: the requirement that admissible solutions satisfy a designated entropy inequality; and the principle that admissible solutions should be limits of families of solutions to systems containing diffusive terms, as the diffusion asymptotically vanishes. A preliminary comparison of these criteria will be conducted.

A preliminary discussion on the issue of identifying mathematically and physically meaningful boundary conditions will be presented in Section 4.7.

The final section 4.8 of this chapter collects a representative sample of results on the Euler equations of (isentropic) gas dynamics, in three spatial dimensions, emerging from work of older or recent vintage and highlighting current research trends in that important area of hyperbolic conservation laws.

4.1 The Cauchy Problem: Classical Solutions

To avoid trivial complications, we focus the investigation on homogeneous hyperbolic systems of conservation laws in canonical form,

$$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) = 0, \tag{4.1.1}$$

even though the analysis can be extended in a routine manner to general hyperbolic systems of balance laws (3.1.1). The spatial variable x takes values in $\mathbb{R}^m$ and time

C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*,
Grundlehren der mathematischen Wissenschaften 325,
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t takes values in $[0,T)$, for some $T>0$ or possibly $T=\infty$. The state vector U takes values in some open subset $\mathscr{O}$ of $\mathbb{R}^n$ and $G=(G_1,\dots,G_m)$ is a given smooth function from $\mathscr{O}$ to $\mathbb{M}^{n\times m}$. The system (4.1.1) is hyperbolic when, for every fixed $U\in\mathscr{O}$ and $\nu\in\mathbb{S}^{m-1}$, the $n\times n$ matrix

$$\Lambda(\nu;U)=\sum_{\alpha=1}^{m}\nu_\alpha \mathrm{D}G_\alpha(U) \tag{4.1.2}$$

has real eigenvalues $\lambda_1(\nu;U),\dots,\lambda_n(\nu;U)$ and n linearly independent eigenvectors $R_1(\nu;U),\dots,R_n(\nu;U)$.

To formulate the Cauchy problem, we assign initial conditions

$$U(x,0)=U_0(x), \qquad x\in\mathbb{R}^m, \tag{4.1.3}$$

where U_0 is a function from $\mathbb{R}^m$ to $\mathscr{O}$.

A *classical solution* of (4.1.1) is a locally Lipschitz function U, defined on $\mathbb{R}^m\times[0,T)$ and taking values in $\mathscr{O}$, which satisfies (4.1.1) almost everywhere. This function solves the Cauchy problem, with initial data U_0, if it also satisfies (4.1.3) for all $x\in\mathbb{R}^m$.

As we shall see, the theory of the Cauchy problem is greatly enriched when the system is endowed with an entropy η with associated entropy flux Q, related by

$$\mathrm{D}Q_\alpha(U)=\mathrm{D}\eta(U)\mathrm{D}G_\alpha(U), \qquad \alpha=1,\cdots,m. \tag{4.1.4}$$

In that case, any classical solution of (4.1.1) will satisfy the additional conservation law

$$\partial_t\eta(U(x,t))+\operatorname{div}Q(U(x,t))=0. \tag{4.1.5}$$

As we proceed with the development of the theory, it will become clear that convex entropy functions exert a stabilizing influence on solutions. As a first indication of that effect, the following proposition shows that for systems endowed with a convex entropy, the range of influence of the initial data on solutions of the Cauchy problem is finite.

4.1.1 Theorem. *Assume* (4.1.1) *is a hyperbolic system, with characteristic speeds $\lambda_1(\nu;U)\le\cdots\le\lambda_n(\nu;U)$, which is endowed with an entropy $\eta(U)$ and associated entropy flux $Q(U)$. Suppose $U(x,t)$ is a classical solution of* (4.1.1) *on* $\mathbb{R}^m\times[0,T)$, *with initial data* (4.1.3)*, where U_0 is constant on a half-space: For some $\xi\in\mathbb{S}^{m-1}$, $U_0(x)=\bar U=$ constant whenever $x\cdot\xi\ge 0$. Assume, further, that $\mathrm{D}^2\eta(\bar U)$ is positive definite. Then, for any $t\in[0,T)$, $U(x,t)=\bar U$ whenever $x\cdot\xi\ge\lambda_n(\xi;\bar U)t$.*

Proof. Without loss of generality, we may assume that $\eta(\bar U)=0$, $\mathrm{D}\eta(\bar U)=0$, $Q_\alpha(\bar U)=0$, $\mathrm{D}Q_\alpha(\bar U)=0$, $\alpha=1,\dots,m$, since otherwise we just replace the given entropy-entropy flux pair with the new pair

$$\bar\eta(U)=\eta(U)-\eta(\bar U)-\mathrm{D}\eta(\bar U)[U-\bar U], \tag{4.1.6}$$

(4.1.7) $$\bar{Q}(U) = Q(U) - Q(\bar{U}) - \mathrm{D}\eta(\bar{U})\big[G(U) - G(\bar{U})\big].$$

For each $s \in \mathbb{R},\ \nu \in \mathbb{S}^{m-1}$ and $U \in \mathscr{O}$, we define

(4.1.8) $$\Phi(s,\nu;U) = s\eta(U) - Q(U)\nu,$$

noting that $\Phi(s,\nu;\bar{U}) = 0$ and $\mathrm{D}\Phi(s,\nu;\bar{U}) = 0$. Furthermore, upon using (4.1.4) and (4.1.2),

(4.1.9) $$\mathrm{D}^2\Phi(s,\nu;\bar{U}) = \mathrm{D}^2\eta(\bar{U})[sI - \Lambda(\nu;\bar{U})].$$

Hence, for $j,k = 1,\dots,n$,

(4.1.10)
$$R_j(\nu;\bar{U})^\top \mathrm{D}^2\Phi(s,\nu;\bar{U})R_k(\nu;\bar{U}) = [s - \lambda_k(\nu;\bar{U})]R_j(\nu;\bar{U})^\top \mathrm{D}^2\eta(\bar{U})R_k(\nu;\bar{U}).$$

The right-hand side of (4.1.10) vanishes for $j \neq k$, by virtue of (3.2.5), and has the same sign as $s - \lambda_k(\nu;\bar{U})$ for $j = k$, since $\mathrm{D}^2\eta(\bar{U})$ is positive definite.

For $\varepsilon > 0$, we set $\bar{s} = \max\limits_{\nu \in S^{m-1}} \lambda_n(\nu;\bar{U}) + \varepsilon$ and $\hat{s} = \lambda_n(\xi;\bar{U}) + \varepsilon$. Then there exists $\delta = \delta(\varepsilon)$ such that

(4.1.11) $$\begin{cases} \Phi(\bar{s},\nu;U) > 0, & \text{for } 0 < |U - \bar{U}| < \delta(\varepsilon),\ \nu \in \mathbb{S}^{m-1} \\ \Phi(\hat{s},\xi;U) > 0, & \text{for } 0 < |U - \bar{U}| < \delta(\varepsilon). \end{cases}$$

To establish the assertion of the theorem, it suffices to show that for each fixed $\varepsilon > 0$ and $t \in [0,T)$, $U(x,t) = \bar{U}$ whenever $x \cdot \xi \geq \hat{s}t$.

With any point (y,τ), where $\tau \in (0,T)$ and $y \cdot \xi \geq \hat{s}\tau$, we associate the cone

(4.1.12) $$\mathscr{K}_{y,\tau} = \{(x,t) : 0 \leq t \leq \tau,\ |x - y| \leq \bar{s}(\tau - t),\ x \cdot \xi \geq y \cdot \xi - \hat{s}(\tau - t)\},$$

which is contained in the set $\{(x,t) : 0 \leq t < T,\ x \cdot \xi \geq \hat{s}t\}$. Thus, the boundary of the t-section of $\mathscr{K}_{y,\tau}$ is the union $\mathscr{P}_t \cup \mathscr{S}_t$ of a subset $\mathscr{P}_t$ of a hyperplane perpendicular to ξ, and a subset $\mathscr{S}_t$ of the sphere with center y and radius $\bar{s}(\tau - t)$. The outward unit normal to the t-section at a point x is $-\xi$ if $x \in \mathscr{P}_t$, and $\bar{s}^{-1}(\tau - t)^{-1}(x - y)$ if $x \in \mathscr{S}_t$. Therefore, integrating (4.1.5) over $\mathscr{K}_{y,\tau}$, applying Green's theorem and using the notation (4.1.8) we obtain

(4.1.13)
$$\int_0^\tau\!\!\int_{\mathscr{P}_t} \Phi(\hat{s},\xi;U)\,d\mathscr{H}^{m-1}(x)dt + \int_0^\tau\!\!\int_{\mathscr{S}_t} \Phi\big(\bar{s},-\bar{s}^{-1}(\tau - t)^{-1}(x - y);U\big)\,d\mathscr{H}^{m-1}(x)dt = 0.$$

After this preparation, assume that the assertion of the theorem is false. Since $U(x,t)$ is continuous, and $U(x,0) = \bar{U}$ for $x \cdot \xi \geq 0$, one can find (y,τ), with $\tau \in (0,T)$ and

$y \cdot \xi \geq \hat{s}\tau$, such that $U(y,\tau) \neq \bar{U}$ and $|U(x,t) - \bar{U}| < \delta(\varepsilon)$ for all $(x,t) \in \mathscr{K}_{y,\tau}$. In that case, (4.1.13) together with (4.1.11) yields a contradiction. The proof is complete.

It is interesting that in the above proof a crude, "energy", estimate provides the sharp value of the rate of growth of the range of influence of the initial data.

As we shall see in Chapter V, the Cauchy problem is well-posed in the class of classical solutions, so long as U_0 is sufficiently smooth and T is sufficiently small. In the large, however, the situation is quite different. This will be demonstrated in the following section.

4.2 Breakdown of Classical Solutions

Here we shall make the acquaintance of two distinct mechanisms, namely "wave breaking" and "mass confinement," that may induce the breakdown of classical solutions of the Cauchy problem for nonlinear hyperbolic conservation laws.

We shall see first that nonlinearity forces compressive wave profiles to become steeper and eventually break, so that a derivative of the solution blows up. This will be demonstrated in the context of the simplest example of a nonlinear hyperbolic conservation law in one spatial variable, namely the *Burgers equation*

$$\partial_t u(x,t) + \partial_x \left(\tfrac{1}{2} u^2(x,t)\right) = 0. \tag{4.2.1}$$

This deceptively simple-looking equation pervades the theory of hyperbolic conservation laws, as it repeatedly emerges, spontaneously, in the analysis of general systems; see for instance Section 7.6.

Suppose $u(x,t)$ is a smooth solution of the Cauchy problem for (4.2.1), with initial data $u_0(\cdot)$, defined on some time interval $[0,T)$. The characteristics of (4.2.1) associated with this solution are integral curves of the ordinary differential equation $dx/dt = u(x,t)$. Letting an overdot denote differentiation, $\cdot = \partial_t + u\partial_x$, in the characteristic direction, we may rewrite (4.2.1) as $\dot{u} = 0$, which shows that u stays constant along characteristics. This implies, in turn, that characteristics are straight lines.

Setting $\partial_x u = v$ and differentiating (4.2.1) with respect to x yields the equation $\partial_t v + u\partial_x v + v^2 = 0$, or $\dot{v} + v^2 = 0$. Therefore, along the characteristic issuing from any point $(\bar{x},0)$ where $u_0'(\bar{x}) < 0$, $|\partial_x u|$ will be an increasing function which blows up at $\bar{t} = [-u_0'(\bar{x})]^{-1}$. It is thus clear that $u(x,t)$ must break down, as a classical solution, at or before time $\bar{t}$.

For an alternative, instructive, perspective on wave breaking, let us associate with any point (x,t) on the domain of the above solution the number $y = x - tu(x,t)$. Thus y marks the interceptor on the x-axis of the characteristic associated with u that passes through the point (x,t). Then $(x,t) \mapsto (y,t)$ induces a new coordinate system. In the spirit of continuum physics, one may regard (x,t) as "Eulerian coordinates" and (y,t) as "Lagrangian coordinates". Expressed in Lagrangian coordinates, the solution takes the simple form $u(y,t) = u_0(y)$, with bounded derivatives: $\partial_y u(y,t) = u_0'(y)$ and $\partial_t u(y,t) = 0$. However, the problem arises when one switches back to Eulerian coordinates, since $x = y + tu_0(y)$ implies that the transformation becomes singular when $x_y = 1 + tu_0'(y)$ vanishes.

One may stop at the critical time where the earliest singularity develops, or else seek the so called *maximal development region*, namely the largest subset of the upper half-plane on which u exists, as a classical solution. In Lagrangian coordinates this is the set of (y,t) with $tu_0'(y) > -1$.

For future reference, we shall compare and contrast the behavior of solutions of (4.2.1) with the behavior of solutions to Burgers's equation with damping:

$$\text{(4.2.2)} \qquad \partial_t u(x,t) + \partial_x\left(\tfrac{1}{2}u^2(x,t)\right) + u(x,t) = 0.$$

The arguments employed above for (4.2.1), adapted to (4.2.2), yield that the evolution of classical solutions u, and their derivatives $v = \partial_x u$, along characteristics $dx/dt = u(x,t)$, is now governed by the equations $\dot{u} + u = 0$, and $\dot{v} + v^2 + v = 0$. The last equation exemplifies the competition between the destabilizing action of nonlinear response and the smoothing effect of damping: When the initial data u_0 satisfy $u_0'(x) \geq -1$, for all $x \in (-\infty,\infty)$, then damping prevails, $\partial_x u$ remains bounded, and a global classical solution exists for the Cauchy problem. By contrast, if $u_0'(\bar{x}) < -1$, for some $\bar{x} \in (-\infty,\infty)$, then waves break in finite time, as $v = \partial_x u$ must blow up along the characteristic issuing from the point $(\bar{x},0)$.

To see an alternative scenario for breaking down of classical solutions, consider the Cauchy problem for the Burgers equation (4.2.1), with initial data $u_0(\cdot)$ supported in the interval $[0,1]$. Suppose a classical solution $u(x,t)$ exists on some time interval $[0,T)$. In that case, by virtue of Theorem 4.1.1, $u(\cdot,t)$ will be supported in $[0,1]$, for any $t \in [0,T)$. We define the weighted total mass

$$\text{(4.2.3)} \qquad M(t) = \int_0^1 xu(x,t)dx,$$

and use (4.2.1) and Schwarz's inequality to derive the differential inequality

$$\text{(4.2.4)} \qquad \dot{M}(t) = -\tfrac{1}{2}\int_0^1 x\partial_x\left(u^2(x,t)\right)dx = \tfrac{1}{2}\int_0^1 u^2(x,t)dx \geq \tfrac{3}{2}M^2(t).$$

Thus, if $M(0) > 0$, $M(t)$ must blow up no later than at time $t^* = \frac{2}{3}M(0)^{-1}$. The interpretation is that the constraints on the rate of growth of the size of the range of influence of classical solutions, imposed by Theorem 4.1.1, confines the "mass", hampering dispersion. This results in segregation of the positive from the negative part of the mass, eventually leading to blowup. However, it is not difficult to see that in the present context waves will start breaking no later than at time $\bar{t} = \frac{1}{6}M(0)^{-1}$, i.e., the wave-breaking catastrophe will occur before the mass confinement catastrophe may materialize.

As we shall see in Section 7.8, the wave breaking catastrophe occurs generically to solutions of genuinely nonlinear systems of hyperbolic conservation laws in one spatial dimension, as waves propagating along characteristics are confined to a plane and thus cannot avoid colliding with each other. By contrast, in several space dimensions wave breaking may be averted as it competes with dispersion. Nevertheless, as we shall see in Sections 6.1 and 4.8, waves still break for scalar conservation laws in several space dimensions and for the Euler equations in three spatial dimensions. The discussion of systems in which dispersion prevails and prevents the breaking of waves lies beyond the scope of this book. An example of such a system, arising in elastodynamics, will be exhibited in Section 5.5.

4.3 The Cauchy Problem: Weak Solutions

In view of the examples of breakdown of classical solutions presented in the previous section—and many more that will be encountered throughout the text—it becomes imperative to consider weak solutions to systems of conservation laws (4.1.1). The natural notion for a weak solution should be determined in conjunction with an existence theory. The issue of existence of weak solutions has been settled in a definitive manner for scalar conservation laws, in any number of spatial variables (see Chapter VI), and at least partially for systems in one spatial variable (see Chapters XIII–XVII); it remains totally unsettled, however, for systems in several spatial variables. In the absence of a definitive existence theory, and in order to introduce a number of relevant notions, without imposing technical growth conditions on the flux function G, we shall define here as *weak solutions* of (4.1.1) locally bounded, measurable functions U, defined on $\mathbb{R}^m \times [0,T)$ and taking values in $\mathscr{O}$, which satisfy (4.1.1) in the sense of distributions.

Recalling Lemma 1.3.3, we normalize any weak solution U of (4.1.1) so that the map $t \mapsto U(\cdot,t)$ becomes continuous on $[0,T)$ in $L^\infty(\mathbb{R}^m)$ weak*. A normalized weak solution of (4.1.1) will then solve the Cauchy problem (4.1.1), (4.1.3) if it also satisfies (4.1.3) almost everywhere on $\mathbb{R}^m$. Lemma 1.3.3 also implies

$$(4.3.1)\qquad \int_\tau^T \int_{\mathbb{R}^m} \left[\partial_t \Phi U + \sum_{\alpha=1}^m \partial_\alpha \Phi G_\alpha(U)\right] dx\,dt + \int_{\mathbb{R}^m} \Phi(x,\tau)U(x,\tau)dx = 0,$$

for every Lipschitz test function $\Phi(x,t)$, with compact support in $\mathbb{R}^m \times [0,T)$ and values in $\mathbb{M}^{1\times n}$, and any $\tau \in [0,T)$. In particular,

$$(4.3.2)\qquad \int_0^T \int_{\mathbb{R}^m} \left[\partial_t \Phi U + \sum_{\alpha=1}^m \partial_\alpha \Phi G_\alpha(U)\right] dx\,dt + \int_{\mathbb{R}^m} \Phi(x,0)U_0(x)dx = 0,$$

which may serve as an equivalent definition of a weak solution of (4.1.1), (4.1.3). The continuity of $t \mapsto U(\cdot,t)$ also induces the desirable semigroup property: if $U(x,t)$ is a weak solution of (4.1.1) on $[0,T)$, with initial values $U(x,0)$, then for any $\tau \in [0,T)$

the function $U_\tau(x,t) = U(x,t+\tau)$ is a weak solution of (4.1.1) with initial values $U_\tau(x,0) = U(x,\tau)$.

An important class of weak solutions are those in which U is a function of locally bounded variation on $\mathbb{R}^m \times [0,T)$. Such solutions satisfy the system (4.1.1) in the sense of measures and the initial conditions (4.1.3) as the trace of U at $t=0$. On the set of points of approximate jump discontinuity, BV solutions satisfy the Rankine-Hugoniot jump conditions (3.1.3), which here take the following form:

$$-s\,[U_+ - U_-] + [G(U_+) - G(U_-)]\,\nu = 0. \tag{4.3.3}$$

As the system is in divergence form, there is a mechanism of regularity transfer from the spatial to the temporal variables, which can be illustrated in the context of BV solutions:

4.3.1 Theorem. *Let U be a bounded weak solution of* (4.1.1) *on $[0,T)$ such that, for any fixed $t \in [0,T)$, $U(\cdot,t) \in BV(\mathbb{R}^m)$ and $TV_{\mathbb{R}^m} U(\cdot,t) \le V$, for all $t \in [0,T)$. Then $t \mapsto U(\cdot,t)$ is Lipschitz continuous in $L^1(\mathbb{R}^m)$ on $[0,T)$,*

$$\|U(\cdot,\sigma) - U(\cdot,\tau)\|_{L^1(\mathbb{R}^m)} \le aV|\sigma - \tau|, \quad 0 \le \tau < \sigma < T, \tag{4.3.4}$$

where a depends solely on the Lipschitz constant of G. In particular, U is in BV_{loc} on $\mathbb{R}^m \times [0,T)$.

Proof. Fix $0 \le \tau < \sigma < T$ and any $\Psi \in C_0^\infty\left(\mathbb{R}^m; \mathbb{M}^{1\times n}\right)$, with $|\Psi(x)| \le 1$ for $x \in \mathbb{R}^m$. Using the test function $\Phi(x,t) = f(t)\Psi(x)$, where $f \in C_0^\infty[0,T)$, with $f(t) = 1$ for $t \in [0,\sigma]$, write (4.3.1), first for $\tau = \sigma$, then for $\tau = \tau$, and subtract to get

$$\int_{\mathbb{R}^m} \Psi(x)[U(x,\sigma) - U(x,\tau)]dx = \int_\tau^\sigma \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \Psi(x) G_\alpha(U(x,t))dx\,dt. \tag{4.3.5}$$

The spatial integral on the right-hand side is majorized by the total variation of $G(U(\cdot,t))$, which in turn is bounded by aV. Taking the supremum of (4.3.5) over all Ψ with $|\Psi(x)| \le 1$, we arrive at (4.3.4).

Theorem 1.7.2 together with (4.3.4) implies that U is in BV_{loc} on $\mathbb{R}^m \times [0,T)$. The proof is complete.

4.4 Nonuniqueness of Weak Solutions

Extending the notion of solution from classical to weak introduces a new difficulty: nonuniqueness. To see this, consider the Cauchy problem for the Burgers equation (4.2.1), with initial data

$$u(x,0) = \begin{cases} -1, & x < 0 \\ \;\;\,1, & x > 0. \end{cases} \tag{4.4.1}$$

This is an example of the celebrated Riemann problem, which will be discussed at length in Chapter IX. The problem (4.2.1), (4.4.1) admits infinitely many weak solutions, including the family

$$
u_\alpha(x,t) = \begin{cases} -1, & -\infty < x \le -t \\[2ex] \dfrac{x}{t}, & -t < x \le -\alpha t \\[2ex] -\alpha, & -\alpha t < x \le 0 \\[2ex] \alpha, & 0 < x \le \alpha t \\[2ex] \dfrac{x}{t}, & \alpha t < x \le t \\[2ex] 1, & t < x < \infty, \end{cases} \tag{4.4.2}
$$

for any $\alpha \in [0,1]$. Indeed, $u_\alpha(x,t)$ satisfies (4.2.1), in the classical sense, provided $x/t \notin \{0, \pm\alpha, \pm 1\}$. The lines $x/t = \pm 1$, for $\alpha \in [0,1]$, and $x/t = \pm\alpha$, for α in $(0,1)$, are just weak fronts, across which u_α is continuous. Finally, for $\alpha \neq 0$, the line $x = 0$ is a stationary shock front across which the Rankine-Hugoniot jump condition (4.3.5) holds.

To resolve the issue of nonuniqueness, additional restrictions, in the form of *admissibility conditions*, shall be imposed on weak solutions. At the outset, reasonable admissibility criteria should meet at least some of the following requirements:
(a) They should be dictated, or at least motivated, by physics.
(b) They should be compatible with other established admissibility conditions.
(c) They should be broad enough to allow for existence of admissible solutions, and yet sufficiently narrow to disqualify spurious solutions. Ideally, they should be capable of singling out a unique admissible solution.

The issue of admissibility of weak solutions to hyperbolic conservation laws will be a central theme in this book, requiring lengthy discussions, which will commence in the following two sections and culminate in Chapters VIII and IX. In particular, it will turn out that $u_0(x,t)$ is the sole admissible solution of the simple problem (4.2.1), (4.4.1) considered in this section.

4.5 Entropy Admissibility Condition

As we saw in Chapter III, every system of balance laws arising in continuum physics is accompanied by an entropy inequality that must be satisfied by any physically meaningful process, as it expresses, explicitly or implicitly, the Second Law of thermodynamics. This motivates the following procedure for characterizing admissible weak solutions of hyperbolic systems of conservation laws.

Assume our system (4.1.1) is endowed with a designated entropy η, associated with an entropy flux Q, so that (4.1.4) holds. A weak solution of (4.1.1), in the sense of Section 4.3, defined on $\mathbb{R}^m \times [0,T)$, will satisfy the *entropy admissibility criterion,* relative to η, if

$$\partial_t \eta(U(x,t)) + \operatorname{div} Q(U(x,t)) \leq 0 \tag{4.5.1}$$

holds, in the sense of distributions, on $\mathbb{R}^m \times [0,T)$.

Clearly, any classical solution of (4.1.1) is admissible, as it satisfies the equality (4.1.5). Another relevant remark is that the entropy admissibility criterion induces a time *irreversibility* condition on solutions: if $U(x,t)$ is an admissible weak solution of (4.1.1) that satisfies (4.5.1) as a strict inequality, then $\bar{U}(x,t) = U(-x,-t)$, which is also a solution, is not admissible.

A natural question is how one may designate an appropriate entropy for the admissibility criterion. For instance, it is clear that a weak solution that is admissible relative to an entropy η fails to be admissible relative to the entropy $\bar{\eta} = -\eta$. When the system derives from physics, then it is physics that should designate the natural entropy. In the absence of guidance from physics, one has to use mathematical arguments. It is, of course, desirable that the admissibility criterion induced by the designated entropy should be compatible with admissibility conditions induced by alternative criteria, to be introduced later. Another natural condition is that admissible weak solutions should enjoy reasonable stability properties. As we shall see, all of the above requirements are met when the designated entropy $\eta(U)$ is convex, or at least "convexlike".

The reader should bear in mind that convexity is a relevant property of the entropy only when the system is in canonical form. In the general case, convexity of η should be replaced by the condition that the ($n \times n$ matrix-valued) derivative $\mathrm{D}B(U,x,t)$ of the (n-vector-valued) function $B(U,x,t)$ in (3.2.2) is positive definite.

A review of the examples considered in Section 3.3 reveals that the entropy, as a function of the state vector that converts the system of balance laws into canonical form, is indeed convex in the case of the thermoelastic fluid (example 3.3.5), the isentropic thermoelastic fluid (example 3.3.6) and magnetohydrodynamics (example 3.3.9). This may raise expectations that in the equations of continuum physics entropy is generally convex. However, as we shall see, this is not always the case; hence the necessity to consider a broader class of entropy functions.

For any weak solution U satisfying the entropy admissibility criterion, the left-hand side of (4.5.1) is a nonpositive distribution, and thereby a measure, which shall be dubbed the *entropy production measure.* Then Lemma 1.3.3 implies that the map $t \mapsto \eta(U(\cdot,t))$ is continuous on $[0,T)\backslash\mathscr{F}$ in $L^\infty(\mathbb{R}^m)$ weak*, where $\mathscr{F}$ is at most countable. Furthermore, for every nonnegative Lipschitz test function $\psi(x,t)$, with compact support in $\mathbb{R}^m \times [0,T)$, and any $\tau \in [0,T)\backslash\mathscr{F}$,

$$\int_\tau^T \int_{\mathbb{R}^m} \left[\partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U) \right] dx\,dt + \int_{\mathbb{R}^m} \psi(x,\tau)\, \eta(U(x,\tau)) dx \geq 0. \tag{4.5.2}$$

It would be important to determine conditions under which the set $\mathscr{F}$ is actually empty, but the presence of wildly oscillating, exotic solutions for the Euler equations, which will be demonstrated in Section 4.8, is an indication that answering this question will not be easy. Indeed, at the time of this writing, there is a rigorous proof that $\mathscr{F} = \emptyset$ only in the scalar case, $n = 1$ (see Section 6.8). As we shall see, it is a great help to the analysis if at least $0 \notin \mathscr{F}$, in which case (4.5.2), with $\tau = 0$, becomes

$$\int_0^T \int_{\mathbb{R}^m} \left[\partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U) \right] dx\,dt + \int_{\mathbb{R}^m} \psi(x,0)\, \eta(U_0(x))dx \geq 0. \tag{4.5.3}$$

Accordingly, it is (4.5.3), rather than the slightly weaker condition (4.5.1), that is often postulated in the literature as the entropy admissibility criterion for the weak solution U. It should be noted, however, that admissible weak solutions characterized through (4.5.3) do not necessarily possess the desirable semigroup property, i.e., $U(x,t)$ admissible does not generally imply that $U_\tau(x,t) = U(x,t+\tau)$ is also admissible, for all $\tau \in [0,T)$. Thus, in the author's opinion, admissibility should be defined either through (4.5.1) alone or through (4.5.2), for all $\tau \in [0,T)$. An eventual proof that, at least in certain systems, $\mathscr{F}$ is empty will render the distinction moot.

A first indication of the enhanced regularity enjoyed by admissible weak solutions when the entropy is convex is provided by the following

4.5.1 Theorem. *Assume $U(x,t)$ is a weak solution of* (4.1.1) *on $\mathbb{R}^m \times [0,T)$, which satisfies the entropy admissibility condition* (4.5.1) *relative to a uniformly convex entropy η. Then $t \mapsto U(\cdot,t)$ is continuous on $[0,T) \backslash \mathscr{F}$ in $L^p_{loc}(\mathbb{R}^m)$, for any $p \in [1,\infty)$, where $\mathscr{F}$ is at most countable. Moreover,* (4.5.2) *holds for some τ in $[0,T)$, if and only if $t \mapsto U(\cdot,t)$ is continuous from the right at τ in $L^p_{loc}(\mathbb{R}^m)$, for any $p \in [1,\infty)$.*

Proof. Since both $t \mapsto U(\cdot,t)$ and $t \mapsto \eta(U(\cdot,t))$ are continuous on $[0,T)\backslash\mathscr{F}$ in $L^\infty(\mathbb{R}^m)$ weak*, and η is uniformly convex, it follows that $t \mapsto U(\cdot,t)$ is strongly continuous on $[0,T)\backslash\mathscr{F}$ in $L^p(\mathscr{D})$, for any compact subset $\mathscr{D}$ of $\mathbb{R}^m$ and $p \in [1,\infty)$.

Assume now (4.5.2) holds, for some $\tau \in [0,T)$. Fix $\varepsilon > 0$ and apply (4.5.2) for $\psi(x,t) = \phi(x)g(t)$, where $\varphi \in C_0^\infty(\mathbb{R}^m)$, with $\varphi(x) \geq 0$ for $x \in \mathbb{R}^m$, while g is defined by $g(t) = 1 - \varepsilon^{-1}(t-\tau)$, for $0 \leq t < \tau + \varepsilon$, and $g(t) = 0$, for $t + \varepsilon \leq t < \infty$. This gives

$$\frac{1}{\varepsilon} \int_\tau^{\tau+\varepsilon} \int_{\mathbb{R}^m} \varphi(x) \left[\eta(U(x,\tau)) - \eta(U(x,t)) \right] dx\,dt \geq O(\varepsilon). \tag{4.5.4}$$

By Lemma 1.3.3, letting $\varepsilon \downarrow 0$, we deduce $\operatorname*{ess\,lim}_{t\downarrow\tau} \eta(U(\cdot,t)) \leq \eta(U(\cdot,\tau))$, where the limit is taken in $L^\infty(\mathscr{D})$ weak*, for any compact subset $\mathscr{D}$ of $\mathbb{R}^m$. Recalling that $\lim_{t\downarrow\tau} U(\cdot,t) = U(\cdot,\tau)$, again in $L^\infty(\mathscr{D})$ weak*, and that $\eta(U)$ is uniformly convex, it follows that, as $t \downarrow \tau$, $U(\cdot,t) \to U(\cdot,\tau)$, strongly in $L^p(\mathscr{D})$, for any $p \in [1,\infty)$.

Conversely, assuming $t \mapsto U(\cdot,t)$ is right-continuous in $L^p_{\mathrm{loc}}(\mathbb{R}^m)$ at τ, we fix any nonnegative Lipschitz function $\psi(x,t)$ with compact support in $\mathbb{R}^m \times [0,T)$ and set $\phi(x,t) = h(t)\psi(x,t)$, where $h(t) = 0$, for $0 \leq t \leq \tau$, $h(t) = \varepsilon^{-1}(t-\tau)$, for

$\tau < t \leq \tau + \varepsilon$, and $h(t) = 1$, for $\tau + \varepsilon < t < T$. Upon applying (4.5.1) to the test function ϕ, and then letting $\varepsilon \downarrow 0$, we arrive at (4.5.2). The proof is complete.

The implications of (4.5.3) or (4.5.2) are further elucidated by the following

4.5.2 Theorem. *Let $U(x,t)$ be a weak solution of* (4.1.1), (4.1.3), *on* $\mathbb{R}^m \times [0,T)$, *with $U_0(\cdot) - \bar{U} \in L^2(\mathbb{R}^m)$, for some fixed state $\bar{U}$. Assume U satisfies the entropy admissibility condition* (4.5.1) *relative to a uniformly convex entropy η, normalized*[1] *so that $\eta(\bar{U}) = 0$, $\mathrm{D}\eta(\bar{U}) = 0$. Then U satisfies* (4.5.3) *if and only if*

$$\int_{\mathbb{R}^m} \eta(U(x,\tau))dx \leq \int_{\mathbb{R}^m} \eta(U_0(x))dx, \qquad 0 < \tau < T. \tag{4.5.5}$$

It thus follows that (4.5.2) *holds for all $\tau \in [0,T)$ if and only if the function $t \mapsto \int_{\mathbb{R}^m} \eta(U(x,t))dx$ is nonincreasing on $[0,T)$.*

Proof. Assume (4.5.3) holds and fix any $\tau \in (0,T)$. For $\varepsilon > 0$ small, $r > 0$ and $s > 0$, set

$$h(t) = \begin{cases} 1 & 0 \leq t < \tau \\ \varepsilon^{-1}(\tau - t) + 1 & \tau \leq t < \tau + \varepsilon \\ 0 & \tau + \varepsilon \leq t < T\,, \end{cases} \tag{4.5.6}$$

$$\phi(x,t) = \begin{cases} 1 & |x| - r - s(\tau - t) < 0 \\ \varepsilon^{-1}[r + s(\tau - t) - |x|] + 1 & 0 \leq |x| - r - s(\tau - t) < \varepsilon \\ 0 & |x| - r - s(\tau - t) \geq \varepsilon\,, \end{cases} \tag{4.5.7}$$

and write (4.5.3) for the test function $\psi(x,t) = h(t)\phi(x,t)$ to get

$$\begin{aligned} \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} \int_{|x|<r} \eta(U(x,\tau))dxdt \leq \int_{|x|<r+s\tau} \eta(U_0(x))dx \\ - \frac{1}{\varepsilon}\int_0^\tau \int_{r+s(\tau-t)<|x|<r+s(\tau-t)+\varepsilon} \left[s\eta(U) + \frac{Q(U)\cdot x}{|x|} \right] dxdt + O(\varepsilon). \end{aligned} \tag{4.5.8}$$

Recalling that η is uniformly convex, $\eta(\bar{U}) = 0$, $\mathrm{D}\eta(\bar{U}) = 0$, and thereby we have $\mathrm{D}Q_\alpha(\bar{U}) = 0$, $\alpha = 1, \cdots, m$, normalize Q by $Q(\bar{U}) = 0$, and fix s sufficiently large for $s\eta \geq |Q|$ to hold on the range of the solution U. In (4.5.8), let $\varepsilon \downarrow 0$, use the weak lower semicontinuity of $\int \eta(U)dx$, and then let $r \uparrow \infty$, to arrive at (4.5.5).

[1] As explained in the proof of Theorem 4.1.1.

Conversely, assume (4.5.5) holds. Since $U(\cdot,\tau) \to U_0(\cdot)$, in $L^\infty(\mathbb{R}^m)$ weak*, as $\tau \downarrow 0$, (4.5.5) and the weak lower semicontinuity of $\int \eta(U)dx$ imply $U(\cdot,\tau) \to U_0(\cdot)$ in $L^p_{\text{loc}}(\mathbb{R}^m)$. This in turn yields (4.5.3), by virtue of Theorem 4.5.1. The proof is complete.

The monotonic decay of total entropy, manifested in (4.5.5), is physically appealing. It hinges on the admissibility condition (4.5.2) in conjunction with the weak lower semicontinuity of the map $V \mapsto \int \eta(V)dx$, for which convexity of $\eta(V)$ is both necessary and sufficient, so long as V is unrestricted. However, in Chapter III we encountered important systems arising in mechanics and electrodynamics, in which the entropy function is not allowed to be convex. What saves the day is that the state vector for such systems must satisfy side conditions that may render $\int \eta(V)dx$ weakly lower semicontinuous even for $\eta(V)$ that are not convex.

For illustration, let us consider the system (3.3.19) that governs the isentropic motion of thermoelastic media. Recall that the entropy function $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ fails to be convex, because the internal energy function $\varepsilon(F)$ is not allowed to be convex. Nevertheless, it is known (references in Section 4.9) that since F is constrained to be a gradient, the map $F \mapsto \int \varepsilon(F)dx$ is lower semicontinuous in L^∞ weak* if and only if $\varepsilon(F)$ is *quasiconvex* in the sense of Morrey, namely, letting $\mathscr{K}$ denote the standard hypercube in $\mathbb{R}^3$,

$$\varepsilon(\hat{F}) \leq \int_{\mathscr{K}} \varepsilon(\hat{F} + \nabla\chi)dx \tag{4.5.9}$$

holds for every constant matrix $\hat{F}$ and any Lipschitz vector field χ with compact support in $\mathscr{K}$. In physical terms, (4.5.9) stipulates that ε is quasiconvex when any homogeneous deformation of $\mathscr{K}$ minimizes the internal energy stored in $\mathscr{K}$ among all placements of $\mathscr{K}$ with the same boundary values.

Convexity of $\varepsilon(F)$ is a sufficient condition for quasiconvexity, but it is not necessary. On the other hand, rank-one convexity (3.3.7), with s constant, is a necessary condition for quasiconvexity of $\varepsilon(F)$ but it is not sufficient. In fact, since (4.5.9) is nonlocal, it is not easy to test whether any particular rank-one convex, but not convex, function is quasiconvex.

A method for constructing physically admissible, nonconvex but quasiconvex internal energies is based on that (4.5.9) holds as equality when $\varepsilon(F) = \varphi(F)$, where $\varphi(F)$ is any null Lagrangian in the form (2.2.9). Thus null Lagrangians are continuous in L^∞ weak*. It follows that internal energies with constitutive equations

$$\varepsilon(F) = \theta(F, F^*, \det F), \tag{4.5.10}$$

where $\theta(F,H,\delta)$ is convex in $\mathbb{R}^{19}$, are lower semicontinuous in L^∞ weak* and thereby quasiconvex. Constitutive equations of this type are termed *polyconvex*. They provide realistic models for actual elastic materials and are playing an important role in elastostatics. Their role is elastodynamics will be elucidated in Section 5.4. For present purposes, when the internal energy function is polyconvex, the assertions of Theorem 4.5.2 hold for the system (3.3.19) of isentropic thermoelasticity.

As we shall see later, the entropy admissibility condition eliminates some, but not necessarily all, of the undesirable, spurious weak solutions of hyperbolic systems of conservation laws. A potential remedy is to require that (4.5.3) hold simultaneously for every convex entropy of the system. However, this appears promising only for the very special class of systems that are endowed with a rich family of entropies and in particular, as we shall see in Chapter VI, for scalar conservation laws. In that connection, one should be aware of the following

4.5.3 Remark. Assume (4.1.1) is endowed with an entropy-entropy flux pair $(\bar{\eta},\bar{Q})$ with $\mathrm{D}^2\bar{\eta}$ positive definite on $\mathscr{O}$, and consider the stronger admissibility condition on solutions U of (4.1.1), namely, that the inequality (4.5.3) must hold for every entropy-entropy flux pair (η,Q) with η convex. In that case, $\partial_t\eta(U)+\mathrm{div}Q(U)$ will be a measure for any entropy-entropy flux pair (η,Q). This follows from the observation that any C^2 entropy-entropy flux pair (η,Q) may be written as the difference of the entropy-entropy flux pairs $(k\bar{\eta}+\eta,k\bar{Q}+Q)$ and $(k\bar{\eta},k\bar{Q})$, where the Hessians of both $k\bar{\eta}+\eta$ and $k\bar{\eta}$ are positive definite on any compact subset of $\mathscr{O}$, for k sufficiently large.

In the absence of a rich family of entropies and when (4.5.1) for a single entropy does not suffice for weeding out all spurious solutions, one may attempt to single out the physically admissible solution by adopting a more selective admissibility condition, always in the spirit of the Second Law of thermodynamics. As (4.5.5) manifests that admissible solutions must be dissipative, it is natural to inquire whether the solution exhibiting dissipation at the highest rate has special status. One may experiment with various characterizations of maximal dissipativeness. Thus, in the setting of Theorem 4.5.2, a solution U may be termed *maximally dissipative* if for every $t\in[0,T)$ and any other solution $\hat{U}$ that coincides with U on $\mathbb{R}^m\times[0,t]$, either

$$D^+\int_{\mathbb{R}^m}\eta(U(x,t))dx\le D_+\int_{\mathbb{R}^m}\eta(\hat{U}(x,t))dx, \tag{4.5.11}$$

where D^+ and D_+ denote the upper and lower right Dini derivatives; or, alternatively, if there is a decreasing sequence $\{t_k\}$, $t_k\to t$ as $k\to\infty$, such that

$$\int_{\mathbb{R}^m}\eta(U(x,t_k))dx\le\int_{\mathbb{R}^m}\eta(\hat{U}(x,t_k))dx,\qquad k=1,2,\dots. \tag{4.5.12}$$

Notice that (4.5.11) and (4.5.12) are in the same spirit, but neither one implies the other. We will return to these considerations, briefly in Section 4.8 and more thoroughly in Section 9.7.

Whenever the admissible solution U is of class BV_{loc} , Theorem 1.8.2 implies that the entropy production measure is concentrated on the set of points of approximate jump discontinuity of U, i.e., on the shock fronts. In that case, (4.5.1) reduces to the local condition (1.8.5), which in the present notation takes the form

$$-s\,[\eta(U_+)-\eta(U_-)]+[Q(U_+)-Q(U_-)]\,\nu\leq 0. \tag{4.5.13}$$

For admissibility of U relative to the entropy η, (4.5.12) has to be tested at any point of a shock that propagates in the direction $\nu\in S^{m-1}$ with speed s.

As an application, let us test the admissibility of the family $u_\alpha(x,t)$ of solutions to (4.2.1), (4.4.1), defined by (4.4.2), relative to the entropy-entropy flux pair $\left(\frac{1}{2}u^2,\frac{1}{3}u^3\right)$. It is clear that, for any $\alpha\in(0,1]$, the stationary shock $x=0$ violates (4.5.12). Thus, the sole admissible solution in that family is $u_0(x,t)$, which is Lipschitz continuous, away from the origin.

4.6 The Vanishing Viscosity Approach

According to the *viscosity criterion,* a solution U of (4.1.1) is admissible provided it is the $\mu\downarrow 0$ limit of solutions U_μ to a system of conservation laws with diffusive terms:

$$\partial_t U(x,t)+\sum_{\alpha=1}^{m}\partial_\alpha G_\alpha(U(x,t))=\mu\sum_{\alpha,\beta=1}^{m}\partial_\alpha[B_{\alpha\beta}(U(x,t))\partial_\beta U(x,t)], \tag{4.6.1}$$

where the $B_{\alpha\beta}$ are $n\times n$ matrix-valued functions defined on $\mathscr{O}$.

The motivation for this principle and the term "vanishing viscosity" derive from continuum physics: as we saw in earlier chapters, the balance laws for thermoelastic materials under adiabatic conditions induce first order systems of hyperbolic type. By contrast, the balance laws for thermoviscoelastic, heat-conducting materials, introduced in Section 2.6, generate systems of second order in the spatial variables, containing diffusive terms akin to those appearing on the right-hand side of (4.6.1). In nature, every material has viscous response and conducts heat, to a certain degree. Classifying a particular material as an elastic nonconductor of heat simply means that viscosity and heat conductivity are negligible, albeit not totally absent. Consequently, the theory of adiabatic thermoelasticity may be physically meaningful only as a limiting case of thermoviscoelasticity, with viscosity and heat conductivity tending to zero. It is this premise that underlies the vanishing viscosity approach.

In laying down (4.6.1), the first task is to select the $n\times n$ matrices $B_{\alpha\beta}(U)$, for $\alpha,\beta=1,\dots,m$. In the case of systems of physical origin, the natural choice for these matrices is dictated, or at least suggested, by physics. For example, thermoelastic fluid nonconductors of heat should be regarded as Newtonian fluids with constitutive equations (2.6.16), (2.6.17) having vanishingly small viscosity and heat conductivity. Accordingly, when (4.1.1) stands for the system (3.3.29) of balance laws of mass, momentum and energy for thermoelastic fluids that do not conduct heat (with zero body force and heat supply), the appropriate choice for the corresponding system (4.6.1), with diffusive terms, should be[2]

[2] We write this system in components form, let ∂_i denote $\partial/\partial x_i$ and employ the summation convention: repeated indices are summed over the range 1,2,3.

$$
(4.6.2) \qquad \begin{cases} \partial_t \rho + \partial_j(\rho v_j) = 0 \\ \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) + \partial_i p(\rho, s) = \lambda \partial_i \partial_j v_j + \mu \partial_j (\partial_i v_j + \partial_j v_i) \\ \partial_t \left[\rho \varepsilon(\rho, s) + \frac{1}{2}\rho |v|^2\right] + \partial_j \left[\left(\rho \varepsilon(\rho, s) + \frac{1}{2}\rho |v|^2 + p(\rho, s)\right) v_j\right] \\ \qquad\qquad = \lambda \partial_i (v_i \partial_j v_j) + \mu \partial_j \left[\left(\partial_i v_j + \partial_j v_i\right) v_i\right] + \kappa \partial_i \partial_i \theta. \end{cases}
$$

The reader should take notice that (4.6.2) contains three independent physical parameters, namely the bulk viscosity $\lambda + \frac{2}{3}\mu$, the shear viscosity μ and the thermal conductivity κ, which might all be very small albeit of different orders of magnitude. Thus, one should be prepared to consider formulations of the vanishing viscosity principle, more general than (4.6.1), involving several independent small parameters. However, this generalization will not be pursued here.

Physics suggests the natural form for (4.6.1) in every example considered in Section 3.3, including electromagnetism, magnetohydrodynamics, etc. On the other hand, in the absence of guidance from physics, or for mere analytical and computational convenience, one may experiment with *artificial viscosity* added to the right-hand side of (4.1.1). For example, one may add artificial viscosity to (4.2.1) to derive the *Burgers equation with viscosity:*

$$
(4.6.3) \qquad \partial_t u(x,t) + \partial_x\left(\tfrac{1}{2}u^2(x,t)\right) = \mu \partial_x^2 u(x,t).
$$

It is clear that artificial viscosity should be selected in such a way that the $B_{\alpha\beta}$ induce dissipation and thus render the Cauchy problem for (4.6.1) well-posed. The temptation is to use for $B_{\alpha\beta}$ matrices that would render (4.6.1) parabolic; and in particular the zero matrix if $\alpha \neq \beta$ and the identity matrix if $\alpha = \beta$, which would reduce the right-hand side to $\mu \Delta U$. The physically motivated example (4.6.2) demonstrates, however, that confining attention to the parabolic case would be ill-advised. In general, one has to deal with systems of intermediate parabolic-hyperbolic type, in which case establishing the well-posedness of the Cauchy problem may require considerable effort. See Section 5.5.

Assuming a vanishing viscosity mechanism has been selected, rendering the Cauchy problem (4.6.1), (4.1.3) well-posed, the question arises as to the sense of convergence of the family $\{U_\mu\}$ of solutions, as $\mu \downarrow 0$. This is of course a serious issue: requiring very strong convergence may raise unreasonable expectations for compactness of the family $\{U_\mu\}$. On the other hand, if the sense of convergence is too weak, it is not clear that $\lim U_\mu$ will be a solution of (4.1.1). Various aspects of this problem will be discussed later, mainly in Chapters VI, XV and XVII.

Another important task is to compare admissibility of solutions in the sense of the vanishing viscosity approach and admissibility in the sense of a designated entropy inequality (4.5.1), as discussed in Section 4.5. In continuum thermodynamics, presented in Chapter II, whenever (4.6.1) results from actual constitutive equations compatible with the Clausius-Duhem inequality, and (4.5.1) is, or derives from, the Clausius-Duhem inequality, solutions of (4.1.1) obtained by the vanishing viscosity approach will automatically satisfy (4.5.1). For example, solutions of (3.3.29) obtained as the $(\lambda,\mu,\kappa)\downarrow 0$ limit of solutions of (4.6.2) will satisfy automatically the inequality (3.3.32).

If $\eta(U)$ is an entropy for (4.1.1), associated with the entropy flux $Q(U)$, then any (classical) solution U_μ of (4.6.1) satisfies the identity

$$\begin{aligned}(4.6.4)\qquad \partial_t\eta(U_\mu)+\sum_{\alpha=1}^{m}\partial_\alpha Q_\alpha(U_\mu)&=\mu\sum_{\alpha,\beta=1}^{m}\partial_\alpha\left[\mathrm{D}\eta(U_\mu)B_{\alpha\beta}(U_\mu)\partial_\beta U_\mu\right]\\ &\quad-\mu\sum_{\alpha,\beta=1}^{m}(\partial_\alpha U_\mu)^\top\mathrm{D}^2\eta(U_\mu)B_{\alpha\beta}(U_\mu)\partial_\beta U_\mu\,.\end{aligned}$$

The second term on the right-hand side should be dissipative, so that the quadratic form associated with $\mathrm{D}^2\eta B_{\alpha\beta}$ must be positive semidefinite. Beyond that, however, this term is entrusted with the responsibility of dominating the first term on the right-hand side of (4.6.4) as well as the right-hand side of (4.6.1). A sufficient, though not necessary, condition for this will be

$$(4.6.5)\qquad \sum_{\alpha,\beta=1}^{m}\xi_\alpha^\top\mathrm{D}^2\eta(U)B_{\alpha\beta}(U)\xi_\beta\geq a\sum_{\alpha=1}^{m}\left|\sum_{\beta=1}^{m}B_{\alpha\beta}(U)\xi_\beta\right|^2,$$

for some positive constant a, any $U\in\mathscr{O}$ and all $\xi_\alpha\in\mathbb{R}^n$, $\alpha=1,\cdots,m$. Notice that when $B_{\alpha\beta}$ vanishes for $\alpha\neq\beta$, and is the identity for $\alpha=\beta$, (4.6.5) reduces to the statement that $\eta(U)$ is uniformly convex.

Suppose now that the initial data U_0 and the solution U_μ of (4.6.1), (4.1.3) tend sufficiently fast, as $|x|\to\infty$, to a constant state $\bar{U}$. Without loss of generality we may assume that $\eta(\bar{U})=0$ and $\mathrm{D}\eta(\bar{U})=0$, since otherwise we may replace $\eta(U)$ by $\bar{\eta}(U)$, defined by (4.1.6). We make the further assumption that actually $\bar{U}$ is the minimum of η over $\mathscr{O}$. This of course will automatically be the case when $\eta(U)$ is convex. Under these hypotheses, integrating (4.6.4) over $\mathbb{R}^m\times[0,T)$ yields the estimate

$$(4.6.6)\quad \mu\int_0^T\int_{\mathbb{R}^m}\sum_{\alpha,\beta=1}^{m}(\partial_\alpha U_\mu)^\top\mathrm{D}^2\eta(U_\mu)B_{\alpha\beta}(U_\mu)\partial_\beta U_\mu\,dx\,dt\leq\int_{\mathbb{R}^m}\eta(U_0(x))\,dx.$$

We have now laid the groundwork for showing that the viscosity admissibility criterion implies the entropy admissibility condition.

4.6.1 Theorem. *Under the assumptions on $\eta(U)$ and $\{U_\mu\}$ stated above, suppose that a sequence $\{U_{\mu_k}\}$, with $\mu_k\to 0$ as $k\to\infty$, converges boundedly almost everywhere on $\mathbb{R}^m\times[0,T)$ to some function U. Then U is a weak solution of* (4.1.1), (4.1.3) *on $\mathbb{R}^m\times[0,T)$, which satisfies the entropy admissibility condition* (4.5.3).

Proof. We multiply (4.6.1) by any Lipschitz test function $\Phi(x,t)$, taking values in $\mathbb{M}^{1\times n}$ with compact support in $\mathbb{R}^m\times[0,T)$ and integrate the resulting equation over

$\mathbb{R}^m \times [0,T)$. After an integration by parts, with respect to the time and spatial variables, we deduce

$$\int_0^T \int_{\mathbb{R}^m} \Big[\partial_t \Phi\, U_\mu + \sum_{\alpha=1}^m \partial_\alpha \Phi\, G_\alpha \big(U_\mu\big) \Big] dx\,dt + \int_{\mathbb{R}^m} \Phi(x,0)\, U_0(x) dx \tag{4.6.7}$$
$$= \mu \int_0^T \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \Phi\, B_{\alpha\beta} \big(U_\mu\big) \partial_\beta U_\mu \, dx\,dt.$$

By virtue of (4.6.5) and (4.6.6), as $\mu_k \to 0$, the right-hand side tends to zero, and hence the limit function U satisfies the equation (4.3.2).

Next we multiply (4.6.4) by any nonnegative Lipschitz test function $\psi(x,t)$, with compact support in $\mathbb{R}^m \times [0,T)$, and integrate the resulting equation over the strip $\mathbb{R}^m \times [0,T)$. Integrating by parts with respect the spatial and time variables, we end up with the identity

$$\int_0^T \int_{\mathbb{R}^m} \Big[\partial_t \psi\, \eta\big(U_\mu\big) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Q_\alpha \big(U_\mu\big) \Big] dx\,dt + \int_{\mathbb{R}^m} \psi(x,0)\, \eta\big(U_0(x)\big) dx \tag{4.6.8}$$
$$= \mu \int_0^T \int_{\mathbb{R}^m} \partial_\alpha \psi \mathrm{D}\eta\big(U_\mu\big) B_{\alpha\beta}\big(U_\mu\big) \partial_\beta U_\mu \, dx\,dt$$
$$+ \mu \int_0^T \int_{\mathbb{R}^m} \psi \sum_{\alpha,\beta=1}^m \big(\partial_\alpha U_\mu\big)^\top \mathrm{D}^2 \eta\big(U_\mu\big) B_{\alpha\beta}\big(U_\mu\big) \partial_\beta U_\mu \, dx\,dt.$$

On account of (4.6.5) and (4.6.6), the first term on the right-hand side tends to zero, as $\mu_k \to 0$, while the second term is nonnegative. Therefore, the limit function U satisfies the inequality (4.5.3). This completes the proof.

More general admissibility conditions, of the same genre as the viscosity criterion, may be formulated by replacing (4.6.1) with a system of the form

$$\partial_t U + \sum_{\alpha=1}^m \partial_\alpha G_\alpha(U) = \mu \sum_{\alpha,\beta=1}^m \partial_\alpha [B_{\alpha\beta}(U) \partial_\beta U] + \nu \sum_{\alpha,\beta,\gamma=1}^m \partial_\alpha [H_{\alpha\beta\gamma}(U) \partial_\beta \partial_\gamma U], \tag{4.6.9}$$

involving third, and sometimes even fourth, order differential operators and two "vanishing" parameters μ and ν. For example, in the place of (4.6.3) one may take

$$\partial_t u(x,t) + \partial_x\big(\tfrac{1}{2} u^2(x,t)\big) = \mu \partial_x^2 u(x,t) + \nu \partial_x^3 u(x,t). \tag{4.6.10}$$

The approach to admissibility via (4.6.9) is suggested by physics when the dissipative effect of viscosity coexists with some dispersive mechanism induced, for instance, by capillarity. Accordingly, the admissibility condition associated with (4.6.9) is termed the *viscosity-capillarity criterion.* Which solutions of (4.1.1) pass this test of admissibility will generally depend not only on the choice of $B_{\alpha\beta}$ and $H_{\alpha\beta\gamma}$, but also on the relative rate by which μ and ν tend to zero. As a minimum requirement, (4.6.9) must be compatible with the Second Law of thermodynamics, i.e., a proposition analogous to Theorem 4.6.1 must hold for the entropy-entropy flux pair provided by physics.

4.7 Initial-Boundary Value Problems

Suppose that the hyperbolic system of conservation laws (4.1.1) is posed on a proper, open subset $\mathscr{D}$ of $\mathbb{R}^m$, with Lipschitz boundary $\partial\mathscr{D}$ and outward unit normal field ν. To formulate a well-posed problem for (4.1.1) on the cylinder $\mathscr{X} = \mathscr{D} \times (0,T)$, in addition to assigning initial data $U(x,0) = U_0(x)$ on the base $\mathscr{D} \times \{0\}$, one must also prescribe boundary conditions on the lateral boundary $\mathscr{B} = \partial\mathscr{D} \times (0,T)$.

Typically, homogeneous boundary conditions associated with the systems of conservation laws in continuum physics may be cast in the form

$$P(U(x,t))\nu(x) = 0, \qquad (x,t) \in \mathscr{B}, \tag{4.7.1}$$

where P is a smooth function defined on $\mathscr{O}$ and taking values in $\mathbb{M}^{n\times m}$. Classical examples include the clamped boundary condition

$$v(x,t) = 0, \qquad (x,t) \in \mathscr{B}, \tag{4.7.2}$$

or the traction-free boundary condition

$$S(F(x,t))\nu(x) = 0, \qquad (x,t) \in \mathscr{B}, \tag{4.7.3}$$

for the system (3.3.19) of isentropic elastodynamics, in Lagrangian coordinates, and the corresponding no-penetration (slip boundary)

$$\rho(x,t)v(x,t) \cdot \nu(x) = 0, \qquad (x,t) \in \mathscr{B}, \tag{4.7.4}$$

or constant pressure

$$p(\rho(x,t)) = p_0, \qquad (x,t) \in \mathscr{B}, \tag{4.7.5}$$

boundary conditions for the Euler equations (3.3.36). The natural question of characterizing the class of P that render the initial-boundary value problem well-posed, in the regime of classical solutions, will be addressed later, in Section 5.6.

A separate issue, which will be discussed here briefly, is how to interpret the boundary condition (4.7.1) in the context of weak solutions. Whenever the weak solution U is a BV function on $\mathscr{X}$, its inner trace U_- is well-defined on $\mathscr{B}$ (cf. Section 1.7). Consequently, within the BV framework, (4.7.1) may be interpreted in a virtually classical, pointwise sense. The situation is different when the weak solution U is merely in L^∞, so that its trace on $\mathscr{B}$ cannot be identified. Nevertheless, by Theorem 1.3.4, one may still define on $\mathscr{B}$ the normal component of vector fields whose space-time divergence is a bounded measure on $\mathscr{X}$. In particular, since the space-time divergence of $(G_1(U),\cdots,G_n(U),U)$ vanishes on $\mathscr{X}$, one may define the trace $G_{\mathscr{B}} \in L^\infty(\mathscr{B};\mathbb{R}^n)$ of $G(U)\nu$ on $\mathscr{B}$, by means of Equation (1.3.14), which here takes the form

(4.7.6)
$$\int_0^T\int_{\partial\mathscr{D}} \Phi G_{\mathscr{B}}\, d\mathscr{H}^{m-1}(x)dt - \int_{\mathscr{D}} \Phi(x,0)\,U_0(x)dx$$
$$= \int_0^T\int_{\mathscr{D}} \Big[\partial_t\Phi U + \sum_{\alpha=1}^m \partial_\alpha\Phi\, G_\alpha(U)\Big] dx\,dt,$$

for any Lipschitz test function Φ compactly supported in $\mathbb{R}^m\times[0,T)$ and taking values in $\mathbb{M}^{1\times n}$.

More generally, when U satisfies an entropy admissibility condition (4.5.1), one may define the trace $Q_{\mathscr{B}} \in L^\infty(\mathscr{B})$ of $Q(U)\nu$ on $\mathscr{B}$ by means of

(4.7.7)
$$\int_0^T\int_{\partial\mathscr{D}} \psi Q_{\mathscr{B}}\, d\mathscr{H}^{m-1}(x)dt = \int_0^T\int_{\mathscr{D}} \Big[\partial_t\psi\,\eta(U) + \sum_{\alpha=1}^m \partial_\alpha\psi\, Q_\alpha(U)\Big] dx\,dt + \langle\mathscr{P},\psi\rangle_{\mathscr{X}}\,,$$

where $\mathscr{P}$ is the nonpositive entropy production measure and ψ is any Lipschitz test function with compact support in $\mathbb{R}^m\times(0,T)$. Moreover, under the conditions described in Remark 4.5.3, (4.7.7) will hold for all smooth entropy-entropy flux pairs (η,Q).

We conclude that boundary conditions (4.7.1) may be defined for L^∞ weak solutions, provided that the rows of the matrix $P(U)$ are entropy fluxes. In particular, this is the case in the examples (4.7.2), (4.7.3), (4.7.4) and (4.7.5), recorded above, in which the rows of $P(U)$ are linear combinations of the rows of $G(U)$:

(4.7.8)
$$P(U) = BG(U), \qquad U\in\mathscr{O},$$

for some $n\times n$ matrix B. For $P(U)$ of the form (4.7.8), it follows from (4.7.6) that L^∞ weak solutions to the initial-boundary value problem are fully characterized by the equation

(4.7.9)
$$\int_0^T\int_{\mathscr{D}} \Big[\partial_t\Phi U + \sum_{\alpha=1}^m \partial_\alpha\Phi G_\alpha(U)\Big] dxdt + \int_{\mathscr{D}} \Phi(x,0)U_0(x)dx = 0,$$

for every Lipschitz test function $\Phi(x,t)$, with compact support in $\mathbb{R}^m\times[0,T)$, values in $\mathbb{M}^{1\times n}$, and trace on $\partial\mathscr{D}$ that lies in the orthogonal complement of the kernel of B.

An alternative approach to boundary value problems stems from the viewpoint, presented in Section 4.6, that the hyperbolic system (4.1.1) should be regarded as a system with diffusion, such as (4.6.1), with vanishing viscosity coefficient μ. On the basis of this premise, one should consider boundary conditions suitable for the parabolic system (4.6.1) and let the limiting process dictate how these boundary conditions relate to the hyperbolic system (4.1.1). Boundary layers may form, as $\mu \to 0$, on parts of the boundary $\mathscr{B}$, so that one should not expect that the resulting solution to the hyperbolic system will satisfy the assigned boundary conditions everywhere. Nevertheless, when the system (4.1.1) is endowed with an entropy-entropy flux pair compatible with (4.6.1), as described in Section 4.6, then it is possible to derive useful information on the boundary behavior of solutions. As an illustration, consider the initial-boundary value problem for the system (4.6.1), with initial conditions $U = U_0$ on $\mathscr{D}$ and boundary conditions $U = \bar{U}$ on $\mathscr{B}$, where $\bar{U}$ is some fixed state. As in Section 4.6, assume that, for any $\mu > 0$, this problem possesses a classical solution U_μ on $\mathscr{X}$, and that some sequence $\{U_{\mu_k}\}$, with $\mu_k \to 0$ as $k \to \infty$, converges boundedly almost everywhere on $\mathscr{X}$ to a (weak) solution U of (4.1.1). Suppose (η, Q) is an entropy-entropy flux pair satisfying (4.6.7). We write (4.6.6) for the normalized entropy-entropy flux pair $(\bar{\eta}, \bar{Q})$, defined by (4.1.6), (4.1.7), multiply by any nonnegative Lipschitz test function ψ with compact support in $\mathbb{R}^m \times [0,T)$, integrate over $\mathscr{D} \times (0,T)$, integrate by parts, and use the initial and boundary conditions thus obtaining the following equation:

$$\int_0^T \int_{\mathscr{D}} \left[\partial_t \psi \, \bar{\eta}(U_\mu) + \sum_{\alpha=1}^m \partial_\alpha \psi \, \bar{Q}_\alpha(U_\mu) \right] dx\,dt + \int_{\mathscr{D}} \psi(x,0)\, \bar{\eta}(U_0(x))\,dx \tag{4.7.10}$$
$$= \mu \int_0^T \int_{\mathscr{D}} \partial_\alpha \psi \, \mathrm{D}\bar{\eta}(U_\mu) B_{\alpha\beta}(U_\mu) \partial_\beta U_\mu \,dx\,dt$$
$$+ \mu \int_0^T \int_{\mathscr{D}} \psi \sum_{\alpha,\beta=1}^m \left(\partial_\alpha U_\mu\right)^\top \mathrm{D}^2 \bar{\eta}(U_\mu) B_{\alpha\beta}(U_\mu) \partial_\beta U_\mu \,dx\,dt.$$

The argument employed in Section 4.6 shows that, as $\mu \to 0$, the first term on the right-hand side of (4.7.10) tends to zero while the second term stays nonnegative. Therefore,

$$\int_0^T \int_{\mathscr{D}} \left[\partial_t \psi \, \bar{\eta}(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, \bar{Q}_\alpha(U) \right] dx\,dt + \int_{\mathscr{D}} \psi(x,0)\, \bar{\eta}(U_0(x))\,dx \geq 0. \tag{4.7.11}$$

To return to the original entropy-entropy flux pair (η, Q), we write (4.7.6) for $\Phi = \psi \mathrm{D}\eta(\bar{U})$. Upon combining the resulting equation with the inequality (4.7.11), we obtain

(4.7.12)
$$\int_0^T \int_{\mathscr{D}} \left[\partial_t \psi \, \eta(U) + \sum_{\alpha=1}^m \partial_\alpha \psi \, Q_\alpha(U) \right] dx\,dt + \int_{\mathscr{D}} \psi(x,0)\, \eta(U_0(x))dx$$
$$\geq \int_0^T \int_{\partial\mathscr{D}} \psi \left\{ \bar{Q}_{\mathscr{B}} - \mathrm{D}\eta(\bar{U})[\bar{G}_{\mathscr{B}} - G_{\mathscr{B}}] \right\} d\mathscr{H}^{m-1}(x)dt,$$

where we have set

(4.7.13)
$$\bar{G}_{\mathscr{B}} = G(\bar{U})\nu, \qquad \bar{Q}_{\mathscr{B}} = Q(\bar{U})\nu.$$

Finally, we combine (4.7.12) with (4.7.7),

(4.7.14)
$$\int_0^T \int_{\partial\mathscr{D}} \psi \left\{ Q_{\mathscr{B}} - \bar{Q}_{\mathscr{B}} - \mathrm{D}\eta(\bar{U}) \left[G_{\mathscr{B}} - \bar{G}_{\mathscr{B}} \right] \right\} d\mathscr{H}^{m-1}(x)dt \geq \langle \mathscr{P}, \psi \rangle_{\mathscr{X}} ,$$

assuming $\psi(x,0) = 0$, $x \in \mathbb{R}^m$. By letting the support of ψ shrink about points of $\mathscr{B}$, we deduce the pointwise condition

(4.7.15)
$$Q_{\mathscr{B}} - \bar{Q}_{\mathscr{B}} - \mathrm{D}\eta(\bar{U})[G_{\mathscr{B}} - \bar{G}_{\mathscr{B}}] \geq 0.$$

The quantity on the left-hand side of (4.7.15) may be interpreted as the density of a surface measure that represents the entropy loss in the boundary layer.

The inequality (4.7.15) furnishes some information on the boundary conditions induced by the vanishing viscosity approach. Naturally, this information becomes more precise when the system (4.1.1) is endowed with multiple independent entropies compatible with (4.6.1). In particular, as we shall see in Section 6.9, for the scalar conservation law a sufficiently large collection of inequalities (4.7.15) characterizes completely the solution to the initial-boundary value problem constructed by the vanishing viscosity approach.

4.8 Euler Equations

The Euler equations (3.3.36), governing isentropic gas flow in one, two or three spatial dimensions, offer the primordial, and still most important, example of a hyperbolic system of conservation laws. They have long served as the paradigm for the entire class, and they command an enormous literature, addressing properties tied to their special structure as well as generic properties shared by other hyperbolic systems of conservation laws. So as to set the stage for the issues of concern in the remainder of the book, we present in this section a representative sample of properties of the Euler equations, in three spatial dimensions, emerging from research work of recent or older vintage. The analysis will only be sketched here – for the details, the reader may consult the bibliography cited in Section 4.9.

We write (3.3.36), with zero body force, $b = 0$, as a system in canonical form,

$$
\begin{cases} \partial_t \rho + \operatorname{div} m^\top = 0 \\[2ex] \partial_t m + \operatorname{div}\left(\rho^{-1} m m^\top\right) + \operatorname{grad} p(\rho) = 0, \end{cases} \tag{4.8.1}
$$

using as state variables the mass density ρ and the momentum density $m = \rho v$.

Recalling (3.3.39), we infer that the system (4.8.1) is endowed with the entropy-entropy flux pair

$$
\eta = \rho\varepsilon(\rho) + \frac{1}{2}\rho^{-1}|m|^2\,, \qquad Q = \eta\rho^{-1}m + p(\rho)\rho^{-1}m, \tag{4.8.2}
$$

where $\varepsilon'(\rho) = \rho^{-2}p(\rho)$.

In the realm of classical solutions, one may combine the two equations in (4.8.1) and write the system in the form

$$
\begin{cases} \partial_t \rho + (v\cdot\operatorname{grad})\rho + \rho \operatorname{div} v = 0 \\[2ex] \partial_t v + (v\cdot\operatorname{grad})v + p'(\rho)\operatorname{grad}\rho = 0, \end{cases} \tag{4.8.3}
$$

with state variables (ρ, v).

The systems (4.8.1), (4.8.3) are hyperbolic, and the entropy η is a uniformly convex function of (ρ, m), as long as $p'(\rho) > 0$. Throughout this section we shall be assuming, for simplicity, that the gas is ideal, with equations of state (2.5.31), namely $p = \kappa\rho^\gamma$, where, in accordance with basic kinetic theory, $\gamma \in (1, 5/3]$. The sonic speed is $c(\rho) = [\kappa\gamma\rho^{\gamma-1}]^{\frac{1}{2}}$. Thus our system is hyperbolic for $\rho > 0$, but hyperbolicity breaks down at $\rho = 0$. The presence of vacuum complicates the analysis of the Euler equations.

One may restore hyperbolicity to the full physical range, including vacuum, by replacing ρ by the new state variable $\omega = \beta^{-1}c(\rho)$, with $\beta = \frac{\gamma-1}{2}$, thus transforming (4.8.3) into

$$
\begin{cases} \partial_t \omega + (v\cdot\operatorname{grad})\omega + \beta\omega \operatorname{div} v = 0 \\[2ex] \partial_t v + (v\cdot\operatorname{grad})v + \beta\omega\operatorname{grad}\omega = 0, \end{cases} \tag{4.8.4}
$$

which is a symmetric hyperbolic system, even for $\omega = 0$. As we shall see in Section 5.1, this guarantees that the Cauchy problem for (4.8.4), with initial data (ω_0, v_0) in a Sobolev space of sufficiently high order and $\omega_0 \geq 0$, possesses a unique classical solution (ω, v) on some maximal time interval $[0, T)$. Furthermore, if T is finite, then some derivative of ω and/or v must blow up as t tends to T. Clearly, the solution (ω, v) induces classical solutions (ρ, m) and (ρ, v) to the systems (4.8.1) and (4.8.3), on the same time interval $[0, T)$.

We now employ an argument, similar in spirit to that encountered in Section 4.2 for the Burgers equation, to demonstrate that the lifespan T of the aforementioned classical solution is typically finite, as a result of mass confinment. We consider the Cauchy problem for (4.8.1), with initial values $\rho(x, 0) = \rho_0 = \text{constant}$, for $x \in \mathbb{R}^3$, and $m(x,0)$ supported in the unit ball, $m(x,0) = 0$ if $|x| \geq 1$. Suppose there exists a classical solution $(\rho(x,t), m(x,t))$ on some time interval $[0,T)$. The fast characteristic speed in the direction $\nu \in \mathbb{S}^2$ is $v \cdot \nu + c(\rho)$. Then, by virtue of Theorem 4.1.1, we have $\rho(x,t) = \rho_0$ and $m(x,t) = 0$, for any $t \in [0,T)$ and $|x| \geq r(t)$, where $r(t) = 1 + c(\rho_0)t$. In particular, from $(4.8.1)_1$,

$$\int_{|x|<r(t)} [\rho(x,t) - \rho_0]\, dx = 0, \qquad 0 \leq t < T. \tag{4.8.5}$$

We will monitor the evolution of the weighted radial momentum

$$M(t) = \int_{|x|<r(t)} x \cdot m(x,t) dx. \tag{4.8.6}$$

We differentiate (4.8.6) with respect to t, express the time derivative $\partial_t m$ in terms of spatial derivatives, through $(4.8.1)_2$, and integrate by parts to get

$$\dot{M}(t) = \int_{|x|<r(t)} \left[\rho^{-1}|m|^2 + 3\kappa\left(\rho^\gamma - \rho_0^\gamma\right)\right] dx. \tag{4.8.7}$$

Since $\gamma > 1$, (4.8.5) and Jensen's inequality imply

$$\int_{|x|<r(t)} \left(\rho^\gamma - \rho_0^\gamma\right) dx \geq 0. \tag{4.8.8}$$

Furthermore, by (4.8.6), (4.8.5) and Schwarz's inequality,

$$M^2(t) \leq \int_{|x|<r(t)} \rho |x|^2 dx \int_{|x|<r(t)} \rho^{-1}|m|^2 dx \leq \frac{4\pi}{3}\rho_0 r^5(t) \int_{|x|<r(t)} \rho^{-1}|m|^2 dx. \tag{4.8.9}$$

Upon combining (4.8.7) with (4.8.8) and (4.8.9), we end up with the differential inequality

$$\dot{M}(t) \geq \frac{3}{4\pi\rho_0}[1 + c(\rho_0)t]^{-5} M^2(t). \tag{4.8.10}$$

After an elementary integration, recalling that $c(\rho_0) = \left[\kappa\gamma\rho_0^{\gamma-1}\right]^{\frac{1}{2}}$, we conclude that if

$$M(0) > \frac{16\pi}{3}(\kappa\gamma)^{\frac{1}{2}} \rho_0^{\frac{\gamma+1}{2}}, \tag{4.8.11}$$

then $M(t)$ will blow up in finite time. Thus, classical solutions of the Cauchy problem for the system of isentropic gas dynamics, with large initial data, generally break down in finite time.

More refined analysis, reported in the literature cited in Section 4.9, shows that the above arguments may be extended for establishing breakdown of classical solutions in nonisentropic gas dynamics, and even when the initial data are not necessarily large.

The above argument verifies that the lifespan of classical solutions to the Cauchy problem for the Euler equations (4.8.1) is generally finite, but it does not pinpoint when and how catastrophe occurs. Typically, classical solutions blow up as the result of wave breaking, but before turning to that issue, we shall discuss an alternative manifestation of mass confining, in the presence of vacuum.

Let us consider the Cauchy problem for the system (4.8.1), with initial data (ρ_0, m_0), $m_0 = \rho_0 v_0$, that vanish outside a bounded subset Ω of $\mathbb{R}^3$. Assume that there exists a (generally weak) solution (ρ, m) on some time interval $[0, T)$, which satisfies the entropy admissibility condition (4.5.2) for the entropy-entropy flux pair (4.8.2). The gas will disperse into the vacuum, with finite speed, so that, for each $t \in [0, T)$, (ρ, m) will be supported in some bounded set Ω_t. We shall estimate the size of Ω_t with the help of an interesting estimate, derived as follows.

We start out from the identity

$$\frac{1}{2}\rho|tv - x|^2 + t^2\rho\varepsilon = \frac{1}{2}|x|^2\rho - tx\cdot m + t^2\eta(\rho, m). \tag{4.8.12}$$

After a long but straightforward calculation, using (4.8.1) and the entropy inequality, we deduce

(4.8.13)
$$\partial_t\left(\frac{1}{2}\rho|tv - x|^2 + t^2\rho\varepsilon\right) + \operatorname{div}\left([\frac{1}{2}\rho|tv - x|^2 + t^2\rho\varepsilon]v + tp[tv - x]\right) \leq t[2\rho\varepsilon - 3p].$$

In particular, when (ρ, m) is a classical solution, (4.8.13) holds as equality.

For the ideal gas (2.5.31), $2\rho\varepsilon - 3p = (5 - 3\gamma)\rho\varepsilon$. Thus, integrating (4.8.13) over $\mathbb{R}^3$ yields the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|tv - x|^2 + t^2\rho\varepsilon\right)dx \leq \frac{5 - 3\gamma}{t}\int_{\mathbb{R}^3}\left(\frac{1}{2}\rho|tv - x|^2 + t^2\rho\varepsilon\right)dx, \tag{4.8.14}$$

whence

$$\int_{\mathbb{R}^3}\rho\varepsilon\, dx \leq Ct^{3(1-\gamma)}. \tag{4.8.15}$$

The physical interpretation of (4.8.15) is that gas expansion converts internal to kinetic energy, and as a result the internal energy decays.

By Hölder's inequality,

$$\int_{\mathbb{R}^3}\rho dx \leq \left(\int_{\mathbb{R}^3}\rho^\gamma dx\right)^{\frac{1}{\gamma}}|\Omega_t|^{\frac{1-\gamma}{\gamma}} \leq Ct^{\frac{3(1-\gamma)}{\gamma}}|\Omega_t|^{\frac{1-\gamma}{\gamma}}, \tag{4.8.16}$$

where $|\Omega_t|$ denotes the volume of Ω_t. The left-hand side of (4.8.16) is a positive constant, namely the conserved mass of the gas. Therefore, $|\Omega_t| \geq At^3$, for some positive constant A. We now discuss whether classical solutions are capable of sustaining, in the long term, growth of $|\Omega_t|$ at the above rate.

Assume then that the above solution is smooth, in which case Ω and Ω_t are open sets with smooth boundaries denoted by Γ and Γ_t. On Γ_t, (4.8.3) and (4.8.4) reduce to the transport equation

$$\partial_t v + (v \cdot \mathrm{grad})v = 0. \tag{4.8.17}$$

Consequently, for any $x \in \Gamma$, v stays constant, equal to $v_0(x)$, along the straight line $y = x + tv_0(x)$, and

$$\Gamma_t = \{y \in \mathbb{R}^3 \,:\, y = x + tv_0(x),\ \ x \in \Gamma\}, \tag{4.8.18}$$

for all $t \in [0,T)$. Therefore, for t sufficiently small,

$$\Omega_t = \{y \in \mathbb{R}^3 \,:\, y = x + tv_0(x),\ \ x \in \Omega\}, \tag{4.8.19}$$

whence

$$|\Omega_t| = \int_\Omega \det\left[I + t\,\mathrm{grad}\, v_0(x)\right] dx. \tag{4.8.20}$$

Thus, for t small, $|\Omega_t|$ is a polynomial:

$$|\Omega_t| = Jt^3 + Kt^2 + Lt + N, \tag{4.8.21}$$

where J, K, L depend on $\mathrm{grad}\, v_0$, and $N = |\Omega|$. Actually, the constants J, K and L are integrals over Ω of null Lagrangians of v_0, and thus fully determined by the values of v_0 and $\mathrm{grad}\, v_0$ on Γ. In particular,

$$J = \int_\Omega \det\left[\mathrm{grad}\, v_0(x)\right] dx. \tag{4.8.22}$$

Since the above argument may be repeated after substituting 0 by any $\bar{t} \in [0,T)$, we conclude that (4.8.21) holds for all $t \in [0,T)$. It is now clear that if $J \leq 0$, then the solution cannot accommodate the requirement $|\Omega_t| \geq At^3$, for t large. Thus, the lifespan of any classical solution with initial data satisfying $J \leq 0$ is necessarily finite.

In the present situation, the demise of the classical solution may occur either as a result of wave breaking, or because a singularity forms at the interface between gas and vacuum. Let us discuss the latter possibility, within the following setting: The interface Γ_t is still a smooth surface and the solution (ρ, v) is still smooth on Ω_t, but allowed to be singular across Γ_t.

First we test whether the interface may be a shock. To that end, we apply the Rankine-Hugoniot conditions (3.3.40), with ν the unit exterior normal on Ω_t, to infer that the jump in the pressure p, and thereby also in the density ρ, vanishes.

Thus the singularity at the interface must be milder than a jump discontinuity. We seek conditions that would allow the interface to accelerate at a controlled rate. For that purpose, the natural assumption is that the square of the sonic speed c must be Lipschitz, with normal derivative that jumps across the interface:

$$\left[\left[\frac{\partial c^2}{\partial \nu}\right]\right] = g, \tag{4.8.23}$$

where g is a bounded function that is necessarily nonnegative. Indeed, when (4.8.23) holds, the second equation of the system (4.8.4), restricted to the interface, yields

$$\partial_t v + (v \cdot \mathrm{grad})v = \frac{g}{\gamma - 1}\nu, \tag{4.8.24}$$

which should be compared and contrasted to (4.8.17). Since $g \geq 0$, (4.8.24) signals acceleration of the interface. In particular, for g suitably large, the volume of Ω_t may grow at a rate compatible with the requirement $|\Omega_t| \geq At^3$, thus allowing for the existence of solutions in the large with no singularities, beyond those lying on the interface with vacuum. More on that in Section 5.6.

We now turn to the question of the breakdown of classical solutions due to wave breaking. As we saw in Section 4.2, in the context of Burgers's equation, the physical description of the phenomenon is that, due to the nonlinearity, wave speed depends on the wave amplitude and this may generate steep wave profiles that eventually break. The same mechanism is present in the Euler equations. Indeed, as we shall see in Section 7.8, compressive waves for the Euler equations, in one space dimension, must break. However, in three dimensions the situation is more delicate. To begin with, the steepening of wave profiles due to compression competes with dispersion, which has the opposite effect. This delays, and may even thwart, the breaking of waves. The difficulty is compounded by the presence of vorticity, whose potential contribution to the development of singularities is not yet fully understood. So as to focus on the effects of compression, we shall eliminate the fallout of vorticity by limiting our discussion to irrotational flow.

In irrotational flow, the velocity derives from a potential ϕ, through (3.3.48), and the Euler equations reduce to the system (3.3.49), (3.3.50), where h is the enthalpy. We assume, for simplicity, that the body force vanishes, $g = 0$. Since $h'(\rho) = p'(\rho)/\rho > 0$, one may realize the density as function of enthalpy, $\rho = \rho(h)$.

In what follows, we shall be employing the summation convention. Inserting in (3.3.50) ρ as a function of h, with h given by (3.3.49), reduces the mass conservation equation to the quasilinear wave equation

$$\partial_t^2 \phi - c^2(\rho)\Delta\phi = -2(\partial_j\phi)(\partial_t\partial_j\phi) - (\partial_i\phi)(\partial_j\phi)(\partial_i\partial_j\phi), \tag{4.8.25}$$

for the potential ϕ, where $c(\rho)$ stands for the sonic speed. Notice that linearization of (4.8.25) about any rest state $\rho = \rho_0$, $v = 0$ yields the classical wave equation.

A remarkable feature of (4.8.25) is that it may be realized as an Euler-Lagrange equation. Indeed, recall that $\partial_i\phi = v_i$, for $i = 1,2,3$, and set $\partial_t\phi = -v_0$. Introducing the Lagrangian

$$L(v_0, v_1, v_2, v_3) = p(\rho(h)), \qquad h = v_0 - \frac{1}{2}v_i v_i \tag{4.8.26}$$

and using $\frac{dp}{d\rho}\frac{d\rho}{dh} = \rho$, we deduce

$$\frac{\partial L}{\partial v_0} = \rho, \qquad \frac{\partial L}{\partial v_i} = -\rho v_i\,, \quad i = 1,2,3, \tag{4.8.27}$$

which implies

$$\partial_t \frac{\partial L}{\partial v_0} - \partial_i \frac{\partial L}{\partial v_i} = \partial_t \rho + \partial_i(\rho v_i) = 0. \tag{4.8.28}$$

We consider the Cauchy problem for (4.8.25), under assigned initial conditions

$$\phi(x,0) = \phi_0(x), \quad \partial_t \phi(x,0) = \phi_1(x), \qquad x \in \mathbb{R}^3. \tag{4.8.29}$$

As a corollary to Theorem 5.1.1, which will be proved in Chapter V, whenever $\phi_0 \in W^{4,2}(\mathbb{R}^3)$ and $\phi_1 \in W^{3,2}(\mathbb{R}^3)$, there exists a unique classical C^2 solution ϕ, locally in time. Considerable effort has been expended in monitoring classical solutions, under the assumption that the initial data reside in Sobolev spaces of sufficiently high order, with norms of small size delimited by a parameter ε. The task requires powerful analysis, utilizing to a full extent the underlying geometric structure of (4.8.25) and in particular its manifestation as an Euler-Lagrange equation. Unfortunately, this work is too technical and laborious to be presented here, even in abridged form, so the reader should consult the bibliography cited in Section 4.9. The next paragraph provides a sketchy summary of the main conclusions.

In three spatial dimensions, dispersion induces $O(t^{-1})$ decay rate on derivatives of solutions to the Cauchy problem for the classical wave equation. The solutions to the quasilinear wave equation (4.8.25) inherit that property at the level of first derivatives. However, at the level of derivatives of second order there is an even contest between nonlinearity and dispersion. Dispersion manages to prolong the lifespan of classical solutions to $O(\exp\frac{1}{\varepsilon})$, a major improvement over the one-dimensional situation, where the lifespan is merely $O(\frac{1}{\varepsilon})$. Nevertheless, nonlinearity eventually prevails, driving second derivatives to infinity. The insightful proof proceeds by introducing special coordinates, adapted to the wave profiles, identifying the principal direction along which second derivatives grow and eventually break, in contrast to transversal directions along which dispersion dominates, keeping the size of derivatives under control. The analysis in these coordinates is quite explicit, so that in addition to exposing the breaking of waves it also provides a description of the maximal development of the solution.

We close this section with certain surprising, and perhaps disturbing, facts concerning weak solutions of the Euler equations. The analysis in the forthcoming Section 5.2 will establish that whenever the Cauchy problem for (4.8.1) possesses a classical solution on some time interval $[0,T)$, this solution is unique, not only among other classical solutions, but even within the broader class of L^∞ weak solutions that satisfy the entropy admissibility condition (4.5.3), for the entropy-entropy flux pair (4.8.2). However, in the absence of a classical solution, the above entropy inequality is no longer sufficiently selective for singling out a unique admissible solution. This remarkable fact will be demonstrated by considering the Cauchy problem for (4.8.1) under initial data that are periodic in each component x_i of x, with period 1. It will thus be convenient to regard the solutions at time t as functions defined on the standard torus $\mathbb{T}^3$. The existence of multiple solutions is established by the following

4.8.1 Theorem. *There exist m_0 in $L^\infty(\mathbb{T}^3)$ such that the Cauchy problem for the Euler equations* (4.8.1) *with initial data*

$$\rho(x,0) = 1, \qquad m(x,0) = m_0(x), \qquad x \in \mathbb{T}^3, \tag{4.8.30}$$

admits infinitely many L^∞ weak solutions (ρ, m) on $[0,\infty)$, satisfying the admissibility condition (4.5.2) *for the entropy-entropy flux pair* (4.8.2).

The proof is lengthy and technical, so only a rough sketch will be presented here. The details are found in the bibliography cited in Section 4.9.

A surprising feature of the Euler equations is that, in the setting of L^∞ weak solutions, the Cauchy problem is underdetermined to the extent that one may prescribe the density field ρ together with the length $|m|$ of the momentum field and still leave room for constructing infinitely many solutions satisfying the entropy admissibility condition. Accordingly, let us prescribe $\rho \equiv 1$, in which case (4.8.1) reduces to

$$\begin{cases} \operatorname{div} m^\top = 0 \\ \partial_t m + \operatorname{div}\left(mm^\top\right) = 0. \end{cases} \tag{4.8.31}$$

Sidestepping, for the time being, the requirement of entropy admissibility, we fix $T > 0$ and a C^1 function M with positive values on $[0,T]$ and seek L^∞ solutions of (4.8.31) on $[0,T]$ that satisfy initial and terminal conditions

$$m(\cdot,0) = m_0(\cdot), \qquad m(\cdot,T) = 0, \tag{4.8.32}$$

together with the constraint

$$|m(x,t)| = M(t), \qquad \text{a.e. on } \mathbb{T}^3 \times (0,T). \tag{4.8.33}$$

An obvious compatibility condition is $|m_0(x)| \le M(0)$, a.e. on $\mathbb{T}^3$.

Assuming (4.8.33), it is instructive to rewrite (4.8.31) in the equivalent form

$$\begin{cases} \operatorname{div} m^\top(x,t) = 0 \\ \partial_t m(x,t) + \operatorname{div} U(x,t) = 0, \end{cases} \tag{4.8.34}$$

$$U(x,t) = m(x,t)m^\top(x,t) - \frac{1}{3}|m(x,t)|^2 I. \tag{4.6.35}$$

Thus, in the spirit of continuum physics, we are regarding (4.8.31) as the composition of a system of conservation laws (4.8.34) with a constitutive equation (4.8.35). Notice that the 3×3 matrix-valued function U is symmetric and traceless. In what follows, $\mathbb{N}$ stands for the space of symmetric and traceless 3×3 matrices.

The first step in the analysis is to introduce a class of functions deemed subsolutions to the above system.

A *subsolution* of (4.8.32), (4.8.33), (4.8.34) and (4.8.35) is a function u defined on $\mathbb{T}^3 \times [0,T]$, taking values in $\mathbb{R}^3$, and having the following properties:

(a) The function $t \mapsto u(\cdot,t)$ is continuous on $[0,T]$, in $L^\infty(\mathbb{T}^3;\mathbb{R}^3)$ weak*, and

$$u(\cdot,0) = m_0(\cdot), \qquad u(\cdot,T) = 0. \tag{4.8.36}$$

(b) u is C^1 on $\mathbb{T}^3 \times (0,T)$ and satisfies

$$\begin{cases} \operatorname{div} u^\top(x,t) = 0 \\ \partial_t u(x,t) + \operatorname{div} U(x,t) = 0, \end{cases} \tag{4.8.37}$$

for some C^1 function U on $\mathbb{T}^3 \times (0,T)$ with values in $\mathbb{N}$.

(c) For any $x \in \mathbb{T}^3$ and $t \in (0,T)$,

$$u(x,t)\,u^\top(x,t) - U(x,t) \le \frac{1}{3}M^2(t)I. \tag{4.8.38}$$

Let it be noted that, for any fixed $w \in \mathbb{R}^3$ and $W \in \mathbb{N}$,

$$ww^\top - W \ge \frac{1}{3}|w|^2 I. \tag{4.8.39}$$

Thus (4.8.38) implies $|u(x,t)| \le M(t)$, for all $x \in \mathbb{T}^3$ and $t \in (0,T)$.

We proceed under the assumption that the set of subsolutions associated with m_0, M and T is nonempty. We denote this set by $\mathscr{X}$. In particular, $\mathscr{X}$ associated with $m_0 \equiv 0$ is nonempty, for any choice of positive M, as it contains the zero function. We let $\overline{\mathscr{X}}$ denote the closure of $\mathscr{X}$ in the weak topology of $L^2(\mathbb{T}^3 \times [0,T])$.

On $\overline{\mathscr{X}}$ we define the functional

$$J[u] = \int_0^T \int_{\mathbb{T}^3} \left[|u(x,t)|^2 - M^2(t)\right] dxdt, \tag{4.8.40}$$

with nonpositive values. Members of $\overline{\mathscr{X}}$ inherit from $\mathscr{X}$ the properties (4.8.37), in the sense of distributions, and (4.8.38), almost everywhere, since the maximum eigenvalue of symmetric matrices is a convex function. From this observation and (4.8.39) follows that $m \in \overline{\mathscr{X}}$ is a solution of (4.8.32), (4.8.33), (4.8.34) and (4.8.35) if and only if $J[m] = 0$.

Since $|u(x,t)| \le M(t)$, for all $u \in \mathscr{X}$, it follows that any $m \in \overline{\mathscr{X}}$ with $J[m] = 0$ must be a point of continuity of J. It turns out that the converse is also true: if J is continuous at $m \in \overline{\mathscr{X}}$, then $J[m] = 0$. The reason is that if $J[u] < 0$, for some $u \in \mathscr{X}$, then there exist $v \in \mathscr{X}$ that are weakly close to u, and yet $J[v]$ differs substantially from $J[u]$. The precise statement is provided by the following proposition whose technical proof is found in the literature cited in Section 4.9.

4.8.2 Lemma. *Let u be any subsolution and (τ_1, τ_2) any subinterval of $(0,T)$. Then there exist sequences $\{v_n\}$ in $\mathscr{X}$ such that $u - v_n$ is supported in the time interval (τ_1, τ_2), $v_n \to u$ in L^∞weak* and*

(4.8.41)
$$\liminf_{n\to\infty} \int_0^T \int_{\mathbb{T}^3} |v_n(x,t) - u(x,t)|^2 dxdt \geq a \int_{\tau_1}^{\tau_2} \int_{\mathbb{T}^3} \left[|u(x,t)|^2 - M^2(t) \right]^2 dxdt$$

holds, with $a > 0$ independent of u.

The perturbations of the subsolution u, within $\mathscr{X}$, that are "slight", in the sense of the weak topology, and yet incur sizable changes to the entropy typically involve rapid oscillations. Indeed the presence of wild oscillations is the trademark of the theory at hand.

We view $\overline{\mathscr{X}}$ as a complete metric space. The functional J is of Baire class 1 (pointwise limit of continuous functions), whence the set of its points of continuity is dense in $\overline{\mathscr{X}}$. Each and everyone of these points furnishes a solution to (4.8.31), (4.8.32), (4.8.33). Henceforth, we call these solutions *exotic*.

We now turn to the question of admissibility of exotic solutions. By virtue of (4.8.2) and (4.8.33),

$$\eta(x,t) = \frac{1}{2}M^2(t) + \varepsilon(1), \quad Q(x,t) = \left[\frac{1}{2}M^2(t) + \varepsilon(1) + p(1) \right] m(x,t). \tag{4.8.42}$$

Notice that the divergence of $Q^\top$ vanishes. Thus, when $\dot{M}(t) \leq 0$, the solution $(1,m)$ satisfies the entropy admissibility criterion (4.5.1) on $\mathbb{T}^3 \times (0,\infty)$. In particular, if M is constant, the entropy is conserved on $\mathbb{T}^3 \times (0,T)$.

The above properties underscore the difference between exotic and standard solutions, say of class BV. In the latter case entropy is produced exclusively by jump discontinuities. As noted in (sub)section 3.3.6, solutions to the Euler equations may support two types of jump discontinuities, namely compressive shocks and vortex sheets (contact discontinuities). Compressive shocks, which produce entropy, cannot take part in the exotic solutions, because they involve jumps in the density. By contrast, vortex sheets, which are compatible with uniform density, are probably present in exotic solutions and may serve as building blocks for the oscillatory profiles of these solutions. However, vortex sheets do not produce entropy and hence the decay of η encoded in (4.8.42) cannot be attributed to the presence of jump discontinuities, but is due to an alternative mechanism. We shall discuss related issues in Section 6.8 and 11.13.

Even though $\dot{M}(t) \leq 0$ guarantees that the exotic solutions satisfy (4.5.1), the more selective entropy admissibility criterion (4.5.2) will not hold, unless $t = 0$ is a point of right continuity of the function $t \mapsto m(\cdot,t)$, in the strong $L^1(\mathbb{T}^3;\mathbb{R}^3)$ topology. It turns out that given m in $\mathscr{X}$ one may construct, with the help of the properties of $\mathscr{X}$ encoded in Lemma 4.8.2, $\bar{m}$ in $\mathscr{X}$ such that $\bar{m}(\cdot,\tau) = M(\tau)$, for some $\tau \in (0,T)$, and $\bar{m}(\cdot,t) = m(\cdot,t)$, for t near T. It is now clear that if one replaces M by $\bar{M}$ and m_0 by $\bar{m}_0$, where $\bar{M}(t) = M(t+\tau)$ and $\bar{m}_0(\cdot) = \bar{m}(\cdot,\tau)$, the resulting exotic solutions will satisfy (4.5.2).

Refined, technical, analysis shows that, in fact, one may even construct entropy dissipating exotic solutions that are Hölder (though not Lipschitz) continuous.

The discovery of exotic solutions, with massive non uniqueness, for the Euler equations, is alarming, so one hopes that they will be disqualified as being physically inadmissible. Nevertheless, we saw above that these solutions pass the entropy admissibility test. On the other hand, because of the afforded flexibility in setting $M(t)$, none of the exotic solutions may satisfy the maximal dissipativeness criterion (4.5.11). Another risk is that standard solutions that were formerly pronounced maximally dissipative may lose this status in competition with exotic solutions. The reader may find relevant comments on this issue in Section 9.11.

4.9 Notes

To a great extent, it is the breaking of waves catastrophe that sets the tone for the theory of nonlinear hyperbolic systems of conservation laws. This effect was introduced, in Section 4.2, through the paradigm of the Burgers equation, but, as we shall see in Section 7.8, it pervades all (genuinely) nonlinear systems of conservation laws in one spatial dimension. The drive to wave breaking is still present in several spatial dimensions, but it has to compete with dispersion, which may delay or even prevent outright the breakdown of classical solutions, in systems that satisfy the so called *null condition*. In that direction, out of a voluminous literature, see for instance Christodoulou [1], Klainerman [1], Klainerman and Sideris [1], Sideris [2,3,4], Agemi [1], Chae and Huh [1], and Ta-tsien Li [1].

In particular, for the three-dimensional Euler equations the competing mechanisms of wave breaking and dispersion are nearly evenly matched. As a result, the proof that wave breaking eventually prevails, noted in Section 4.8, requires very delicate analysis. Following the pioneering work of John [2], the breakdown of classical solutions was established by Alinhac [1,2,3]. However, the definitive treatment that provides a detailed description of the breaking of waves, is due to Christodoulou [2], for the relativistic Euler equations, and to Christodoulou and Miao [1], for the classical Euler equations. These proofs are very technical, occupying several hundred pages of text. The recent paper by Holzegel, Klainerman, Speck and Wong [1] provides a very readable survey of the above work, placed in the context of the historical developement of the subject. Speck [1] is a monograph extending the results to general quasilinear wave equations, in three spatial dimensions.

The proof outlined in Section 4.8 that mass confinement is an alternative motor for the breakdown of classical solutions to the Euler equations, has been adapted from Sideris [1]. The argument, in the same section, that the volume of a gas mass is expanding in vacuum at cubic order and that growth at such rate cannot be sustained unless singularities develop at the interface, is taken from Serre [31]. In that connection, see also Chemin [1], Liu and Yang [1], and Yang and Zhu [2]. For a proof of breakdown of classical solutions to the Euler equations in the presence of vacuum, by the method of characteristics, see Chae and Ha [1]. An interesting class of global (dubbed "eternal") classical solutions to the Euler equations in two-space dimensions is presented in Serre [13,31] and in Grassin and Serre [1]. See also Chen and Young [3]. For time periodic solutions, see Georgiev and LeFloch [1].

Nonuniqueness of weak solutions to the Cauchy problem and the need of developing selection criteria poses another challenge to the theory of hyperbolic conservation laws, with many facets that will be discussed extensively in ensuing chapters of the book. In particular, as we saw in the historical introduction and in Chapters II and III, the entropy admissibility criterion, introduced in Section 4.5, is an abstraction of the Clausius-Duhem inequality, expressing the Second Law of thermodynamics, and it has been applied in concrete situations, at least since the turn of the twentieth century. The earliest explicit reference to this criterion, in its abstract form, is found in Kruzkov [1], but its central importance was recognized after the publication of the seminal paper by Lax [4], which, inter alia, introduced the term "entropy" in the present context. Considering the direction of the inequality (4.5.1), which is opposite to the direction of the Clausius-Duhem inequality (2.3.13), the term "free energy" rather than "entropy" would have been more appropriate, from the standpoint of continuum physics.

As we saw in Section 4.5, convexity of the entropy, which is a necessary prerequisite according to the definition of the concept by Lax [4], induces a modicum of stability to admissible L^∞ weak solutions, but is not always satisfied in the systems arising in continuum physics. For the weaker, but still sufficient, condition of quasiconvexity, noted in Section 4.5, see Morrey [1], Dacorogna [1], and Müller and Fonseca [1]. Šverak [1] shows that rank-one convexity is not generally sufficient for quasiconvexity. Polyconvexity was introduced by Ball [1], in the context of elastostatics. We shall return to this notion in Section 5.4. Remark 4.5.3 is due to Gui-Qian Chen [9].

For various experimentations with the idea of maximal dissipativeness, see Dafermos [32], Demoulini, Stuart and Tzavaras [1], Gangbo and Westdickenberg [1], Westdickenberg [1], Chiodaroli and Kreml [1], and Feireisl [2]. We will return to this issue in greater detail, albeit within a more narrow scope, in Section 9.7.

In later chapters, we shall have frequent encounters with the vanishing viscosity approach, both as a method for constructing solutions and as a means of identifying admissible shocks. It is for the latter purpose that the method was originally introduced by Rayleigh [4] and G.I. Taylor [1]; see the historical introduction.

An exposition of the theory of intermediate parabolic-hyperbolic type systems is presented in the monographs by Songmu Zheng [1] and Hsiao [3], as well as in the survey article by Hsiao and Jiang [1], where the reader will find an extensive list of references.

For the viscosity-capillarity admissibility condition on weak solutions, see Slemrod [3] and LeFloch [5].

For a more detailed discussion of initial-boundary value problems, and related bibliography, the reader should consult Chapters V and VI. The inequalities (4.7.8) were first derived by Bardos, Leroux and Nédélec [1], for scalar conservation laws, and were then extended to systems, in one spatial dimension, by DuBois and LeFloch [1]. As we shall see in Section 6.9, these inequalities completely characterize admissible boundary conditions in the scalar case.

Following the pioneering work of De Lellis and Szekelyhidi [1,2], and Buckmaster, De Lellis, Isett and Szekelyhidi [1], the theory of exotic solutions for the Euler equations, outlined here in Section 4.8, was further developed in Chiodaroli [1], Chiodaroli and Kreml [1,2], Chiodaroli, Feireisl and Kreml [1], Feireisl [2], Feireisl and Kreml [1], Feireisl, Kreml and Vasseur [1], Chiodaroli, De Lellis and Kreml [1], and Villani [2]. In particular, Lemma 4.8.2 is taken from Feireisl [2].

V

Entropy and the Stability of Classical Solutions

It is a tenet of continuum physics that the Second Law of thermodynamics is essentially a statement of stability. In the examples discussed in the previous chapters, the Second Law manifests itself in the presence of companion balance laws, to be satisfied identically, as equalities, by classical solutions, and to be imposed as thermodynamic admissibility inequality constraints on weak solutions of the systems of balance laws. A recurring theme in the exposition of the theory of hyperbolic systems of balance laws in this book will be that companion balance laws induce stability under various guises. Here the reader will get a glimpse of the implications of entropy inequalities on the stability of classical solutions.

It will be shown that when the system of balance laws is endowed with a companion balance law induced by a convex entropy, the initial value problem is locally well-posed in the context of classical solutions: sufficiently smooth initial data generate a classical solution defined on a maximal time interval, typically of finite duration. However, in the presence of damping induced by relaxation or other dissipative mechanisms, and when the initial data are sufficiently small, the classical solution exists globally in time. Classical solutions are unique and depend continuously on their initial values, not only within the class of classical solutions but even within the broader class of weak solutions that satisfy the companion balance law as an inequality admissibility constraint.

Similar existence and stability results will be established, even when the entropy fails to be convex, in the following two situations: (a) the entropy is convex only in the direction of a certain cone in state space but the system is equipped with special companion balance laws, called involutions, whose presence compensates for the lack of convexity in complementary directions; or (b) the system is endowed with complementary entropies and the principal entropy is polyconvex. This structure arises in elastodynamics and electromagnetism.

The chapter will close with a brief discussion of the existence of classical solutions to the initial-boundary value problems.

From the standpoint of analytical technique, this chapter presents the aspects of the theory of quasilinear hyperbolic systems of balance laws that can be tackled by the methodology of the linear theory, namely energy estimates and Fourier analysis.

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5.1 Convex Entropy and the Existence of Classical Solutions

The aim in this section is to establish local existence of classical solutions to the Cauchy problem

$$(5.1.1)\qquad \partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, \qquad x \in \mathbb{R}^m,\ t > 0,$$

$$(5.1.2)\qquad U(x,0) = U_0(x), \qquad x \in \mathbb{R}^m,$$

for a homogeneous system of conservation laws endowed with a convex entropy η. The flux G, entropy η and associated entropy flux Q are smooth functions defined on a closed ball $\overline{\mathscr{B}_\rho}$ in $\mathbb{R}^n$, centered at the origin.

Throughout this chapter, we will employ the following notation. A *multi-index* r is an m-tuple of nonnegative integers: $r = (r_1, \dots, r_m)$. We put $|r| = r_1 + \cdots + r_m$, for the order of r, and $\partial^r = \partial^{r_1} \dots \partial^{r_m}$. Thus ∂^r is a differential operator of order $|r|$. For $\ell = -1, 0, 1, \dots,$ H_ℓ and H_ℓ^m will denote the Sobolev spaces $W^{l,2}(\mathbb{R}^m;\mathbb{R}^n)$ and $W^{l,2}(\mathbb{R}^m;\mathbb{M}^{n\times m})$, with respective norms $\|\cdot\|_\ell$ and $|\!|\!|\cdot|\!|\!|_\ell$. In particular, H_ℓ^m is identical to the Cartesian product space $[H_\ell]^m$. We will also use the symbol ∇ for the gradient operator $(\partial_1, \dots, \partial_m)$. Hence $V \in H_\ell$ implies $\nabla V \in H_{\ell-1}^m$ and $|\!|\!|\nabla V|\!|\!|_{\ell-1} \le \|V\|_\ell$. By the Sobolev embedding theorem, for $\ell > \frac{m}{2} + 1$, H_ℓ is continuously embedded in the space $C^1(\mathbb{R}^m;\mathbb{R}^n)$ of continuously differentiable n-vector fields on $\mathbb{R}^m$.

For $U \in \overline{\mathscr{B}_\rho}$, we introduce the $n \times n$ matrices

$$(5.1.3)\qquad A(U) = \mathrm{D}^2\eta(U),$$

$$(5.1.4)\qquad J_\alpha(U) = A(U)\mathrm{D}G_\alpha(U), \qquad \alpha = 1, \cdots, m,$$

which are symmetric, by virtue of (3.2.4).

The main result of this section is the following

5.1.1 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with a convex entropy η, so that $A(U)$ is positive definite for any $U \in \overline{\mathscr{B}_\rho}$. Suppose the initial data U_0 lie in H_ℓ, for some $\ell > \frac{m}{2} + 1$, and take values in a ball $\overline{\mathscr{B}_{\rho_0}}$ with radius $\rho_0 < \rho$. Then there exist $T_\infty \le \infty$ and a unique continuously differentiable function U on $\mathbb{R}^m \times [0, T_\infty)$, taking values in $\overline{\mathscr{B}_\rho}$, which is a classical solution to the Cauchy problem* (5.1.1), (5.1.2), *on the time interval $[0, T_\infty)$. Furthermore,*

$$(5.1.5)\qquad U(\cdot,t) \in \bigcap_{k=0}^{\ell} C^k([0,T_\infty); H_{\ell-k}).$$

The interval $[0, T_\infty)$ is maximal in that if $T_\infty < \infty$ then

$$(5.1.6)\qquad \int_0^{T_\infty} |\!|\!|\nabla U(\cdot,t)|\!|\!|_{L^\infty} dt = \infty$$

and/or $\limsup_{t\to T_\infty} \|U(\cdot,t)\|_{L^\infty} = \rho$.

The traditional proof of the above theorem, found in the literature cited in Section 5.7, and even in the second edition of the present book, determines the solution of (5.1.1), (5.1.2), in a suitable function space $\mathscr{F}$, as a fixed point of the map that carries $V \in \mathscr{F}$ to the solution $U \in \mathscr{F}$ of the linearized system

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(V(x,t))\partial_\alpha U(x,t) = 0, \tag{5.1.7}$$

with initial conditions (5.1.2). This approach is effective when the entropy η is convex, because in that case multiplication by $A(V)$ renders the system (5.1.7) symmetric; however, it is inapplicable under the conditions to be encountered in Sections 5.3 and 5.4, where the entropy fails to be convex and the compensatory estimates are inexorably tied to the geometric structure of (5.1.1) and do not carry over to the linearized form (5.1.7).

Accordingly, we shall employ here the vanishing viscosity method, which determines solutions to (5.1.1) as the $\varepsilon \to 0$ limit of solutions of the parabolic system

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = \varepsilon \Delta U(x,t). \tag{5.1.8}$$

This approach may not lead to the proof of Theorem 5.1.1 via the most direct route, but it has the advantage of rendering the passage to the following sections of this chapter as effortless as possible. Another benefit of the vanishing viscosity method is that it starts out at an elementary level. The sole prerequisite is knowing how to solve the Cauchy problem for the classical heat equation.

The first step is to establish local existence for the Cauchy problem for (5.1.8), (5.1.2), with fixed $\varepsilon > 0$. The dominant term in (5.1.8) is the Laplacian, so the entropy will not play any role at this stage.

Throughout this chapter, we shall employ c to denote some generic positive constant that may depend at most on ρ and on bounds of G, η and their derivatives on $\overline{\mathscr{B}_\rho}$.

5.1.2 Lemma. *As in the statement of Theorem* 5.1.1, *assume that* U_0 *takes values in* $\overline{\mathscr{B}_{\rho_0}}$, $\rho_0 < \rho$, and belongs to H_ℓ, *with* $\ell > \frac{m}{2} + 1$. *Set* $\omega_0 = \|U_0\|_\ell$. *Then for any fixed* $\omega > \omega_0$ *and* $\varepsilon > 0$, *there exist* $T_{\omega,\varepsilon}$, $0 < T_{\omega,\varepsilon} \leq \infty$, *and a solution* U *of* (5.1.8), (5.1.2) *on the time interval* $[0, T_{\omega,\varepsilon})$, *taking values in* $\overline{\mathscr{B}_\rho}$ *and such that*

$$U(\cdot,t) \in C^0([0,T_{\omega,\varepsilon});H_\ell) \bigcap L^2([0,T_{\omega,\varepsilon});H_{\ell+1}), \tag{5.1.9}$$

with

$$\|U(\cdot,t)\|_\ell < \omega, \qquad 0 \leq t < T_{\omega,\varepsilon}. \tag{5.1.10}$$

Moreover, if $T_{\omega,\varepsilon} < \infty$, *then* $\limsup_{t\to T_{\omega,\varepsilon}} \|U(\cdot,t)\|_\ell = \omega$ *and/or* $\limsup_{t\to T_{\omega,\varepsilon}} \|U(\cdot,t)\|_{L^\infty} = \rho$.

Proof. Fix ω_1, with $\omega_0 < \omega_1 < \omega$. With $T > 0$, to be specified below, we associate the class $\mathscr{V}$ of Lipschitz functions V defined on $\mathbb{R}^m \times [0,T]$, taking values in $\overline{\mathscr{B}_\rho}$ and satisfying

$$V(\cdot,t) \in L^\infty([0,T];H_\ell), \qquad \sup_{[0,T]} \|V(\cdot,t)\|_\ell \le \omega_1 . \tag{5.1.11}$$

By standard weak lower semicontinuity of L^p norms, $\mathscr{V}$ is a complete metric space under the metric

$$d(V,\bar{V}) = \sup_{[0,T]} \|V(\cdot,t) - \bar{V}(\cdot,t)\|_0 . \tag{5.1.12}$$

For any given $V \in \mathscr{V}$, we construct the solution U on $\mathbb{R}^m \times [0,T]$ of the linear parabolic system (coupled heat equations)

$$\partial_t U(x,t) - \varepsilon \Delta U(x,t) = -\sum_{\alpha=1}^m \partial_\alpha G_\alpha(V(x,t)) , \tag{5.1.13}$$

with initial condition (5.1.2). Thus

$$\begin{aligned}(4\pi\varepsilon)^{\frac{m}{2}} U(x,t) &= \int_{\mathbb{R}^m} t^{-\frac{m}{2}} \exp\left[-\frac{|x-y|^2}{4\varepsilon t}\right] U_0(y)dy \\ &- \int_0^t \int_{\mathbb{R}^m} (t-\tau)^{-\frac{m}{2}} \exp\left[-\frac{|x-y|^2}{4\varepsilon(t-\tau)}\right] \sum_{\alpha=1}^m \partial_\alpha G_\alpha(V(y,\tau))dyd\tau .\end{aligned} \tag{5.1.14}$$

We proceed to establish a priori bounds on U and in particular to show that if T is suitably small, then $U \in \mathscr{V}$ and the map that carries V to U is a contraction. The unique fixed point of that map will be the solution to (5.1.8), (5.1.2) on the time interval $[0,T]$.

To begin with, by virtue of (5.1.14) and (5.1.11),

$$\|U(\cdot,t) - U_0(\cdot)\|_{L^\infty} \le c\omega(\sqrt{\varepsilon t} + t), \tag{5.1.15}$$

for $0 \le t \le T$, which shows, in particular, that when T is sufficiently small, $U(\cdot,t)$ takes values in $\mathscr{B}_\rho$, for any $t \in [0,T]$.

We fix any multi-index r of order $|r| \le \ell$, set $U_r = \partial^r U$, $U_{0r} = \partial^r U_0$ and apply ∂^r to (5.1.13) to get

$$\partial_t U_r(x,t) - \varepsilon \Delta U_r(x,t) = -\sum_{\alpha=1}^m \partial_\alpha \partial^r G_\alpha(V(x,t)). \tag{5.1.16}$$

Since $\|V(\cdot,t)\|_{L^\infty} < \rho$ and $\|V(\cdot,t)\|_\ell < \omega$, familiar interpolation estimates from the theory of Sobolev spaces yield

$$\|\partial^r G_\alpha(V(\cdot,t))\|_0 \le c\|V(\cdot,t)\|_\ell \le c\omega, \tag{5.1.17}$$

for any r, with $|r| \le \ell$, and any $t \in [0,T]$. Then by standard theory of the heat equation, U_r as solution to (5.1.16) with initial values U_{0r}, belongs to the spaces $C^0([0,T];H_0) \bigcap L^2([0,T];H_1)$. This yields two "energy" integrals, namely

$$\begin{aligned}(5.1.18)\qquad &\int_{\mathbb{R}^m} |U_r(x,t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla U_r|^2 dx d\tau \\ &= \int_{\mathbb{R}^m} |U_{0r}(x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha U_r^\top \partial^r G_\alpha(V) dx d\tau \\ &\le \int_{\mathbb{R}^m} |U_{0r}(x)|^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla U_r|^2 dx d\tau + \frac{ct\omega^2}{\varepsilon} ,\end{aligned}$$

which is derived formally by multiplying (5.1.16) by $2U_r^\top$, integrating the resulting equation over $\mathbb{R}^m \times [0,t]$ and integrating by parts, and holds for any r with $|r| \le \ell$, and

$$\begin{aligned}(5.1.19)\qquad &2\int_0^t \int_{\mathbb{R}^m} |\partial_\tau U_r|^2 dx d\tau + \varepsilon \int_{\mathbb{R}^m} |\nabla U_r(x,t)|^2 dx \\ &= \varepsilon \int_{\mathbb{R}^m} |\nabla U_{0r}(x)|^2 dx - 2\int_0^t \int_{\mathbb{R}^m} \partial_\tau U_r^\top \sum_{\alpha=1}^m \partial_\alpha \partial^r G_\alpha(V) dx d\tau \\ &\le \int_0^t \int_{\mathbb{R}^m} |\partial_\tau U_r|^2 dx d\tau + c(t+\varepsilon)\omega^2,\end{aligned}$$

which is derived formally by multiplying (5.1.16) by $2\partial_t U_r^\top$, integrating the resulting equation over $\mathbb{R}^m \times [0,t]$ and integrating by parts, and holds for any r with $|r| \le \ell - 1$.

Upon summing (5.1.18) over all r of order $|r| \le \ell$, and (5.1.19) over all r of order $|r| \le \ell - 1$, we deduce the estimates

$$(5.1.20)\qquad \|U(\cdot,t)\|_\ell^2 \le \omega_0^2 + \frac{ct\omega^2}{\varepsilon},$$

$$(5.1.21)\qquad \int_0^t \|\partial_\tau U(\cdot,\tau)\|_{\ell-1}^2 d\tau \le (\varepsilon + ct)\omega^2.$$

In particular, (5.1.20) implies that when T is sufficiently small, $\sup_{[0,T]} \|U(\cdot,t)\|_\ell < \omega_1$ and thus $U \in \mathscr{V}$.

We now fix V and $\bar{V}$ in $\mathscr{V}$ and consider the solutions U and $\bar{U}$ of (5.1.13), (5.1.2) induced by them. Then

$$(5.1.22)\qquad \partial_t(U-\bar{U}) - \varepsilon\Delta(U-\bar{U}) = -\sum_{\alpha=1}^m \partial_\alpha[G_\alpha(V) - G_\alpha(\bar{V})].$$

Multiplying (5.1.22) by $2(U-\bar{U})^\top$, integrating over $\mathbb{R}^m \times [0,t],\ t\in(0,T)$, and integrating by parts we deduce

$$
\begin{aligned}
(5.1.23)\quad & \int_{\mathbb{R}^m} |U(x,t)-\bar{U}(x,t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla(U-\bar{U})|^2 dx d\tau \\
&= 2\int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha (U-\bar{U})^\top [G_\alpha(V)-G_\alpha(\bar{V})] dx d\tau \\
&\le \varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla(U-\bar{U})|^2 dx d\tau + \frac{cT}{\varepsilon} \sup_{[0,T]} \int_{\mathbb{R}^m} |V(x,\tau)-\bar{V}(x,\tau)|^2 dx.
\end{aligned}
$$

Recalling (5.1.12), we conclude that if $\dfrac{cT}{\varepsilon} = \mu^2 < 1$, then $d(U,\bar{U}) \le \mu d(V,\bar{V})$, which establishes that the map $V \mapsto U$ possesses a unique fixed point, which is the unique solution U to (5.1.8), (5.1.2) on the time interval $[0,T]$.

Let $\rho_1 = \|U(\cdot,T)\|_0 \le \rho$. If $\rho_1 = \rho$ the lemma has been proved with $T_{\omega,\varepsilon} = T$. On the other hand, if $\rho_1 < \rho$ and since $\|U(\cdot,T)\|_\ell \le \omega_1 < \omega$, we may extend the solution U of (5.1.8), (5.1.2) beyond T by solving, as above, a new Cauchy problem for (5.1.8) with initial data $U(\cdot,T)$ and (ρ_1,ω_1) in the role of (ρ_0,ω_0). By iterating this process, one extends U to a maximal time interval $[0,T_{\omega,\varepsilon})$, with either $T_{\omega,\varepsilon} = \infty$ or $T_{\omega,\varepsilon} < \infty$, in which case $\limsup_{t\to T_{\omega,\varepsilon}} \|U(\cdot,t)\|_{L^\infty} = \rho$ and/or $\limsup_{t\to T_{\omega,\varepsilon}} \|U(\cdot,t)\|_\ell = \omega$. This completes the proof of Lemma 5.1.2.

The next step is to derive bounds on the solution to (5.1.8), (5.1.2) that are sustained even as $\varepsilon \to 0$. Notice that (5.1.15) and (5.1.21) hold for $t \in [0,T_{\omega,\varepsilon})$ and meet this requirement. The next proposition establishes the main estimate in that direction, with the convex entropy moving to center stage and the viscosity term reduced to a merely supporting role.

5.1.3 Lemma. *Assume $A(U)$, defined by* (5.1.3), *is positive definite for $U \in \overline{\mathscr{B}_\rho}$. Then there exists $c_0 > 1$, depending solely on ρ, ℓ and on bounds of G, η and their derivatives on $\overline{\mathscr{B}_\rho}$, with the following property. Let $U_0 \in H_\ell$, $\ell > \frac{m}{2}+1$, taking values in $\overline{\mathscr{B}_{\rho_0}}$, $\rho_0 < \rho$. Fix any $\omega \ge c_0\|U_0\|_\ell$ and, with reference to Lemma* 5.1.2, *consider the lifespan $T_{\omega,\varepsilon}$ of the solution U to* (5.1.8), (5.1.2) *that satisfies* (5.1.10). *Then $T_{\omega,\varepsilon} > T_\omega$, where T_ω is a positive constant independent of $\varepsilon \in (0,1)$, and for any $t \in [0,T_\omega]$,*

$$(5.1.24)\qquad \|U(\cdot,t)\|_\ell \le c_0 \|U_0(\cdot)\|_\ell \exp \int_0^t g(\tau) d\tau,$$

where

$$(5.1.25)\qquad g(t) \le c \left[\, |||\nabla U(\cdot,t)|||_{L^\infty} + \varepsilon |||\nabla U(\cdot,t)|||_{L^\infty}^2 \,\right].$$

Proof. For any multi-index r, of order $|r| \le \ell$, we consider (5.1.16) for $V = U$ and write it in the form

$$(5.1.26)\qquad \begin{aligned}\partial_t U_r + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(U)\partial_\alpha U_r \\ = \sum_{\alpha=1}^{m} \{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\} + \varepsilon \Delta U_r\,.\end{aligned}$$

In the summation on the right-hand side of the above equation, the terms with derivatives of (the highest) order $l+1$ cancel out. Hence, by familiar interpolation estimates in Sobolev space (commonly referred to as Moser estimates), which can be found in the literature cited in Section 5.7, one obtains

$$(5.1.27)\qquad \|\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\|_0 \le c\|\nabla U\|_{L^\infty}\|\nabla U\|_{\ell-1}\,.$$

Then (5.1.26) induces the following “energy” integral:

$$(5.1.28)\qquad \begin{aligned}&\int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx - \int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx \\ &= -2\varepsilon\int_0^t\int_{\mathbb{R}^m} \partial_\alpha U_r^\top A(U)\partial_\alpha U_r\,dxd\tau - 4\varepsilon\int_0^t\int_{\mathbb{R}^m}\sum_{\alpha=1}^{m} U_r^\top \partial_\alpha A(U)\partial_\alpha U_r\,dxd\tau \\ &-\varepsilon\int_0^t\int_{\mathbb{R}^m}\sum_{\alpha=1}^{m} U_r^\top[\partial_\alpha \mathrm{D}A(U)\partial_\alpha U]U_r\,dxd\tau \\ &+\int_0^t\int_{\mathbb{R}^m}\sum_{\alpha=1}^{m} U_r^\top[\partial_\alpha J_\alpha(U) - \mathrm{D}A(U)\partial_\alpha G_\alpha(U)]U_r\,dxd\tau \\ &+2\int_0^t\int_{\mathbb{R}^m}\sum_{\alpha=1}^{m} U_r^\top A(U)[\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]]dxd\tau,\end{aligned}$$

which may be derived formally upon multiplying (5.1.26) by $2U_r^\top A(U)$, integrating the resulting equation over $\mathbb{R}^m \times [0,t]$, and integrating by parts. In the process, one uses the following identities, which manifest the symmetry of the matrices $A(U)$ and $J_\alpha(U)$:

$$(5.1.29)\qquad 2U_r^\top A(U)\partial_t U_r = \partial_t[U_r^\top A(U)U_r] - U_r^\top \partial_t A(U)U_r\,,$$

$$(5.1.30)\qquad U_r^\top \partial_t A(U)U_r = -U_r^\top[\sum_{\alpha=1}^{m} \mathrm{D}A(U)\partial_\alpha G_\alpha(U)]U_r + \varepsilon U_r^\top[\mathrm{D}A(U)\Delta U]U_r\,,$$

$$U_r^\top[\mathrm{D}A(U)\Delta U]U_r = \sum_{\alpha=1}^m \partial_\alpha[U_r^\top[\partial_\alpha A(U)]U_r] \tag{5.1.31}$$
$$-2\sum_{\alpha=1}^m U_r^\top[\partial_\alpha A(U)]\partial_\alpha U_r - \sum_{\alpha=1}^m U_r^\top[\partial_\alpha \mathrm{D}A(U)\partial_\alpha U]U_r\,,$$

$$U_r^\top A(U)\Delta U_r = \sum_{\alpha=1}^m \partial_\alpha[U_r^\top A(U)\partial_\alpha U_r] \tag{5.1.32}$$
$$-\sum_{\alpha=1}^m \partial_\alpha U_r^\top A(U)\partial_\alpha U_r - \sum_{\alpha=1}^m U_r^\top \partial_\alpha A(U)\partial_\alpha U_r\,,$$

$$\sum_{\alpha=1}^m 2U_r^\top A(U)\mathrm{D}G_\alpha(U)\partial_\alpha U_r \tag{5.1.33}$$
$$= \sum_{\alpha=1}^m \partial_\alpha[U_r^\top J_\alpha(U)U_r] - \sum_{\alpha=1}^m U_r^\top \partial_\alpha J_\alpha(U)U_r\,.$$

Since $A(U)$ is positive definite,

$$\int_{\mathbb{R}^m} \partial_\alpha U_r^\top(x,\tau)A(U(x,\tau))\partial_\alpha U_r(x,\tau)dx \geq \mu\int_{\mathbb{R}^m}|\partial_\alpha U_r(x,\tau)|^2dx. \tag{5.1.34}$$

By the Cauchy-Schwarz inequality,

$$2\int_{\mathbb{R}^m} U_r^\top(x,\tau)\partial_\alpha A(U(x,\tau))\partial_\alpha U_r(x,\tau)dx \tag{5.1.35}$$
$$\leq \mu\int_{\mathbb{R}^m}|\partial_\alpha U_r(x,\tau)|^2dx + c\int_{\mathbb{R}^m}|\partial_\alpha U(x,\tau)|^2|U_r(x,\tau)|^2dx.$$

Therefore, summing (5.1.28) over all r with $|r|\leq\ell$ and using (5.1.34), (5.1.35) and (5.1.27), we obtain

$$\sum_{|r|\leq\ell}\int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx \tag{5.1.36}$$
$$\leq \sum_{|r|\leq\ell}\int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx + c\int_0^t g(\tau)\|U(\cdot,\tau)\|_\ell^2 d\tau,$$

with g bounded as in (5.1.25).

Again, since $A(U)$ is positive definite,

$$\sum_{|r|\leq\ell}\int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx \geq \mu\|U(\cdot,t)\|_\ell^2\,. \tag{5.1.37}$$

Therefore, (5.1.36) yields

$$\|U(\cdot,t)\|_\ell^2 \le c_0^2\|U_0(\cdot)\|_\ell^2 + 2\int_0^t g(\tau)\|U(\cdot,\tau)\|_\ell^2 d\tau, \tag{5.1.38}$$

whence (5.1.24) follows by the Gronwall lemma.

Since $|||\nabla U(\cdot,\tau)|||_{L^\infty} \le c\|U(\cdot,\tau)\|_\ell \le c\omega$, it is clear from (5.1.24) that there is $T_\omega > 0$ such that $c_0\|U_0\|_\ell < \omega$ implies $\|U(\cdot,t)\|_\ell < \omega$ for all $t \in [0,T_\omega]$, i.e. $T_\omega < T_{\omega,\varepsilon}$, for all $\varepsilon \in (0,1)$. This completes the proof of the lemma.

We have now set the stage for dealing with the hyperbolic system (5.1.1) by letting $\varepsilon \to 0$ in (5.1.8).

5.1.4 Lemma. *Assume $A(U)$ is positive definite for $U \in \overline{\mathscr{B}_\rho}$, and take $U_0 \in H_\ell$, for $\ell > \frac{m}{2}+1$, with values in $\overline{\mathscr{B}_{\rho_0}}$, $\rho_0 < \rho$. Fix $\omega \ge c_0\|U_0\|_\ell$, where c_0 is the constant introduced in Lemma* 5.1.3, *and identify the corresponding T_ω. Then there exists a classical solution U to* (5.1.1), (5.1.2), *defined on $\mathbb{R}^m \times [0,T_\omega]$ and taking values in $\overline{\mathscr{B}_\rho}$. For any $t \in [0,T_\omega]$, $U(\cdot,t) \in H_\ell$, $\|U(\cdot,t)\|_\ell \le \omega$ and*

$$\|U(\cdot,t)\|_\ell \le c_0\|U_0(\cdot)\|_\ell \exp\int_0^t c|||\nabla U(\cdot,\tau)|||_{L^\infty} d\tau. \tag{5.1.39}$$

Furthermore, the function $t \mapsto U(\cdot,t)$ is continuous in H_ℓ on $[0,T_\omega]$.

Proof. Take any sequence $\{\varepsilon_k\}$, with $\varepsilon_k \to 0$, as $k \to \infty$, and let U_k be the solution of (5.1.8), (5.1.2), with $\varepsilon = \varepsilon_k$, on the time interval $[0,T_\omega]$. By Lemma 5.1.3, $\{U_k\}$ is bounded in $L^\infty([0,T_\omega];H_\ell)$, with $\|U_k(\cdot,t)\|_\ell < \omega$, for $0 \le t \le T_\omega$. Furthermore, it follows from (5.1.21) that $\{U_k\}$ is also bounded in $W^{1,2}([0,T_\omega];H_{\ell-1})$. Therefore, by standard theory of Sobolev spaces, $\{U_k\}$ is equicontinuous and thereby contains a subsequence, denoted again by $\{U_k\}$, which converges, uniformly on compact sets, to some continuous function U on $\mathbb{R}^m \times [0,T_\omega]$, taking values in $\overline{\mathscr{B}_\rho}$ and satisfying (5.1.1), (5.1.2), in the sense of distributions. For any $t \in [0,T_\omega]$, the bound $\|U_k(\cdot,t)\|_\ell < \omega$ implies that $U_k(\cdot,t) \to U(\cdot,t)$, weakly in H_ℓ, and $\|U(\cdot,t)\|_\ell \le \omega$. This in turn gives $U(\cdot,t) \in C^1(\mathbb{R}^m)$, with $|||\nabla U(\cdot,t)|||_{L^\infty} \le c\omega$, and since U is a solution of (5.1.1), $\partial_t U(\cdot,t) \in C^0(\mathbb{R}^m)$, with $\|\partial_t U(\cdot,t)\|_{L^\infty} \le c\omega$. Thus U is Lipschitz.

Next we show that $U_k(\cdot,t) \to U(\cdot,t)$, strongly in H_0, as $k \to \infty$, for any $t \in [0,T_\omega]$. To that end, we notice that $V_k = U_k - U$ solves the equation

$$\partial_t V_k + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(U)\partial_\alpha V_k = \varepsilon\Delta U_k - \sum_{\alpha=1}^m [\mathrm{D}G_\alpha(U_k) - \mathrm{D}G_\alpha(U)]\partial_\alpha U_k, \tag{5.1.40}$$

on $\mathbb{R}^m \times [0,T_\omega]$, with initial condition $V_k(\cdot,0) = 0$. Hence, multiplying (5.1.40) by $2V_k^\top A(U)$, integrating the resulting equation over $\mathbb{R}^m \times [0,t]$, for $t \in [0,T_\omega]$, and integrating by parts yields

$$\int_{\mathbb{R}^m} V_k^\top(x,t)A(U(x,t))V_k(x,t)dx \tag{5.1.41}$$
$$= \int_0^t \int_{\mathbb{R}^m} V_k^\top \{\partial_t A(U) + \sum_{\alpha=1}^m \partial_\alpha J_\alpha(U)\} V_k \, dxd\tau$$
$$-2\int_0^t \int_{\mathbb{R}^m} V_k^\top A(U)[\mathrm{D}G_\alpha(U_k) - \mathrm{D}G_\alpha(U)]\partial_\alpha U_k \, dxd\tau$$
$$-2\varepsilon_k \int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha [V_k^\top A(U)]\partial_\alpha U_k \, dxd\tau.$$

Since $A(U)$ is positive definite,

$$\int_{\mathbb{R}^m} V_k^\top(x,t)A(U(x,t))V_k(x,t)dx \geq \mu \|V_k(\cdot,t)\|_0^2 . \tag{5.1.42}$$

Therefore, (5.1.41) induces an inequality of the form

$$\|V_k(\cdot,t)\|_0^2 \leq c\omega \int_0^t \|V_k(\cdot,\tau)\|_0^2 d\tau + c\omega^2 \varepsilon_k t, \tag{5.1.43}$$

whence we conclude that $V_k(\cdot,t) \to 0$, strongly in H_0, as $k \to \infty$, for any $t \in [0,T_\omega]$.

Since $\{U_k(\cdot,t)\}$ converges to $U(\cdot,t)$, strongly in H_0 and weakly in H_ℓ, we infer by interpolation that the convergence is strong in $H_{\ell-1}$ and also uniform in $\mathbb{R}^m$. Moreover, $\nabla U_k(\cdot,t) \to \nabla U(\cdot,t)$, uniformly in $\mathbb{R}^m$, for any $t \in [0,T_\omega]$. In particular, recalling that (5.1.24) holds for the U_k, with g bounded as in (5.1.25), and letting $k \to \infty$, we verify that U satisfies (5.1.39).

It remains to prove that $t \mapsto U(\cdot,t)$ is continuous in H_ℓ on $[0,T_\omega]$. Considering that (5.1.1) is invariant under time translations and reflections, it will suffice to show that $t \mapsto U(\cdot,t)$ is right-continuous at $t=0$, i.e., $U(\cdot,t) \to U_0(\cdot)$ in H_ℓ, as $t \to 0$.

We begin with the identity

$$\sum_{|r|\leq \ell} \int_{\mathbb{R}^m} U_r^\top A(U) U_r \, dx - \sum_{|r|\leq \ell} \int_{\mathbb{R}^m} U_{kr}^\top A(U_k) U_{kr} \, dx \tag{5.1.44}$$
$$= -\sum_{|r|\leq \ell} \int_{\mathbb{R}^m} U_{kr}^\top [A(U_k) - A(U)] U_{kr} \, dx$$
$$- \sum_{|r|\leq \ell} \int_{\mathbb{R}^m} 2(U_{kr} - U_r)^\top A(U) U_r \, dx$$
$$- \sum_{|r|\leq \ell} \int_{\mathbb{R}^m} (U_{kr} - U_r)^\top A(U)(U_{kr} - U_r) dx,$$

which holds for any fixed $t \in [0,T_\omega]$, and let $k \to \infty$. On the right-hand side, the first two terms tend to zero, while the last term has a definite sign:

$$(5.1.45) \qquad \sum_{|r|\le\ell} \int_{\mathbb{R}^m} (U_{kr} - U_r)^\top A(U)(U_{kr} - U_r)dx \ge \mu \|U_k - U\|_\ell^2 .$$

It follows that, as $k \to \infty$, the limit inferior of the left-hand side of (5.1.44) is non-positive. Recalling that the U_k satisfy (5.1.36), we conclude that

$$(5.1.46) \qquad \sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx - \sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx \le c\omega^3 t.$$

We now write the identity

$$(5.1.47) \qquad \sum_{|r|\le\ell} \int_{\mathbb{R}^m} [U_r(x,t) - U_{0r}(x)]^\top A(U_0(x))[U_r(x,t) - U_{0r}(x)]dx$$
$$= \sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx$$
$$- \sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx$$
$$- \sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_r^\top(x,t)[A(U(x,t)) - A(U_0(x))]U_r(x,t)dx$$
$$- \sum_{|r|\le\ell} \int_{\mathbb{R}^m} 2[U_r(x,t) - U_{0r}(x)]^\top A(U_0(x))U_{0r}(x)dx$$

and let $t \to 0$. On the right-hand side, the last term tends to zero, because U is a continuous function and $\|U(\cdot,t)\|_\ell \le \omega$, whence it follows that $t \mapsto U(\cdot,t)$ is at least weakly continuous in H_ℓ. Similarly, the penultimate term tends to zero, because the U_k satisfy (5.1.15), and this estimate is then passed on to U, as $k \to \infty$. Finally, the contribution of the remaining two terms is non-positive, by virtue of (5.1.46). We thus conclude that, as $t \to 0$, the limit inferior of the left-hand side of (5.1.47) is nonpositive. On the other hand, since $A(U)$ is positive definite,

$$(5.1.48) \qquad \sum_{|r|\le\ell} \int_{\mathbb{R}^m} [U_r(x,t) - U_{0r}(x)]^\top A(U_0(x))[U_r(x,t) - U_{0r}(x)]dx \ge \mu \|U(\cdot,t) - U_0(\cdot)\|_\ell^2 .$$

Hence, $t \mapsto U(\cdot,t)$ is continuous in H_ℓ. In particular, U is a continuously differentiable, classical solution to (5.1.1), (5.1.2). On account of (5.1.15), we may reduce, if necessary, the size of T_ω so as to secure that $\|U(\cdot,t)\|_{L^\infty} < \rho$ for $t \in [0, T_\omega]$. This completes the proof of the lemma.

Proof of Theorem 5.1.1. We start out with the solution U to (5.1.1), (5.1.2) on $[0, T_\omega]$ constructed in Lemma 5.1.4, and set $\rho_1 = \|U(\cdot,T_\omega)\|_\infty < \rho$, $\omega_1 = \|U(\cdot,T_\omega)\|_\ell$. Thus we may extend U beyond T_ω by solving a new Cauchy problem for (5.1.1) with initial data $U(\cdot,T_\omega)$ and $(\rho_1, c_0\omega_1)$ in the role of (ρ_0,ω). By iterating this process, one extends U to a maximal time interval $[0,T_\infty)$, with either $T_\infty=\infty$ or $T_\infty<\infty$, in which case $\limsup_{t\to T_\infty}\|U(\cdot,t)\|_{L^\infty}=\rho$ or $\limsup_{t\to T_\infty}\|U(\cdot,t)\|_\ell=\infty$. On account of (5.1.39), $\|U(\cdot,t)\|_\ell$ may become unbounded only if (5.1.6) holds.

Time, and mixed space-time, derivatives of U may be determined from space derivatives by employing the system (5.1.1). Hence (5.1.5) follows as a result of $U(\cdot,t)\in C^0([0,T_\infty];H_\ell)$, which has already been established.

The uniqueness of the solution will be established in Section 5.2, under quite weak hypotheses of regularity. This completes the proof of the theorem.

As noted in Chapter IV, $T_\infty=\infty$ is a rare occurrence: Generically, smooth solutions break down in finite time, as shocks develop.

The proof of Theorem 5.1.1 hinges on the presence of the symmetric positive definite matrix-valued function $A(U)$ that acts as *symmetrizer* by rendering the matrix-valued functions $J_\alpha(U)$, defined by (5.1.4), symmetric. It is fortuitous that here $A(U)$ is the Hessian matrix of $\eta(U)$, as there are systems endowed with symmetrizers that do not derive from an entropy. In fact, existence of solutions to the Cauchy problem has been established even for systems equipped with so-called symbolic symmetrizers. These include, in particular, all hyperbolic systems in which the multiplicity of each characteristic speed $\lambda_i(\nu;U)$ does not vary with ν or U.

5.2 Relative Entropy and the Stability of Classical Solutions

The aim here is to show that the presence of a convex entropy guarantees that classical solutions of the initial value problem depend continuously on the initial data, even within the broader class of admissible bounded weak solutions.

5.2.1 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with an entropy-entropy flux pair* (η,Q), *where* $\mathrm{D}^2\eta(U)$ *is positive definite on* $\overline{\mathscr{B}_\rho}$. *Suppose* $\bar U$ *is a classical solution of* (5.1.1) *on* $[0,T)$, *taking values in* $\overline{\mathscr{B}_\rho}$, *with initial data* $\bar U_0$. *Let* U *be any weak solution of* (5.1.1) *on* $[0,T)$, *taking values in* $\overline{\mathscr{B}_\rho}$, *which satisfies the entropy admissibility condition* (4.5.3), *and has initial data* U_0. *Then*

$$\int_{|x|<r}|U(x,t)-\bar U(x,t)|^2dx\le ae^{bt}\int_{|x|<r+st}|U_0(x)-\bar U_0(x)|^2dx \tag{5.2.1}$$

holds for any $r>0$ *and* $t\in[0,T)$, *with positive constants* s,a, *depending solely on bounds on* G,η,Q *and their derivatives on* $\overline{\mathscr{B}_\rho}$, *and* b *that also depends on the Lipschitz constant of* $\bar U$. *In particular,* $\bar U$ *is the unique admissible weak solution of* (5.1.1) *with initial data* $\bar U_0$ *and values in* $\overline{\mathscr{B}_\rho}$.

Proof. On $\overline{\mathscr{B}_\rho}\times\overline{\mathscr{B}_\rho}$ we define the functions

$$h(U,\bar U)=\eta(U)-\eta(\bar U)-\mathrm{D}\eta(\bar U)[U-\bar U], \tag{5.2.2}$$

$$Y_\alpha(U,\bar{U}) = Q_\alpha(U) - Q_\alpha(\bar{U}) - \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})], \tag{5.2.3}$$

$$Z_\alpha(U,\bar{U}) = A(\bar{U})\big\{G_\alpha(U) - G_\alpha(\bar{U}) - \mathrm{D}G_\alpha(\bar{U})[U-\bar{U}]\big\}, \tag{5.2.4}$$

all of quadratic order in $U - \bar{U}$ (recall (4.1.4) and (5.1.3)). Consequently, since $\mathrm{D}^2\eta(U)$ is positive definite on $\overline{\mathscr{B}_\rho}$, there is a positive constant s such that

$$|Y(U,\bar{U})| \le sh(U,\bar{U}). \tag{5.2.5}$$

Let us fix any nonnegative, Lipschitz continuous test function ψ with compact support on $\mathbb{R}^m \times [0,T)$ and evaluate h, Y and Z along the two solutions $U(x,t)$, $\bar{U}(x,t)$. Since U satisfies the inequality (4.5.3), while $\bar{U}$, being a classical solution, satisfies identically (4.5.3) as an equality, we deduce

(5.2.6)

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi Y_\alpha(U,\bar{U})]dxdt + \int_{\mathbb{R}^m} \psi(x,0)\, h(U_0(x),\bar{U}_0(x))\, dx \\
&\ge -\int_0^T \int_{\mathbb{R}^m} \{\partial_t \psi \mathrm{D}\eta(\bar{U})[U-\bar{U}] + \sum_{\alpha=1}^m \partial_\alpha \psi \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})]\}dxdt \\
&\qquad - \int_{\mathbb{R}^m} \psi(x,0)\, \mathrm{D}\eta(\bar{U}_0(x))[U_0(x) - \bar{U}_0(x)]dx.
\end{aligned}$$

Next we write (4.3.2) for both solutions U and $\bar{U}$, using the Lipschitz continuous vector field $\psi \mathrm{D}\eta(\bar{U})$ as test function Φ, to get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} \{\partial_t[\psi \mathrm{D}\eta(\bar{U})][U-\bar{U}] + \sum_{\alpha=1}^m \partial_\alpha[\psi \mathrm{D}\eta(\bar{U})][G_\alpha(U) - G_\alpha(\bar{U})]\}dxdt \\
&\qquad + \int_{\mathbb{R}^m} \psi(x,0)\mathrm{D}\eta(\bar{U}_0(x))[U_0(x) - \bar{U}_0(x)]dx = 0.
\end{aligned} \tag{5.2.7}$$

Since $\bar{U}$ is a classical solution of (5.1.1), and by virtue of (5.1.3), (5.1.4),

(5.2.8)

$$\partial_t \mathrm{D}\eta(\bar{U}) = \partial_t \bar{U}^\top A(\bar{U}) = -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top J_\alpha(\bar{U})^\top = -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top A(\bar{U})\mathrm{D}G_\alpha(\bar{U})$$

so that, recalling (5.2.4),

$$\partial_t \mathrm{D}\eta(\bar{U})[U-\bar{U}] + \sum_{\alpha=1}^m \partial_\alpha \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})] = \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}). \tag{5.2.9}$$

Combining (5.2.6), (5.2.7) and (5.2.9) yields

(5.2.10)
$$\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi Y_\alpha(U,\bar{U})] dxdt + \int_{\mathbb{R}^m} \psi(x,0)\, h(U_0(x),\bar{U}_0(x))\, dx$$
$$\geq \int_0^T \int_{\mathbb{R}^m} \psi \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}) dxdt.$$

We now fix $t \in (0,T)$ and $r > 0$. For any $\sigma \in (0,t]$ and ε positive small, write (5.2.10) for the test function $\psi(x,\tau) = \chi(x,\tau)\omega(\tau)$, with

$$\omega(\tau) = \begin{cases} 1 & 0 \leq \tau < \sigma \\ \varepsilon^{-1}(\sigma - \tau) + 1 & \sigma \leq \tau < \sigma + \varepsilon \\ 0 & \sigma + \varepsilon \leq \tau < \infty \end{cases} \tag{5.2.11}$$

$$\chi(x,\tau) = \begin{cases} 1 & |x| - r - s(\sigma - \tau) < 0 \\ \varepsilon^{-1}[r + s(t-\tau) - |x|] + 1 & 0 \leq |x| - r - s(t-\tau) < \varepsilon \\ 0 & |x| - r - s(t-\tau) \geq \varepsilon \end{cases} \tag{5.2.12}$$

where s is the constant appearing in (5.2.5). The calculation gives

(5.2.13)
$$\frac{1}{\varepsilon}\int_\sigma^{\sigma+\varepsilon} \int_{|x|<r+s(t-\sigma)} h(U(x,\tau),\bar{U}(x,\tau)) dxd\tau \leq \int_{|x|<r+st} h(U_0(x),\bar{U}_0(x)) dx$$
$$-\frac{1}{\varepsilon}\int_0^\sigma \int_{r+s(t-\tau)<|x|<r+s(t-\tau)+\varepsilon} \left[sh(U,\bar{U}) + \frac{Y(U,\bar{U})x}{|x|} \right] dxd\tau$$
$$-\int_0^\sigma \int_{|x|<r+s(t-\tau)} \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}) dxd\tau + O(\varepsilon).$$

We let $\varepsilon \downarrow 0$. The second integral on the right-hand side of (5.2.13) is nonnegative on account of (5.2.5). Hence,

$$\int_{|x|<r+s(t-\sigma)} h(U(x,\sigma),\bar{U}(x,\sigma)) dx \leq \int_{|x|<r+st} h(U_0(x),\bar{U}_0(x)) dx \tag{5.2.14}$$
$$-\int_0^\sigma \int_{|x|<r+s(t-\tau)} \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}) dxd\tau,$$

for all points σ of L^∞ weak* continuity of $\eta(U(\cdot,\tau))$ in $(0,t)$. As noted above, $h(U,\bar{U})$ and the $Z_\alpha(U,\bar{U})$ are of quadratic order in $U-\bar{U}$ and, in addition, $h(U,\bar{U})$ is positive definite, due to the convexity of η. Thus, upon setting

$$u(\tau) = \int_{|x|<r+s(t-\tau)} |U(x,\tau) - \bar{U}(x,\tau)|^2 dx\,, \tag{5.2.15}$$

(5.2.14) implies

$$u(\sigma) \le au(0) + b\int_0^\sigma u(\tau)d\tau\,, \tag{5.2.16}$$

for almost all $\sigma \in (0,t)$. Since $u(\cdot)$ is weakly lower semicontinuous, (5.2.16) holds for all $\sigma \in [0,t]$. Then Gronwall's inequality yields $u(t) \le au(0)e^{bt}$, which is (5.2.1). Notice that a and s depend solely on bounds on G,η,Q and their derivatives on $\overline{\mathscr{B}_\rho}$ while b also depends on the Lipschitz constant of $\bar{U}$. This completes the proof.

It is remarkable that a single entropy inequality, with convex entropy, manages to weed out all but one solution of the initial value problem, so long as a classical solution exists. As we shall see, however, when no classical solution exists, just one entropy inequality is no longer generally sufficient to single out any particular weak solution. In particular, as we saw in Section 4.8, the Cauchy problem for the Euler equations (3.3.36), under specially constructed initial data, admits infinitely many weak solutions satisfying the entropy admissibility condition (4.5.3), relative to the entropy $\rho\varepsilon(\rho)+\frac{1}{2}\rho|v|^2$, as an equality. The issue of uniqueness of weak solutions is knotty and will be a major topic for discussion in subsequent chapters.

The functions $h(U,\bar{U})$ and $Y(U,\bar{U})$ of U, defined by (5.2.2) and (5.2.3), are commonly called the *relative entropy* and associated *relative entropy flux*, with respect to the state $\bar{U}$.

5.2.2 Remark. In the proof of Theorem 5.2.1 one only needs that $h(U,\bar{U})$ is positive definite for all $\bar{U}$ in the range of the classical solution. This may well hold, even for η that fails to be convex, when the classical solution is special, e.g., it is a constant state $\bar{U}$ which is a strong minimum of η.

5.3 Involutions and Contingent Entropies

The previous three sections have illustrated the beneficent role of convex entropies. Nevertheless, the entropy associated with systems of balance laws in continuum physics is not always convex. Indeed, we have already encountered, in Chapter III, the cases of isentropic elastodynamics (Section 3.3.3) and electrodynamics (Section 3.3.8), in which invariance, dictated by physics, is incompatible with global convexity of the entropy. The objective in this and the following section is to identify special structure in such systems that may compensate for lack of convexity in the entropy.

Recall that solutions of the system (3.3.19) with relevance to elastodynamics should also satisfy the equations (3.3.10). Notice that (3.3.10) is not independent of (3.3.19). Indeed, in a Cauchy problem, $(3.3.19)_1$ implies that when (3.3.10) is satisfied by the initial data, then it will hold for all $t > 0$.

The equations of electrodynamics exhibit similar behavior: in addition to the hyperbolic system (3.3.66), the magnetic induction and the electric displacement must also satisfy (3.3.67). However, in a Cauchy problem, by virtue of (3.3.66) and (3.3.68), both equalities in (3.3.67) will hold automatically for all $t > 0$, so long as they are satisfied by the initial data.

One recognizes a similar structure in many other systems arising in continuum physics, and so an examination of its implications in a general framework is warranted.

We consider the class of hyperbolic systems (5.1.1) with the property that the symmetry condition

$$M_\alpha G_\beta(U) + M_\beta G_\alpha(U) = 0, \quad U \in \overline{\mathscr{B}_\rho}, \ \alpha, \beta = 1, \dots, m \tag{5.3.1}$$

holds for some family of $k \times n$ matrices M_α, $\alpha = 1, \dots, m$. A direct consequence of (5.3.1) is that any (generally weak) solution U to the Cauchy problem for (5.1.1) will satisfy the additional equation

$$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0, \tag{5.3.2}$$

so long as the initial data do so. We call (5.3.2) an *involution* for (5.1.1). Thus (3.3.10) is an involution for both (3.3.19) and (3.3.4), while (3.3.67) is an involution for (3.3.66). Typically, for systems in this class arising in physics, the only relevant solutions are those that also satisfy the involution.

With any $\nu \in \mathbb{S}^{m-1}$ we associate the $k \times n$ matrix

$$N(\nu) = \sum_{\alpha=1}^{m} \nu_\alpha M_\alpha \, . \tag{5.3.3}$$

By virtue of (4.1.2) and (5.3.1),

$$N(\nu)\Lambda(\nu; U) = 0, \quad \nu \in \mathbb{S}^{m-1}, \ U \in \overline{\mathscr{B}_\rho}, \tag{5.3.4}$$

which shows that, in the presence of involutions, zero must be an eigenvalue of $\Lambda(\nu; U)$, with geometric multiplicity at least equal to the rank of $N(\nu)$, and any eigenvector $R_i(\nu; U)$ of $\Lambda(\nu; U)$ with nonzero eigenvalue $\lambda_i(\nu; U)$ must lie in the kernel of $N(\nu)$.

It should also be noted that any shock associated with a solution of (5.1.1), compatible with the involution (5.3.2), propagating in the direction ν must satisfy the jump condition

$$N(\nu)[U_+ - U_-] = 0, \tag{5.3.5}$$

so that its amplitude $U_+ - U_-$ must lie in the kernel of $N(\nu)$.

In this section, we will be operating under the assumption

$$\dim\ker \Lambda(\nu;U) = \operatorname{rank} N(\nu), \quad \nu \in \mathbb{S}^{m-1}, \; U \in \overline{\mathscr{B}_\rho} \tag{5.3.6}$$

in which case the kernel of $N(\nu)$ is spanned by the eigenvectors of $\Lambda(\nu;U)$ with nonzero eigenvalue and thus coincide with the range of $\Lambda(\nu;U)$. This is indeed the case for the system (3.3.19) of isentropic elastodynamics, where the kernel of $\Lambda(\nu;U)$ is six-dimensional and the rank of $N(\nu)$ is six, though not for the system (3.3.4) of nonisentropic thermoelasticity, in which the kernel of $\Lambda(\nu;U)$ is seven-dimensional. Systems with involutions, such as (3.3.4) in which the dimension of $\ker\Lambda(\nu;U)$ is larger than the rank of $N(\nu)$ may be handled by the same methodology at the expense of introducing additional structure – the reader should be spared from such complications.

We now introduce the *involution cone*

$$\mathscr{C} = \bigcup_{\nu\in S^{m-1}} \ker N(\nu), \tag{5.3.7}$$

embedded in the state space $\mathbb{R}^n$. In particular, for the system (3.3.19) of isentropic elastodynamics $\mathscr{C} = \{(F,v) \,:\, F = uw^\top,\, u,w,v \in \mathbb{R}^3\}$, while for the system (3.3.66) of electrodynamics $\mathscr{C}$ occupies the entire state space $\mathbb{R}^6$. As we shall see, all interesting action takes place on $\mathscr{C}$.

The presence of involutions affords a natural broadening of the notion of entropy. Recall that the definition of an entropy-entropy flux pair (η,Q) for the system (5.1.1) has been crafted so that the extra conservation law

$$\partial_t\eta(U(x,t)) + \sum_{\alpha=1}^{m} \partial_\alpha Q_\alpha(U(x,t)) = 0 \tag{5.3.8}$$

is automatically satisfied by any C^1 solution U of (5.1.1). However, in the present setting, it is reasonable to require that (5.3.8) holds identically just for C^1 solutions of (5.1.1) that also satisfy the involution (5.3.2). This motivates the following

5.3.1 Definition. In a system of conservation laws (5.1.1), endowed with the involution (5.3.2), a smooth, scalar-valued function η on $\overline{\mathscr{B}_\rho}$ is a *contingent entropy*, associated with the $1\times k$ matrix-valued *contingent entropy flux* $Q(U)$, if there is a k-vector-valued function $\Xi(U)$ on $\overline{\mathscr{B}_\rho}$ such that

$$\mathrm{D}Q_\alpha(U) = \mathrm{D}\eta(U)\mathrm{D}G_\alpha(U) + \Xi(U)^\top M_\alpha\,, \quad \alpha = 1,\dots,m. \tag{5.3.9}$$

In particular, any entropy is a contingent entropy, with $\Xi = 0$. On the other hand, by virtue of (3.3.11) and (3.3.12), $\det F$ and the nine entries of the matrix F^* are contingent entropies for the system (3.3.19) of isentropic elastodynamics, which are not entropies. Similarly, on account of (3.3.74), the three components of $B\wedge D$ are contingent entropies for the system (3.3.66), which are not entropies. The useful role of these particular contingent entropies will be exposed in the next section. Another interesting example of contingent entropies for (3.3.19), which will not be used here, are the components of the 3-vector $F^\top v$. The associated contingent entropy fluxes are the corresponding rows of the 3×3 matrix $F^\top S + (\varepsilon + \frac{1}{2}|v|^2)I$.

The integrability condition for (5.3.9), which generalizes the symmetry relation (3.2.4), is that the $n \times n$ matrices

$$J_\alpha(U) = A(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Xi(U)^\top M_\alpha\,, \quad \alpha = 1,\dots,m, \tag{5.3.10}$$

are symmetric.

The results of Section 5.1 are not applicable to the system (3.3.19) of isentropic elastodynamics, because, as we have seen, global convexity of the entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ is incompatible with the laws of physics. Nevertheless, the physically legitimate assumption (3.3.7) that the internal energy $\varepsilon(F)$ is rank-one convex guarantees that η is convex at least on the involution cone $\mathscr{C}$. The aim in this section is to demonstrate that, in the presence of the involution (5.3.2), local existence and stability of classical solutions to the Cauchy problem may be established under the assumption that the system (5.1.1) is endowed with a contingent entropy η that is convex merely on the involution cone, i.e.,

$$X^\top A(U)X \geq 2\mu|X|^2, \quad X \in \mathscr{C},\ U \in \overline{\mathscr{B}_\rho}\,, \tag{5.3.11}$$

with $\mu > 0$.

In the place of Theorems 5.1.1 and 5.2.1 we here have the following propositions:

5.3.2 Theorem. *Assume the hyperbolic system* (5.1.1) *of conservation laws satisfies* (5.3.1), (5.3.6) *and is endowed with a contingent entropy η that is convex on the involution cone $\mathscr{C}$, so that* (5.3.11) *holds. Suppose the initial data U_0 lie in H_ℓ, for some $\ell > \frac{m}{2} + 1$, take values in a ball $\overline{\mathscr{B}_{\rho_0}}$ with radius $\rho_0 < \rho$, and satisfy the involution* (5.3.2)*. Then there exist $T_\infty \leq \infty$ and a unique continuously differentiable function U on $\mathbb{R}^m \times [0,T_\infty)$, taking values in $\overline{\mathscr{B}_\rho}$, which is a classical solution to the Cauchy problem* (5.1.1), (5.1.2) *on the time interval $[0,T_\infty)$. Furthermore,*

$$U(\cdot,t) \in \bigcap_{k=0}^{\ell} C^k([0,T_\infty)\,;\,H_{\ell-k}). \tag{5.3.12}$$

The interval $[0,T_\infty)$ is maximal in that if $T_\infty < \infty$, then

$$\int_0^{T_\infty} \|\!|\nabla U(\cdot,t)|\!\|_{L^\infty} dt = \infty\,, \tag{5.3.13}$$

and/or $\limsup_{t\to T_\infty} \|U(\cdot,t)\|_{L^\infty} = \rho$.

5.3.3 Theorem. *Assume the hyperbolic system* (5.1.1) *of conservation laws satisfies* (5.3.1), (5.3.6) *and is endowed with a contingent entropy-entropy flux pair (η,Q), where η is convex on the involution cone $\mathscr{C}$, so that* (5.3.11) *holds. Suppose $\bar{U}$ is a classical solution of* (5.1.1) *on $[0,T)$, taking values in $\overline{\mathscr{B}_\rho}$, with initial values $\bar{U}_0$ satisfying the involution* (5.3.2)*. Furthermore, assume $\bar{U}(x,t) \to 0$, $|x| \to \infty$, uniformly in $t \in [0,T)$. Let U be any weak solution of* (5.1.1) *on $[0,T)$, taking values in $\mathscr{B}_\rho$, which satisfies the entropy admissibility condition* (4.5.3) *and has initial values*

U_0 satisfying the involution (5.3.2). *Moreover, assume that $U(x,t) \to 0$, as $|x| \to \infty$, uniformly in $t \in [0,T)$. Finally, let $\|U(\cdot,t) - \bar{U}(\cdot,t)\|_0$ be bounded on $[0,T)$. Under the above hypotheses, there are constants a and κ, depending on ρ and on bounds of G, η, Q and their derivatives on $\overline{\mathscr{B}_\rho}$, and b that also depends on the Lipschitz constant of $\bar{U}$, such that if*

$$\limsup_{x \to y, t \to \tau} |U(x,t) - U(y,\tau)| < \kappa, \quad y \in \mathbb{R}^m, \ \tau \in [0,T), \tag{5.3.14}$$

then

$$\int_{\mathbb{R}^m} |U(x,t) - \bar{U}(x,t)|^2 dx \le a e^{bt} \int_{\mathbb{R}^m} |U_0(x) - \bar{U}_0(x)|^2 dx, \quad a.e \text{ on } [0,T]. \tag{5.3.15}$$

In particular, $\bar{U}$ is the unique solution of the Cauchy problem for (5.1.1), *with initial data $\bar{U}_0$, within the class of admissible weak solutions with sufficiently small local oscillation* (5.3.14) *and the same asymptotic behavior as $\bar{U}$ at $|x| = \infty$.*

The following lemma, which manifests how involutions compensate for the lack of convexity of the entropy outside the involution cone, will play a pivotal role in the proof of the above two propositions.

5.3.4 Lemma. *Let P be a bounded measurable symmetric $n \times n$ matrix-valued function on $\mathbb{R}^m$, such that*

$$X^\top P(x) X \ge 2\mu |X|^2, \quad X \in \mathscr{C}, \ x \in \mathbb{R}^m. \tag{5.3.16}$$

Assume further that there is a finite covering of $\mathbb{R}^m$ by the union of open sets $\Omega_0, \Omega_1, \dots, \Omega_K$ with the property that for $J = 0, 1, \dots, K$,

$$|P(x) - P(y)| < \frac{1}{2}\mu, \quad x, y \in \Omega_J. \tag{5.3.17}$$

Then there is δ, depending solely on the covering, such that

$$\int_{\mathbb{R}_m} S(x)^\top P(x) S(x) dx \ge \mu \|S\|_0^2 - \delta \|S\|_{-1}^2, \tag{5.3.18}$$

holds for any $S \in L^2(\mathbb{R}^m \,;\, \mathbb{R}^n)$ that satisfies the involution

$$\sum_{\alpha=1}^m M_\alpha \partial_\alpha S = 0, \tag{5.3.19}$$

in the sense of distributions on $\mathbb{R}^m$.

Proof. Fix $U \in \overline{\mathscr{B}_\rho}$ and consider the linear differential operator

$$\mathscr{L} = \sum_{\alpha=1}^m \mathrm{D}G_\alpha(U) \partial_\alpha. \tag{5.3.20}$$

We seek a solution $\Phi \in H_1$ to the equation

$$\mathscr{L}\Phi + \Phi = S, \tag{5.3.21}$$

with the help of the Fourier transform:

$$[i|\xi|\Lambda(\nu;U) + I]\hat{\Phi}(\xi) = \hat{S}(\xi), \quad \xi \in \mathbb{R}^m, \tag{5.3.22}$$

where $\nu = |\xi|^{-1}\xi$ and $\Lambda(\nu;U)$ is defined by (4.1.2). The $n \times n$ matrix on the left-hand side of (5.3.22) is nonsingular, since $\Lambda(\nu;U)$ has only real eigenvalues. Furthermore, by virtue of (5.3.19), $\hat{S}(\xi)$ lies in the kernel of $N(\nu)$, which is spanned by the eigenvectors of $\Lambda(\nu;U)$ with nonzero eigenvalue. It follows that (5.3.22) admits a solution $\hat{\Phi}(\xi)$ with

$$|\hat{\Phi}(\xi)|^2 \leq c^2(1+|\xi|^2)^{-1}|\hat{S}(\xi)|^2, \quad \xi \in \mathbb{R}^m, \tag{5.3.23}$$

whence

$$\|\Phi\|_0 \leq c\|S\|^{-1}. \tag{5.3.24}$$

Next we consider a partition of unity $\psi_0, \psi_1, \dots, \psi_K$ subordinate to the covering $\Omega_0, \Omega_1, \dots, \Omega_K$, i.e., for $J = 0, 1, \dots, K$, $\psi_J \in C^\infty(\mathbb{R}^m)$, spt $\psi_J \subset \Omega_J$ and

$$\sum_{J=0}^{K} \psi_J^2(x) = 1, \quad x \in \mathbb{R}^m. \tag{5.3.25}$$

We also fix $y_J \in \Omega_J$, for $J = 0, 1, \dots, K$, and write

$$\begin{aligned} \int_{\mathbb{R}^m} S(x)^\top P(x) S(x) dx &= \sum_{J=0}^{K} \int_{\mathbb{R}^m} \psi_J^2(x) S(x)^\top P(x) S(x) dx \\ &= \sum_{J=0}^{K} \int_{\mathbb{R}^m} \psi_J^2(x) S(x)^\top P(y_J) S(x) dx \\ &+ \sum_{J=0}^{K} \int_{\mathbb{R}^m} \psi_J^2(x) S(x)^\top [P(x) - P(y_J)] S(x) dx. \end{aligned} \tag{5.3.26}$$

By virtue of (5.3.17),

$$\sum_{J=0}^{K} \int_{\mathbb{R}^m} \psi_J^2(x) S(x)^\top [P(x) - P(y_J)] S(x) dx \geq -\frac{1}{2}\mu\|S\|_0^2. \tag{5.3.27}$$

For each $J = 0, 1, \dots, K$ we split $\psi_J S$ into

$$\psi_J S = X_J + Y_J, \tag{5.3.28}$$

where

$$X_J = \mathscr{L}(\psi_J \Phi), \tag{5.3.29}$$

$$Y_J = [\psi_J I - \sum_{\alpha=1}^{m} \partial_\alpha \psi_J \mathrm{D}G_\alpha(U)]\Phi. \tag{5.3.30}$$

Setting $\nu = |\xi|^{-1}\xi$ and recalling (5.3.4),

$$N(\nu)\hat{X}_J(\xi) = i|\xi|N(\nu)\Lambda(\nu,U)(\widehat{\psi_J\Phi})(\xi) = 0, \tag{5.3.31}$$

so both the real and the imaginary parts of $\hat{X}_J(\xi)$ lie in $\mathscr{C}$, for any $\xi \in \mathbb{R}^m$ and for $J = 0,1,\dots,K$. Thus, applying Parseval's relation and using (5.3.16) we deduce

(5.3.32)
$$\int_{\mathbb{R}^m} X_J(x)^\top P(y_J) X_J(x)dx = \int_{\mathbb{R}^m} \hat{X}_J(\xi)^* P(y_J)\hat{X}_J(\xi)d\xi \geq 2\mu \int_{\mathbb{R}^m} |X_J(x)|^2 dx.$$

Furthermore, from (5.3.30) and (5.3.24) we infer

$$\int_{\mathbb{R}^m} |Y_J(x)|^2 dx \leq c_1 \|S\|_{-1}^2, \quad J = 0,\dots,K. \tag{5.3.33}$$

We now return to (5.3.26). On account of (5.3.28), (5.3.32) and (5.3.33),

$$\begin{aligned} &\int_{\mathbb{R}^m} \psi_J^2(x)S(x)^\top P(y_J)S(x)dx \\ &\geq \frac{7}{8}\int_{\mathbb{R}^m} X_J(x)^\top P(y_J)X_J(x)dx - 8\int_{\mathbb{R}^m} Y_J(x)^\top P(y_J)Y_J(x)dx \\ &\geq \frac{7}{4}\mu \int_{\mathbb{R}^m} |X_J(x)|^2 dx - c_2\|S\|_{-1}^2 . \end{aligned} \tag{5.3.34}$$

Again by (5.3.28) and (5.3.33),

$$\int_{\mathbb{R}^m} |X_J(x)|^2 dx \geq \frac{6}{7}\int_{\mathbb{R}^m} \psi_J^2(x)|S(x)|^2 dx - c_3\|S\|_{-1}^2 . \tag{5.3.35}$$

Combining (5.3.25), (5.3.26), (5.3.27), (5.3.34) and (5.3.35), we arrive at (5.3.18). This completes the proof of the lemma.

Proof of Theorem 5.3.2. The solution to (5.1.1), (5.1.2), under the current assumptions, will be constructed by the vanishing viscosity method, namely as the $\varepsilon \to 0$ limit of solutions to the parabolic system (5.1.8), with the same initial data (5.1.2). This approach, which was already tested in Section 5.1, is also effective in the present setting because, when (5.3.1) holds, the parabolic system (5.1.8) inherits from the hyperbolic system (5.1.1) the involution (5.3.2).[1] Indeed, if U is a solution of (5.1.8) with initial data (5.1.2) satisfying the involution (5.3.2), then the function

[1] Note that this is not the case for the linearized system (5.1.7) and as a result the traditional approach seems inapplicable under the current assumptions, in the presence of involutions.

$Z = \sum M_\alpha \partial_\alpha U$ is the solution of the heat equation $\partial_t Z = \varepsilon \Delta Z$ with zero initial data, and thus vanishes identically.

The construction of the solution in the present setting will closely parallel the treatment of the classical case, in Theorem 5.1.1. A number of modifications shall be needed to account for the fact that η is no longer a convex entropy but it is merely a contingent entropy which is convex only on the involution cone. In order to keep duplication at a minimum, we shall not write a lengthy self-contained proof, but we will simply retrace the steps in the proof of Theorem 5.1.1, through the Lemmas 5.1.2, 5.1.3 and 5.1.4, interjecting the adjustments, as needed.

To begin with, Lemma 5.1.2 does not rest on the presence of an entropy and hence it applies here, without any modification. We thus know that for any fixed $\varepsilon > 0$ and $\omega > \|U_0\|_\ell$ there exists a solution U of (5.1.8), (5.1.2) on a time interval $[0, T_{\omega,\varepsilon})$, taking values in $\overline{\mathscr{B}_\rho}$ and satisfying (5.1.10). We proceed to derive bounds for U, independent of ε, by retracing the steps in the proof of Lemma 5.1.3.

Equations (5.1.26), (5.1.27), (5.1.29), (5.1.30), (5.1.31) and (5.1.32) are still valid. We may no longer count on symmetry of the matrices $A(U)\mathrm{D}G_\alpha(U)$. Nevertheless, upon switching to (5.3.10) as definition of $J_\alpha(U)$, (5.1.33) still holds, since U, and thereby U_r, satisfy the involution (5.3.2). With that modification, the basic "energy" integral (5.1.28) remains in force.

The next obstacle is that we no longer have (5.1.34) and (5.1.37) , because $A(U)$ is not necessarily positive definite. We shall compensate for the loss of convexity of $\eta(U)$, with the help of Lemma 5.3.4, as follows. Since $U_0 \in H_\ell$,

$$|A(U_0(x)) - A(U_0(y))| < \frac{1}{4}\mu, \tag{5.3.36}$$

for all x and y in the set $\Omega_0 = \{z \in \mathbb{R}^m \,:\, |z| > \alpha\}$, for α sufficiently large. Next we cover the compact set Ω_0^c by the union of balls $\Omega_1, \dots, \Omega_K$ with radii so small that (5.3.36) also holds for any x and y in Ω_I, $I = 1, \dots, K$. Finally, recalling (5.1.15), we restrict U to a time interval $[0, T]$, with $T < T_{\omega,\varepsilon}$ so small that

$$|A(U(x,t)) - A(U_0(x))| < \frac{1}{8}\mu, \tag{5.3.37}$$

for all $x \in \mathbb{R}^m$ and $t \in [0, T]$. It follows that, for any fixed $t \in [0, T]$, the covering $\Omega_0, \Omega_1, \dots, \Omega_K$ of $\mathbb{R}^m$ meets the conditions in Lemma 5.3.4, with $A(U(\cdot,t))$ in the role of $P(\cdot)$. In particular, in the place of (5.1.34) and (5.1.37) we now have

$$\begin{aligned} &\int_{\mathbb{R}^m} \partial_\alpha U_r^\top(x,\tau) A(U(x,\tau)) \partial_\alpha U_r(x,\tau) dx \\ &\qquad \geq \mu \int_{\mathbb{R}^m} |\partial_\alpha U_r(x,\tau)|^2 dx - \delta \int_{\mathbb{R}^m} |U_r(x,\tau)|^2 dx, \end{aligned} \tag{5.3.38}$$

$$\sum_{r \leq \ell} \int_{\mathbb{R}^m} U_r^\top(x,t) A(U(x,t)) U_r(x,t) dx \geq \mu \|U(\cdot,t)\|_\ell^2 - c\|U(\cdot,t)\|_{\ell-1}^2 . \tag{5.3.39}$$

Therefore, (5.3.38) in conjunction with (5.1.28), (5.1.35) and (5.1.27), demonstrates that (5.1.36) is still valid here, with

$$g(\tau) \le c[1 + \|\nabla U(\cdot,\tau)\|_{L^\infty} + \varepsilon \|\nabla U(\cdot,\tau)\|^2_{L^\infty}]. \tag{5.3.40}$$

With reference to (5.3.39), in order to estimate the term $\|U(\cdot,t)\|_{\ell-1}$, we multiply (5.1.26) by $2U_r^\top$, integrate the resulting equation over $\mathbb{R}^m \times [0,t]$, $t \in [0,T]$, integrate by parts the term $2\varepsilon U_r^\top \Delta U_r$ and sum over all multi-indices r of order $|r| \le \ell - 1$. Upon using (5.1.27), we deduce

$$\|U(\cdot,t)\|^2_{\ell-1} \le \|U_0(\cdot)\|^2_{\ell-1} + c\int_0^t [1 + \|\nabla U(\cdot,\tau)\|_{L^\infty}]\|U(\cdot,\tau)\|^2_\ell d\tau. \tag{5.3.41}$$

Combining (5.1.36) with (5.3.39) and (5.3.41), we infer that (5.1.38), and thereby (5.1.24), hold in the present setting, with g bounded through (5.3.40).

As in Lemma 5.1.3, we conclude that for any fixed $\omega > c_0\|U_0\|_\ell$, solutions U to (5.1.8), (5.1.2), with $\|U(\cdot,t)\|_\ell < \omega$, exist on a time interval $[0,T_\omega]$, for any $\varepsilon > 0$.

By retracing the steps in the proof of Lemma 5.1.4, we now construct a solution U to the Cauchy problem (5.1.1), (5.1.2) as the limit of a sequence $\{U_k\}$ of solutions U_k to (5.1.8), (5.1.2) with $\varepsilon = \varepsilon_k$, $\varepsilon_k \to 0$, as $k \to \infty$. The convergence is uniform on compact subsets of $\mathbb{R}^m \times [0,T_\omega]$. Furthermore, $U_k(\cdot,t) \to U(\cdot,t)$, weakly in H_ℓ, for any fixed $t \in [0,T_\omega]$.

Next we demonstrate that $U_k(\cdot,t) \to U(\cdot,t)$, strongly in H_0, for any $t \in [0,T_\omega]$. To that end, we set $V_k = U_k - U$ and appeal to (5.1.41), which still holds in the present setting. However, we may no longer use (5.1.42), since $A(U)$ is not necessarily positive definite. In its place, we employ

$$\int_{\mathbb{R}^m} V_k^\top(x,t)A(U(x,t))V_k(x,t)dx \ge \mu\|V_k(\cdot,t)\|_0^2 - \delta\|V_k(\cdot,t)\|^2_{-1}, \tag{5.3.42}$$

which follows from Lemma 5.3.4. We can handle the term $\|V_k(\cdot,t)\|_{-1}$ by noting that since U satisfies (5.1.1) and U_k satisfies (5.1.8), with the same initial data,

$$V_k(x,t) = \sum_{\alpha=1}^m \partial_\alpha \int_0^t [G_\alpha(U(x,\tau)) - G_\alpha(U_k(x,\tau)) + \varepsilon_k \partial_\alpha U_k(x,\tau)]d\tau \tag{5.3.43}$$

whence

$$\|V_k(\cdot,t)\|_{-1} \le c\int_0^t [\|V_k(\cdot,\tau)\|_0 + \varepsilon\|U_k(\cdot,\tau)\|_1]d\tau. \tag{5.3.44}$$

Combining (5.1.41) with (5.3.42) and (5.3.44), we deduce

$$\|V_k(\cdot,t)\|_0^2 \le c(\omega + t)\int_0^t \|V_k(\cdot,\tau)\|_0^2 d\tau + c\omega^2\varepsilon_k t(1+\varepsilon_k t), \tag{5.3.45}$$

which is comparable to (5.1.43) and implies $V_k(\cdot,t) \to 0$, strongly in H_0, for every t in $[0,T_\omega]$. Since $U_k(\cdot,t) \to U(\cdot,t)$, weakly in H_ℓ, this yields $U_k(\cdot,t) \to U(\cdot,t)$, strongly in $H_{\ell-1}$ and uniformly in $\mathbb{R}^m$. Moreover, $\nabla U_k(\cdot,t) \to \nabla U(\cdot,t)$, uniformly in $\mathbb{R}^m$, for any $t \in [0, T_\omega]$. In particular, since the U_k satisfy (5.1.24), with g bounded through (5.3.40), we infer

$$\|U(\cdot,t)\|_\ell \le c_0 \|U_0\|_\ell \exp \int_0^t c[1 + \|\!|\nabla U(\cdot,\tau)|\!\|_{L^\infty}] d\tau. \tag{5.3.46}$$

The next step is to verity that $t \mapsto U(\cdot,t)$ is continuous on $[0, T_\omega]$. As in the proof of Lemma 5.1.4, we consider the identity (5.1.44), which is still valid in the present setting, and let $k \to \infty$. On the right-hand side of (5.1.44), the first two terms tend to zero, while the limit superior of the third term is nonpositive. Indeed, even though we no longer have (5.1.45), in its place, by Lemma 5.3.4,

$$\sum_{|r|\le\ell} \int_{\mathbb{R}^m} (U_{kr} - U_r)^\top A(U)(U_{kr} - U_r) dx \ge \mu \|U_k - U\|_\ell^2 - c\|U_k - U\|_{\ell-1}^2 , \tag{5.3.47}$$

and, as noted above, $\|U_k - U\|_{\ell-1} \to 0$. Since the limit superior of the left-hand side of (5.1.44) is nonpositive and the U_k satisfy (5.1.36), with g bounded through (5.3.40), we deduce the inequality

$$\begin{aligned} &\sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_r^\top(x,t) A(U(x,t)) U_r(x,t) dx \\ &\qquad - \sum_{|r|\le\ell} \int_{\mathbb{R}^m} U_{0r}^\top(x) A(U_0(x)) U_{0r}(x) dx \le c\omega^2(1+\omega)t, \end{aligned} \tag{5.3.48}$$

which is the analog of (5.1.46).

Continuing along the road map of the proof of Lemma 5.1.4, we observe that the identity (5.1.47) remains in force in the present setting. We no longer have (5.1.48), but instead, by virtue of Lemma 5.3.4,

$$\begin{aligned} &\sum_{|r|\le\ell} \int_{\mathbb{R}^m} [U_r(x,t) - U_{0r}(x)]^\top A(U_0(x)) [U_r(x,t) - U_{0r}(x)] dx \\ &\qquad \ge \mu \|U(\cdot,t) - U_0(\cdot)\|_\ell^2 - c\|U(\cdot,t) - U_0(\cdot)\|_{\ell-1}^2 . \end{aligned} \tag{5.3.49}$$

On account of (5.1.21), with $\varepsilon = 0$, $\|U(\cdot,t) - U_0(\cdot)\|_{\ell-1} = O(t)$, as $t \to 0$. On the other hand, as $t \to 0$, the limit superior of the right-hand side of (5.1.47) is nonpositive, in consequence of $U(\cdot,t) \to U_0(\cdot)$, weakly in H_ℓ, together with (5.3.48) and (5.1.15). It thus follows that $\|U(\cdot,t) - U_0(\cdot)\|_\ell \to 0$, as $t \to 0$, i.e., $t \mapsto U(\cdot,t)$ is right-continuous at $t = 0$, and thereby continuous on $[0, T_\omega]$.

The remainder of the proof of the theorem just redoubles the final steps in the proof of Theorem 5.1.1: One extends the solution U of (5.1.1), (5.1.2), constructed above, beyond T_ω by solving a new Cauchy problem for (5.1.1) with initial data $U(\cdot, T_\omega)$, and iterates this process until reaching a maximal time interval $[0, T_\infty)$. The lifespan T_∞ may be finite only if $\limsup_{t\to T_\infty} \|U(\cdot,t)\|_{L^\infty} = \rho$ and/or

$\limsup_{t\to T_\infty} \|U(\cdot,t)\|_\ell = \infty$. On account of (5.3.46), $\|U(\cdot,t)\|_\ell$ cannot become unbounded unless (5.3.13) holds. Finally, (5.3.12) follows from $U(\cdot,t) \in C^0([0,T_\infty];H_\ell)$ and (5.1.1). This completes the proof of the theorem.

Proof of Theorem 5.3.3. It will suffice to retrace the steps in the proof of Theorem 5.2.1, making the necessary adjustments.

The definitions (5.3.2) and (5.3.4) for $h(U,\bar{U})$ and $Z_\alpha(U,\bar{U})$ will remain intact. However, so as to account for the assumption that η is now merely a contingent entropy, the definition (5.3.3) for $Y_\alpha(U,\bar{U})$ must be replaced by

(5.3.50)
$$Y_\alpha(U,\bar{U}) = Q_\alpha(U) - Q_\alpha(\bar{U}) - \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})] - \Xi(\bar{U})^\top M_\alpha[U - \bar{U}].$$

As a result of this change, in the place of (5.2.6) we now have

$$\begin{aligned}(5.3.51)\qquad &\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi Y_\alpha(U,\bar{U})]dxdt \\ &+ \int_{\mathbb{R}^m} \psi(x,0)h(U_0(x),\bar{U}_0(x))dx \geq -\int_0^T \int_{\mathbb{R}^m} \partial_t \psi \mathrm{D}\eta(\bar{U})[U-\bar{U}]dxdt \\ &- \int_0^T \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \psi \{\mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})] + \Xi(\bar{U})^\top M_\alpha[U-\bar{U}]\}dxdt.\end{aligned}$$

Equation (5.2.7) is still in force, but (5.2.8) must be modified to reflect the symmetry of the new $J_\alpha(U)$, defined by (5.3.10). Taking into account that $\bar{U}$ satisfies the involution (5.3.2), we get

$$\begin{aligned}(5.3.52)\qquad \partial_t \mathrm{D}\eta(\bar{U}) &= \partial_t \bar{U}^\top A(\bar{U}) = -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top \mathrm{D}G_\alpha(\bar{U})^\top A(\bar{U}) \\ &= -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top J_\alpha(\bar{U})^\top = -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top J_\alpha(\bar{U}) \\ &= -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top A(\bar{U})\mathrm{D}G_\alpha(\bar{U}) - \sum_{\alpha=1}^m \partial_\alpha \Xi(\bar{U})^\top M_\alpha\,.\end{aligned}$$

Combining (5.3.51) with (5.2.7) and (5.3.52), and using that $U-\bar{U}$ satisfies the involution (5.3.2), we conclude that (5.2.10) is still in force in the present setting. We fix any point $\sigma \in (0,T)$ of L^∞ weak* continuity of $\eta(U(\cdot,t))$ and ε positive small, and apply (5.2.10) for the test function $\psi(x,t) = \chi(x)\omega(t)$, where ω is defined by (5.2.11) and χ is any $C_0^\infty(\mathbb{R}^m)$ function such that $\chi(x) = 1$ for all x with $|x| < \varepsilon^{-1}$. Letting $\varepsilon \to 0$, we deduce

$$\int_{\mathbb{R}^m} h(U(x,\sigma),\bar{U}(x,\sigma))dx \tag{5.3.53}$$
$$\leq \int_{\mathbb{R}^m} h(U_0(x),\bar{U}_0(x))dx - \int_0^\sigma \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U})dxdt.$$

By virtue of (5.2.2),

$$h(U,\bar{U}) = \frac{1}{2}(U-\bar{U})^\top H(U,\bar{U})(U-\bar{U}), \tag{5.3.54}$$

where

$$H(U,\bar{U}) = 2\int_0^1 \int_0^z A(sU+(1-s)\bar{U})dsdz. \tag{5.3.55}$$

On account of (5.3.11),

$$X^\top H(U,\bar{U})X \geq 2\mu|X|^2, \qquad X \in \mathscr{C}, \quad U \in \overline{\mathscr{B}_\rho}\,, \;\; \bar{U} \in \overline{\mathscr{B}_\rho}\,. \tag{5.3.56}$$

Since $\bar{U}$ is Lipschitz, when (5.3.14) holds with κ sufficiently small there is $\varepsilon > 0$ such that $|x-y| < 2\varepsilon$ implies

$$|H(U(x,t),\bar{U}(x,t)) - H(U(y,t),\bar{U}(y,t))| < \frac{1}{2}\mu, \tag{5.3.57}$$

for all $t \in [0,T)$. Furthermore, in view of the prescribed behavior of $U(x,t)$ and $\bar{U}(x,t)$ as $|x| \to \infty$, (5.3.57) will also hold for all $t \in [0,T)$ and all x and y in the set $\Omega_0 = \{z \in \mathbb{R}^m : |z| > \alpha\}$, for large α. Let us fix some covering of the compact set Ω_0^c by balls $\Omega_1,\dots,\Omega_K$ of radius ε. Then, for any fixed $t \in [0,T)$, $\Omega_0,\Omega_1,\dots,\Omega_K$ provide a covering of $\mathbb{R}^m$ which meets the conditions laid down in Lemma 5.3.4, with $H(U(\cdot,t),\bar{U}(\cdot,t))$ in the role of $P(\cdot)$. Hence

$$\int_{\mathbb{R}^m} h(U(x,\sigma),\bar{U}(x,\sigma))dx \geq \frac{1}{2}\mu\|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_0^2 - \frac{1}{2}\delta\|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_{-1}^2\,. \tag{5.3.58}$$

From

$$U(\cdot,\sigma)-\bar{U}(\cdot,\sigma) = U_0(\cdot)-\bar{U}_0(\cdot) - \sum_{\alpha=1}^m \partial_\alpha \int_0^\sigma [G_\alpha(U(\cdot,t)) - G(\bar{U}(\cdot,t))]dt, \tag{5.3.59}$$

it follows that

$$\|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_{-1} \leq \|U_0(\cdot)-\bar{U}_0(\cdot)\|_0 + c\int_0^\sigma \|U(\cdot,t)-\bar{U}(\cdot,t)\|_0 dt. \tag{5.3.60}$$

Combining (5.3.53), (5.3.58), (5.3.60) and (5.2.4) we deduce that the function u defined by $u(t) = \|U(\cdot,t) - \bar{U}(\cdot,t)\|_0$ satisfies an integral inequality

$$u(\sigma) \leq au(0) + b\int_0^\sigma u(t)dt, \tag{5.3.61}$$

for almost all $\sigma \in [0,T)$, with a depending solely on bounds of G, η and their derivatives on $\overline{\mathscr{B}_\rho}$, and b that also depends on T and on the Lipschitz constant of $\bar{U}$. Applying Gronwall's inequality to (5.3.61), we arrive at (5.3.15). This completes the proof.

It should be noted that in certain systems (5.1.1) of conservation laws relations of the form (5.3.2), introduced through the initial data, may be preserved by (at least classical) solutions even though the M_α are not induced by symmetry relations (5.3.1). A case in point is the condition $\omega = \operatorname{curl} v = 0$ of irrotational flow which is preserved by smooth solutions of the Euler equations (3.3.16) but breaks down after discontinuities develop.

There are also systems of conservation laws equipped with involutions involving nonlinear functions of the state vector and its spatial derivatives. For example, (2.2.12) and (2.2.13) may be regarded as nonlinear involutions for the system (3.3.19) of isentropic elastodynamics. Furthermore, if one writes (3.3.19) in Eulerian coordinates, then the linear involution (3.3.10) becomes nonlinear: $F_{j\beta}\partial_j F_{i\alpha} - F_{j\alpha}\partial_j F_{i\beta} = 0$ (with summation convention). However, the most celebrated example of a system with nonlinear involutions is provided by the Einstein equations of general relativity.

Theorem 5.3.2, 5.3.3 and their proofs serve as confirmation that, from the standpoint of analysis, the notion of contingent entropy is the natural extension of the notion of entropy for systems of conservation laws endowed with involutions. Nevertheless, in the applications of the above theorems to systems arising in continuum physics, such as (3.3.19), η is an actual entropy. The importance of contingent entropies that are not entropies will become clear in the following section.

The preconditions for applying Theorems 5.3.2 and 5.3.3 are not met by every system with non-convex entropy and involutions encountered in physics. For example, as noted above, in the system (3.3.66) of electrodynamics (with $J = 0$), endowed with the involutions (3.3.67) (with $\rho = 0$), the involution cone is the entire state space $\mathbb{R}^6$, on which the electromagnetic filed energy η fails to be convex. Even when the conditions for applying these theorems are present, as in the case of the system (3.3.19) of elastodynamics, it should be noted that, in comparison to Theorem 5.2.1, Theorem 5.3.3 requires more and delivers less. In the following section we will identify special structure, associated with the presence of contingent entropies in certain systems encountered in continuum physics, that may remedy the above shortcomings.

5.4 Contingent Entropies and Polyconvexity

We consider here systems of conservation laws with involutions and a nonconvex contingent entropy, as in the previous section, but also endowed with supplementary contingent entropies rendering existence and stability of solutions to the Cauchy problem as strong as that inferred by Theorems 5.1.1 and 5.2.1.

The systems (3.3.19), of elastodynamics, and (3.3.66), of electrodynamics, will serve as models. As noted in the previous section, (3.3.19) is endowed with the involution (3.3.10), the entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ and ten contingent entropies, namely $\det F$ and the nine entries of the adjugate matrix F^*. Similarly, (3.3.66) is equipped with the involution (3.3.67), the electromagnetic field energy $\eta(B,D)$ and three contingent entropies, namely the components of the vector $B \wedge D$. Accordingly, throughout this section we will be operating under the following

5.4.1 Assumptions. In the system of the conservation laws (5.1.1), the flux G satisfies the symmetry condition (5.3.1), which induces the involution (5.3.2). The system is endowed with the *principal contingent entropy-entropy flux pair* (η, Q) and a family of N *supplementary contingent entropy-entropy flux pairs* (W^I, X^I), $I = 1, \dots, N$.

As a contingent entropy pair, (η, Q) must satisfy (5.3.9), for some k-vector-valued function $\Xi(U)$. Similarly, for $I = 1, \dots, N$,

$$\mathrm{D}X_\alpha^I(U) = \mathrm{D}W^I(U)\mathrm{D}G_\alpha(U) + \Omega^I(U)^\top M_\alpha\,, \qquad \alpha = 1, \dots, m, \tag{5.4.1}$$

for some k-vector-valued function $\Omega^I(U)$. It will be convenient to assemble the W^I into a N-vector W, the X^I into a $N \times m$ matrix X and the Ω^I into a $N \times k$ matrix Ω, in which case (5.4.1) may be written as the matrix equation

$$\mathrm{D}X_\alpha(U) = \mathrm{D}W(U)\mathrm{D}G_\alpha(U) + \Omega(U)^\top M_\alpha\,, \qquad \alpha = 1, \dots, m. \tag{5.4.2}$$

Every component U^i of the state vector U may be regarded as an entropy, and thereby as a contingent entropy, with associated flux the i-th row G^i of the matrix G. For convenience, we embed these entropy pairs in the list of supplementary pairs recorded in the Assumption 5.4.1. Thus, we assume that $N \geq n$ and for $I = 1, \dots, n$, $W^I(U) = U^I$, $X_\alpha^I(U) = G_\alpha^I(U)$ and $\Omega^I(U) = 0$.

The equation (5.3.8), together with

$$\partial_t W(U(x,t)) + \sum_{\alpha=1}^m \partial_\alpha X_\alpha(U(x,t)) = 0 \tag{5.4.3}$$

must hold for any classical solution U of (5.1.1) that satisfies the involution (5.3.2). On the other hand, admissible weak solutions U must satisfy the inequality (4.5.3) for the principal contingent entropy-entropy flux pair (η, Q). Typically, in systems with involutions, such as (3.3.19) and (3.3.66), encountered in continuum physics, the principal contingent entropy is an actual entropy.

The objective of this section is to demonstrate that in the above setting the requirement of convexity on the principal entropy may be relaxed into the following weaker condition:

5.4.2 Definition. The principal contingent entropy η is called *polyconvex*, relative to the contingent entropies W, if it admits a representation

$$\eta(U) = \theta(W(U)), \quad U \in \overline{\mathscr{B}_\rho}, \tag{5.4.4}$$

where θ is a smooth function defined on an open neighborhood $\mathscr{F}$ of $W(\mathscr{B}_\rho)$ in $\mathbb{R}^N$ whose Hessian matrix is positive definite on every $W \in \mathscr{F}$.

In the example of elastodynamics, with $W = (F, v, F^*, \det F)$ arranged as a 22-vector, the principal entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ will be polyconvex when the internal energy function $\varepsilon(F)$ admits a representation

$$\varepsilon(F) = \theta(F, F^*, \det F), \tag{5.4.5}$$

where $\theta(F, H, \delta)$ is a smooth function with positive definite Hessian on an open neighborhood of the manifold $\{(F, H, \delta) : \det F > 0, H = F^*, \delta = \det F\}$, embedded in $\mathbb{R}^{19}$. This is a physically reasonable hypothesis which has been discussed thoroughly in the literature, especially in the context of elastostatics. We have already encountered it in Section 4.5, where it was noted that it implies that $\varepsilon(F)$ is rank-one convex and thereby η is convex on the involution cone. However, as we shall see here, the implications of polyconvexity on stability of solutions are much stronger than the consequences of mere convexity on the involution cone, discussed in the previous section.

The situation is similar with the system (3.3.66) of electrodynamics, in which case $W = (B, D, B \wedge D)$, arranged as a 9-vector. Polyconvexity is a natural condition for the electromagnetic field energy η, which serves as principal entropy. Indeed, in the Born-Infeld case (3.3.73), η is polyconvex.

We introduce the following notation for the function $\theta(W)$: θ_{W^I} for the partial derivative $\partial\theta/\partial W^I$; θ_W for the differential $[\theta_{W^1} \cdots \theta_{W^N}]$, treated as a $1 \times n$ matrix; and θ_{WW} for the $N \times N$ Hessian matrix.

For $U \in \overline{\mathscr{B}_\rho}$ we define the $n \times n$ matrices

$$A(U) = \mathrm{D}^2\eta(U) - \sum_{I=1}^{N} \theta_{W^I}(W(U))\mathrm{D}^2 W^I, \tag{5.4.6}$$

$$J_\alpha(U) = A(U)\mathrm{D}G_\alpha(U) + \Gamma(U)^\top M_\alpha, \qquad \alpha = 1, \dots, m, \tag{5.4.7}$$

where

$$\Gamma(U) = \mathrm{D}\Xi(U) - \sum_{I=1}^{N} \theta_{W^I}(W(U))\mathrm{D}\Omega^I(U). \tag{5.4.8}$$

It is clear that $A(U)$ is symmetric. As already noted in Section 5.3, (5.3.9) implies that the matrices

$$\mathrm{D}^2\eta(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Xi(U)^\top M_\alpha\,, \qquad \alpha = 1,\dots,m, \tag{5.4.9}$$

are symmetric. Similarly, (5.4.1) implies that the matrices

$$\mathrm{D}^2 W^I(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Omega^I(U)^\top M_\alpha\,, \qquad \alpha = 1,\dots,m \tag{5.4.10}$$

are also symmetric. It thus follows that the matrices $J_\alpha(U)$, for $\alpha = 1,\dots,m$, are symmetric.

From (5.4.5) and (5.4.6),

$$A(U) = \mathrm{D}W(U)^\top \theta_{WW}(W(U))\mathrm{D}W(U), \tag{5.4.11}$$

so that if η is polyconvex then $A(U)$ is positive definite.

The following proposition establishes the local existence of classical solutions to the Cauchy problem for systems with polyconvex entropy.

5.4.3 Theorem. *Let the hyperbolic system* (5.1.1) *of conservation laws satisfy the Assumptions* 5.4.1, *with a principal contingent entropy* η *that is polyconvex* (5.4.4). *Suppose the initial data* U_0 *lie in* H_ℓ, *for some* $\ell > \frac{m}{2} + 1$, *take values in a ball* $\overline{\mathscr{B}_{\rho_0}}$ *with radius* $\rho_0 < \rho$, *and satisfy the involution* (5.3.2). *Then there exist* $T_\infty \le \infty$ *and a unique continuously differentiable function* U *on* $\mathbb{R}^m \times [0,T_\infty)$, *taking values in* $\overline{\mathscr{B}_\rho}$, *which is a classical solution to the Cauchy problem* (5.1.1), (5.1.2) *on the time interval* $[0,T_\infty)$. *Furthermore*,

$$U(\cdot,t) \in \bigcap_{k=0}^{\ell} C^k([0,T_\infty);H_{\ell-k}). \tag{5.4.12}$$

The interval $[0,T_\infty)$ *is maximal in that if* $T_\infty < \infty$, *then*

$$\int_0^{T_\infty} \|\!|\nabla U(\cdot,t)|\!\|_{L^\infty} dt = \infty\,, \tag{5.4.13}$$

and/or $\limsup_{t\to T_\infty} \|U(\cdot,t)\|_{L^\infty} = \rho$.

Proof. Following the vanishing viscosity approach, we construct the solution to (5.1.1), (5.1.2) as the $\varepsilon \to 0$ limit of solutions to (5.1.8), (5.1.2), by retracing the steps in the proof of Theorem 5.1.1, through the Lemmas 5.1.2, 5.1.3 and 5.1.4. In fact, the notation here has been designed so that, upon substituting (5.4.6) and (5.4.7) for (5.1.3) and (5.1.4) as definitions of $A(U)$ and $J_\alpha(U)$, one may transfer virtually verbatim the text and the equations from Section 5.1 to the present setting. The straightforward verification is left to the reader.

We now turn to the question of uniqueness and stability of classical solutions within a class of weak solutions that will be dubbed mild.

5.4.4 Definition. A measurable function U, defined on $\mathbb{R}^m \times [0,T)$ and taking values in $\overline{\mathscr{B}_\rho}$, is a *mild solution* to (5.1.1), (5.1.2) if

(5.4.14)
$$\int_0^T \int_{\mathbb{R}^m} [\partial_t V^\top W(U) + \sum_{\alpha=1}^m \partial_\alpha V^\top X_\alpha(U)] dxdt + \int_{\mathbb{R}^m} V^\top(x,0) W(U_0(x)) dx = 0$$

holds for all Lipschitz N-vector-valued test functions V, with compact support in $\mathbb{R}^m \times [0,T)$.

Notice that (5.4.14) holds when U satisfies (5.4.3), in the sense of distributions, together with the initial condition $W(U(\cdot,t)) \to W(U_0(\cdot))$ in L^∞ weak*, as $t \to 0$. In particular, any mild solution of (5.1.1), (5.1.2) is a weak solution, since (5.1.1) is embedded in (5.4.3). Clearly, any classical solution of (5.1.1), (5.1.2) is a mild solution, because (5.4.3) and the initial conditions are automatically satisfied in that case. However, it comes as a surprise that in the applications one often encounters even discontinuous mild solutions. For example, any weak solution (F,v) of the system (3.3.19) of isentropic elastodynamics is mild. Indeed, as we saw in Section 2.3, (3.3.11) and (3.3.12) hold for any L^∞ fields that satisfy $(3.3.19)_1$ and the involution (3.3.10). Moreover, as noted in Section 4.5, null Lagrangians (2.2.9) are continuous functions in L^∞ weak*, and hence $F(\cdot,t) \to F_0(\cdot)$, in L^∞ weak*, as $t \to 0$, implies $F^*(\cdot,t) \to F_0^*(\cdot)$ and $\det F(\cdot,t) \to \det F_0(\cdot)$, in L^∞ weak*, as $t \to 0$. Similarly, BV weak solutions (B,D) of the system (3.3.66) of electrodynamics, with Born-Infeld constitutive relations (3.3.73) and involutions (3.3.67), are necessarily mild solutions, because all shocks satisfy (3.3.80). Thus, (3.3.74) will hold for such solutions. Moreover, in the BV setting there is sufficient regularity so that $B(\cdot,t) \to B_0(\cdot)$ and $D(\cdot,t) \to D_0(\cdot)$, as $t \to 0$, implies $B(\cdot,t) \wedge D(\cdot,t) \to B_0(\cdot) \wedge D_0(\cdot)$, as $t \to 0$.

A mild solution U will be *admissible* if it is admissible as a weak solution, i.e., if (4.5.3) is satisfied for the principal contingent entropy-entropy flux pair. In particular, any BV solution of (3.3.66), under the Born-Infeld constitutive relation, is admissible, because shocks do not incur energy production. Of course, this is not the case with the system (3.3.19) of elastodynamics.

The following proposition is the analog of Theorem 5.2.1:

5.4.5 Theorem. *Let the hyperbolic system* (5.1.1) *of conservation laws satisfy the Assumptions* 5.4.1, *with a principal contingent entropy* η *that is polyconvex* (5.4.4). *Suppose* $\bar{U}$ *is a classical solution of* (5.1.1) *on* $[0,T)$, *taking values in* $\overline{\mathscr{B}_\rho}$, *with initial data* $\bar{U}_0$ *satisfying the involution* (5.3.2). *Let* U *be any admissible mild solution of* (5.1.1) on $[0,T)$, *taking values in* $\overline{\mathscr{B}_\rho}$, *with initial data* U_0 *satisfying the involution* (5.3.2). *Then*

$$\int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \leq a e^{bt} \int_{|x|<r+st} |U_0(x) - \bar{U}_0(x)|^2 dx \tag{5.4.15}$$

holds for any $r > 0$ *and* $t \in [0,T)$, *with positive constants* s, a, *depending solely on bounds of* G, η, Q *and their derivatives on* $\overline{\mathscr{B}_\rho}$, *and* b *that also depends on the*

Lipschitz constant of $\bar{U}$. In particular, $\bar{U}$ is the unique admissible mild solution of (5.1.1) *with initial data $\bar{U}_0$.*

Proof. We retrace the steps in the proof of Theorem 5.2.1, with the needed modifications. On $\overline{\mathscr{B}_\rho} \times \overline{\mathscr{B}_\rho}$ we define

$$h(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - \theta_W(W(\bar{U}))[W(U) - W(\bar{U})], \tag{5.4.16}$$

$$\begin{aligned} Y_\alpha(U,\bar{U}) = {} & Q_\alpha(U) - Q_\alpha(\bar{U}) - \theta_W(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})] \\ & + [\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top]M_\alpha[U - \bar{U}], \end{aligned} \tag{5.4.17}$$

$$\begin{aligned} Z_\alpha(U,\bar{U}) = {} & -\mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))[W(U) - W(\bar{U})] \\ & + \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})] \\ & - \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\Omega(\bar{U})^\top M_\alpha[U - \bar{U}] \\ & + \Gamma(\bar{U})^\top M_\alpha[U - \bar{U}], \end{aligned} \tag{5.4.18}$$

where Γ is given by (5.4.8).

Recalling Definition 5.4.2, we see that $h(U,\bar{U})$ is of quadratic order in $U - \bar{U}$ and positive definite. Upon using (5.3.9), (5.4.2) and (5.4.4), we deduce

$$\begin{aligned} \mathrm{D}Y_\alpha(U,\bar{U}) = {} & [\theta_W(W(U)) - \theta_W(W(\bar{U}))]\mathrm{D}W(U)\mathrm{D}G_\alpha(U) \\ & + [\Xi(U) - \Xi(\bar{U})]^\top M_\alpha - \theta_W(W(\bar{U}))[\Omega(U) - \Omega(\bar{U})]^\top M_\alpha, \end{aligned} \tag{5.4.19}$$

which vanishes at $U = \bar{U}$, so that $Y(U,\bar{U})$ is also of quadratic order in $U - \bar{U}$. In particular, for s large, (5.2.5) holds.

Turning to $Z(U,\bar{U})$, and by virtue of (5.4.2),

$$\begin{aligned} \mathrm{D}Z_\alpha(U,\bar{U}) = {} & -\mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\mathrm{D}W(U) \\ & + \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\mathrm{D}W(U)\mathrm{D}G_\alpha(U) \\ & + \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))[\Omega(U) - \Omega(\bar{U})]^\top M_\alpha \\ & + \Gamma(\bar{U})^\top M_\alpha . \end{aligned} \tag{5.4.20}$$

Recalling (5.4.6), (5.4.7) and since J_α is symmetric, we conclude that

(5.4.21)
$$\mathrm{D}Z_\alpha(\bar{U},\bar{U}) = -\mathrm{D}G_\alpha(\bar{U})^\top A(\bar{U}) + A(\bar{U})\mathrm{D}G_\alpha(\bar{U}) + \Gamma(\bar{U})^\top M_\alpha = M_\alpha^\top \Gamma(\bar{U}).$$

As in the proof of Theorem 5.2.1, we fix a nonnegative, Lipschitz continuous test function ψ with compact support in $\mathbb{R}^m \times [0,T)$, and evaluate h, Y and Z along the two solutions $U(x,t)$ and $\bar{U}(x,t)$. As an admissible weak solution, U must satisfy the inequality (4.5.3), while $\bar{U}$ being a classical solution, will satisfy (4.5.3) as an equality. We thus deduce

(5.4.22)
$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi\, h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi Y_\alpha(U,\bar{U})]dxdt + \int_{\mathbb{R}^m} \psi(x,0)\, h(U_0(x),\bar{U}_0(x))\, dx \\
&\qquad \geq \; -\int_0^T \int_{\mathbb{R}^m} \Big\{ \partial_t \psi\, \theta_W(W(\bar{U}))[W(U) - W(\bar{U})] \\
&\qquad\qquad + \sum_{\alpha=1}^m \partial_\alpha \psi \{ \theta_W(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})] \\
&\qquad\qquad - [\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top] M_\alpha [U - \bar{U}] \} \Big\} dxdt \\
&\qquad - \int_{\mathbb{R}^m} \psi(x,0)\theta_W(W(\bar{U}_0(x)))[W(U_0(x)) - W(\bar{U}_0(x))]dx.
\end{aligned}$$

Next we write (5.4.14) for both U and $\bar{U}$, with test function $V^T = \psi\theta_W(W(\bar{U}))$. This yields

$$\begin{aligned}
(5.4.23) \quad &\int_0^T \int_{\mathbb{R}^m} \Big\{ \partial_t [\psi\theta_W(W(\bar{U}))][W(U) - W(\bar{U})] \\
&\qquad + \sum_{\alpha=1}^m \partial_\alpha [\psi\theta_W(W(\bar{U}))][X_\alpha(U) - X_\alpha(\bar{U})] \Big\} dxdt \\
&\quad + \int_{\mathbb{R}^m} \psi(x,0)\theta_W(W(\bar{U}_0(x)))[W(U_0(x)) - W(\bar{U}_0(x))]dx = 0.
\end{aligned}$$

Furthermore, since both U and $\bar{U}$ satisfy the involution (5.3.2),

$$(5.4.24) \quad \int_0^T \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \Big\{ \psi[\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top] \Big\} M_\alpha [U - \bar{U}] dxdt = 0.$$

By virtue of (5.4.2) and $\sum M_\alpha \partial_\alpha \bar{U} = 0$,

$$(5.4.25)\quad \begin{aligned}\partial_t \theta_W(W(\bar{U})) &= \partial_t W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\\ &= \sum_{\alpha=1}^m \partial_\alpha X_\alpha(\bar{U})^\top \theta_{WW}(W(\bar{U}))\\ &= -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top \mathrm{D}X_\alpha(\bar{U})^\top \theta_{WW}(W(\bar{U}))\\ &= -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top [\mathrm{D}W(\bar{U})\mathrm{D}G_\alpha(\bar{U}) + \Omega(\bar{U})^\top M_\alpha]^\top \theta_{WW}(W(\bar{U}))\\ &= -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top \mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U})).\end{aligned}$$

Similarly,

$$(5.4.26)\qquad \partial_\alpha \theta_W(W(\bar{U})) = \partial_\alpha \bar{U}^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U})),$$

$$(5.4.27)\qquad \begin{aligned}&\partial_\alpha[\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top]\\ &\qquad \partial_\alpha \bar{U}^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\Omega(\bar{U})^\top - \partial_\alpha \bar{U}^\top \Gamma(\bar{U})^\top.\end{aligned}$$

Therefore, recalling (5.4.18),

$$(5.4.28)\qquad \begin{aligned}&\partial_t \theta_W(W(\bar{U}))[W(U) - W(\bar{U})] + \sum_{\alpha=1}^m \partial_\alpha \theta_W(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})]\\ &- \sum_{\alpha=1}^m \partial_\alpha[\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top]M_\alpha[U-\bar{U}] = \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}).\end{aligned}$$

On account of (5.4.21),

$$(5.4.29)\qquad \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top \mathrm{D}Z_\alpha(\bar{U},\bar{U}) = \left[\sum_{\alpha=1}^m M_\alpha \partial_\alpha \bar{U}\right]^\top \Gamma(\bar{U}) = 0.$$

Consequently, the right-hand side of (5.4.28) is of quadratic order in $U - \bar{U}$.

By combining (5.4.22), (5.4.23), (5.4.24) and (5.4.28), we recover (5.2.10). The remainder of the proof follows along the lines of the proof of Theorem 5.2.1: departing from (5.2.10) and fixing any $t \in (0,T)$, we derive (5.2.13), for $\sigma \in (0,t)$, and then (5.2.12), for any σ of L^∞ weak* continuity of $\eta(U(\cdot,\tau))$. This in turn yields (5.2.14), for u defined by (5.2.15), and thereby (5.4.15). The proof is complete.

In particular, Theorems 5.4.3 and 5.4.5 apply to the class of systems of conservation laws that are endowed with an involution and are equipped with a convex contingent entropy $\eta(U)$ (just take $W(U) \equiv U$). One may attempt to reduce the more general class of systems endowed with an involution and equipped with a polyconvex contingent entropy to the above special class by means of the following procedure. Assume that the system (5.1.1) is endowed with the involution (5.3.2) and is equipped with a principal contingent entropy-entropy flux pair $(\eta(U), Q(U))$ which is polyconvex (5.4.4), relative to the contingent entropies W. We seek functions $S(\Psi)$ and $\Pi(\Psi)$, defined on $\mathbb{R}^N$ and taking values in $\mathbb{M}^{N\times m}$ and $\mathbb{M}^{1\times m}$, respectively, such that

$$S(W(U)) = W(U), \qquad \Pi(W(U)) = Q(U) \tag{5.4.30}$$

and, in addition, $(\theta(\Psi), \Pi(\Psi))$ is a (generally contingent) entropy-entropy flux pair for the *extended system*

$$\partial_t \Psi(x,t) + \operatorname{div} S(\Psi(x,t)) = 0. \tag{5.4.31}$$

When functions satisfying the above specifications can be found, one may construct solutions to the Cauchy problem (5.1.1), (5.1.2) by first solving (5.4.31) with initial conditions

$$\Psi(x,0) = W(U_0(x)), \tag{5.4.32}$$

and then getting U from the equation $W(U) = \Psi$. The merit of this approach lies in that (5.4.31) is now equipped with a convex (possibly contingent) entropy θ.

The above program has been implemented successfully for the systems of elastodynamics and electrodynamics.

In elastodynamics, $U = (F,v)^\top, \Psi = (F,v,\Theta,\omega)^\top$, $\sigma = \sigma(F,\Theta,\omega)$, the extended system reads

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & \alpha = 1,2,3; \qquad i = 1,2,3 \\ \partial_t v_i - \partial_\alpha \left(\dfrac{\partial \sigma}{\partial F_{i\alpha}} + \dfrac{\partial \sigma}{\partial \Theta_{\beta j}} \dfrac{\partial F^*_{\beta j}}{\partial F_{i\alpha}} + \dfrac{\partial \sigma}{\partial \omega} \dfrac{\partial \det F}{\partial F_{i\alpha}} \right) = 0, & i = 1,2,3 \\ \partial_t \Theta_{\beta i} - \partial_\alpha \left(\dfrac{\partial F^*_{\beta i}}{\partial F_{j\alpha}} v_j \right) = 0, & \beta = 1,2,3; \qquad i = 1,2,3 \\ \partial_t \omega - \partial_\alpha \left(\dfrac{\partial \det F}{\partial F_{j\alpha}} v_j \right) = 0, & \end{cases} \tag{5.4.33}$$

and the entropy-entropy flux pair is

$$\theta = \tfrac{1}{2}|v|^2 + \sigma(F,\Theta,\omega), \tag{5.4.34}$$

$$\Pi_\alpha = -\left(\frac{\partial \sigma}{\partial F_{i\alpha}} + \frac{\partial \sigma}{\partial \Theta_{\beta j}} \frac{\partial F^*_{\beta j}}{\partial F_{i\alpha}} + \frac{\partial \sigma}{\partial \omega} \frac{\partial \det F}{\partial F_{i\alpha}} \right) v_i \,. \tag{5.4.35}$$

On the "manifold" $\Psi = W(U) = (F,v,F^*,\det F)^\top$, (5.4.33) reduces to the system (3.3.19) (with $b = 0$) together with the kinematic conservation laws (3.3.11), (3.3.12), while (θ, Π) reduces to the classical entropy-entropy flux pair recorded in Section 3.3.3.

In electrodynamics, for the Born-Infeld constitutive relations, where $U = (B,D)^\top$, $\Psi = (B,D,P)^\top$, the extended system reads

$$
\begin{cases}
\partial_t B + \operatorname{curl}[\theta^{-1}(D + B \wedge P)] = 0 \\[2ex]
\partial_t D - \operatorname{curl}[\theta^{-1}(B - D \wedge P)] = 0 \\[2ex]
\partial_t P - \operatorname{div}[\theta^{-1}(I + BB^\top + DD^\top - PP^\top)] = 0,
\end{cases}
\tag{5.4.36}
$$

and the entropy-entropy flux pair is

$$
\theta = (1 + |B|^2 + |D|^2 + |P|^2)^{\frac{1}{2}},
\tag{5.4.37}
$$

$$
\Pi = P - \theta^{-2}[P - D\lambda B - (D \cdot P)D - (B \cdot P)B].
\tag{5.4.38}
$$

Again, on the “manifold” $\Psi = W(U) = (B,D,D \wedge B)^\top$ (5.4.36) reduces to Maxwell’s equations (3.3.66) (with $J = 0$), together with the supplementary conservation law (3.3.74), while (θ, Π) reduces to the entropy-entropy flux pair (η, Q) recored in (3.3.73).

5.5 The Role of Damping and Relaxation

This section discuss the Cauchy problem

$$
\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) + P(U(x,t)) = 0, \qquad x \in \mathbb{R}^m, \;\; t > 0,
\tag{5.5.1}
$$

$$
U(x,0) = U_0(x), \qquad x \in \mathbb{R}^m,
\tag{5.5.2}
$$

for a homogeneous system of balance laws endowed with a convex entropy. As in the previous sections of this chapter, the flux G, source P, entropy η and associated entropy flux Q are smooth functions defined on a closed ball $\overline{\mathscr{B}_\rho}$ in $\mathbb{R}^n$, centered at the origin.

We assume $P(0) = 0$, so $U = 0$ is an equilibrium state for (5.5.1). Furthermore, as in (4.1.6), (4.1.7), we normalize the entropy by $\eta(0) = 0$, $Q(0) = 0$, $\mathrm{D}\eta(0) = 0$, and $\mathrm{D}Q(0) = 0$. The source is a lower order term in (5.5.1). Consequently, a straightforward adaptation of the analysis from Section 5.1 yields the following extension of Theorem 5.1.1:

5.5.1 Theorem. *Assume $A(U)$, defined by* (5.1.3), *is positive definite for any $U \in \overline{\mathscr{B}_\rho}$. Suppose the initial data U_0 lie in H_ℓ, for some $\ell > \frac{m}{2} + 1$, and take values in a ball $U \in \overline{\mathscr{B}_{\rho_0}}$, with radius $\rho_0 < \rho$. Then there exist $T_\infty \leq \infty$ and a unique continuously* *differentiable function U on $\mathbb{R}^m \times [0,T_\infty)$, taking values in $\overline{\mathscr{B}_\rho}$, which is a classical solution to the Cauchy problem* (5.5.1), (5.5.2) *on the time interval* $[0,T_\infty)$. *Furthermore,*

$$U(\cdot,t) \in \bigcap_{k=0}^{\ell} C^k([0,T_\infty);H_{\ell-k}). \tag{5.5.3}$$

The interval $[0,T_\infty)$ is maximal in that if $T_\infty < \infty$, then

$$\int_0^{T_\infty} |\!|\!|\nabla U(\cdot,t)|\!|\!|_{L^\infty} dt = \infty \tag{5.5.4}$$

and/or $\limsup_{t\to T_\infty} \|U(\cdot,t)\|_{L^\infty} = \rho$.

The aim here is to identity conditions on the source and the initial data that would render the solution to the Cauchy problem (5.5.1), (5.5.2) global in time, i.e., $T_\infty = \infty$. The simple example (4.2.2), discussed in Section 4.2, raises the expectation that this may be achieved when the source exerts a damping effect and the initial data take values close to the equilibrium state.

Classical solutions of (5.5.1) satisfy the extra balance law

$$\partial_t \eta(U(x,t)) + \sum_{\alpha=1}^m \partial_\alpha Q_\alpha(U(x,t)) + \mathrm{D}\eta(U(x,t))P(U(x,t)) = 0. \tag{5.5.5}$$

We shall call the source *dissipative* if it incurs a nonnegative entropy production:

$$\mathrm{D}\eta(U)P(U) \geq 0, \qquad U \in \overline{\mathscr{B}_\rho}. \tag{5.5.6}$$

We proceed to estimate the effect of the presence of a dissipative source. We fix initial data U_0 in $H_{\ell+1}$, for $\ell > \frac{m}{2}+1$, and consider the solution U to the Cauchy problem (5.5.1), (5.5.2), on the maximal time interval $[0,T_\infty)$.

The first step is to integrate (5.5.5) over $\mathbb{R}^m \times [0,t]$, $t \in [0,T_\infty)$, which yields

$$\int_{\mathbb{R}^m} \eta(U(x,t))dx + \int_0^t \int_{\mathbb{R}^m} \mathrm{D}\eta(U)P(U)dxd\tau = \int_{\mathbb{R}^m} \eta(U_0(x))dx. \tag{5.5.7}$$

Next we fix any multi-index r, of order $1 \leq |r| \leq \ell$, set $U_r = \partial^r U$, $U_{0r} = \partial^r U_0$ and apply ∂^r to (5.5.1) to get

$$\begin{aligned}\partial_t U_r + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(U)\partial_\alpha U_r + \mathrm{D}P(U)U_r &= \mathrm{D}P(U)U_r - \partial^r P(U)\\ &+ \sum_{\alpha=1}^m \{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\},\end{aligned} \tag{5.5.8}$$

which holds in H_0, for any $t \in [0,T_\infty)$.

We multiply (5.5.8) by $2U_r^\top A(U)$ and then integrate the resulting equation over $\mathbb{R}^m \times [0,t]$, $t \in [0,T_\infty)$. After an integration by parts, using (5.1.29), (5.1.30) for $\varepsilon = 0$, and (5.1.31), we deduce

$$
\begin{aligned}
(5.5.9)\qquad & \int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx + 2\int_0^t \int_{\mathbb{R}^m} U_r^\top A(0)\mathrm{D}P(0)U_r\,dxd\tau \\
&= \int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx \\
&+ \int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m U_r^\top[\partial_\alpha J_\alpha(U) - \mathrm{D}A(U)\partial_\alpha G_\alpha(U)]U_r\,dxd\tau \\
&+ \int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m 2U_r^\top A(U)\{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\}dxd\tau \\
&+ \int_0^t \int_{\mathbb{R}^m} 2U_r^\top A(U)[\mathrm{D}P(U)U_r - \partial^r P(U)]dxd\tau \\
&- \int_0^t \int_{\mathbb{R}^m} 2U_r^\top[A(U)\mathrm{D}P(U) - A(0)\mathrm{D}P(0)]U_r\,dxd\tau.
\end{aligned}
$$

We sum (5.5.9) over all r with $1 \le |r| \le \ell$, and combine the resulting equation with (5.5.7). We use that $A(U)$ is positive definite and estimate the right-hand side with the help of (5.1.27) and the similar bound

$$
(5.5.10)\qquad \|\mathrm{D}P(U)U_r - \partial^r P(U)\|_0 \le c\|\nabla U\|_{L^\infty}\|\nabla U\|_{\ell-2}\,.
$$

We thus end up with the estimate

$$
\begin{aligned}
(5.5.11)\qquad & \mu\|U(\cdot,t)\|_\ell^2 + 2\sum_{1\le|r|\le\ell}\int_0^t \int_{\mathbb{R}^m} U_r^\top A(0)\mathrm{D}P(0)U_r\,dxd\tau \\
&\le c\|U_0(\cdot)\|_\ell^2 + c\int_0^t [\|U\|_{L^\infty} + \|\nabla U\|_{L^\infty}]\|\nabla U\|_{\ell-1}^2 d\tau,
\end{aligned}
$$

where $\mu > 0$.

In the above estimate, it is the second term on the right-hand side that may be responsible for the growing, and eventual blowing up, of $\|U(\cdot,t)\|_\ell$. This, however, may be offset by the second term on the left-hand side, which manifests the damping action of the dissipative source. Indeed, this term is nonnegative since the matrix $A(0)\mathrm{D}P(0)$ is at least positive semidefinite, by virtue of (5.5.6). In fact, the damping definitely prevails when $\|U_0\|_\ell$ is small and the source is *dissipative definite* in the sense

$$
(5.5.12)\qquad \mathrm{D}\eta(U)P(U) \ge a|U|^2, \qquad U \in \overline{\mathscr{B}_\rho},
$$

with $a > 0$. Indeed, (5.5.12) implies that the matrix $A(0)\mathrm{D}P(0)$ is positive definite, and hence

$$\text{(5.5.13)} \qquad \sum_{1\le|r|\le\ell}\int_0^t\int_{\mathbb{R}^m} U_r^\top A(0)\mathrm{D}P(0)U_r dxd\tau \ge a\int_0^t |||\nabla U|||_{\ell-1}^2 d\tau.$$

It is now clear that $\|U(\cdot,\tau)\|_{L^\infty}$ and $|||\nabla U(\cdot,\tau)|||_{L^\infty}$ small on the interval $[0,t]$ imply $\|U(\cdot,t)\|_\ell \le c\|U_0(\cdot)\|_\ell$, and this last inequality, with $\|U_0(\cdot)\|_\ell$ small, implies in turn that $\|U(\cdot,t)\|_{L^\infty}$ and $|||\nabla U(\cdot,\tau)|||_{L^\infty}$ are small. It thus follows that when the source is dissipative definite (5.5.12), $\|U_0(\cdot)\|_\ell$ small renders $\|U(\cdot,t)\|_\ell$, and thereby $|||\nabla U(\cdot,\tau)|||_{L^\infty}$, uniformly bounded on $[0,T_\infty)$, whence $T_\infty=\infty$, by virtue of (5.5.4).

Recall that the identity (5.5.9), and thereby the estimate (5.5.11), were derived under the assumption $U_0\in H_{\ell+1}$. Nevertheless, the size of $\|U_0\|_{\ell+1}$ does not enter in the final estimate $\|U(\cdot,t)\|_\ell\le c\|U_0(\cdot)\|_\ell$. Therefore, by completion of $H_{\ell+1}$ in H_ℓ and on account of the weak lower semicontinuity property of norms, we conclude that under the assumption (5.5.12), we have $T_\infty=\infty$ even when U_0 is merely in H_ℓ, with $\|U_0\|_\ell$ sufficiently small.

For the balance laws arising in the applications, dissipative sources are ubiquitous, but it is only on rare occasions that they may turn out to be dissipative definite. Indeed, systems (5.5.1) encountered in physics commonly result from the coupling of conservation laws with balance laws, and thereby appear in the special form

$$\text{(5.5.14)} \qquad \begin{cases} \partial_t V + \sum_{\alpha=1}^m \partial_\alpha F_\alpha(V,W) = 0 \\[2ex] \partial_t W + \sum_{\alpha=1}^m \partial_\alpha H_\alpha(V,W) + \Pi(V,W) = 0. \end{cases}$$

An illustrative example is provided by the system governing isentropic gas flow through a porous medium, namely (3.3.6) with body force $-v$:

$$\text{(5.5.15)} \qquad \begin{cases} \partial_t\rho + \operatorname{div}(\rho v^\top) = 0 \\[2ex] \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) + \rho v = 0. \end{cases}$$

It is now clear that a source in a system of the form (5.5.14) can only be partially dissipative and at best it may satisfy

$$\text{(5.5.16)} \qquad \mathrm{D}\eta(U)P(U) \ge a|P(U)|^2, \qquad U\in\overline{\mathscr{B}_\rho},$$

with $a>0$. When (5.5.16) holds, we shall term the source *dissipative semidefinite*.

When (5.5.16) replaces (5.5.12), one may no longer rely on (5.5.11) alone, for bounding $\|U(\cdot,t)\|_\ell$ on $[0,T_\infty)$, but needs supplementary estimates, manifesting the synergy between source and flux. Such estimates are in force on condition that the system

$$\text{(5.5.17)} \qquad \partial_t V + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(0)\partial_\alpha V + \mathrm{D}P(0)V = 0,$$

resulting from linearization of (5.5.1) about the equilibrium state $U=0$, does not admit traveling wave front solutions

$$V(x,t) = \varphi(x \cdot \nu - \lambda_i(\nu;0)t)R_j(\nu;0), \tag{5.5.18}$$

which are not attenuated by entropy dissipation. Equivalently, the above requirement is expressed by the *Kawashima condition*

$$\mathrm{D}P(0)R_i(\nu;0) \neq 0, \qquad \nu \in \mathbb{S}^{m-1}, \quad i = 1,\dots,n. \tag{5.5.19}$$

As shown in the references cited in Section 5.7, (5.5.19) implies that, for any $\nu \in \mathbb{S}^{m-1}$, there exists a $n \times n$ skew symmetric matrix $K(\nu)$ that renders the matrix

$$M(\nu) = K(\nu)\Lambda(\nu;0) + A(0)\mathrm{D}P(0) \tag{5.5.20}$$

positive definite. This has the following important implications:

5.5.2 Lemma. *Assume that the source is dissipative semidefinite* (5.5.16), *and the matrices $M(\nu)$, defined by* (5.5.20), *with $K(\nu)$ skew-symmetric, are positive definite for all $\nu \in \mathbb{S}^{m-1}$. Let $V \in C([0,T];H_1)$ be a solution to the linear system*

$$\partial_t V + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(0)\partial_\alpha V + \kappa\, \mathrm{D}P(0)V = Z, \tag{5.5.21}$$

on the time interval $[0,T]$, for some $Z \in L^2([0,T];H_0)$. Then, for any $t \in [0,T]$,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^m} |\nabla V|^2 dx d\tau \leq c \int_0^t \int_{\mathbb{R}^m} \nabla V^\top A(0)\mathrm{D}P(0)\nabla V dx d\tau \\ + c|\kappa| \int_0^t \int_{\mathbb{R}^m} V^\top A(0)\mathrm{D}P(0)V dx d\tau \\ + c\|V(\cdot,t)\|_1^2 + c\|V(\cdot,0)\|_1^2 + c\int_0^t \int_{\mathbb{R}^m} |Z|^2 dx d\tau.\end{aligned} \tag{5.5.22}$$

Proof. We introduce the Fourier transform $V(x,t) \mapsto \hat{V}(\xi,t)$ of V with respect to the spatial variable. Applying the Fourier transform to the system (5.5.21) and setting $\xi = |\xi|\nu$, with $\nu \in \mathbb{S}^{m-1}$, we obtain the equation

$$\partial_t \hat{V}(\xi,t) + i|\xi|\Lambda(\nu;0)\hat{V}(\xi,t) + \kappa \mathrm{D}P(0)\hat{V}(\xi,t) = \hat{Z}(\xi,t). \tag{5.5.23}$$

We now multiply (5.5.23) by $i|\xi|\hat{V}^*(\xi,t)K(\nu)$. Since $iK(\nu)$ is Hermitian, and upon using (5.5.20), we deduce

$$\begin{aligned}\frac{1}{2}\partial_t[i|\xi|\hat{V}^*K(\nu)\hat{V}] - |\xi|^2\hat{V}^*M(\nu)\hat{V} + |\xi|^2\hat{V}^*A(0)\mathrm{D}P(0)\hat{V} \\ -i\kappa|\xi|\hat{V}^*K(\nu)\mathrm{D}P(0)\hat{V} = i|\xi|\hat{V}^*K(\nu)\hat{Z}.\end{aligned} \tag{5.5.24}$$

By (5.5.16), both matrices $A(0)\mathrm{D}P(0)$ and $A(0)\mathrm{D}P(0) - a\mathrm{D}P(0)^\top\mathrm{D}P(0)$ are positive semidefinite. It follows that the eigenspace of the symmetric part of $A(0)\mathrm{D}P(0)$, associated with the zero eigenvalue, coincides with the kernel of the matrix $\mathrm{D}P(0)$. Hence

$$|\mathrm{D}P(0)\hat{V}|^2 \leq c\mathrm{Re}[\hat{V}^*A(0)\mathrm{D}P(0)\hat{V}]. \tag{5.5.25}$$

We now integrate the real part of (5.5.24) over $\mathbb{R}^m \times [0,t]$. Upon using that $M(\nu)$ is positive definite, together with Parseval's relation, (5.5.25) and the Cauchy-Schwarz inequality, we arrive at (5.5.22). This completes the proof.

We have now laid the groundwork for establishing the existence of global classical solution to the Cauchy problem (5.5.1), (5.5.2), under the assumption that the source is dissipative semidefinite and it satisfies the Kawashima condition (5.5.19).

5.5.3 Theorem. *Assume $A(U)$ is positive definite, the source is dissipative semidefinite* (5.5.16), and the matrices $M(\nu)$, *defined by* (5.5.20), *with $K(\nu)$ skew-symmetric, are positive definite, for all $\nu \in \mathbb{S}^{m-1}$. When the initial data U_0, taking values in a ball $\overline{\mathscr{B}_{\rho_0}}$*, with radius $\rho_0 < \rho$, *lie in H_ℓ, for some $\ell > \frac{m}{2}+1$, and $\|U_0\|_\ell$ is sufficiently small, then the Cauchy problem* (5.5.1), (5.5.2) *admits a global classical solution U, on the time interval* $[0,\infty)$.

Proof. We write (5.5.1) as

$$\partial_t U + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(0)\partial_\alpha U = -\sum_{\alpha=1}^m [\mathrm{D}G_\alpha(U) - \mathrm{D}G_\alpha(0)]\partial_\alpha U - P(U), \tag{5.5.26}$$

which is in the form (5.5.21), for $\kappa = 0$. By virtue of (5.5.5) and (5.5.16),

$$\int_0^t \int_{\mathbb{R}^m} |P(U)|^2 dx d\tau \leq c \int_{\mathbb{R}^m} |U_0(x)|^2 dx. \tag{5.5.27}$$

Therefore, by (5.5.22), with $\kappa = 0$,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^m} |\nabla U|^2 dx d\tau &\leq c \int_0^t \int_{\mathbb{R}^m} \nabla U^\top A(0)\mathrm{D}P(0)\nabla U dx d\tau \\ &\quad + c\|U(\cdot,t)\|_1^2 + c\|U_0(\cdot)\|_1^2 + c\int_0^t \|U\|_{L^\infty} |||\nabla U|||_0^2 d\tau.\end{aligned} \tag{5.5.28}$$

Next we fix any multi-index r of order $1 \leq |r| \leq \ell - 1$ and write (5.5.8) as

$$\begin{aligned}\partial_t U_r + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(0)\partial_\alpha U_r + \mathrm{D}P(0)U_r = &-\sum_{\alpha=1}^m [\mathrm{D}G_\alpha(U) - \mathrm{D}G_\alpha(0)]\partial_\alpha U_r \\ &-[\mathrm{D}P(U) - \mathrm{D}P(0)]U_r + \mathrm{D}P(U)U_r - \partial^r P(U) \\ &+\sum_{\alpha=1}^m \{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\},\end{aligned} \tag{5.5.29}$$

in the form (5.5.21), with $\kappa = 1$. Therefore, on account of (5.5.22), (5.1.27) and (5.5.10),

$$\int_0^t \int_{\mathbb{R}^m} |\nabla U_r|^2 dxd\tau \leq c \int_0^t \int_{\mathbb{R}^m} \nabla U_r^\top A(0) \mathrm{D}P(0) \nabla U_r \, dxd\tau \tag{5.5.30}$$
$$+c \int_0^t \int_{\mathbb{R}^m} U_r^\top A(0) \mathrm{D}P(0) U_r \, dxd\tau + c\|U(\cdot,t)\|_\ell^2 + c\|U_0(\cdot)\|_\ell^2$$
$$+c \int_0^t [\|U\|_{L^\infty}^2 + |||\nabla U|||_{L^\infty}^2] |||\nabla U|||_{\ell-1}^2 d\tau.$$

We sum (5.5.30) over all r, with $1 \leq |r| \leq \ell - 1$, and combine the result with (5.5.28) to get

$$\int_0^t |||\nabla U|||_{\ell-1}^2 d\tau \leq c \sum_{1 \leq |r| \leq \ell} \int_0^t \int_{\mathbb{R}^m} U_r^\top A(0) \mathrm{D}P(0) U_r \, dxd\tau \tag{5.5.31}$$
$$+c\|U(\cdot,t)\|_\ell^2 + c\|U_0(\cdot)\|_\ell^2 + c \int_0^t [\|U\|_{L^\infty} + |||\nabla U|||_{L^\infty}] |||\nabla U|||_{\ell-1}^2 d\tau.$$

Next we multiply (5.5.11) by a large positive number and add the resulting inequality to (5.5.31). This yields the following estimate:

$$\|U(\cdot,t)\|_\ell^2 + \int_0^t |||\nabla U|||_{\ell-1}^2 d\tau \tag{5.5.32}$$
$$\leq c\|U_0(\cdot)\|_\ell^2 + c \int_0^t [\|U\|_{L^\infty} + \|U\|_{L^\infty}^2 + |||\nabla U|||_{L^\infty} + |||\nabla U|||_{L^\infty}^2] |||\nabla U|||_{\ell-1}^2 d\tau.$$

We are now back in the situation we were before: As long as $\|U(\cdot,\tau)\|_{L^\infty}$ and $|||\nabla U(\cdot,\tau)|||_{L^\infty}$ stay small for $\tau \in [0,t]$, (5.5.32) yields $\|U(\cdot,t)\|_\ell \leq c\|U_0(\cdot)\|_\ell$. In return, this last inequality, with $\|U_0(\cdot)\|_\ell$ small, keeps $\|U(\cdot,t)\|_\ell$, and thereby also $\|U(\cdot,t)\|_{L^\infty}$ and $|||\nabla U(\cdot,t)|||_{L^\infty}$, small. We thus conclude that if $\|U_0(\cdot)\|_\ell$ is sufficiently small, then $|||\nabla U(\cdot,t)|||_{L^\infty}$ stays bounded, uniformly on $[0,T_\infty)$, in which case, by (5.5.4), $T_\infty = \infty$ and the classical solution U to (5.5.1), (5.5.2) is global. This completes the proof.

In the literature cited in Section 5.7, the reader will find alternative, often technical, treatments of the Cauchy problem, under slightly different hypotheses and/or in different function spaces. There is also extensive bibliography on the long time behavior of classical solutions. Clearly, (5.5.7) indicates that solutions must tend to "equilibrium", where the entropy production vanishes. The asymptotic behavior of solutions has been established either by use of "energy" type estimates in time-weighted Sobolev spaces, or by treating (5.5.1) as a perturbation of its linearized form (5.5.17). The large time behavior of solutions of the latter system, in various L^p spaces, can be determined quite precisely with the help of its Green function. Because of the synergy between dissipation and dispersion, the rate of decay of solutions to equilibrium in L^p depends on the value of p. Out of a host of very technical theorems, we record below one of the simplest:

5.5.4 Theorem. *Consider the Cauchy problem* (5.5.1), (5.5.2) *in* $\mathbb{R}^m$, *for* $m \geq 2$, *assuming that the system has the following properties:*
(a) *An entropy* $\eta(U)$ *exists and* $A(U)$ *is positive definite.*
(b) *If* $\mathscr{E}$ *denotes the set of zeros of the source* P *and* $\mathscr{M}$ *is the orthogonal complement of the range of* P, *then* $U \in \mathscr{E}$ *if and only if* $\mathrm{D}\eta(U) \in \mathscr{M}$.
(c) *For* $U \in \mathscr{E}$, $\mathrm{D}\eta(U)A^{-1}(U)$ *is a symmetric positive semidefinite matrix whose kernel is* $\mathscr{M}$.
Under the above conditions, for any $U_0 \in H_\ell$, $\ell > \frac{m}{2} + 1$, *with* $\|U_0\|_\ell$ *sufficiently small, there exists a global classical solution* U *to* (5.5.1), (5.5.2) *on* $[0,\infty)$ *and*

$$\|\partial^r U(\cdot,t)\|_0 \leq c\|U_0(\cdot)\|_\ell (1+t)^{-\frac{|r|}{2}}, \qquad 0 \leq |r| \leq \ell. \tag{5.5.33}$$

If, in addition, $U_0 \in L^1$, *with* $\|U_0\|_{L^1}$ *sufficiently small, then*

$$\|\partial^r U(\cdot,t)\|_0 \leq c[\|U_0\|_{L^1} + \|U_0\|_\ell](1+t)^{-\frac{|r|}{2}-\frac{m}{4}}, \qquad 0 \leq |r| \leq \ell - 1. \tag{5.5.34}$$

The stretching of the space-time coordinates $(x,t) \mapsto (\mu x, \mu t)$, $\mu > 0$, transforms the systems (5.5.1) into

$$\partial_t U(x,t) + \sum_{\alpha=1}^m \partial_\alpha G_\alpha(U(x,t)) + \frac{1}{\mu} P(U(x,t)) = 0, \tag{5.5.35}$$

so that there is an intimate relation between the long time behavior of solutions to (5.5.1) and the asymptotic behavior of solutions to (5.5.35), as $\mu \to 0$.

Relaxation phenomena are often governed by systems in the form (5.5.35), with μ being the *relaxation parameter*. The special structure of such systems has been variously abstracted in the literature. Here we outline a simple popular framework, related to (5.5.14), which captures numerous application.

We thus consider systems consisting of k conservation laws coupled with $n-k$ balance laws:

$$\begin{cases} \partial_t V + \sum_{\alpha=1}^m \partial_\alpha F_\alpha(V,W) = 0 \\ \\ \partial_t W + \sum_{\alpha=1}^m \partial_\alpha H_\alpha(V,W) + \frac{1}{\mu}\Pi(V,W) = 0, \end{cases} \tag{5.5.36}$$

equipped with a uniformly convex entropy $\eta(V,W)$ and associated entropy flux $Q(V,W)$. Since the entropy is convex, (5.5.36) is hyperbolic, with characteristic speeds $\lambda_1(\nu;V,W) \leq \cdots \leq \lambda_n(\nu;V,W)$, for any $\nu \in S^{m-1}$.

We assume that the source is dissipative semidefinite, so that

$$\mathrm{D}_W \eta(V,W)\Pi(V,W) \geq a|\Pi(V,W)|^2, \tag{5.5.37}$$

with $a > 0$. We also assume that the $(n-k) \times (n-k)$ matrix $\mathrm{D}_W\Pi(0,0)$ has rank $n-k$, whence, for $|V| < \delta$, the equation $\Pi(V,W) = 0$ yields $W = \Phi(V)$, depicting the $(n-k)$-dimensional *local equilibrium manifold*, embedded in $\mathbb{R}^n$.

The natural conjecture is that, as the relaxation parameter tends to zero, the stiff term in the system (5.5.36) will force the state (V,W) to relax to the equilibrium manifold $(V,\Phi(V))$, on which V will be governed by the *relaxed system*

$$\partial_t V + \sum_{\alpha=1}^{m} \partial_\alpha \hat{G}_\alpha(V) = 0, \tag{5.5.38}$$

where $\hat{G}(V) = G(V,\Phi(V))$. Indeed, this has been verified in various settings, as reported in the references cited in Section 5.7. It should be noted, however, that the convergence of (V,W) to $(V,\Phi(V))$, as $\mu \to 0$, cannot be uniform in time unless the initial data (V_0,W_0) for (5.5.36) happen to lie on the equilibrium manifold, $W_0 = \Phi(V_0)$. In the opposite case, a boundary layer must form across $t = 0$, joining the initial data to the equilibrium manifold.

The following proposition summarizes the special structure that the relaxed system (5.5.38) inherits from its parent system (5.5.36).

5.5.5 Theorem. *The relaxed system* (5.5.38) *is equipped with the entropy function* $\hat{\eta}(V) = \eta(V,\Phi(V))$, *with associated entropy flux* $\hat{Q}(V) = Q(V,\Phi(V))$. *Furthermore,* $\hat{\eta}(V)$ *is uniformly convex, so that* (5.5.38) *is hyperbolic, with characteristic speeds* $\hat{\lambda}_1(V;\nu),\dots,\hat{\lambda}_k(V;\nu)$. *Finally, the so-called subcharacteristic condition*

$$\lambda_1(\nu;V,\Phi(V)) \le \hat{\lambda}_1(\nu;V) \le \cdots \le \hat{\lambda}_k(\nu;V) \le \lambda_n(\nu;V,\Phi(V)) \tag{5.5.39}$$

holds for all $\nu \in S^{m-1}$ *and* $|V| < \delta$.

Proof. By (5.5.37), the entropy production is minimized on the equilibrium manifold, and hence

$$\mathrm{D}_W \eta(V,\Phi(V)) = 0. \tag{5.5.40}$$

Therefore, for any fixed V, with $|V| < \delta$, and $\alpha = 1,\dots,m$, (3.2.3) and (5.5.40) imply

$$\begin{aligned} \mathrm{D}_V \hat{Q}_\alpha &= \mathrm{D}_V Q_\alpha + (\mathrm{D}_W Q_\alpha)\mathrm{D}_V \Phi \\ &= (\mathrm{D}_V \eta)\mathrm{D}_V G_\alpha + (\mathrm{D}_V \eta)(\mathrm{D}_W G_\alpha)\mathrm{D}_V \Phi = (\mathrm{D}_V \hat{\eta})\mathrm{D}_V \hat{G}_\alpha\,, \end{aligned} \tag{5.5.41}$$

with $\mathrm{D}_V Q_\alpha$, $\mathrm{D}_W Q_\alpha$, $\mathrm{D}_V G_\alpha$, $\mathrm{D}_W G_\alpha$ and $\mathrm{D}_V \eta$ all evaluated at $(V,\Phi(V))$. We conclude that $(\hat{\eta},\hat{Q})$ is an entropy-entropy flux pair for (5.5.38).

By virtue of (5.5.40), we obtain

$$\mathrm{D}_V^2 \hat{\eta} = \mathrm{D}_V^2 \eta + (\mathrm{D}_{VW}\eta)\mathrm{D}_V \Phi, \tag{5.5.42}$$

$$\mathrm{D}_{WV}\eta + (\mathrm{D}_W^2 \eta)\mathrm{D}_V \Phi = 0, \tag{5.5.43}$$

with $\mathrm{D}_V^2\eta$, $\mathrm{D}_{VW}\eta$, $\mathrm{D}_{WV}\eta$ and $\mathrm{D}_W^2\eta$ evaluated at $(V,\Phi(V))$.

For $\xi \in \mathbb{R}^k$ and $\zeta \in \mathbb{R}^{n-k}$, we set

$$B(V,W;\xi,\zeta)=\xi^\top(\mathrm{D}_V^2\eta)\xi+\xi^\top(\mathrm{D}_{VW}\eta)\zeta+\zeta^\top(\mathrm{D}_{WV}\eta)\xi+\zeta^\top(\mathrm{D}_W^2\eta)\zeta \tag{5.5.44}$$

$$\begin{aligned} I_\alpha(V,W;\xi,\zeta)=\xi^\top&[(\mathrm{D}_V^2\eta)(\mathrm{D}_V F_\alpha)+(\mathrm{D}_{VW}\eta)(\mathrm{D}_V H_\alpha)]\xi\\ +\xi^\top&[(\mathrm{D}_V^2\eta)(\mathrm{D}_W F_\alpha)+(\mathrm{D}_{VW}\eta)(\mathrm{D}_W H_\alpha)]\zeta\\ +\zeta^\top&[(\mathrm{D}_{WV}\eta)(\mathrm{D}_V F_\alpha)+(\mathrm{D}_W^2\eta)(\mathrm{D}_V H_\alpha)]\xi\\ +\zeta^\top&[(\mathrm{D}_{WV}\eta)(\mathrm{D}_W F_\alpha)+(\mathrm{D}_W^2\eta)(\mathrm{D}_W H_\alpha)]\zeta, \end{aligned} \tag{5.5.45}$$

$$\hat{B}(V;\xi)=\xi^\top(\mathrm{D}_V^2\hat{\eta})\xi, \tag{5.5.46}$$

$$\hat{I}_\alpha(V;\xi)=\xi^\top(\mathrm{D}_V^2\hat{\eta})(\mathrm{D}_V\hat{G}_\alpha)\xi. \tag{5.5.47}$$

Upon using (5.5.42) and (5.5.43),

$$\hat{B}(V;\xi)=B(V,\Phi(V);\xi,(\mathrm{D}_V\Phi)\xi), \tag{5.5.48}$$

$$\hat{I}_\alpha(V;\xi)=I_\alpha(V,\Phi(V);\xi,(\mathrm{D}_V\Phi)\xi). \tag{5.5.49}$$

Since η is uniformly convex, (5.5.48) implies that $\hat{\eta}$ is also uniformly convex.

For any $\nu\in\mathbb{S}^{m-1}$, $\lambda_1(\nu;V,W)$ and $\lambda_n(\nu;V,W)$ are the minimum and the maximum of the Rayleigh quotient

$$\frac{\sum_{\alpha=1}^m \nu_\alpha I_\alpha(V,W;\xi,\zeta)}{B(V,W;\xi,\zeta)}, \tag{5.5.50}$$

over all $\xi\in\mathbb{R}^k\backslash\{0\}$ and $\zeta\in\mathbb{R}^{n-k}$. Similarly, $\hat{\lambda}_1(\nu;V)$ and $\hat{\lambda}_k(\nu;V)$ are the minimum and the maximum of the Rayleigh quotient

$$\frac{\sum_{\alpha=1}^m \nu_\alpha \hat{I}_\alpha(V;\xi)}{\hat{B}(V;\xi)}, \tag{5.5.51}$$

over all $\xi\in\mathbb{R}^k\backslash\{0\}$. Thus, the subcharacteristic condition (5.5.39) follows from (5.5.48) and (5.5.49). The proof is complete.

For illustration, consider the system

$$\begin{cases} \partial_t u(x,t)+\partial_x v(x,t)=0\\ \partial_t v(x,t)+\partial_x p(u(x,t))+\frac{1}{\mu}[v(x,t)-f(u(x,t))], \end{cases} \tag{5.5.52}$$

in one spatial dimension, which has served as a paradigm in the literature. Assuming $p'(u)=a^2(u)$, with $a(u)>0$, (5.5.52) is strictly hyperbolic, with characteristic speeds $\lambda_1=-a(u)$, $\lambda_2=a(u)$. The local equilibrium manifold is the curve $v=f(u)$, embedded in the u-v plane, and the relaxed system is the scalar conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \tag{5.5.53}$$

with characteristic speed $\hat{\lambda} = f'(u)$. As noted above, the subcharacteristic condition

$$-a(u) \le f'(u) \le a(u) \tag{5.5.54}$$

is a necessary prerequisite for the existence of a convex entropy $\eta(u,v)$ for the system (5.5.52) that renders the source dissipative. We will return to the system (5.5.52) in Chapters XVI and XVII, where we shall see, in particular, that (5.5.54) in strict inequality form is also sufficient for the existence of a convex entropy.

In Section 4.6, we encountered an alternative mechanism inducing entropy production, namely viscosity. The relevant systems are in the form (4.6.1) and the entropy production is encoded in the second term on the right-hand side of (4.6.4). In analogy with the terminology introduced above in the context of systems with source, we shall call the viscosity term *dissipative* when the quadratic form associated with $\mathrm{D}^2\eta B_{\alpha\beta}$ is positive semidefinite, and in particular *dissipative definite* when the quadratic form is positive definite or *dissipative semidefinite* when (4.6.5) holds with $a > 0$. Dissipative definite viscosity renders the system (4.6.1) parabolic, in which case the Cauchy problem is well-posed and admits very smooth solutions. However, viscosity terms commonly encountered in systems arising in continuum physics are only dissipative semidefinite. Indeed, such systems often appear in the form

$$\begin{cases} \partial_t V + \sum_{\alpha=1}^m \partial_\alpha F_\alpha(V,W) = 0 \\ \\ \partial_t W + \sum_{\alpha=1}^m \partial_\alpha H_\alpha(V,W) = \mu \sum_{\alpha,\beta=1}^m \partial_\alpha [B_{\alpha\beta}(V,W)\partial_\beta W], \end{cases} \tag{5.5.55}$$

to be compared with (5.5.36). For an example, see (4.6.2).

The existence and long time behavior of solutions to the Cauchy problem for the systems with semidefinite dissipative viscosity has been investigated extensively, in various settings. The emerging theory closely parallels, in scope, methodology and conclusions, the theory of the Cauchy problems for systems of balance laws with dissipative semidefinite source, outlined above. In particular, for well-posedness of the Cauchy problem, when the viscosity is merely dissipative semidefinite, one needs the supplementary condition that the system

$$\partial_t V + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(0)\partial_\alpha V = \mu \sum_{\alpha,\beta=1}^m B_{\alpha\beta}(0)\partial_\alpha\partial_\beta V \tag{5.5.56}$$

resulting from linearization of (4.6.1) about the equilibrium state $U = 0$ does not admit undamped traveling wave front solutions (5.5.18). This requirement is met when the *Kawashima condition*

$$\sum_{\alpha,\beta=1}^m \nu_\alpha \nu_\beta B_{\alpha\beta}(0) R_i(\nu;0) \neq 0, \qquad \nu \in \mathbb{S}^{m-1}, \quad i = 1,\dots,n \tag{5.5.57}$$

holds. This is the counterpart of the Kawashima condition (5.5.19), for systems of balance laws with dissipative semidefinite source.

An intimate relation between viscous relaxation and relaxation induced by a stiff source emerges if one writes a formal expansion of solutions to (5.5.36) in powers of the (small) relaxation parameter μ. At the zero degree level, μ^0, this yields the relaxed hyperbolic system (5.5.38). At the next level, μ^1, the result is a system with viscosity:

$$\partial_t V + \sum_{\alpha=1}^{m} \partial_\alpha \hat{G}(V) = \mu \sum_{\alpha,\beta=1}^{m} \partial_\alpha [\hat{B}_{\alpha\beta}(V)\partial_\beta V]. \tag{5.5.58}$$

Furthermore, the viscosity is dissipative with respect to the entropy $\hat{\eta}(V)$ of (5.5.38). The subcharacteristic condition (5.5.39) is an alternative, though related, manifestation of stability.

In order to avoid the cumbersome calculation for determining the coefficients $\hat{B}_{\alpha\beta}$ in (5.5.58), for general systems, let us illustrate the above in the context of the simple system (5.5.52). We have seen already that the scalar conservation law (5.5.53) is the relaxed system. In order to get to the next level, of degree μ, we substitute $v = f(u) + \mu w$ into (5.5.52), eliminate $\partial_t u$ with the help of $(5.5.52)_1$ and then drop all terms of order μ, which yields

$$w = [f'(u)^2 - a(u)^2]\partial_x u. \tag{5.5.59}$$

Finally, we substitute $v = f(u) + \mu w$ into $(5.5.52)_1$ to get

$$\partial_t u + \partial_x f(u) = \mu \partial_x \{[a(u)^2 - f'(u)^2]\partial_x u\}, \tag{5.5.60}$$

which is of the form (5.5.56). The relation between the subcharacteristic condition (5.5.54) and disspativeness of viscosity in (5.5.60) is now quite clear.

In continuum physics, one encounters a host of evolutionary systems with the feature that wave amplification induced by nonlinear advection cohabits and competes with some kind of dissipation; and the former is in control far from equilibrium, while the latter prevails in the vicinity of equilibrium, securing the existence of smooth solutions in the large. Such systems are generally treated by methods akin to those employed in this section, namely "energy" type estimates that bring out the balance between amplification and damping. This subject, which already commands a large body of literature, lies beyond the scope of the present book. Nevertheless, in order to give a taste of the wide diversity of systems with such features, a few representative examples will be recorded below, and a small sample of relevant references will be listed in Section 5.7.

We begin with the so-called Euler-Poisson system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) = a\rho \operatorname{grad} \psi \\ \Delta \psi = b(\rho - \bar{\rho}). \end{cases} \tag{5.5.61}$$

For $a = -1$, $b = 4\pi G$ and $\bar{\rho} = 0$, (5.5.61) governs the flow of a gas in the gravitational field generated by its own mass. This is the "attractive" case. In the opposite, "repulsive" case, where both a and b are positive constants, this system models the movement of electrons in a plasma. In that connection, the aggregate of the electrons is regarded as an elastic fluid with density ρ and pressure $p(\rho)$, flowing with velocity v; while the much heavier ions are assumed stationary, merely providing a uniform background of positive charge, proportional to $\bar{\rho}$. The combined charge of electrons and ions, which is proportional to $\rho - \bar{\rho}$, generates the electrostatic potential ψ, and thereby the electric field $\operatorname{grad}\psi$ that sets the electrons in motion. As in (5.5.15), we are dealing here with the hyperbolic system of the Euler equations, with a source induced by some feedback mechanism, which derives from the Poisson equation $(5.5.61)_3$ and is dissipative at least when the flow of electrons is irrotational, $\operatorname{curl} v = 0$. Recall from Section 3.3.6 that flows starting out irrotational stay irrotational for as long as they are smooth. It has been shown that sufficiently smooth, irrotational Cauchy data, close to equilibrium $\rho = \bar{\rho}$, $v = 0$, $\psi = 0$, generate globally defined smooth solutions. On the other hand, solutions starting out far from equilibrium generally develop singularities in a finite time.

The situation is similar for the system

$$(5.5.62) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\[2ex] \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) + \mu^{-1}\rho v = a\rho \operatorname{grad}\psi \\[2ex] \Delta\psi = b(\rho - \bar{\rho}), \end{cases}$$

associated with the hydrodynamic model of semiconductors. Here $\mu > 0$ is a relaxation parameter. Notice that (5.5.62) combines the dissipative mechanisms encountered in (5.5.15) and (5.5.61).

The balance laws for continuous media with internal friction, such as viscosity and heat conductivity, yield systems exhibiting similar behavior. The reason is that one may trace the lineage of these media back to elasticity, and hence, even though the resulting systems are not hyperbolic, they inherit features of hyperbolicity, giving rise to a destabilizing wave amplification mechanism that competes with the damping induced by the internal friction.

A first example is the system (4.6.2), which governs the flow of heat conducting thermoviscoelastic fluids with Newtonian viscosity. Internal friction manifests itself on the right-hand side of the second and the third equation, while the first equation retains its hyperbolic character.

Still another example with similar features is the system

$$(5.5.63) \qquad \begin{cases} \partial_t u - \partial_x v = 0 \\[2ex] \partial_t v - \partial_x \sigma(u,\theta) = 0 \\[2ex] \partial_t[\varepsilon(u,\theta) + \frac{1}{2}v^2] - \partial_x[\sigma(u,\theta)v] = \partial_x q(u,\theta,\partial_x\theta), \end{cases}$$

which governs rectilinear motion, in Lagrangian coordinates, of a heat-conducting thermoelastic medium. Here u is the strain (deformation gradient), v is the velocity,

θ is the (absolute) temperature, σ is the stress, ε is the internal energy, q is the heat flux, and the reference density is taken to be one. For compliance with (2.5.28) and (2.5.29), the material response functions ε, σ and q must satisfy $\varepsilon_u = \sigma - \theta\sigma_\theta$ and $q(u,\theta,g)g \geq 0$. These should be supplemented with the natural assumptions $\sigma_u > 0$, $\varepsilon_\theta > 0$ and $q_g > 0$. Here internal friction is provided by thermal diffusion.

Internal friction of yet another nature, but with similar effects, is induced by fading memory, encountered in viscoelastic continuous media in which the stress σ at the particle x and time t is no longer solely determined, as in elastic materials, by the deformation gradient at (x,t), but also depends on the past history of the deformation gradient at x. The balance laws are then expressed by functional-partial differential equations. A simple, one-dimensional model system that captures the damping effect of memory reads

$$(5.5.64) \qquad \begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \\ \partial_t v(x,t) + \partial_x p(u(x,t)) + \displaystyle\int_{-\infty}^{t} k'(t-\tau)\partial_x q(u(x,\tau))d\tau = 0, \end{cases}$$

where k is a smooth integrable relaxation kernel on $[0,\infty)$, with $k(\tau) > 0$, $k'(\tau) < 0$ and $k''(\tau) \geq 0$, for $\tau \in [0,\infty)$, and $p'(u) > k(0)q'(u) > 0$. Notice that (5.5.64) is intimately related to (5.5.52), as the latter system, for $f \equiv 0$, may be rewritten in the form

$$(5.5.65) \qquad \begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \\ \partial_t v(x,t) + \partial_x p(u(x,t)) + \displaystyle\int_{-\infty}^{t} [\exp(-\frac{t-\tau}{\mu})]'\partial_x p(u(x,\tau))d\tau = 0. \end{cases}$$

The above systems, (5.5.63) and (5.5.64), share the property that smooth initial data near equilibrium generate globally defined smooth solutions, while smooth solutions starting out from "large" initial values generally blow up in finite time. See the relevant references in Section 5.7.

An alternative decay mechanism acting on the systems of balance laws of continuum physics is dispersion. It is particularly effective when the dimension of the space is large and solutions stay close to equilibrium. As we saw in Section 4.8, in systems that are fully nonlinear, such as the Euler equations, dispersion may delay but not prevent the breaking of waves. However, in systems with gentler nonlinearity, satisfying the so-called *null condition*, dispersion renders the existence of globally defined smooth solutions to the Cauchy problem, with initial data close to equilibrium. As a typical example, consider the system (3.3.19) of equations of isentropic elastodynamics. For convenience, assume that the reference space coincides with the physical space, and that the reference configuration, with $F \equiv I$, is an isotropic equilibrium state, so that the internal energy $\varepsilon(F)$ is a function (2.5.21) of the principal invariants (J_1, J_2, J_3) of the right stretch tensor (2.1.7). Assume, further, that $\varepsilon(F)$ is rank-one convex and satisfies the null condition

$$(5.5.66) \qquad \sum_{i,j,k=1}^{3} \sum_{\alpha,\beta,\gamma=1}^{3} \frac{\partial^3 \varepsilon(F)}{\partial F_{i\alpha}\partial F_{j\beta}\partial F_{k\gamma}} \nu_\alpha \nu_\beta \nu_\gamma \nu_i \nu_j \nu_k = 0,$$

at $F = I$, for any vector $\nu \in \mathbb{R}^3$. Then the Cauchy problem with initial data (F_0, v_0) close to $(I, 0)$, in an appropriate Sobolev space, admits a unique, globally defined classical solution. For isotropic incompressible elastic media, the relevant null condition is automatically satisfied. There is voluminous literature on these issues, a sample of which is cited in Section 5.7.

5.6 Initial-Boundary Value Problems

The issue of properly formulating the initial-boundary value problem for systems of hyperbolic conservation laws and establishing local existence of classical solutions has been the object of intensive study in recent years. A fairly definitive, albeit highly technical and complicated, theory has emerged, which lies beyond the scope of this book. Fortunately, detailed expositions are now available, in books and survey articles, referenced in Section 5.7. In order to convey to the reader a taste of the current state of this theory, a representative result will be recorded here, along the lines of the formulation of initial-boundary value problems presented in Section 4.7.

We begin by fixing as domain the half-space

(5.6.1) $$\mathscr{D} = \{x \in \mathbb{R}^m : \nu \cdot x < 0\},$$

with outward unit normal $\nu \in \mathbb{S}^{m-1}$. We seek solutions to the system

(5.6.2) $$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, \qquad x \in \mathscr{D},\ t \in (0,T),$$

satisfying initial conditions

(5.6.3) $$U(x,0) = U_0(x), \qquad x \in \mathscr{D},$$

and boundary conditions in the special form (4.7.1), (4.7.8), namely,

(5.6.4) $$BG(U(x,t))\nu = 0, \qquad x \in \partial\mathscr{D},\ t \in (0,T),$$

where B is a constant $n \times n$ matrix.

We make the following assumptions on the system (5.6.2). The flux $G(U)$ is a smooth $n \times m$ matrix-valued function defined on $\overline{\mathscr{B}_\rho}$. For normalization, $G(0) = 0$. Furthermore, (5.6.2) is endowed with a smooth entropy $\eta(U)$ such that $\mathrm{D}^2\eta(U)$ is positive definite on $\overline{\mathscr{B}_\rho}$. This implies, in particular, that (5.6.2) is hyperbolic, so that for any $U \in \overline{\mathscr{B}_\rho}$ and $\xi \in \mathbb{S}^{m-1}$, the matrix $\Lambda(\xi;U)$, defined by (4.1.2), possesses real eigenvalues (characteristic speeds) $\lambda_1(\xi;U) \le \cdots \le \lambda_n(\xi;U)$ and associated linearly independent eigenvectors $R_1(\xi;U), \cdots, R_n(\xi;U)$. We require that each eigenvalue has constant multiplicity on $\mathbb{S}^{m-1} \times \overline{\mathscr{B}_\rho}$.

Turning to the boundary conditions (5.6.4), we introduce the "manifold" of boundary data

(5.6.5) $$\mathscr{M} = \{U \in \overline{\mathscr{B}_\rho} : BG(U)\nu = 0\}$$

and assume that the boundary is *noncharacteristic*, in the sense that, for a certain $k = 0, \cdots, n$ and all $U \in \mathscr{M}$,

$$\lambda_k(\nu;U) < 0 < \lambda_{k+1}(\nu;U), \tag{5.6.6}$$

where $\lambda_0(\nu;U) = -\infty$ and $\lambda_{n+1}(\nu;U) = \infty$. Thus k characteristic fields are incoming to $\mathscr{D}$ and $n-k$ characteristic fields are outgoing from $\mathscr{D}$, through $\partial\mathscr{D}$.

We assume, further, that for any $U \in \mathscr{M}$ the rank of $B\Lambda(\nu;U)$ is k and

$$\mathbb{E}^k(\nu;U) \oplus \ker[B\Lambda(\nu;U)] = \mathbb{R}^n, \tag{5.6.7}$$

where $\mathbb{E}^k(\nu;U)$ denotes the subspace of $\mathbb{R}^n$ spanned by $R_1(\nu;U), \cdots, R_k(\nu;U)$. To motivate this condition, we linearize the system (5.6.2) and the boundary condition (5.6.4) about any constant state $U \in \mathscr{M}$:

$$\partial_t V(x,t) + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(U)\partial_\alpha V(x,t) = 0, \qquad x \in \mathscr{D},\ t \in (0,T), \tag{5.6.8}$$

$$B\Lambda(\nu;U)V(x,t) = 0, \qquad x \in \partial\mathscr{D},\ t \in (0,T). \tag{5.6.9}$$

Thus, roughly speaking, the role of (5.6.7) is to ensure that the trace of V on $\partial\mathscr{D}$ is determined by combining the boundary conditions with the information carried to the boundary by the $n-k$ outgoing characteristic fields.

The final assumption on the boundary conditions is the *uniform Kreiss-Lopatinski condition*, which is formulated as follows. For each state $U \in \mathscr{M}$, vector $\xi \in \mathbb{S}^{m-1}$ tangent to the boundary, i.e., $\xi \cdot \nu = 0$, and complex number z with $\mathrm{Re}\, z > 0$, we define the matrix

$$M(z,\xi;U) = \Lambda(\nu;U)^{-1}[zI + i\Lambda(\xi;U)]. \tag{5.6.10}$$

We denote by $\mathbb{E}(z,\xi;U)$ the subspace of $\mathbb{R}^n$ spanned by the eigenvectors associated with the eigenvalues of $M(z,\xi;U)$ with negative real part and require that

$$|W| \leq c|B\Lambda(\nu;U)W|, \qquad \text{for all } W \in \mathbb{E}(z,\xi;U), \tag{5.6.11}$$

where c is a positive constant, independent of U, ξ and z. To interpret this assumption, notice that the linear system (5.6.8) admits solutions of the form

$$V(x,t) = \exp(i\xi \cdot x + zt)W(\nu \cdot x), \tag{5.6.12}$$

where the function $W(\tau)$ satisfies the ordinary differential equation

$$\dot{W} + M(z,\xi;U)W = 0. \tag{5.6.13}$$

The role of (5.6.11) is to rule out solutions (5.6.12) that satisfy the boundary condition (5.6.9) and exhibit "tame" growth in the spatial directions but grow exponentially with time.

Finally, we turn to the initial condition (5.6.3). For $j=0,1,\cdots$, we let H_j denote the Sobolev space $W^{j,2}(\mathscr{D};\mathbb{R}^n)$, and assume $U_0 \in H_\ell$, for $\ell > \frac{m}{2}+1$. One may then calculate formally, from (5.6.2), the initial values $U_1(x),\cdots,U_{\ell-1}(x)$ of the time derivatives $\partial_t U(x,0),\cdots,\partial_t^{\ell-1}U(x,0)$ of solutions. Thus

$$U_1 = -\sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(U_0)\partial_\alpha U_0\,, \tag{5.6.14}$$

$$U_2 = -\sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(U_0)\partial_\alpha U_1 - \sum_{\alpha=1}^{m} \mathrm{D}^2G_\alpha(U_0)[U_1,\partial_\alpha U_0], \tag{5.6.15}$$

and so on. Moreover, $U_j \in H_{\ell-j}$, $j=0,\cdots,\ell-1$. In particular, the trace of U_j on the hyperplane $\partial\mathscr{D}$ is well-defined, for $j=0,\cdots,\ell-1$. We then require that the initial data be compatible with the boundary condition, in the sense

$$B\partial_t^j G(U(x,t))\nu = 0, \qquad t=0,\ x\in\partial\mathscr{D}, \quad j=0,\cdots,\ell-1, \tag{5.6.16}$$

namely,

$$BG(U_0(x))\nu = 0, \qquad x\in\partial\mathscr{D}, \tag{5.6.17}$$

$$B\Lambda(\nu;U_0(x))U_1(x) = 0, \qquad x\in\partial\mathscr{D}, \tag{5.6.18}$$

and so on.

We have now laid the preparation for stating the existence theorem:

5.6.1 Theorem. *Under the above assumptions on the system, the boundary conditions and the initial data, there exists a unique classical solution $U\in C^1(\overline{\mathscr{D}}\times[0,T_\infty))$ of the initial-boundary value problem* (5.6.2), (5.6.3), (5.6.4), *for some $0<T_\infty\le\infty$. Furthermore,*

$$U(\cdot,t) \in \bigcap_{j=0}^{\ell} C^j([0,T_\infty);H_{\ell-j}). \tag{5.6.19}$$

The interval $[0,T_\infty)$ is maximal in that if $T_\infty<\infty$ then

$$\limsup_{t\to T_\infty} \|\nabla U(\cdot,t)\|_{L^\infty} = \infty \tag{5.6.20}$$

and/or $\limsup_{t\to T_\infty} \|U(\cdot,t)\|_{L^\infty} = \rho$.

The (lengthy and technical) proof proceeds from linear systems with constant coefficients to linear systems with variable coefficients, and then passes to quasilinear systems via linearization (5.6.8), (5.6.9) and a fixed point argument, for the map $U\mapsto V$.

It should be noted that the assumptions in the above theorem are too restrictive for dealing with many natural initial-boundary value problems arising in continuum physics. In the Euler equations, for isentropic or nonisentropic gas flow, the assumption that the characteristic speeds have constant multiplicity is indeed valid (see Sections 3.3.5 and 3.3.6); but the assumption that the boundary is noncharacteristic is often violated, for instance in the case of no-penetration (or slip) boundary conditions $v \cdot \nu = 0$. In the equations of isentropic or nonisentropic elastodynamics, the condition that the characteristic speeds have constant multiplicity is often violated, for example in the vicinity of the natural state of an isotropic elastic solid where the multiplicity of the characteristic speed associated with shear waves undergoes a transition. Moreover, the boundary is always characteristic, as the system possesses zero characteristic speeds. Beyond that, one needs to consider more general domains $\mathscr{D}$ and homogeneous or inhomogeneous boundary conditions on $\partial\mathscr{D}$ of more general form than (5.6.4). These issues are addressed by more sophisticated versions of Theorem 5.6.1. References are cited in Section 5.7.

Instead of appealing to the general theory, outlined above, it is often advantageous to treat initial-boundary value problems for particular systems of conservation laws arising in continuum physics ab initio, taking advantage of their special structure. As an illustrative example, let us consider the system (3.3.19), which governs the isentropic motion of an elastic solid, in Lagrangian coordinates, assuming for simplicity that the body force vanishes, so that

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i, \alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = 0, & i = 1,2,3. \end{cases} \tag{5.6.21}$$

In (5.6.21) and in what follows, we employ the summation convention.

As noted in (sub)section 3.3.3, the Piola-Kirchhoff stress S derives from a potential (3.3.20), so that $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ is an entropy for the system. We assume that the internal energy function $\varepsilon(F)$ satisfies $\varepsilon(F) \geq 0$, $\varepsilon(I) = 0$ and is rank-one convex, i.e.

$$A_{i\alpha j\beta}(F)\nu_\alpha \nu_\beta \xi_i \xi_j > 0, \quad \text{for all } \nu \text{ and } \xi \text{ in } \mathbb{S}^2, \tag{5.6.22}$$

where

$$A_{i\alpha j\beta}(F) = \frac{\partial S_{i\alpha}(F)}{\partial F_{j\beta}} = \frac{\partial^2 \varepsilon(F)}{\partial F_{i\alpha}\partial F_{j\beta}}\,. \tag{5.6.23}$$

Notice the important symmetry relations

$$A_{i\alpha j\beta}(F) = A_{j\beta i\alpha}(F)\,. \tag{5.6.24}$$

We also define

$$B_{i\alpha j\beta k\gamma}(F) = \frac{\partial A_{i\alpha j\beta}(F)}{\partial F_{k\gamma}} = \frac{\partial^3 \varepsilon(F)}{\partial F_{i\alpha}\partial F_{j\beta}\partial F_{k\gamma}}\,. \tag{5.6.25}$$

The reference configuration of the elastic body is a bounded domain Ω of $\mathbb{R}^3$, with smooth boundary Γ, which is assumed clamped, so that the boundary conditions read

(5.6.26) $$v(x,t) = 0, \qquad x \in \Gamma,\ t > 0.$$

We also assign initial conditions

(5.6.27) $$F(x,0) = F^0(x), \quad v(x,0) = v^0(x), \qquad x \in \Omega,$$

assuming that

(5.6.28) $$F^0 \in W^{m-1,2}(\Omega;\mathbb{M}^{3\times 3}), \qquad v^0 \in W^{m,2}(\Omega;\mathbb{R}^3),$$

with $m \geq 4$. For $r = 0,1,\cdots,m$, we shall denote by $\|\cdot\|_r$ and $|||\cdot|||_r$ the norms of the Sobolev spaces $W^{r,2}(\Omega;\mathbb{R}^3)$ and $W^{r,2}(\Omega;\mathbb{M}^{3\times 3})$. Also $\overset{(r)}{v}$ will denote the time derivative $\partial_t^r v$ of v, of order r. On account of (5.6.28), one may derive candidates $v^r \in W^{m-r,2}(\Omega;\mathbb{R}^3)$ for the initial values of $\overset{(r)}{v}$, by combining (5.6.21) with (5.6.27):

(5.6.29) $$v_i^1(x) = \partial_\alpha S_{i\alpha}(F^0(x)), \qquad i = 1,2,3, \quad x \in \Omega,$$

(5.6.30) $$v_i^2(x) = \partial_\alpha[A_{i\alpha j\beta}(F^0(x))\partial_\beta v_j^0(x)], \qquad i = 1,2,3, \quad x \in \Omega,$$

and so on. In that setting:

5.6.2 Theorem. *Assume the initial data* (F^0, v^0) *are compatible with the boundary condition* (5.6.26), *in the sense that* $v^r \in W_0^{1,2}(\Omega;\mathbb{R}^3)$, *for* $r = 0,\cdots,m-1$. *Then there exists a unique classical solution* (F,v) *to* (5.6.21), (5.6.26), (5.6.27) *on a time interval* $[0,T]$, *and*

(5.6.31) $$v(\cdot,t) \in \bigcap_{r=0}^{m} C^r([0,T];W^{m-r,2}(\Omega;\mathbb{R}^3)).$$

The regularity of F and its time derivatives may be inferred by combining the first equation of (5.6.21) with (5.6.31). The condition $\det F > 0$, necessary for physical admissibility of the solution, can be secured by taking $\det F^0 > 0$ and T sufficiently small.

Detailed proofs of the above proposition are found in the references cited in Section 5.7. A formal, sketchy, derivation of the key estimate will suffice for the present purposes. Our strategy is to monitor the pointwise behavior of the solution and its derivatives with the help of L^2 bounds on derivatives of higher order. Thus, assuming (F,v) is a solution of (5.6.21), (5.6.26), (5.6.27), on the time interval $[0,T]$, with the properties recounted in Theorem 5.6.2, the aim is to show that, for T sufficiently small, the functional

(5.6.32)
$$E(t) = \sum_{r=0}^{m} \| \overset{(r)}{v}(\cdot,t) \|_{m-r}$$

is bounded on $[0,T]$. Setting

(5.6.33)
$$M = \left|\!\left|\!\left| F^0(\cdot) \right|\!\right|\!\right|_{m-1}, \qquad N = \sum_{r=0}^{m-1} \| v^r(\cdot) \|_{m-r-1},$$

we will be operating under the ansatz

(5.6.34)
$$|\!|\!| F(\cdot,t) |\!|\!|_{m-1} \leq 2M, \qquad \sum_{r=0}^{m-1} \| \overset{(r)}{v}(\cdot,t) \|_{m-r-1} \leq 2N, \qquad t \in [0,T],$$

which shall be verified a posteriori.

The approach followed in earlier sections for establishing such estimates for the Cauchy problem, by differentiating the equations of the system with respect to the spatial variables, cannot be applied here, because the boundary behavior of spatial derivatives is not known a priori. We only know that, by virtue of (5.6.26), time derivatives of v, of any order, vanish on Γ. We shall take advantage of that in estimating the simpler functional

(5.6.35)
$$G(t) = \sum_{r=0}^{m-1} \| \overset{(r)}{v}(\cdot,t) \|_1 + \| \overset{(m)}{v}(\cdot,t) \|_0 .$$

For $r = 0,\cdots,m-1$, we apply ∂_t^{r+1} to (5.6.21) to get

(5.6.36)
$$\overset{(r+2)}{v_i}(x,t) = \partial_\alpha [A_{i\alpha j\beta}(F(x,t)) \partial_\beta \overset{(r)}{v_j}(x,t)] + Z_i^r(x,t),$$

where

(5.6.37)
$$Z_i^r(x,t) = \partial_\alpha [\partial_t^{r+1} S_{i\alpha}(F(x,t)) - A_{i\alpha j\beta}(F(x,t)) \partial_\beta \overset{(r)}{v_j}(x,t)].$$

We multiply (5.6.36) by $2\overset{(r+1)}{v_i}$ and integrate the resulting equation on $\Omega \times [0,t]$. Recalling the symmetry condition (5.6.24), and after an integration by parts,

(5.6.38)
$$\int_\Omega [\,| \overset{(r+1)}{v}(x,t)|^2 + A_{i\alpha j\beta}(F(x,t)) \partial_\alpha \overset{(r)}{v_i}(x,t) \partial_\beta \overset{(r)}{v_j}(x,t)] dx$$
$$= \int_\Omega [\,|v^{r+1}(x)|^2 + A_{i\alpha j\beta}(F^0(x)) \partial_\alpha v_i^r(x) \partial_\beta v_j^r(x)] dx$$
$$+ \int_0^t \int_\Omega [2Z_i^r(x,\tau) \overset{(r+1)}{v_i}(x,\tau) + B_{i\alpha j\beta k\gamma}(F) \partial_\alpha \overset{(r)}{v_i}(x,\tau) \partial_\beta \overset{(r)}{v_j}(x,\tau) \partial_\gamma v_k(x,\tau)] dx d\tau.$$

We supplement (5.6.38) with the total energy (entropy) conservation law:

$$\int_\Omega [\varepsilon(F(x,t)) + \frac{1}{2}|v(x,t)|^2]dx = \int_\Omega [\varepsilon(F^0(x)) + \frac{1}{2}|v^0(x)|^2]dx. \tag{5.6.39}$$

In order to avoid proliferation of symbols, we adopt the following convention: In the sequel, a, b and K will stand for generic positive constants, with a depending on M, b depending on M and on N, and K depending on $|||F^0|||_{m-1}$ and $\|v^0\|_m$. In particular, for $r = 0, \cdots, m$, $\|v^r\|_{m-r} \leq K$.

By virtue of (5.6.22) and since $\overset{(r)}{v}(\cdot,t) \in W_0^{1,2}(\Omega;\mathbb{R}^3)$, Gårding's inequality applies:

$$\int_\Omega A_{i\alpha j\beta}(F(x,t))\partial_\alpha \overset{(r)}{v}_i(x,t)\partial_\beta \overset{(r)}{v}_j(x,t)dx \geq \mu \|\overset{(r)}{v}(\cdot,t)\|_1^2 - a\|\overset{(r)}{v}(\cdot,t)\|_0^2, \tag{5.6.40}$$

with $\mu > 0$. Moreover, since $m \geq 4$, standard calculus inequalities for Sobolev spaces yield

$$\|Z^r(\cdot,t)\|_0 \leq bE(t), \qquad r = 1, \cdots, m-1. \tag{5.6.41}$$

Upon combining (5.6.38) with (5.6.39), (5.6.40) and (5.6.41), one arrives at an estimate in the form

$$G(t) \leq K + b\int_0^t E(\tau)d\tau. \tag{5.6.42}$$

To estimate spatial derivatives of $\overset{(r)}{v}$, of order greater than one, we view (5.6.36) as a strongly elliptic system

$$A_{i\alpha j\beta}(x,t)\partial_\alpha\partial_\beta \overset{(r)}{v}_j(x,t) = \overset{(r+2)}{v_i}(x,t) - Y_i^r(x,t), \tag{5.6.43}$$

where

$$Y_i^r(x,t) = Z_i^r(x,t) + B_{i\alpha j\beta k\gamma}(F(x,t))\partial_\alpha F_{k\gamma}(x,t)\partial_\beta \overset{(r)}{v}_j(x,t). \tag{5.6.44}$$

For $s = 0, \cdots, m-r-2$, we employ the standard estimates from the theory of elliptic systems:

$$\|\overset{(r)}{v}(\cdot,t)\|_{s+2} \leq a\left[\|\overset{(r+2)}{v}(\cdot,t)\|_s + \|\overset{(r)}{v}(\cdot,t)\|_s + \|Y^r(\cdot,t)\|_s\right]. \tag{5.6.45}$$

By virtue of calculus inequalities for Sobolev spaces,

$$\|Y^r(\cdot,t)\|_s \leq bE(t), \qquad \|\partial_t Y^r(\cdot,t)\|_s \leq bE(t), \qquad r = 1, \cdots, m-2. \tag{5.6.44}$$

Hence

$$\|Y^r(\cdot,t)\|_s \leq K + b\int_0^t E(\tau)d\tau. \tag{5.6.45}$$

For $r = 0,\cdots,m-2$, one may estimate recursively, with the help of (5.6.45), $\|\overset{(r)}{v}\|_{m-r}$ in terms of $\|\overset{(r)}{v}\|_0,\cdots,\|\overset{(m)}{v}\|_0$, when $m-r$ is even, or of $\|\overset{(r)}{v}\|_1,\cdots,\|\overset{(m-1)}{v}\|_1$, when $m-r$ is odd, all of which are parts of G. We thus end up with an estimate in the form

$$E(t) \leq b\left[K + G(t) + \int_0^t E(\tau)d\tau\right]. \tag{5.6.47}$$

Combining (5.6.47) with (5.6.42), yields $E(t) \leq K\exp(bt)$, which establishes that $E(t)$ is bounded on $[0,T]$, so long as the ansatz (5.6.34) holds. This is indeed the case, since

$$F_{i\alpha}(x,t) = F^0_{i\alpha}(x) + \int_0^t \partial_\alpha v_i(x,\tau)d\tau, \quad \overset{(r)}{v}_i(x,t) = v^r_i(x) + \int_0^t \overset{(r+1)}{v_i}(x,\tau)d\tau \tag{5.6.48}$$

yield

$$|||F(\cdot,t)|||_{m-1} \leq M + K(e^{bt}-1), \qquad \sum_{r=0}^{m-1} \|\overset{(r)}{v}(\cdot,t)\|_{m-r-1} \leq N + K(e^{bt}-1), \tag{5.6.49}$$

which in turn imply (5.6.34), when T is sufficiently small.

For a complete, rigorous proof of Theorem 5.6.2, one may construct the solution to (5.6.21), (5.6.26), (5.6.27) by establishing the existence of a fixed point of the map that carries Φ, in a suitable function class, to F, where (F,v) is the solution to the linear system

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i,\alpha = 1,2,3 \\ \partial_t v_i - A_{i\alpha j\beta}(\Phi)\partial_\alpha F_{j\beta} = 0, & i = 1,2,3, \end{cases} \tag{5.6.50}$$

with boundary conditions (5.6.26) and initial conditions (5.6.27). Alternatively, one may employ the vanishing viscosity method, obtaining the solution to (5.6.21), (5.6.26), (5.6.27) as the $\varepsilon \to 0$ limit of solutions to the system

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & i,\alpha = 1,2,3 \\ \partial_t v_i - \partial_\alpha S_{i\alpha}(F) = \varepsilon\Delta v_i\,, & i = 1,2,3, \end{cases} \tag{5.6.51}$$

under the same boundary and initial conditions. In either approach, the crucial task is to demonstrate that the functional $E(t)$ is bounded, uniformly in Φ, for the system (5.6.50), or uniformly in ε, for the system (5.6.51).

Unless it is clamped, the boundary of moving bodies varies with time. Thus, tracking the evolution of continuous media in Eulerian coordinates often leads to initial-boundary value problems with a free boundary that is to be determined as part of the solution. This usually raises serious technical complications. One may attempt to circumvent that obstacle by switching to Lagrangian coordinates, in which case the domain becomes the fixed reference configuration of the body. However, the price to pay is that typically the equations in Lagrangian form are complicated and lack the symmetries of their Eulerian counterparts. In particular, this is encountered in the Euler equations of gas dynamics.

For illustration, let us consider the isentropic expansion of a gaseous mass that is confined, at $t = 0$, in a bounded domain of $\mathbb{R}^3$, surrounded by vacuum. The evolution of the gas in that setting has been modeled, in Section 4.6, as a Cauchy problem for the Euler equations (4.8.1), by visualizing the vacuum as a gas with zero density. It was pointed out that, insofar as the focus stays on classical solutions, it is preferable to switch state variables from density ρ to the weighted sonic speed ω, because, unlike the Euler equations (4.8.3), the symmetric system (4.8.4) is hyperbolic even at the vacuum state. When the initial data (ω_0, v_0), extended to all of $\mathbb{R}^3$, lie in the Sobolev space $W^{3,2}$, Theorem 5.1.1 guarantees the existence of a classical solution to the Cauchy problem for (4.8.4), and thereby to the Euler equations (4.8.3), at least locally in time. However, as already discussed in Section 4.8, the lifespan of this classical solution is generally finite, because either waves break inside the gas cloud or singularities develop on the interface between gas and vacuum. The breaking of waves is associated with the transition from a classical to a weak solution, containing shocks, and falls outside the scope of the present discussion, which will focus on interfacial singularities.

The nature of interfacial singularities was discussed in Section 4.8. The balance laws force the pressure, and thereby the density, to be continuous across the interface. It was argued that the natural condition, encoded in (4.8.23), is that the normal derivative of the square of the sonic speed must experience a jump across the interface. To test whether solutions may stay smooth in the interior, even after the onset of interfacial singularities, one introduces, through the initial data, singularities of this type at $t = 0$, and attempts to construct classical solutions to the resulting initial-boundary value problem. In that setting, the interface must be treated as a free boundary, to be determined as part of the solution. A result in that direction is the following

5.6.3 Theorem. *Consider the isentropic flow of an ideal gas, with equation of state* $p = \kappa\rho^\gamma$, $1 < \gamma \leq 2$, *which occupies, at* $t = 0$, *a bounded set* Ω *of* $\mathbb{R}^3$, *with smooth boundary* Γ, *surrounded by vacuum. At* $t = 0$, *the density of the gas is* $\rho_0 \in C^\infty(\overline{\Omega})$, *such that*

(5.6.52) $$\rho_0(x) > 0, \quad x \in \Omega, \qquad \rho_0(x) = 0, \quad x \in \Gamma,$$

(5.6.53) $$\rho_0^{\gamma-1}(x) \geq A\,\mathrm{dist}(x, \Gamma), \qquad x \in \Omega,$$

with $A > 0$. *The initial velocity is* $v_0 \in C_0^\infty(\Omega; \mathbb{R}^3)$. *Then, for some* $T > 0$, *there exists a* C^2 *function* χ, *defined on* $\overline{\Omega} \times [0, T]$ *and taking values in* $\mathbb{R}^3$, *with the following properties: (a)* $\chi(\cdot, 0)$ *is the identity map on* Ω; *(b) for any* $t \in [0, T]$, $\chi(\cdot, t)$ *maps diffeomorphically* Ω *to an open set* Ω_t, *and also maps* Γ *to the boundary* Γ_t *of* Ω_t; *(c)* χ *induces a unique* C^1 *solution* (ρ, v) *to the Euler equations* (4.8.3), *defined on* $\{(x,t) : t \in [0,T], x \in \overline{\Omega_t}\}$ *and satisfying the initial conditions*

(5.6.54) $$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega$$

and the boundary conditions

(5.6.55) $$p(\rho(x,t)) = 0, \qquad v(x,t)\cdot\nu(x,t) = V(x,t), \qquad t \in [0,T], \;\; x \in \Gamma_t,$$

where ν *denotes the external unit normal on* Γ_t and V *stands for the normal speed of the surface* Γ_t.

The nature of the singularity across the interface is encoded in (5.6.53) – recall that the square of the sonic speed is proportional to $\rho^{\gamma-1}$. The regularity restrictions imposed on the initial data in Ω are excessive. Solutions exist under considerably weaker hypotheses (initial data in a Sobolev space of sufficiently high order, satisfying compatibility conditions, which however are rather awkward to state.)

The proof of the above proposition, which is found in the bibliography cited in Section 5.7, is lengthy and technical. The following remarks will convey a taste of the methodology.

To overcome the obstacle raised by the presence of the free boundary, we reformulate the problem in Lagrangian coordinates. We employ the notational conventions introduced in Sections 2.1-2.3, with slight modifications, for compatibility with the notation of Section 4.8, for the Euler equations. Thus, material particles will be identified by y (Lagrangian coordinates), while position in physical space will be labelled by x (Eulerian coordinates). For any particular field, we shall be using the same symbol for denoting both its Lagrangian representation, as function of (y,t), and its Eulerian representation, as function of (x,t). To prevent confusion, material time derivatives (holding y fixed) will be denoted by an overdot, while spatial time derivatives (holding x fixed) will be denoted by ∂_t.

We choose the physical space $\mathbb{R}^3$ as reference space, and the physical placement Ω of the gas, at time $t = 0$, as reference configuration. Thus the reference density is $\rho_0(y)$. The function $\chi(y,t)$, appearing in the statement of the theorem, represents the motion of the gas. Thus $\chi(y,0) = y$, for all $y \in \Omega$. The motion induces the deformation gradient field $F(y,t)$, by $F = \nabla\chi$, and the velocity field $v(y,t)$, in Lagrangian coordinates, by $v = \dot{\chi}$. On account of (2.3.3), the density field in Lagrangian coordinates is $\rho(y,t) = \rho_0(y)J^{-1}(y,t)$, where $J = \det F$.

In Lagrangian coordinates, the Euler equations assume the form (2.3.4), namely (when the body force vanishes),

(5.6.56) $$\rho_0(y)\dot{v}(y,t) = \operatorname{Div} S(F(y,t)),$$

where S is the Piola-Kirchhoff stress. For the ideal gas, the Cauchy stress tensor is $-\kappa\rho^\gamma I$ and hence, by virtue of (2.3.6),

(5.6.57) $$S = -\kappa\rho^\gamma J(F^{-1})^\top = -\kappa\rho^\gamma (F^*)^\top,$$

where F^* is the adjugate matrix of F, and hence $(F^*)^\top$ is the cofactor matrix of F. On account of (2.2.13), the divergence of the cofactor matrix of F vanishes, whence

$$\operatorname{Div} S(F(y,t)) = -\kappa (F^*)^\top(y,t)\operatorname{Grad}[\rho_0^\gamma(y) J^{-\gamma}(y,t)]. \tag{5.6.58}$$

The solution to (5.6.56), (5.6.58) is obtained by the vanishing viscosity method, as the $\varepsilon \to 0$ limit of solutions to the system

$$\rho_0(y)\dot{v}(y,t) = \operatorname{Div} S(F(y,t)) + \varepsilon \operatorname{Div} \dot{S}(F(y,t)). \tag{5.6.59}$$

One recognizes in (5.6.59) the equations of motion of a viscoelastic material with a special constitutive equation for the Piola-Kirchhoff stress, of the type introduced in Section 2.6.

In broad terms, the construction of the solution follows here the path taken earlier in treating the elastic solid with clamped boundary (Theorem 5.6.2), and proceeds by establishing L^2 estimates on derivatives of the solution, extracted from "energy" integrals. These combine bounds on the quantity

$$E(t) = \sum_{k=0}^{4} \| \overset{(2k)}{\chi}(t) \|_{4-k}^2 , \tag{5.6.60}$$

with L^2 estimates on certain derivatives of the vorticity field. The degeneracy at the boundary renders the task of deriving these estimates much harder than what was encountered in the case of the clamped boundary. In particular, some of these estimates are tied to the nonlinear structure of the system. As a result one cannot approach the construction of the solution via linearization, by analogy to (5.6.50). The success of the vanishing viscosity approach hinges on the particular choice $\dot{S}$ for the viscous term, which preserves the delicate features of the Piola-Kirchhoff stress for the Euler equations. The details of the hard and lengthy derivations of the estimates are recorded in the bibliography cited in Section 5.7, albeit only for the case $\Omega = \mathbb{T}^2 \times (0,1)$.

5.7 Notes

A comprehensive treatment of classical solutions to the initial and initial-boundary value problem for hyperbolic systems of conservation laws is found in the monograph by Benzoni-Gavage and Serre [2].

Local existence of classical solutions to the Cauchy problem for symmetrizable systems of conservation laws has been established by a variety of methods, ultimately relying on the hierarchy of "energy" estimates derived by differentiating the system with respect to the spatial variables.

The earliest, and still most popular, approach, expounded in Benzoni-Gavage and Serre [2], constructs solutions to (5.1.1) by an iteration process on the linearized systems (5.1.7). It was originated by Schauder [1], in the context of the quasilinear second-order wave equation, and has attained its present general form through the contributions of several authors, in particular Friedrichs [2], Gårding [1] and Majda [3]. Godunov [3], Makino, Ukai and Kawashima [1], Chemin [1], Lax [1], M.E. Taylor [1,2] and Métivier [1] have used symmetrizers other than the Hessian of a convex entropy, or even symbolic symmetrizers. In that connection, recall the symmetrized form (4.8.4) for the Euler equations, which retains hyperbolicity even at the vacuum state.

For the Euler equations for incompressible fluids, it has been shown by Beale, Kato and Majda [1] that classical solutions to the Cauchy problem persist for as long as the vorticity stays bounded. For the case of compressible fluids, Chemin [2] explains how breakdown of classical solutions arises as a result of explosion in vorticity, compression, or the velocity divergence.

An alternative way of establishing Theorem 5.1.1, by Kato [1], is based on the theory of abstract evolution equations. The method of vanishing viscosity was adopted here because it also applies to the cases where the entropy is convex only in the direction of the involution cone or it is merely polyconvex.

The use of relative entropy for proving, as in Theorems 5.2.1, 5.3.3 and 5.4.5, uniqueness and stability of classical solutions within a broader class of admissible weak solutions (informally referred to as "weak-strong uniqueness") originated in the works of Dafermos [9,10] and DiPerna [7]. This approach has now been extended in several directions: the weak solution may be very weak – just measure-valued – or the system may be of intermediate hyperbolic-parabolic type, or even arising in the kinetic theory. A representative sample, out of a large number of relevant papers, is Brenier, De Lellis and Szekelyhidi [1], LeFloch [8], Christoforou and Tzavaras [1], Lattanzio and Tzavaras [2], Tzavaras [7], Demoulini, Stuart and Tzavaras [3], Berthelin, Tzavaras and Vasseur [1], Luo and Smoller [1,2], Germain [1], Elling [5], Miroshnikov and Trivisa [1], and Feireisl and Novotny [1]. In fact, under certain conditions, the method may even yield uniqueness of weak solutions; see DiPerna [7], Chen, Frid and Li [1], Gui-Qiang Chen and Yachun Li [1,2], Gui-Qiang Chen and Jun Chen [1], Kwon [3], Kwon and Vasseur [2], Choi and Vasseur [1], Serre and Vasseur [1], Feireisl, Kreml and Vasseur [1], and Leger and Vasseur [2]. By using the last paper, Texier and Zumbrun [2] show that the relative entropy condition implies the Lopatinski (stability) condition for extreme shocks of arbitrary strength.

There are interesting examples of classical solutions in which the relative entropy production (second term on the right-hand side of (5.2.14)) happens to be non positive. Then (5.2.1) holds with $b = 0$, yielding stability, uniformly in time. Such conditions may be induced by a suitable selection of state vector. This situation arises, for instance, in the case of the rarefaction wave in rectilinear isentropic gas flow. See Gui-Qiang Chen [7], Chen and Frid [7], and Serre [30].

Hyperbolic systems of conservation laws with involutions were discussed by Boillat [4] and Dafermos [14]. In particular, Boillat [4] presents examples arising in general relativity. The notion of contingent entropy is due to Serre [22]. The analysis in Section 5.3 follows and extends Dafermos [14,27,37]. An intimate relation exists between involutions and the theory of compensated compactness, as formalized by Murat and Tartar; see Tartar [1,2]. In that connection, "involution cone" corresponds to "characteristic cone." In particular, for the equations of elastodynamics, see Hughes, Kato and Marsden [1], and Dafermos and Hrusa [1].

Section 5.4 follows Dafermos [27], which improves upon the treatment of this topic in earlier editions of the book. As already noted in Section 4.9, the notion of polyconvexity in elastostatics was introduced by Ball [1], as a condition rendering the internal energy function weakly lower semicontinuous. It is from P.G. LeFloch that the author originally heard the idea of extending the system of conservation laws in elastodynamics by appending conservation laws for the invariants of the stretch tensor. Explicit extensions were first published by Qin [1] and by Demoulini, Stuart and Tzavaras [2]. See also Lattanzio and Tzavaras [1]. Tzavaras [8] embeds the extended system of elastodynamics in a relaxation scheme. In particular, he discusses the example of the equations of gas dynamics, which are endowed with a convex entropy in their Eulerian formulation, whereas the entropy in their Lagrangian formulation is merely polyconvex. Brenier [2] presents two distinct extensions of the equations of electrodynamics, for the Born-Infeld constitutive relations, including the one recorded here, and discusses its asymptotics in various regimes. This investigation continues in Brenier [4] and Brenier and Yong [1]. See also Neves and Serre [1]. Serre [22] devises the proper extension in electrodynamics, under general constitutive relations, by exploiting the contingent entropy-entropy flux pair (3.3.76).

There is extensive literature on the existence and long time behavior of globally defined classical solutions to the Cauchy problem for systems of balance laws with source satisfying a Kawashima-type condition. Variants of this condition are encountered in several papers by Kawashima and coworkers, but its first appearance is in Shizuta and Kawashima [1], and so it is often referred to as the *Kawashima-Shizuta condition.* The reader may find results with the flavor of Theorems 5.5.3 and 5.5.4 in Hanouzet and Natalini [1], Yong [6], Yang, Zhu and Zhao [3], Bianchini, Hanouzet and Natalini [1], Kawashima and Yong [1,2], Xu and Kawashima [1,2], Yanni Zeng [6], and in the bibliography of these papers. See also Beauchard and Zuazua [1], Hu and Wang [1], Luo, Xin and Zeng [1], Mascia and Natalini [2], and Peng and Wang [1]. A survey on the role of viscous dissipation is found in Tai-Ping Liu [30].

The setting of the general relaxation framework, in Section 5.5, follows Chen, Levermore and Liu [1]. For an interesting alternative framework, see Bouchut [1]. The connection between relaxation and diffusion was first recognized in the kinetic theory of gases, where it is effected by means of the Chapman-Enskog expansion (e.g. Cercignani [1]). Chapman-Enskog type expansions have also been employed in order to relate classes of hyperbolic balance laws (5.5.1) with systems with diffusion in the form (4.6.1); see Kawashima and Yong [1,2].

There is voluminous literature on various aspects of relaxation theory. For a historical retrospective, see Mascia [4]. Surveys and extensive bibliographies are found in Natalini [3] and Yong [4]. Relevant references include Tai-Ping Liu [21], Nishibata and Yu [1], Wei-Cheng Wang and Zhouping Xin [1], Donatelli and Marcati [1], Hsiao and Pan [1], Shen, and Winther [1], Yong [2,3,5], Yang and Zhu [1], Yang, Zhu and Zhao [3], Liu and Yong [1], Natalini and Terracina [1], Xin and Xu [1], DiFrancesco and Lattanzio [1], Fan and Härterich [1], Fan and Luo [1], Bedjaoui, Klingenberg and LeFloch [1], Berthelin and Bouchut [1], Junca and Rascle [1], Tadmor and Tang [2], Carbou and Hanouzet [1], Carbou, Hanouzet and Natalini [1], Chalons and Coulombel [1], Lambert and Marchesin [1], Yanni Zeng [4,5], Miroshnikov and Trivisa [2], and Lattanzio and Tzavaras [1]. In particular, the system (5.2.18) with $p(u) = a^2u$, proposed by Jin and Xin [1], has served widely as a vehicle for understanding and explaining the features of relaxation. We will visit the theory of this system in Section 17.5, and the reader may find the relevant references in Section 17.9. Baudin, Coquel and Tran [1] propose a variant of the above relaxation scheme, which bears a curious relationship to the one-dimensional Born-Infeld system; see Serre [11]. We will also come across relaxation in Section 6.6, with references in Section 6.11.

The intimate relation between relaxation and diffusion also manifests itself in the large time behavior of solutions to hyperbolic systems with "frictional" damping and in particular in the simple system governing the isentropic flow of a gas through a porous medium; see Hsiao and Liu [1], Tai-Ping Liu [25], Serre and Xiao [1], Hsiao and Luo [1], Luo and Yang [1], Nishihara and Yang [1], Hsiao and Pan [2,3], Hsiao, Li and Pan [1], Hsiao and Li [1,2], Nishihara, Wang and Yang [1,2,3], Marcati and Mei [1], He and Li [1], Liu and Natalini [1], Marcati and Pan [1], Marcati and Nishihara [1], Marcati, mei and Rubino [1], Pan [1,2], Li and Saxton [1], Huang and Pan [1,2], Lattanzio and Rubino [1], Huang, Marcati and Pan [1], Di Francesco and Marcati [1], Lan and Lin [1], Dafermos and Pan [1], and Huang, Pan and Wang [1].

Out of a huge literature on nonhyperbolic systems that nevertheless exhibit behavior similar to that of hyperbolic systems with damping, here is a small representative sample: For the Euler-Poisson system, see Poupaud, Rascle and Vila [1], Dehua Wang [1,3], Wang and Chen [1], Guo [1], Engelberg, Liu and Tadmor [1], Li, Markowich and Mei [1], Feldman, Ha and Slemrod [1], Jang [1], Chae and Tadmor [1], and Tadmor and Wei [1]. For the semiconductor equations, see the monograph by Markowich, Ringhofer and Schmeiser [1], which contains a comprehensive list of references; also Guo and Strauss [1]. For the system of radiation hydrodynamics, coupling the Euler equations with an elliptic equation accounting for the flux of radiation energy, see Rohde and Yong [1], Rohde, Wang and Xie [1], and Rohde and Xie [1,2]. The monographs by Lions [2] and Feireisl [1] treat the system of equations for compressible viscoelastic fluids, in several space dimensions, and provide an exhaustive bibliography. Of course, the literature on the incompressible case, which includes the classical Navier-Stokes equations, is vast. The system of magnetohydrodynamics for viscous fluids is discussed in Chen and Wang [4,5], and Dehua Wang [4]. For the equations of radiation magnetohydrodynamics, see Rohde and Yong [2]. For the system of one-dimensional thermoviscoelasticity, see Dafermos and Hsiao [2], and Dafermos [12]. For the equations of one-dimensional thermoelasticity, see Slemrod [1], Dafermos and Hsiao [3], and the detailed survey article by Racke [1]. Finally, for the equations of one-dimensional viscoelasticity, with viscosity induced by fading memory dependence, see MacCamy [1], Dafermos and Nohel [1], Dafermos [15], and the monograph by Renardy, Hrusa and Nohel [1].

A thorough discussion of initial-boundary value problems, including the details on the material sketched in Section 5.6, is found in Benzoni-Gavage and Serre [2]. See also the survey article by Higdon [1]. In particular, Theorem 5.6.2, on the equations of elastodynamics, is taken from Dafermos and Hrusa [1], while Theorem 5.6.3, on the Euler equations, in the presence of vacuum, is due to Coutand and Shkoller [2]. For related results on the last problem, see Coutand and Shkoller [1], Makino

[1], Liu and Yang [1], Liu, Xin and Yang [1], Tong Yang [4], Xu and Yang [1], and Jang and Masmoudi [1].

For perspectives on stability issues see Benzoni-Gavage, Rousset, Serre and Zumbrun [1]. See also Benzoni-Gavage and Coulombel [1]. The vanishing viscosity approach and the related questions on the nature and stability of resulting boundary layers have been actively investigated in recent years; see H.O. Kreiss [1], Benabdallah and Serre [1], Gisclon and Serre [1], Gisclon [1], Grenier and Gues [1], Kreiss and Kreiss [1], Xin [6], Serre and Zumbrun [1], Serre [14, 17, 24], Joseph and LeFloch [1,2,3], Rousset [1,2,3], Métivier and Zumbrun [1,2], and Guès, Métivier, Williams and Zumbrun [5,6].

VI

The L^1 Theory for Scalar Conservation Laws

The theory of the scalar balance law, in several spatial dimensions, has reached a state of virtual completeness. In the framework of classical solutions, the elementary, yet effective, method of characteristics yields a sharper version of Theorem 5.1.1, determining explicitly the life span of solutions with Lipschitz continuous initial data and thereby demonstrating that in general this life span is finite. Thus one must deal with weak solutions, even when the initial data are very smooth.

In regard to weak solutions, the special feature that sets the scalar balance law apart from systems of more than one equation is the size of its family of entropies. It will be shown that the abundance of entropies induces an effective characterization of admissible weak solutions as well as very strong L^1-stability and L^∞-monotonicity properties. Armed with such powerful a priori estimates, one can construct admissible weak solutions in a number of ways. As a sample, construction by the method of vanishing viscosity, the theory of L^1-contraction semigroups, the layering method, a relaxation method and an approach motivated by the kinetic theory will be presented here. The method of vanishing viscosity will also be employed for solving the initial-boundary value problem. When the initial data are functions of locally bounded variation then so are the solutions. Remarkably, however, even solutions that are merely in L^∞ exhibit the same geometric structure as BV functions, with jump discontinuities assembling on "manifolds" of codimension one.

The chapter will close with a description of the seemingly insurmountable obstacles encountered in the study of weak solutions for hyperbolic systems of conservation laws in several spatial dimensions, and an account of current efforts to bypass these obstructions.

In order to expose the elegance of the theory, the discussion will be restricted to the homogeneous scalar conservation law, even though the general, inhomogeneous balance law (3.3.1) may be treated by the same methodology, at the expense of rather minor technical complications.

The issue of stability of weak solutions with respect to the weak* topology of L^∞ will be addressed in Chapter XVI. The special case of a single space variable, $m = 1$, has a very rich theory of its own, certain aspects of which will be presented in later chapters and especially in Chapter XI.

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6.1 The Cauchy Problem: Perseverance and Demise of Classical Solutions

We consider the Cauchy problem for a homogeneous scalar conservation law:

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = 0, \quad x \in \mathbb{R}^m, \ t > 0, \tag{6.1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^m. \tag{6.1.2}$$

The flux $G(u) = (G_1(u), \ldots, G_m(u))$ is a given smooth function on $\mathbb{R}$, taking values in $\mathbb{M}^{1\times m}$.

A *characteristic* of (6.1.1), associated with a continuously differentiable solution u, is an orbit $\xi : [0,T) \to \mathbb{R}^m$ of the system of ordinary differential equations

$$\frac{dx}{dt} = G'(u(x,t))^\top. \tag{6.1.3}$$

With every characteristic ξ we associate the differential operator

$$\frac{d}{dt} = \partial_t + G'(u(\xi(t),t))\operatorname{grad}, \tag{6.1.4}$$

which determines the directional derivative along ξ. In particular, since u satisfies (6.1.1), $du/dt = 0$, i.e., u is constant along any characteristic. By virtue of (6.1.3), this implies that the slope of the characteristic is constant. Thus all characteristics are straight lines along which the solution is constant. With the help of this property, one may study classical solutions of (6.1.1), (6.1.2) in minute detail. In particular, for scalar conservation laws Theorem 5.1.1 admits the following refinement:

6.1.1 Theorem. *Assume that u_0, defined on $\mathbb{R}^m$, is bounded and Lipschitz continuous. Let*

$$\kappa = \operatorname*{ess\,inf}_{y\in\mathbb{R}^m} \operatorname{div} G'(u_0(y)). \tag{6.1.5}$$

Then there exists a classical solution u of (6.1.1), (6.1.2) *on the maximal interval $[0,T_\infty)$, where $T_\infty = \infty$ when $\kappa \geq 0$ and $T_\infty = -\kappa^{-1}$ when $\kappa < 0$. Furthermore, if u_0 is C^k so is u.*

Proof. Assume first that $u_0 \in C_0^\infty(\mathbb{R}^m)$ and let u be the unique smooth classical solution of (6.1.1), (6.1.2), defined on the maximal time interval $[0,T_\infty)$, in accordance to Theorem 5.1.1. By the properties of characteristics, stated above, with any point (x,t) in $\mathbb{R}^m \times [0,T_\infty)$ is associated $y \in \mathbb{R}^m$ such that

$$y = x - tG'(u(x,t))^\top, \quad u(x,t) = u_0(y). \tag{6.1.6}$$

From (6.1.6) one easily gets

(6.1.7) $$\nabla u(x,t) = \frac{\nabla u_0(y)}{1 + t \operatorname{div} G'(u_0(y))}, \quad \partial_t u(x,t) = \frac{-G'(u_0(y)) \cdot \nabla u_0(y)}{1 + t \operatorname{div} G'(u_0(y))},$$

which implies, in particular, that $T_\infty = \infty$ if $\kappa \geq 0$, or $T_\infty = -\kappa^{-1}$ if $\kappa < 0$.

Suppose now u_0 is merely Lipschitz on $\mathbb{R}^m$. Set $T_\infty = \infty$ if $\kappa \geq 0$, or $T_\infty = -\kappa^{-1}$ if $\kappa < 0$. With the help of mollifiers, we construct a sequence $\{u_{0n}\}$ in $C_0^\infty(\mathbb{R}^m)$ that converges to u_0, uniformly on compact sets, and also $\nabla u_{0n}(y)$ tends to $\nabla u_0(y)$ at every Lebesgue point y of ∇u_0.

The classical solution u_n of (6.1.1) with initial value u_{0n} is defined on a maximal time interval $[0, T_n)$. We set $T = \liminf_{n\to\infty} T_n$, noting that $0 < T \leq T_n$. By virtue of (6.1.7), the u_n are equilipschitzean on every compact subset of $\mathbb{R}^n \times [0, T)$, whence some subsequence of $\{u_n\}$ converges, uniformly on compact sets, to a locally Lipschitz function u. Clearly u inherits from $\{u_n\}$ the property (6.1.6). This in turn implies that if u_0 is differentiable at some point y, then u is differentiable along the characteristic $x = y + tG'(u_0(y))$ and the derivatives are given by (6.1.7). In particular, u is the classical solution of (6.1.1), (6.1.2) on the time interval $[0, T)$. If $T = T_\infty$, (6.1.7) implies that $[0, T_\infty)$ is the maximal time interval and the assertion of the theorem has been proved. On the other hand, if $T < T_\infty$, (6.1.7) implies that u may be extended to $t = T$ and $u(\cdot, T)$ is Lipschitz on $\mathbb{R}^m$. We may thus repeat the process and prolong the time interval of existence of u up to $[0, T_\infty)$, which is necessarily maximal.

Finally, the implicit function theorem, applied to (6.1.6), yields that when u_0 is C^k the solution u is also C^k. This completes the proof.

From the above considerations it becomes clear that the lifespan of classical solutions is generally finite. It is thus imperative to deal with weak solutions.

An alternative, instructive way of viewing classical solutions u to (6.1.1) is by realizing them as "level surfaces" of functions $f(v; x, t)$, defined on $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$; that is

(6.1.8) $$f(u(x,t); x, t) = 0$$

whenever u satisfies (6.1.1). It is easy to see that for that purpose f must satisfy the transport equation

(6.1.9) $$\partial_t f(v; x, t) + \sum_{\alpha=1}^{m} \mathrm{G}'_\alpha(v) \partial_\alpha f(v; x, t) = 0.$$

Thus, we have transformed the nonlinear equation (6.1.1) into a linear one, at the price of increasing the number of independent variables from $m+1$ to $m+2$. In particular, to solve the initial value problem (6.1.1), (6.1.2), one should solve a Cauchy problem for (6.1.9) with initial condition $f(v; x, 0) = v - u_0(x)$. Since (6.1.9) is linear, a solution of this Cauchy problem will exist on $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$. The resulting f will in turn induce, through (6.1.8), the classical solution u to (6.1.1), (6.1.2), which will be valid up until f_v vanishes for the first time. We shall return to the transport equation (6.1.9), in the context of weak solutions, in Section 6.7.

6.2 Admissible Weak Solutions and their Stability Properties

In Section 4.2, we saw that the initial value problem for a scalar conservation law may admit more than one weak solution, thus raising the need to impose admissibility conditions. In Section 4.5, we discussed how entropy inequalities may serve that purpose. Recall from Section 3.3.1 that for the scalar conservation law (6.1.1) any smooth function η may serve as an entropy, with associated entropy flux

$$Q(u) = \int^u \eta'(\omega) G'(\omega) d\omega, \tag{6.2.1}$$

and entropy production zero. It will be convenient to relax slightly the regularity condition and allow entropies (and thereby entropy fluxes) that are merely locally Lipschitz continuous. Similarly, G need only be locally Lipschitz continuous. It turns out that in order to properly characterize admissible weak solutions, one has to impose the entropy inequality

$$\partial_t \eta(u(x,t)) + \operatorname{div} Q(u(x,t)) \leq 0 \tag{6.2.2}$$

for every convex entropy-entropy flux pair:

6.2.1 Definition. A bounded measurable function u on $\mathbb{R}^m \times [0,\infty)$ is an *admissible weak solution* of (6.1.1), (6.1.2), with u_0 in $L^\infty(\mathbb{R}^m)$, if the inequality

$$\int_0^\infty \int_{\mathbb{R}^m} [\partial_t \psi \eta(u) + \sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u)] dxdt + \int_{\mathbb{R}^m} \psi(x,0) \eta(u_0(x)) dx \geq 0 \tag{6.2.3}$$

holds for every convex function η, with Q determined through (6.2.1), and all nonnegative Lipschitz continuous test functions ψ on $\mathbb{R}^m \times [0,\infty)$, with compact support.

Applying (6.2.3) with $\eta(u) = \pm u$, $Q(u) = \pm G(u)$ shows that (6.2.3) implies (4.3.2), i.e., any admissible weak solution in the sense of Definition 6.2.1 is in particular a weak solution as defined in Section 4.3. Also note that if u is a classical solution of (6.1.1), (6.1.2), then (6.2.3) holds automatically, as an equality, i.e., all classical solutions are admissible. Several motivations for (6.2.3) will be presented in subsequent sections.

To verify (6.2.3) for all convex η, it would suffice to test it just for some family of convex η with the property that the set of linear combinations of its members, with nonnegative coefficients, spans the entire set of convex functions. To formulate examples, consider the following standard notation: For $w \in \mathbb{R}$, w^+ denotes $\max\{w,0\}$ and $\operatorname{sgn} w$ stands for $-1, 1$ or 0, as w is negative, positive or zero. Notice that any Lipschitz continuous function is the limit of a sequence of piecewise linear convex functions

$$c_0 u + \sum_{i=1}^k c_i (u - u_i)^+ \tag{6.2.4}$$

with $c_i > 0$, $i = 1, \cdots, k$. Consequently, it would suffice to verify (6.2.3) for the entropies $\pm u$, with entropy flux $\pm G$, together with the family of entropy-entropy flux pairs

$$\eta(u;\bar{u}) = (u - \bar{u})^+, \qquad Q(u;\bar{u}) = \operatorname{sgn}(u - \bar{u})^+[G(u) - G(\bar{u})], \tag{6.2.5}$$

where $\bar{u}$ is a parameter taking values in $\mathbb{R}$. Equally well, one may use the celebrated family of entropy-entropy flux pairs of Kruzkov:

$$\eta(u;\bar{u}) = |u - \bar{u}|, \qquad Q(u;\bar{u}) = \operatorname{sgn}(u - \bar{u})[G(u) - G(\bar{u})]. \tag{6.2.6}$$

From Remark 4.5.3 one infers that admissible weak solutions u render the distribution $\partial_t \eta(u) + \operatorname{div} Q(u)$ a measure for any (not necessarily convex) entropy-entropy flux pair.

The fundamental existence and uniqueness theorem, which will be demonstrated by several methods in subsequent sections, is

6.2.2 Theorem. *For each $u_0 \in L^\infty(\mathbb{R}^m)$, there exists a unique admissible weak solution u of* (6.1.1), (6.1.2) *and*

$$u(\cdot,t) \in C^0([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^m)). \tag{6.2.7}$$

The following proposition establishes the most important properties of admissible weak solutions of the scalar conservation law, namely, stability in L^1 and monotonicity in L^∞:

6.2.3 Theorem. *Let u and $\bar{u}$ be admissible weak solutions of* (6.1.1) *with respective initial data u_0 and $\bar{u}_0$ taking values in a compact interval* $[a,b]$. *There is $s > 0$, depending solely on* $[a,b]$, *such that, for any $t > 0$ and $r > 0$*

$$\int_{|x|<r} [u(x,t) - \bar{u}(x,t)]^+ dx \le \int_{|x|<r+st} [u_0(x) - \bar{u}_0(x)]^+ dx, \tag{6.2.8}$$

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathscr{B}_r)} \le \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathscr{B}_{r+st})}. \tag{6.2.9}$$

Furthermore, if

$$u_0(x) \le \bar{u}_0(x), \qquad \text{a.e. on } \mathbb{R}^m, \tag{6.2.10}$$

then

$$u(x,t) \le \bar{u}(x,t), \qquad \text{a.e. on } \mathbb{R}^m \times [0,\infty). \tag{6.2.11}$$

In particular, the (essential) range of both u and $\bar{u}$ is contained in $[a,b]$.

Proof. The salient feature of the scalar conservation law that induces (6.2.8) is that the functions $\eta(u;\bar{u})$, $Q(u;\bar{u})$, defined through (6.2.5), constitute entropy-entropy flux pairs not only in the variable u, for fixed $\bar{u}$, but also in the variable $\bar{u}$, for fixed u.

Consider any nonnegative Lipschitz continuous function $\phi(x,t,\bar{x},\bar{t})$, defined on $\mathbb{R}^m \times [0,\infty) \times \mathbb{R}^m \times [0,\infty)$ and having compact support. Fix $(\bar{x},\bar{t})$ in $\mathbb{R}^m \times [0,\infty)$ and write (6.2.3) for the entropy-entropy flux pair $\eta(u;\bar{u}(\bar{x},\bar{t}))$, $Q(u;\bar{u}(\bar{x},\bar{t}))$, and the test function $\psi(x,t) = \phi(x,t,\bar{x},\bar{t})$:

(6.2.12)
$$\int_0^\infty \int_{\mathbb{R}^m} \{\partial_t \phi(x,t,\bar{x},\bar{t})\, \eta(u(x,t);\bar{u}(\bar{x},\bar{t})) + \sum_{\alpha=1}^m \partial_{x_\alpha} \phi(x,t,\bar{x},\bar{t})\, Q_\alpha(u(x,t);\bar{u}(\bar{x},\bar{t}))\} dxdt$$
$$+ \int_{\mathbb{R}^m} \phi(x,0,\bar{x},\bar{t})\, \eta(u_0(x);\bar{u}(\bar{x},\bar{t}))dx \geq 0.$$

Interchanging the roles of u and $\bar{u}$, we similarly obtain, for any fixed point (x,t) in $\mathbb{R}^m \times [0,\infty)$:

(6.2.13)
$$\int_0^\infty \int_{\mathbb{R}^m} \{\partial_{\bar{t}} \phi(x,t,\bar{x},\bar{t})\, \eta(u(x,t);\bar{u}(\bar{x},\bar{t})) + \sum_{\alpha=1}^m \partial_{\bar{x}_\alpha} \phi(x,t,\bar{x},\bar{t})\, Q_\alpha(u(x,t);\bar{u}(\bar{x},\bar{t}))\} d\bar{x}d\bar{t}$$
$$+ \int_{\mathbb{R}^m} \phi(x,t,\bar{x},0)\, \eta(u(x,t);\bar{u}_0(\bar{x}))\, d\bar{x} \geq 0.$$

Integrating over $\mathbb{R}^m \times [0,\infty)$ (6.2.12), with respect to $(\bar{x},\bar{t})$, and (6.2.13), with respect to (x,t), and then adding the resulting inequalities yields

$$\begin{aligned}(6.2.14)\quad & \int_0^\infty \int_{\mathbb{R}^m} \int_0^\infty \int_{\mathbb{R}^m} \{(\partial_t + \partial_{\bar{t}})\phi(x,t,\bar{x},\bar{t})\, \eta(u(x,t);\bar{u}(\bar{x},\bar{t})) \\ & + \sum_{\alpha=1}^m (\partial_{x_\alpha} + \partial_{\bar{x}_\alpha})\phi(x,t,\bar{x},\bar{t})\, Q_\alpha(u(x,t);\bar{u}(\bar{x},\bar{t}))\} dxdtd\bar{x}d\bar{t} \\ & + \int_0^\infty \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \phi(x,0,\bar{x},\bar{t})\, \eta(u_0(x);\bar{u}(\bar{x},\bar{t}))\, dxd\bar{x}d\bar{t} \\ & + \int_0^\infty \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \phi(x,t,\bar{x},0)\, \eta(u(x,t);\bar{u}_0(\bar{x}))\, dxd\bar{x}dt \geq 0.\end{aligned}$$

We fix a smooth nonnegative function ρ on $\mathbb{R}$ with compact support and total mass one:

$$(6.2.15)\qquad \int_{-\infty}^\infty \rho(\xi)d\xi = 1.$$

Consider any nonnegative Lipschitz test function ψ on $\mathbb{R}^m \times [0,\infty)$, with compact support. For positive small ε, write (6.2.14) with

$$(6.2.16)\qquad \phi(x,t,\bar{x},\bar{t}) = \varepsilon^{-(m+1)} \psi(\frac{x+\bar{x}}{2}, \frac{t+\bar{t}}{2}) \rho(\frac{t-\bar{t}}{2\varepsilon}) \prod_{\beta=1}^m \rho(\frac{x_\beta - \bar{x}_\beta}{2\varepsilon})$$

and then let $\varepsilon \downarrow 0$. Noting that

(6.2.17) $$(\partial_t + \partial_{\bar{t}})\phi(x,t,\bar{x},\bar{t}) = \varepsilon^{-(m+1)}\partial_t\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\rho(\frac{t-\bar{t}}{2\varepsilon})\prod_{\beta=1}^{m}\rho(\frac{x_\beta-\bar{x}_\beta}{2\varepsilon}),$$

(6.2.18)
$$(\partial_{x_\alpha} + \partial_{\bar{x}_\alpha})\phi(x,t,\bar{x},\bar{t}) = \varepsilon^{-(m+1)}\partial_\alpha\psi(\frac{x+\bar{x}}{2},\frac{t+\bar{t}}{2})\rho(\frac{t-\bar{t}}{2\varepsilon})\prod_{\beta=1}^{m}\rho(\frac{x_\beta-\bar{x}_\beta}{2\varepsilon}),$$

(6.2.19) $$|\eta(u(x,t);\bar{u}_0(\bar{x})) - \eta(u_0(x);\bar{u}_0(\bar{x}))| \le |u(x,t) - u_0(x)|,$$

(6.2.20) $$|\eta(u_0(x);\bar{u}(\bar{x},\bar{t})) - \eta(u_0(x);\bar{u}_0(\bar{x}))| \le |\bar{u}(\bar{x},\bar{t}) - \bar{u}_0(\bar{x})|,$$

recalling Theorem 4.5.1, and using standard convergence theorems, we conclude that

(6.2.21)
$$\int_0^\infty\int_{\mathbb{R}^m}\{\partial_t\psi(x,t)\,\eta(u(x,t);\bar{u}(x,t)) + \sum_{\alpha=1}^{m}\partial_\alpha\psi(x,t)\,Q_\alpha(u(x,t);\bar{u}(x,t))\}dxdt$$
$$+\int_{\mathbb{R}^m}\psi(x,0)\,\eta(u_0(x);\bar{u}_0(x))\,dx \ge 0.$$

From (6.2.5) it is clear that there is $s > 0$ such that

(6.2.22) $$|Q(u;\bar{u})| \le s\eta(u;\bar{u}),$$

for all u and $\bar{u}$ in the range of the solutions.

Fix $r > 0$, $t \ge 0$ and $\varepsilon > 0$ small; write (6.2.21) for $\psi(x,\tau) = \chi(x,\tau)\omega(\tau)$, with χ and ω defined by (5.3.12) and (5.3.11) to get

(6.2.23) $$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r}[u(x,\tau)-\bar{u}(x,\tau)]^+ dxd\tau \le \int_{|x|<r+st}[u_0(x)-\bar{u}_0(x)]^+ dx$$
$$-\frac{1}{\varepsilon}\int_0^t\int_{r+s(t-\tau)<x<r+s(t-\tau)+\varepsilon}\left[s\eta(u;\bar{u}) + \frac{Q(u;\bar{u})x}{|x|}\right]dxd\tau + O(\varepsilon).$$

On account of (6.2.22), the second integral on the right-hand side of (6.2.23) is nonnegative. Thus, letting $\varepsilon \downarrow 0$, recalling Theorem 4.5.1, and using that $[\,\cdot\,]^+$ is a convex function, we arrive at (6.2.8).

Interchanging the roles of u and $\bar{u}$ in (6.2.8) we deduce a similar inequality which added to (6.2.8) yields (6.2.9).

Clearly, (6.2.10) implies (6.2.11), by virtue of (6.2.8). In particular, applying this monotonicity property, first for $\bar{u}_0(x) \equiv b$ and then for $u_0(x) \equiv a$, we deduce $u(x,t) \le b$ and $\bar{u}(x,t) \ge a$ a.e. Interchanging the roles of u and $\bar{u}$, we conclude that the essential range of both solutions is contained in $[a,b]$. Thus s in (6.2.22) depends solely on $[a,b]$. This completes the proof.

From (6.2.9) we immediately draw the following conclusion on uniqueness and finite dependence:

6.2.4 Corollary. *There is at most one admissible weak solution of* (6.1.1), (6.1.2).

6.2.5 Corollary. *The value of the admissible weak solution at any point* $(\bar{x},\bar{t})$ *depends solely on the restriction of the initial data to the ball* $\mathscr{B}_{s\bar{t}}(\bar{x})$.

Another important consequence of (6.2.9) is that any admissible weak solution of (6.1.1) with initial data of locally bounded variation is itself a function of locally bounded variation:

6.2.6 Theorem. *Let u be an admissible weak solution of* (6.1.1) *with initial data* $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^m)$ *taking values in an interval* $[a,b]$. *Then* $u \in BV_{\mathrm{loc}}(\mathbb{R}^m \times (0,\infty))$. *For any fixed* $t > 0$, $u(\cdot,t)$ *is in* $BV_{\mathrm{loc}}(\mathbb{R}^m)$ *and*

$$TV_{\mathscr{B}_r} u(\cdot,t) \le TV_{\mathscr{B}_{r+st}} u_0(\cdot), \tag{6.2.24}$$

for every $r > 0$, *where s depends solely on* $[a,b]$.

Proof. Let $\{E_\alpha,\ \alpha = 1,\cdots,m\}$ denote the standard orthonormal basis of $\mathbb{R}^m$. Note that, for $\alpha = 1,\cdots,m$, the function $\bar{u}$, defined by $\bar{u}(x,t) = u(x+hE_\alpha,t)$, $h > 0$, is an admissible weak solution of (6.1.1) with initial data $\bar{u}_0$, $\bar{u}_0(x) = u_0(x+hE_\alpha)$. Therefore, by virtue of (6.2.9), for any $t \in (0,T)$,

$$\int_{|x|<r} |u(x+hE_\alpha,t) - u(x,t)|dx \le \int_{|x|<r+st} |u_0(x+hE_\alpha) - u_0(x)|dx. \tag{6.2.25}$$

Since $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^m)$, Theorem 1.7.2 and (1.7.3) yield that $u(\cdot,t) \in BV_{\mathrm{loc}}(\mathbb{R}^m)$ and (6.2.24) holds.

Thus $\partial_\alpha u(\cdot,t)$ is a Radon measure which is bounded on any ball of radius r in $\mathbb{R}^m$, uniformly on compact time intervals. Since u is bounded, it follows from Theorem 1.7.5 that $\operatorname{div} G(u(\cdot,t))$ has the same property. In particular, the distributions $\partial_\alpha u$ and $\operatorname{div} G(u)$ are locally finite measures on $\mathbb{R}^m \times (0,\infty)$. Because (6.1.1) is satisfied in the sense of distributions, $\partial_t u$ will also be a measure on $\mathbb{R}^m \times (0,\infty)$. Consequently, $u \in BV_{\mathrm{loc}}(\mathbb{R}^m \times (0,\infty))$. This completes the proof.

The trivial, constant, solutions of (6.1.1) are stable, not only in L^1 but also in any L^p. Since u may be renormalized, it suffices to establish L^p-stability for the zero solution.

6.2.7 Theorem. *Let u be an admissible weak solution of* (6.1.1), (6.1.2), *with initial data taking values in a compact interval* $[a,b]$. *There is* $s > 0$, *depending solely on* $[a,b]$, *such that, for any* $1 \le p \le \infty$, $t \ge 0$, *and* $r > 0$,

$$\|u(\cdot,t)\|_{L^p(\mathscr{B}_r)} \le \|u_0(\cdot)\|_{L^p(\mathscr{B}_{r+st})} \, . \tag{6.2.26}$$

Proof. For $1 \le p < \infty$, consider the convex entropy $\eta(u) = |u|^p$, with entropy flux Q determined through (6.2.1). Note that there is $s > 0$, independent of p, such that

$$|Q(u)| \le s\eta(u), \qquad u \in [a,b]. \tag{6.2.27}$$

Fix $r > 0$, $t \ge 0$ and $\varepsilon > 0$ small; write (6.2.3) for the above entropy-entropy flux pair and the test function $\psi(x,\tau) = \chi(x,\tau)\omega(\tau)$, with χ and ω defined by (5.3.12) and (5.3.11). This yields

$$\begin{aligned} \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<r} |u(x,\tau)|^p dx d\tau &\le \int_{|x|<r+st} |u_0(x)|^p dx \\ &- \frac{1}{\varepsilon}\int_0^t \int_{r+s(t-\tau)<|x|<r+s(t-\tau)+\varepsilon} \left[s\eta(u) + \frac{Q(u)x}{|x|} \right] dx d\tau + O(\varepsilon). \end{aligned} \tag{6.2.28}$$

We know that the range of u is contained in $[a,b]$ and so, by (6.2.27), the second integral on the right-hand side of (6.2.28) is nonnegative. Thus, letting $\varepsilon \downarrow 0$ and using that $|u|^p$ is convex, we arrive at (6.2.26). This completes the proof.

The following sections will present various methods of constructing admissible weak solutions of (6.1.1), (6.1.2), inducing alternative proofs of Theorem 6.1.1.

6.3 The Method of Vanishing Viscosity

The aim here is to construct admissible weak solutions of the scalar hyperbolic conservation law (6.1.1) as the $\mu \downarrow 0$ limit of solutions of the family of parabolic equations

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = \mu \Delta u(x,t), \qquad x \in \mathbb{R}^m,\ t \in [0,\infty), \tag{6.3.1}$$

where Δ stands for Laplace's operator with respect to the spatial variables, namely $\Delta = \sum_{\alpha=1}^m \partial_\alpha^2$, and μ is a positive parameter.

The motivation for this approach has already been presented in Section 4.6. Note that (6.3.1) is not necessarily related to any specific physical model and so the term $\mu\Delta u$ should be regarded as "artificial viscosity".

Because (6.3.1) is parabolic, the initial value problem (6.3.1), (6.1.2) always has a unique solution, which is smooth for $t > 0$ (assuming G is regular) even when the initial data u_0 are merely in L^∞. For example, if the derivative G' is Hölder continuous, then the solution u of (6.3.1), (6.1.2) is continuously differentiable with respect to t and twice continuously differentiable with respect to the spatial variables, on $\mathbb{R}^m \times (0,\infty)$.

Espousing the premise that "relevant" solutions of (6.1.1), (6.1.2) are $\mu \downarrow 0$ limits of solutions of (6.3.1), (6.1.2) provides the first justification of the notion of admissible weak solution postulated by Definition 6.2.1:

6.3.1 Theorem. *Let u_μ denote the solution of* (6.3.1), (6.1.2). *Assume that for some sequence $\{\mu_k\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, $\{u_{\mu_k}\}$ converges to some function u, boundedly almost everywhere on $\mathbb{R}^m \times [0,\infty)$. Then u is an admissible weak solution of* (6.1.1), (6.1.2) *on* $\mathbb{R}^m \times [0,\infty)$.

Proof. Consider any smooth convex entropy function η, with associated entropy flux Q determined through (6.2.1). Multiply (6.3.1) by $\eta'(u_\mu(x,t))$ and use (6.2.1) to get

$$\partial_t \eta(u_\mu) + \operatorname{div} Q(u_\mu) = \mu \Delta \eta(u_\mu) - \mu \eta''(u_\mu)|\nabla u_\mu|^2. \tag{6.3.2}$$

Multiply (6.3.2) by any smooth nonnegative test function ψ, with compact support in $\mathbb{R}^m \times [0,\infty)$, integrate over $\mathbb{R}^m \times [0,\infty)$, and integrate by parts. Taking into account that the last term in (6.3.2) is nonnegative yields the inequality

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^m} [\partial_t \psi \eta(u_\mu) + \sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u_\mu)] dxdt + \int_{\mathbb{R}^m} \psi(x,0) \eta(u_0(x))\, dx \\ \geq -\mu \int_0^\infty \int_{\mathbb{R}^m} \Delta \psi \eta(u_\mu)\, dxdt. \end{aligned} \tag{6.3.3}$$

Setting $\mu = \mu_k$ in (6.3.3) and letting $k \to \infty$, we conclude that the limit u of $\{u_{\mu_k}\}$ satisfies (6.2.3) for all smooth convex entropy functions η and all smooth nonnegative test functions ψ. By completion we infer that (6.2.3) holds even when η and ψ are merely Lipschitz continuous. This completes the proof.

That (6.1.1) and (6.3.1) are perfectly matched becomes clear by comparing Theorem 6.2.3 with

6.3.2 Theorem. *Let u_μ and $\bar{u}_\mu$ be solutions of* (6.3.1) *with respective initial data u_0 and $\bar{u}_0$ that are in $L^1(\mathbb{R}^m)$ and take values in a compact interval $[a,b]$. Then, for any $t > 0$,*

$$\int_{\mathbb{R}^m} [u_\mu(x,t) - \bar{u}_\mu(x,t)]^+ dx \leq \int_{\mathbb{R}^m} [u_0(x) - \bar{u}_0(x)]^+ dx, \tag{6.3.4}$$

$$\|u_\mu(\cdot,t) - \bar{u}_\mu(\cdot,t)\|_{L^1(\mathbb{R}^m)} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R}^m)}. \tag{6.3.5}$$

Furthermore, if

$$u_0(x) \leq \bar{u}_0(x), \qquad \textit{a.e. on } \mathbb{R}^m, \tag{6.3.6}$$

then

$$u_\mu(x,t) \leq \bar{u}_\mu(x,t), \qquad \textit{on } \mathbb{R}^m \times (0,\infty). \tag{6.3.7}$$

In particular, the range of both u_μ and $\bar{u}_\mu$ is contained in $[a,b]$.

Proof. To simplify the notation, we drop the subscript μ and denote u_μ and $\bar{u}_\mu$ by u and $\bar{u}$. From standard theory of parabolic equations it follows that when $u_0(\cdot)$ and $\bar{u}_0(\cdot)$ are in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, then $u(\cdot,t)$, $\bar{u}(\cdot,t)$ and their spatial derivatives of any order are also in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, with norms uniformly bounded with respect to t on compact subsets of $(0,\infty)$.

For $\varepsilon > 0$, we define the function η_ε on $\mathbb{R}$ by

$$\eta_\varepsilon(w) = \begin{cases} 0 & -\infty < w \le 0 \\ \dfrac{w^2}{4\varepsilon} & 0 < w \le 2\varepsilon \\ w - \varepsilon & 2\varepsilon < w < \infty. \end{cases} \tag{6.3.8}$$

Since both u and $\bar{u}$ satisfy (6.3.1), one easily verifies the equation

$$\begin{aligned} \partial_t \eta_\varepsilon(u-\bar{u}) &+ \sum_{\alpha=1}^m \partial_\alpha\{\eta'_\varepsilon(u-\bar{u})[G_\alpha(u) - G_\alpha(\bar{u})]\} \\ &- \sum_{\alpha=1}^m \eta''_\varepsilon(u-\bar{u})[G_\alpha(u)-G_\alpha(\bar{u})]\partial_\alpha(u-\bar{u}) \\ &= \mu\Delta\eta_\varepsilon(u-\bar{u}) - \mu\eta''_\varepsilon(u-\bar{u})|\nabla(u-\bar{u})|^2. \end{aligned} \tag{6.3.9}$$

Fix $0 < s < t < \infty$ and integrate (6.3.9) over $\mathbb{R}^m \times (s,t)$. Considering that the last term on the right-hand side of (6.3.9) is nonnegative, we thus obtain the inequality

$$\begin{aligned} \int_{\mathbb{R}^m} \eta_\varepsilon(u(x,t)-\bar{u}(x,t))dx &- \int_{\mathbb{R}^m} \eta_\varepsilon(u(x,s)-\bar{u}(x,s))dx \\ &\le \sum_{\alpha=1}^m \int_s^t \int_{\mathbb{R}^m} \eta''_\varepsilon(u-\bar{u})\,[G_\alpha(u)-G_\alpha(\bar{u})]\,\partial_\alpha(u-\bar{u})\,dxd\tau. \end{aligned} \tag{6.3.10}$$

Notice that $\eta''_\varepsilon(u-\bar{u})[G_\alpha(u)-G_\alpha(\bar{u})]$ is bounded, uniformly for $\varepsilon > 0$. Also, it is clear that as $\varepsilon \downarrow 0$, $\eta_\varepsilon(u(x,t)-\bar{u}(x,t))$ converges pointwise to $[u(x,t)-\bar{u}(x,t)]^+$ while $\eta''_\varepsilon(u(x,t)-\bar{u}(x,t))[G_\alpha(u(x,t))-G_\alpha(\bar{u}(x,t))]$ converges pointwise to zero. Therefore, (6.3.10) and the Lebesgue dominated convergence theorem imply

$$\int_{\mathbb{R}^m} [u(x,t)-\bar{u}(x,t)]^+ dx - \int_{\mathbb{R}^m} [u(x,s)-\bar{u}(x,s)]^+ dx \le 0, \tag{6.3.11}$$

whence we deduce (6.3.4), by letting $s \downarrow 0$.

Interchanging the roles of u and $\bar{u}$ in (6.3.4) we derive a similar inequality which added to (6.3.4) yields (6.3.5).

Clearly, (6.3.6) implies (6.3.7), by virtue of (6.3.4). In particular, applying this monotonicity property, first for $\bar{u}_0(x) \equiv b$ and then for $u_0(x) \equiv a$, we deduce that $u(x,t) \le b$ and $\bar{u}(x,t) \ge a$. Interchanging the roles of u and $\bar{u}$, we conclude that the range of both solutions is contained in $[a,b]$. This completes the proof.

Estimate (6.3.5) may be employed to estimate the modulus of continuity in the mean of solutions of (6.3.1) with initial data in $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$.

6.3.3 Lemma. *Let u_μ be the solution of* (6.3.1), (6.1.2), *where u_0 is in $L^1(\mathbb{R}^m)$ and takes values in a compact interval $[a,b]$. In particular,*

(6.3.12)
$$\int_{\mathbb{R}^m} |u_0(x+y) - u_0(x)| dx \le \omega(|y|), \qquad y \in \mathbb{R}^m,$$

for some nondecreasing function ω on $[0,\infty)$, with $\omega(r) \downarrow 0$ as $r \downarrow 0$. There is a constant c, depending solely on $[a,b]$, such that, for any $t > 0$,

(6.3.13)
$$\int_{\mathbb{R}^m} |u_\mu(x+y,t) - u_\mu(x,t)| dx \le \omega(|y|), \qquad y \in \mathbb{R}^m,$$

(6.3.14)
$$\int_{\mathbb{R}^m} |u_\mu(x,t+h) - u_\mu(x,t)| dx \le c(h^{2/3} + \mu h^{1/3}) \|u_0\|_{L^1(\mathbb{R}^m)} + 2\omega(h^{1/3}), \quad h > 0.$$

Proof. Fix $t > 0$. For any $y \in \mathbb{R}^m$, the function $\bar{u}_\mu(x,t) = u_\mu(x+y,t)$ is the solution of (6.3.1) with initial data $\bar{u}_0(x) = u_0(x+y)$. Applying (6.3.5) yields

(6.3.15)
$$\int_{\mathbb{R}^m} |u_\mu(x+y,t) - u_\mu(x,t)| dx \le \int_{\mathbb{R}^m} |u_0(x+y) - u_0(x)| dx$$

whence (6.3.13) follows.

We now fix $h > 0$. We normalize G by subtracting $G(0)$ so henceforth we may assume, without loss of generality, that $G(0) = 0$. We multiply (6.3.1) by a bounded smooth function ϕ, defined on $\mathbb{R}^m$, and integrate the resulting equation over the strip $\mathbb{R}^m \times (t, t+h)$. Integration by parts yields

(6.3.16)
$$\int_{\mathbb{R}^m} \phi(x)[u_\mu(x,t+h) - u_\mu(x,t)] dx$$
$$= \int_t^{t+h} \int_{\mathbb{R}^m} \{ \sum_{\alpha=1}^m \partial_\alpha \phi(x) G_\alpha(u_\mu(x,\tau)) + \mu \Delta \phi(x) u_\mu(x,\tau) \} dx d\tau.$$

Let us set

(6.3.17)
$$v(x) = u_\mu(x,t+h) - u_\mu(x,t).$$

One may establish (6.3.14) formally by inserting $\phi(x) = \operatorname{sgn} v(x)$ in (6.3.16). However, since the function sgn is discontinuous, we have to mollify it first, with the help of a smooth, nonnegative function ρ on $\mathbb{R}$, with support contained in $[-m^{-1/2}, m^{-1/2}]$ and total mass one, (6.2.15):

(6.3.18)
$$\phi(x) = \int_{\mathbb{R}^m} h^{-m/3} \prod_{\beta=1}^m \rho\left(\frac{x_\beta - z_\beta}{h^{1/3}}\right) \operatorname{sgn} v(z) \, dz.$$

Notice that $|\partial_\alpha \phi| \le c_1 h^{-1/3}$ and $|\Delta\phi| \le c_2 h^{-2/3}$. Moreover, by virtue of (6.3.5), with $\bar{u} \equiv 0$, $\|u(\cdot,\tau)\|_{L^1(\mathbb{R}^m)} \le \|u_0(\cdot)\|_{L^1(\mathbb{R}^m)}$. Therefore, (6.3.16) implies

$$\int_{\mathbb{R}^m} \phi(x)v(x)dx \le c(h^{2/3} + \mu h^{1/3})\|u_0\|_{L^1(\mathbb{R}^m)}, \tag{6.3.19}$$

where c depends solely on $[a,b]$. On the other hand, observing that

(6.3.20)

$$|v(x)| - v(x)\operatorname{sgn} v(z) = |v(x)| - |v(z)| + [v(z) - v(x)]\operatorname{sgn} v(z) \le 2|v(x) - v(z)|,$$

we obtain from (6.3.18):

$$\begin{aligned} |v(x)| - \phi(x)v(x) &= \int_{\mathbb{R}^m} h^{-m/3} \prod_{\beta=1}^m \rho\left(\frac{x_\beta - z_\beta}{h^{1/3}}\right)[|v(x)| - v(x)\operatorname{sgn} v(z)]dz \\ &\le 2\int_{|\xi|<1} \prod_{\beta=1}^m \rho(\xi_\beta)|v(x) - v(x - h^{1/3}\xi)|d\xi. \end{aligned} \tag{6.3.21}$$

Combining (6.3.17), (6.3.21), (6.3.19), and (6.3.13), we arrive at (6.3.14). This completes the proof.

We have now laid the groundwork for presenting a

Proof of Theorem 6.2.2. Assume first that $u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Let u_μ denote the solution of (6.3.1), (6.1.2), with $0 < \mu < 1$. By (6.3.14), $\{u_\mu(\cdot,t)\}$, regarded as a family in $C^0([0,\infty); L^1(\mathbb{R}^m))$, is uniformly equicontinuous. Furthermore, (6.3.13) implies that, for any fixed t in $[0,\infty)$, $\{u_\mu(\cdot,t)\}$ is contained in a compact set of $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Hence, by virtue of the Ascoli theorem, with any sequence $\{\mu_k\}$, $\mu_k \to 0$ as $k \to \infty$, is associated a subsequence, denoted again by $\{\mu_k\}$, and a function u in $C^0([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^m))$ such that $\{u_{\mu_k}(\cdot,t)\}$ converges to $u(\cdot,t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$, uniformly for t in any compact subset of $[0,\infty)$. Passing if necessary to a further subsequence, always denoted by $\{\mu_k\}$, we infer that $\{u_{\mu_k}\}$ converges to u boundedly almost everywhere on $\mathbb{R}^m \times [0,\infty)$, and hence, on account of Theorem 6.3.1, u is an admissible weak solution to (6.1.1), (6.1.2). Since the admissible solution is unique (Corollary 6.2.4), the whole family $\{u_\mu\}$ must converge to u, as $\mu \downarrow 0$. Furthermore, by (6.3.14) and weak lower semicontinuity in $L^1(\mathbb{R}^m)$, $u(\cdot,t)$ lies in $C^0([0,\infty); L^1(\mathbb{R}^m))$.

Suppose now $u_0 \in L^\infty(\mathbb{R}^m)$. For $r > 0$, let χ_r denote the characteristic function of the ball $\mathscr{B}_r(0)$, and u^r denote the admissible weak solution of (6.1.1), with initial data $\chi_r u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. As $r \to \infty$, $\chi_r u_0 \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Therefore, on account of (6.2.9), the family $\{u^r\}$ will converge in L^1_{loc} to some function u. Clearly, u is an admissible weak solution of (6.1.1), (6.1.2). By Corollary 6.2.4, this solution is unique. Now, by Corollary 6.2.5, $u \equiv u^r$ on any compact subset of $\mathbb{R}^m \times [0,\infty)$, if r is sufficiently large. Since $u^r(\cdot,t) \in C^0([0,\infty); L^1(\mathbb{R}^m))$, it follows that $u(\cdot,t)$ is in $C^0([0,\infty); L^1_{\mathrm{loc}}(\mathbb{R}^m))$. This completes the proof.

6.4 Solutions as Trajectories of a Contraction Semigroup and the Large Time Behavior of Periodic Solutions

For $t \in [0,\infty)$, consider the map $S(t)$ that carries $u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ to the admissible weak solution u of (6.1.1), (6.1.2) restricted to t, i.e., $S(t)u_0(\cdot) = u(\cdot,t)$. By virtue of the properties of admissible weak solutions demonstrated in the previous two sections, $S(t)$ maps $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ to $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ and

$$S(0) = I \ \text{(the identity)}, \tag{6.4.1}$$

$$S(t+\tau) = S(t)S(\tau), \qquad \text{for any } t \text{ and } \tau \text{ in } [0,\infty), \tag{6.4.2}$$

$$S(\cdot)u_0 \in C^0([0,\infty); L^1(\mathbb{R}^m)), \tag{6.4.3}$$

$$\|S(t)u_0 - S(t)\bar{u}_0\|_{L^1(\mathbb{R}^m)} \le \|u_0 - \bar{u}_0\|_{L^1(\mathbb{R}^m)}, \qquad \text{for any } t \text{ in } [0,\infty). \tag{6.4.4}$$

Consequently, $S(\cdot)$ is a L^1-contraction semigroup on $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$.

Naturally, the question arises whether one may construct $S(\cdot)$ ab initio, through the theory of nonlinear contraction semigroups in Banach space. This would provide a direct, independent proof of existence of admissible weak solutions of (6.1.1), (6.1.2) as well as an alternative derivation of their properties.

To construct the semigroup, we must realize (6.1.1) as an abstract differential equation

$$\frac{du}{dt} + A(u) \ni 0, \tag{6.4.5}$$

for a suitably defined nonlinear transformation A, with domain $\mathscr{D}(A)$ and range $\mathscr{R}(A)$ in $L^1(\mathbb{R}^m)$. This operator may, in general, be multivalued, i.e., for each $u \in \mathscr{D}(A)$, $A(u)$ will be a nonempty subset of $L^1(\mathbb{R}^m)$ which may contain more than one point.

For u smooth, one should expect $A(u) = \operatorname{div} G(u)$. However, the task of extending $\mathscr{D}(A)$ to u that are not smooth is by no means straightforward, because the construction should somehow reflect the admissibility condition encoded in Definition 6.2.1. First we perform a preliminary extension. For convenience, we normalize G so that $G(0) = 0$.

6.4.1 Definition. The (possibly multivalued) transformation $\hat{A}$, with domain $\mathscr{D}(\hat{A})$ in $L^1(\mathbb{R}^m)$, is defined by $u \in \mathscr{D}(\hat{A})$ and $w \in \hat{A}(u)$ if u, w and $G(u)$ are all in $L^1(\mathbb{R}^m)$ and the inequality

$$\int_{\mathbb{R}^m} \{ \sum_{\alpha=1}^{m} \partial_\alpha \psi(x)\, Q_\alpha(u(x)) + \psi(x)\, \eta'(u(x)) w(x) \} dx \ge 0 \tag{6.4.6}$$

holds for any convex entropy function η, such that η' is bounded on $\mathbb{R}$, with associated entropy flux Q determined through (6.2.1), and for all nonnegative Lipschitz continuous test functions ψ on $\mathbb{R}^m$, with compact support.

Applying (6.4.6) for the entropy-entropy flux pairs $\pm u$, $\pm G(u)$, verifies that

$$\hat{A}(u) = \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(u) \tag{6.4.7}$$

holds, in the sense of distributions, for any $u \in \mathscr{D}(\hat{A})$. In particular, $\hat{A}$ is single-valued. Furthermore, the identity

$$\int_{\mathbb{R}^m} \{\sum_{\alpha=1}^{m} \partial_\alpha \psi Q_\alpha(u) + \psi \eta'(u) \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(u)\} dx = 0, \tag{6.4.8}$$

which is valid for any $u \in C_0^1(\mathbb{R}^m)$ and every entropy-entropy flux pair, implies that $C_0^1(\mathbb{R}^m) \subset \mathscr{D}(\hat{A})$. In particular, $\mathscr{D}(\hat{A})$ is dense in $L^1(\mathbb{R}^m)$. For $u \in C_0^1(\mathbb{R}^m)$, $\hat{A}(u)$ is given by (6.4.7). Thus $\hat{A}$ is indeed an extension of (6.4.7).

The reader may have already noticed the similarity between (6.4.6) and (6.2.3). Similar to (6.2.3), to verify (6.4.6) it would suffice to test it just for the entropies $\pm u$ and the family (6.2.5) or (6.2.6) of entropy-entropy flux pairs.

6.4.2 Definition. The (possibly multivalued) transformation A, with domain $\mathscr{D}(A)$ in $L^1(\mathbb{R}^m)$, is the graph closure of $\hat{A}$, i.e., $u \in \mathscr{D}(A)$ and $w \in A(u)$ if (u,w) is the limit in $L^1(\mathbb{R}^m) \times L^1(\mathbb{R}^m)$ of a sequence $\{(u_k, w_k)\}$ such that $u_k \in \mathscr{D}(\hat{A})$ and $w_k \in \hat{A}(u_k)$.

The following propositions establish properties of A, implying that it is the generator of a contraction semigroup on $L^1(\mathbb{R}^m)$.

6.4.3 Theorem. *The transformation A is accretive, that is if u and $\bar{u}$ are in $\mathscr{D}(A)$, then*

$$\|(u+\lambda w) - (\bar{u}+\lambda \bar{w})\|_{L^1(\mathbb{R}^m)} \geq \|u-\bar{u}\|_{L^1(\mathbb{R}^m)}, \lambda > 0, w \in A(u), \bar{w} \in A(\bar{u}). \tag{6.4.9}$$

Proof. It is the property of accretiveness that renders the semigroup generated by A contractive. Consequently, the proof of Theorem 6.4.3 bears close resemblance to the demonstration of the L^1-contraction estimate (6.2.9) in Theorem 6.2.3.

In view of Definition 6.4.2, it would suffice to show that the "smaller" transformation $\hat{A}$ is accretive. Accordingly, fix some u, $\bar{u}$ in $\mathscr{D}(\hat{A})$ and let $w = \hat{A}(u)$ and $\bar{w} = \hat{A}(\bar{u})$. Consider any nonnegative Lipschitz continuous function ϕ on $\mathbb{R}^m \times \mathbb{R}^m$, with compact support. Fix $\bar{x}$ in $\mathbb{R}^m$ and write (6.4.6) for the entropy-entropy flux pair $\eta(u; \bar{u}(\bar{x}))$, $Q(u; \bar{u}(\bar{x}))$ of the Kruzkov family (6.2.6) and for the test function $\psi(x) = \phi(x, \bar{x})$ to obtain

(6.4.10)

$$\int_{\mathbb{R}^m} \mathrm{sgn}[u(x) - \bar{u}(\bar{x})]\{\sum_{\alpha=1}^{m} \partial_{x_\alpha} \phi(x,\bar{x}) [G_\alpha(u(x)) - G_\alpha(\bar{u}(\bar{x}))] + \phi(x,\bar{x}) w(x)\} dx \geq 0.$$

We may interchange the roles of u and $\bar{u}$ and derive the analog of (6.4.10), for any fixed x in $\mathbb{R}^m$:

(6.4.11)
$$\int_{\mathbb{R}^m} \operatorname{sgn}[\bar{u}(\bar{x}) - u(x)]\{\sum_{\alpha=1}^{m} \partial_{\bar{x}_\alpha}\phi(x,\bar{x})\,[G_\alpha(\bar{u}(\bar{x})) - G_\alpha(u(x))] + \phi(x,\bar{x})\,\bar{w}(\bar{x})\}d\bar{x} \geq 0.$$

Integrating over $\mathbb{R}^m$ (6.4.10), with respect to $\bar{x}$, and (6.4.11), with respect to x, and then adding the resulting inequalities yields

$$\text{(6.4.12)} \quad \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \operatorname{sgn}[u(x) - \bar{u}(\bar{x})]\{\sum_{\alpha=1}^{m}(\partial_{x_\alpha} + \partial_{\bar{x}_\alpha})\phi(x,\bar{x})\,[G_\alpha(u(x)) - G_\alpha(\bar{u}(\bar{x}))]$$
$$+\phi(x,\bar{x})[w(x) - \bar{w}(\bar{x})]\}dxd\bar{x} \geq 0.$$

Fix a smooth nonnegative function ρ on $\mathbb{R}$ with compact support and total mass one, (6.2.15). Take any nonnegative Lipschitz continuous test function ψ on $\mathbb{R}^m$, with compact support. For positive small ε, write (6.4.12) with

$$\text{(6.4.13)} \qquad \phi(x,\bar{x}) = \varepsilon^{-m}\psi(\frac{x+\bar{x}}{2})\prod_{\beta=1}^{m}\rho(\frac{x_\beta - \bar{x}_\beta}{2\varepsilon}),$$

and let $\varepsilon \downarrow 0$. Noting that

$$\text{(6.4.14)} \qquad (\partial_{x_\alpha} + \partial_{\bar{x}_\alpha})\phi(x,\bar{x}) = \varepsilon^{-m}\partial_\alpha\psi(\frac{x+\bar{x}}{2})\prod_{\beta=1}^{m}\rho(\frac{x_\beta - \bar{x}_\beta}{2\varepsilon}),$$

(6.4.15)
$$\int_{\mathbb{R}^m}\sigma(x)\{\sum_{\alpha=1}^{m}\partial_\alpha\psi(x)\,[G_\alpha(u(x)) - G_\alpha(\bar{u}(x))] + \psi(x)[w(x) - \bar{w}(x)]\}dx \geq 0,$$

where σ is some function such that

$$\text{(6.4.16)} \qquad \sigma(x)\begin{cases} = 1 & \text{if } u(x) > \bar{u}(x) \\ \in [-1,1] & \text{if } u(x) = \bar{u}(x) \\ = -1 & \text{if } u(x) < \bar{u}(x). \end{cases}$$

Upon choosing ψ with $\psi(x) = 1$ for $|x| < r$, $\psi(x) = 1 + r - |x|$ for $r \leq |x| < r+1$ and $\psi(x) = 0$ for $r+1 \leq |x| < \infty$, and letting $r \to \infty$, we obtain

$$\text{(6.4.17)} \qquad \int_{\mathbb{R}^m}\sigma(x)[w(x) - \bar{w}(x)]dx \geq 0,$$

for some function σ as in (6.4.16).

Take now any $\lambda > 0$ and use (6.4.17), (6.4.16) to conclude

(6.4.18)
$$\begin{aligned}\|(u+\lambda w)-(\bar{u}+\lambda\bar{w})\|_{L^1(\mathbb{R}^m)} &\geq \int_{\mathbb{R}^m}\sigma(x)\{u(x)-\bar{u}(x)+\lambda[w(x)-\bar{w}(x)]\}dx\\ &\geq \int_{\mathbb{R}^m}\sigma(x)[u(x)-\bar{u}(x)]dx = \|u-\bar{u}\|_{L^1(\mathbb{R}^m)}\,.\end{aligned}$$

This completes the proof.

An immediate consequence (actually an alternative, equivalent restatement) of the assertion of Theorem 6.4.3 is

6.4.4 Corollary. *For any $\lambda > 0$, $(I+\lambda A)^{-1}$ is a well-defined, single-valued, L^1-contractive transformation, defined on the range $\mathscr{R}(I+\lambda A)$ of $I+\lambda A$.*

6.4.5 Theorem. *The transformation A is maximal, that is*

(6.4.19)
$$\mathscr{R}(I+\lambda A) = L^1(\mathbb{R}^m), \quad \textit{for any } \lambda > 0.$$

Proof. By virtue of Definition 6.4.2 and Corollary 6.4.4, it will suffice to show that $\mathscr{R}(I+\lambda\hat{A})$ is dense in $L^1(\mathbb{R}^m)$; for instance that it contains $L^1(\mathbb{R}^m)\cap L^\infty(\mathbb{R}^m)$. We thus fix $f \in L^1(\mathbb{R}^m)\cap L^\infty(\mathbb{R}^m)$ and seek solutions $u \in \mathscr{D}(\hat{A})$ of the equation

(6.4.20)
$$u+\lambda\hat{A}(u) = f.$$

Recall that $\hat{A}(u)$ admits the representation (6.4.7), in the sense of distributions. Thus, solving (6.4.20) amounts to determining an admissible weak solution of a first-order quasilinear partial differential equation, namely the stationary analog of (6.1.1).

Motivated by the method of vanishing viscosity, discussed in Section 6.3, we shall construct solutions to (6.4.20) as the $\mu \downarrow 0$ limit of solutions of the family of elliptic equations

(6.4.21)
$$u(x)+\lambda\,\mathrm{div}\,G(u(x))-\mu\Delta u(x) = f(x), \qquad x \in \mathbb{R}^m.$$

For any fixed $\mu > 0$, (6.4.21) admits a solution in $H^2(\mathbb{R}^m)$. We have to show that, as $\mu \downarrow 0$, the family of solutions of (6.4.21) converges, boundedly almost everywhere, to some function u which is the solution of (6.4.20). The proof will be partitioned into the following steps.

6.4.6 Lemma. *Let u_μ and $\bar{u}_\mu$ be solutions of* (6.4.21) *with respective right-hand sides f and $\bar{f}$ that are in $L^1(\mathbb{R}^m)$ and take values in a compact interval $[a,b]$. Then*

(6.4.22)
$$\int_{\mathbb{R}^m}[u_\mu(x)-\bar{u}_\mu(x)]^+dx \leq \int_{\mathbb{R}^m}[f(x)-\bar{f}(x)]^+dx,$$

(6.4.23)
$$\|u_\mu-\bar{u}_\mu\|_{L^1(\mathbb{R}^m)} \leq \|f-\bar{f}\|_{L^1(\mathbb{R}^m)}\,.$$

Furthermore, if

$$f(x) \le \bar{f}(x), \qquad \text{a.e. on } \mathbb{R}^m, \tag{6.4.24}$$

then

$$u_\mu(x) \le \bar{u}_\mu(x), \qquad \text{on } \mathbb{R}^m. \tag{6.4.25}$$

In particular, the range of both u and $\bar{u}$ is contained in $[a,b]$.

Proof. It is very similar to the proof of Theorem 6.3.2 and so it will be left to the reader.

6.4.7 Lemma. *Let u_μ denote the solution of* (6.4.21), *with right-hand side f in $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Then, as $\mu \downarrow 0, \{u_\mu\}$ converges boundedly a.e. to the solution u of* (6.4.20).

Proof. For any $y \in \mathbb{R}^m$, the function $\bar{u}_\mu$, defined by $\bar{u}_\mu(x) = u_\mu(x+y)$, is a solution of (6.4.21) with right-hand side $\bar{f}, \bar{f}(x) = f(x+y)$. Hence, by (6.4.23),

$$\int_{\mathbb{R}^m} |u_\mu(x+y) - u_\mu(x)| dx \le \int_{\mathbb{R}^m} |f(x+y) - f(x)| dx. \tag{6.4.26}$$

Thus the family $\{u_\mu\}$ is uniformly bounded and uniformly equicontinuous in L^1. It follows that every sequence $\{\mu_k\}$, with $\mu_k \to 0$ as $k \to \infty$, will contain a subsequence, labeled again as $\{\mu_k\}$, such that

$$u_{\mu_k} \to u, \quad \text{boundedly a.e. on } \mathbb{R}^m, \tag{6.4.27}$$

where u is in $L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$.

Consider now any smooth convex entropy function η, with associated entropy flux Q, determined by (6.2.1). Then u_μ will satisfy the identity

$$\eta'(u_\mu)u_\mu + \lambda \operatorname{div} Q(u_\mu) - \mu \Delta \eta(u_\mu) + \mu \eta''(u_\mu)|\nabla u_\mu|^2 = \eta'(u_\mu) f. \tag{6.4.28}$$

Multiplying (6.4.28) by any nonnegative smooth test function ψ on $\mathbb{R}^m$, with compact support, and integrating over $\mathbb{R}^m$ yields

$$\int_{\mathbb{R}^m} \{\lambda \sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u_\mu) + \psi \eta'(u_\mu)(f - u_\mu)\} dx \ge -\mu \int_{\mathbb{R}^m} \Delta \psi \eta \, dx. \tag{6.4.29}$$

From (6.4.27) and (6.4.29),

$$\int_{\mathbb{R}^m} \{\sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u) + \psi \eta'(u) \lambda^{-1} (f - u)\} dx \ge 0, \tag{6.4.30}$$

which shows that u is indeed a solution of (6.4.20).

By virtue of Corollary 6.4.4, the solution of (6.4.20) is unique and so the entire family $\{u_\mu\}$ converges to u, as $\mu \downarrow 0$. This completes the proof.

Once accretiveness and maximality have been established, the Crandall-Liggett theory of semigroups in nonreflexive Banach space ensures that A generates a contraction semigroup $S(\cdot)$ on $\overline{\mathscr{D}(A)} = L^1(\mathbb{R}^m)$. $S(\cdot)u_0$ can be constructed by solving the differential equation (6.4.5) through the implicit difference scheme

$$\text{(6.4.31)} \qquad \begin{cases} \dfrac{1}{\varepsilon}[u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)] + A(u_\varepsilon(t)) \ni 0, & t > 0 \\ u_\varepsilon(t) = u_0\,, & t < 0. \end{cases}$$

For any $\varepsilon > 0$, a unique solution u_ε of (6.4.31) exists on $[0,\infty)$, by virtue of Theorem 6.4.5 and Corollary 6.4.4. It can be shown, further, that Corollary 6.4.4 provides the necessary stability to ensure that, as $\varepsilon \downarrow 0$, $u_\varepsilon(\cdot)$ converges, uniformly on compact subsets of $[0,\infty)$, to some function that we denote by $S(\cdot)u_0$.

The general properties of $S(\cdot)$ follow from the Crandall-Liggett theory: When $u_0 \in \mathscr{D}(A)$, $S(t)u_0$ stays in $\mathscr{D}(A)$ for all $t \in [0,\infty)$. In general, $S(t)u_0$ may fail to be differentiable with respect to t, even when $u_0 \in \mathscr{D}(A)$. Thus $S(\cdot)u_0$ should be interpreted as a weak solution of the differential equation (6.4.5).

The special properties of $S(\cdot)$ are consequences of the special properties of A induced by the propositions recorded above (e.g. Lemma 6.4.6). The following theorem, whose proof can be found in the references cited in Section 6.11, summarizes the properties of $S(\cdot)$ and in particular provides an alternative proof for the existence of a unique admissible weak solution to (6.1.1), (6.1.2) (Theorem 6.2.2) and its basic properties (Theorems 6.2.3 and 6.2.7).

6.4.8 Theorem. *The transformation A generates a contraction semigroup $S(\cdot)$ in $L^1(\mathbb{R}^m)$, namely a family of maps $S(t) : L^1(\mathbb{R}^m) \to L^1(\mathbb{R}^m)$, $t \in [0,\infty)$, which satisfy the semigroup property* (6.4.1), (6.4.2); *the continuity property* (6.4.3), *for any u_0 in $L^1(\mathbb{R}^m)$; and the contraction property* (6.4.4), *for any u_0, $\bar{u}_0$ in $L^1(\mathbb{R}^m)$. If*

$$\text{(6.4.32)} \qquad u_0 \le \bar{u}_0\,, \qquad \textit{a.e. on } \mathbb{R}^m,$$

then

$$\text{(6.4.33)} \qquad S(t)u_0 \le S(t)\bar{u}_0\,, \qquad \textit{a.e. on } \mathbb{R}^m.$$

For $1 \le p \le \infty$, the sets $L^p(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ are positively invariant under $S(t)$ and, for any $t \in [0,\infty)$,

$$\text{(6.4.34)} \qquad \|S(t)u_0\|_{L^p(\mathbb{R}^m)} \le \|u_0\|_{L^p(\mathbb{R}^m)}\,, \qquad \textit{for all } u_0 \in L^p(\mathbb{R}^m) \cap L^1(\mathbb{R}^m).$$

If $u_0 \in L^\infty(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, then $S(\cdot)u_0$ is the admissible weak solution of (6.1.1), (6.1.2), *in the sense of Definition* 6.2.1.

The reader should note that the approach via semigroups suggests a notion of admissible weak solution to (6.1.1), (6.1.2) for any, even unbounded, u_0 in $L^1(\mathbb{R}^m)$. These are not necessarily distributional solutions of (6.1.1), unless the flux G exhibits linear growth at infinity.

The theory of contraction semigroups in Banach space provides the proper setting for describing the long time behavior of solutions to the Cauchy problem (6.1.1), (6.1.2), when the initial data are periodic, say

$$u_0(x+e_i) = u_0(x), \quad x \in \mathbb{R}^m, \ i = 1, \dots, m, \tag{6.4.35}$$

where $(e_1, \dots, e_m)$ is the standard basis of $\mathbb{R}^m$. The spatial periodicity property is passed on to the solution u, for any fixed $t \in [0, \infty)$. It is thus expedient to realize $u_0(\cdot)$ and $u(\cdot, t)$ as functions on the standard torus $\mathbb{T}^m$, in which case, by virtue of (6.2.9), the family of maps $S(t)$, $t \geq 0$, that carry $u_0(\cdot)$ to $u(\cdot, t)$ constitutes a L^1-contraction semigroup on $L^\infty(\mathbb{T}^m)$.

The L^1 contraction property implies that the *orbit* $\gamma(u_0) = \bigcup_{t \geq 0} S(t)u_0$ of the trajectory Su_0 of the semigroup, emanating from u_0, is relatively compact in $L^1(\mathbb{T}^m)$. Furthermore, by the semigroup property, $\gamma(S(t)u_0) \subset \gamma(u_0)$, for all $t \in [0, \infty)$. Thus, the nonempty, compact *omega limit set*

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\gamma(S(t)u_0)}, \tag{6.4.36}$$

of u_0 encodes the long time behavior of the solution of (6.1.1), (6.1.2), because, as $t \to \infty$, $u(\cdot, t) \to \omega(u_0)$, in $L^p(\mathbb{T}^m)$, for any $1 \leq p < \infty$.

The omega limit set is *invariant*, $S(t)\omega(u_0) = \omega(u_0)$, for $0 \leq t < \infty$, and *minimal*, in that $\omega(u_0) = \overline{\gamma(v_0)} = \omega(v_0)$, for any $v_0 \in \omega(u_0)$. To see that, suppose $v_0 = \lim_{n \to \infty} S(t_n)u_0$ and $w_0 = \lim_{k \to \infty} S(\tau_k)u_0$, with $t_n \to \infty$ and $\tau_k \to \infty$. Fix any subsequence $\{\tau_{k_n}\}$ of $\{\tau_k\}$, with $s_n = \tau_{k_n} - t_n > n$. Since

$$\|S(s_n)v_0 - w_0\|_{L^1} \leq \|S(s_n)v_0 - S(s_n)S(t_n)u_0\|_{L^1} + \|S(\tau_{k_n})u_0 - w_0\|_{L^1}, \tag{6.4.37}$$

we conclude that $S(s_n)v_0 \to w_0$, as $n \to \infty$, i.e., $w_0 \in \omega(v_0)$.

The minimality of $\omega(u_0)$, in conjunction with Theorem 6.2.7 and the weak lower semicontinuity of L^p norms, implies that, for any $1 \leq p \leq \infty$, $\omega(u_0)$ lies on a sphere in $L^p(\mathbb{T}^m)$, centered at 0. It is also easy to see (references in Section 6.11) that the semigroup S restricted to $\omega(u_0)$ becomes a semigroup of L^1 isometries, which admits an extension into a group $\hat{S}$ of L^1 isometries. Furthermore, for any $v_0 \in \omega(u_0)$, $\hat{S}(t)v_0$ is an almost periodic function with values in $L^1(\mathbb{T}^m)$.

The case of (6.1.1) with linear flux demonstrates that an omega limit set that satisfies all the constraints listed in the previous paragraph may still be quite large. However, nonlinearity in the flux induces damping that may shrink the omega limit set to a single point, namely the constant function equal to the conserved mean value of the solution.

Linear degeneracy of G in the spatial direction marked by the nonzero vector $\xi \in \mathbb{R}^m$ is encoded in the set

$$\mathscr{N}_\xi = \{u \in \mathbb{R} \,:\, \xi \cdot G''(u) = 0\}. \tag{6.4.38}$$

The following proposition states optimal conditions for the asymptotic decay of periodic solutions to their mean value.

6.4.9 Theorem. *Let u be the admissible solution to the Cauchy problem* (6.1.1), (6.1.2), *with initial data $u_0 \in L^\infty(\mathbb{R}^m)$ satisfying the periodicity condition* (6.4.35). *If the mean value*

$$\int_{\mathbb{T}^m} u_0(x)dx = \bar{u} \tag{6.4.39}$$

is not an interior point of the set $\mathscr{N}_\xi$, for any nonzero $\xi \in \mathbb{Z}^m$, then

$$u(\cdot,t) \to \bar{u}, \qquad \textit{as } t \to \infty, \tag{6.4.40}$$

the convergence being in $L^p(\mathbb{T}^m)$, for every $1 \le p < \infty$.

Sketch of Proof. Assume, without loss of generality, that $u_0 \in BV(\mathbb{T}^m)$, in which case $\omega(u_0) \subset BV(\mathbb{T}^m)$. Fix any $v_0 \in \omega(u_0)$. One needs to show that $v_0 = \bar{u}$, a.e. on $\mathbb{T}^m$. Since the mean value of v_0 is $\bar{u}$, it would suffice to prove that the set $\mathscr{V}$ of x in $\mathbb{T}^m$ with $v_0(x) > \bar{u}$ has measure zero, or equivalently that the BV function ϕ,

$$\phi(x) = \chi_{\mathscr{V}}(x) - \int_{\mathbb{T}^m} \chi_{\mathscr{V}}(y)dy, \qquad x \in \mathbb{T}^m, \tag{6.4.41}$$

vanishes a.e. on $\mathbb{T}^m$. By basic integral geometry, one may demonstrate that ϕ vanishes by establishing the vanishing of its integral over every geodesic hyperplane of codimension one, $\mathbb{P}_{\xi,\rho} = \{x \,:\, \xi \cdot x = \rho\}$, with $\xi \in \mathbb{Z}^m \backslash \{0\}$ and $\rho \in \mathbb{R}$. The proof of that, found in the references cited in Section 6.11, rests on the minimality of $\omega(u_0)$ which in particular implies that $v_0 \in \omega(v_0)$.

6.5 The Layering Method

The admissible weak solution of (6.1.1), (6.1.2) will here be determined as the $h \downarrow 0$ limit of a family $\{u_h\}$ of functions constructed by patching together classical solutions of (6.1.1) in a stratified pattern. In addition to providing another method for constructing solutions and thereby an alternative proof of the existence Theorem 6.2.2, this approach also offers a different justification of the admissibility condition, Definition 6.2.1.

The initial data u_0 are in $L^\infty(\mathbb{R}^m)$, taking values in a compact interval $[a,b]$. The construction of approximate solutions will involve mollification of functions on $\mathbb{R}^m$ by forming their convolution with a kernel λ_h constructed as follows. We start out with a nonnegative, smooth function ρ on $\mathbb{R}$, supported in $[-1,1]$, which is even, $\rho(-\xi) = \rho(\xi)$ for $\xi \in \mathbb{R}$, and has total mass one, (6.2.15). For $h > 0$, we set

(6.5.1) $$\lambda_h(x) = (ph)^{-m}\prod_{\beta=1}^{m}\rho(\frac{x_\beta}{ph}),$$

with

(6.5.2) $$p = \sqrt{m}\,q\gamma\|u_0\|_{L^\infty(\mathbb{R}^m)}\,,$$

where q denotes the total variation of the function ρ and γ is the maximum of $|G''(u)|$ over the interval $[a,b]$. We employ λ_h to mollify functions $f \in L^\infty(\mathbb{R}^m)$:

(6.5.3) $$(\lambda_h * f)(x) = \int_{\mathbb{R}^m}\lambda_h(x-y)f(y)dy, \qquad x\in\mathbb{R}^m.$$

From (6.5.3) and (6.5.1) it follows easily

(6.5.4) $$\inf(\lambda_h * f) \geq \operatorname{ess\,inf} f, \qquad \sup(\lambda_h * f) \leq \operatorname{ess\,sup} f,$$

(6.5.5) $$\|\lambda_h * f\|_{L^1(\mathscr{B}_r)} \leq \|f\|_{L^1(\mathscr{B}_{r+\sqrt{m}ph})}\,, \qquad \text{for any } \ r>0,$$

(6.5.6) $$\|\partial_\alpha(\lambda_h * f)\|_{L^\infty(\mathbb{R}^m)} \leq \frac{q}{ph}\|f\|_{L^\infty(\mathbb{R}^m)}\,, \qquad \alpha = 1,\cdots,m.$$

A somewhat subtler estimate, which depends crucially on λ_h being an even function, and whose proof can be found in the references cited in Section 6.11, is

(6.5.7) $$|\int_{\mathbb{R}^m}\chi(x)[(\lambda_h * f)(x) - f(x)]dx| \leq ch^2\|\chi\|_{C^2(\mathbb{R}^m)}\|f\|_{L^\infty(\mathbb{R}^m)}\,,$$

for all $\chi \in C_0^\infty(\mathbb{R}^m)$.

The construction of the approximate solutions proceeds as follows. After the parameter $h>0$ has been fixed, $\mathbb{R}^m\times[0,\infty)$ is partitioned into layers:

(6.5.8) $$\mathbb{R}^m\times[0,\infty) = \bigcup_{\ell=0}^{\infty}\mathbb{R}^m\times[\ell h, \ell h + h).$$

The initial value $u_h(\cdot,0)$ is determined by

(6.5.9) $$u_h(\cdot,0) = \lambda_h * u_0(\cdot).$$

By virtue of (6.5.6) and (6.5.2), $u_h(\cdot,0)$ is Lipschitz continuous, with Lipschitz constant $\omega = 1/p\gamma$. Hence, by Theorem 6.1.1, (6.1.1) with initial data $u_h(\cdot,0)$ admits a classical solution u_h on the layer $\mathbb{R}^m\times[0,h)$.

Next we determine $u_h(\cdot,h)$ by mollifying the limit $u_h(\cdot,h-)$ of $u_h(\cdot,t)$ as $t\uparrow h$:

(6.5.10) $$u_h(\cdot,h) = \lambda_h * u_h(\cdot,h-).$$

We extend u_h to the layer $\mathbb{R}^m\times[h,2h)$ by solving (6.1.1) with data $u_h(\cdot,h)$ at $t=h$.

Continuing this process, we determine u_h on the general layer $[\ell h, \ell h + h)$ by solving (6.1.1) with data

$$u_h(\cdot, \ell h) = \lambda_h * u_h(\cdot, \ell h-) \tag{6.5.11}$$

at $t = \ell h$. We thus end up with a measurable function u_h on $\mathbb{R}^m \times [0, \infty)$ which takes values in the interval $[a, b]$. Inside each layer $\mathbb{R}^m \times [\ell h, \ell h + h)$, u_h is a classical solution of (6.1.1). However, as one crosses the border $t = \ell h$ between adjacent layers, u_h experiences jump discontinuities, from $u_h(\cdot, \ell h-)$ to $u_h(\cdot, \ell h)$.

6.5.1 Theorem. *As $h \downarrow 0$, the family $\{u_h\}$ constructed above converges boundedly almost everywhere on $\mathbb{R}^m \times [0, \infty)$ to the admissible solution u of* (6.1.1), (6.1.2).

The proof is an immediate consequence of the following two propositions together with uniqueness of the admissible solution, Corollary 6.2.4. The fact that the limit of classical solutions yields the admissible weak solution provides another justification of Definition 6.2.1.

6.5.2 Lemma. (*Consistency*). *Assume that for some sequence $\{h_k\}$, with $h_k \to 0$ as $k \to \infty$,*

$$u_{h_k}(x,t) \to u(x,t), \qquad a.e.\ on\ \mathbb{R}^m \times [0, \infty). \tag{6.5.12}$$

Then u is an admissible weak solution of (6.1.1), (6.1.2).

Proof. Consider any convex entropy function η with associated entropy flux Q determined through (6.2.1). In the interior of each layer, u_h is a classical solution of (6.1.1) and so it satisfies the identity

$$\partial_t \eta(u_h(x,t)) + \operatorname{div} Q(u_h(x,t)) = 0. \tag{6.5.13}$$

Fix any nonnegative smooth test function ψ on $\mathbb{R}^m \times [0, \infty)$, with compact support. Multiply (6.5.13) by ψ, integrate over each layer, integrate by parts, and then sum the resulting equations over all layers to get

$$\int_0^\infty \int_{\mathbb{R}^m} [\partial_t \psi \eta(u_h) + \sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u_h)] dx dt + \int_{\mathbb{R}^m} \psi(x,0) \eta(u_h(x,0)) dx \tag{6.5.14}$$
$$= -\sum_{\ell=1}^\infty \int_{\mathbb{R}^m} \psi(x, \ell h)[\eta(u_h(x, \ell h)) - \eta(u_h(x, \ell h-))] dx.$$

Combining (6.5.11) with Jensen's inequality and using (6.5.7) yields

$$\begin{aligned}(6.5.15)\qquad &\int_{\mathbb{R}^m} \psi(x,\ell h)[\eta(u_h(x,\ell h)) - \eta(u_h(x,\ell h-))]dx \\ &\leq \int_{\mathbb{R}^m} \psi(x,\ell h)[\lambda_h * \eta(u_h)(x,\ell h-)) - \eta(u_h(x,\ell h-))]dx \leq Ch^2.\end{aligned}$$

The summation on the right-hand side of (6.5.14) contains $O(1/h)$ many nonzero terms. Therefore, passing to the $k \to \infty$ limit along the sequence $\{h_k\}$ in (6.5.14) and using (6.5.12), (6.5.9), and (6.5.15), we conclude that u satisfies (6.2.3). This completes the proof.

6.5.3 Lemma. (*Compactness*). *There is a sequence $\{h_k\}$, with $h_k \to 0$ as $k \to \infty$, and a L^∞ function u on $\mathbb{R}^m \times [0,\infty)$ such that* (6.5.12) *holds.*

Proof. The first step is to establish the weaker assertion that for some sequence $\{h_k\}$, with $h_k \to 0$ as $k \to \infty$, and a function u,

$$(6.5.16)\qquad u_{h_k}(\cdot,t) \to u(\cdot,t), \qquad \text{as } k \to \infty, \qquad \text{in } L^\infty(\mathbb{R}^m) \text{ weak}^*,$$

for almost all t in $[0,\infty)$. To this end, fix any smooth test function χ on $\mathbb{R}^m$, with compact support, and consider the function

$$(6.5.17)\qquad v_h(t) = \int_{\mathbb{R}^m} \chi(x) u_h(x,t)dx, \qquad t \in [0,\infty).$$

Notice that v_h is smooth on $[\ell h, \ell h + h)$ and satisfies

$$\begin{aligned}(6.5.18)\qquad \int_{\ell h}^{\ell h+h} \left|\frac{d}{dt} v_h(t)\right| dt &= \int_{\ell h}^{\ell h+h} \left| -\int_{\mathbb{R}^m} \chi(x) \sum_{\alpha=1}^m \partial_\alpha G_\alpha(u(x,t))dx \right| dt \\ &= \int_{\ell h}^{\ell h+h} \left| \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \chi(x)\, G_\alpha(u(x,t))dx \right| dt \leq Ch.\end{aligned}$$

On the other hand, v_h experiences jump discontinuities across the points $t = \ell h$ which can be estimated with the help of (6.5.11) and (6.5.7):

$$(6.5.19)\qquad |v_h(\ell h) - v_h(\ell h-)| = \left| \int_{\mathbb{R}^m} \chi(x)[u_h(x,\ell h) - u_h(x,\ell h-)]dx \right| \leq Ch^2.$$

From (6.5.18) and (6.5.19) it follows that the total variation of v_h over any compact subinterval of $[0,\infty)$ is bounded, uniformly in h. Therefore, by Helly's theorem (cf. Section 1.7), there is a sequence $\{h_k\}$, $h_k \to 0$ as $k \to \infty$, such that $v_{h_k}(t)$ converges for almost all t in $[0,\infty)$.

By Cantor's diagonal process, we may construct a subsequence of $\{h_k\}$, which will be denoted again by $\{h_k\}$, such that the sequence

$$(6.5.20)\qquad \left\{ \int_{\mathbb{R}^m} \chi(x) u_{h_k}(x,t)dx \right\}$$

converges for almost all t, for every member χ of any given countable family of test functions. Consequently, the sequence (6.5.20) converges for any χ in $L^1(\mathbb{R}^m)$. Thus, for almost any t in $[0,\infty)$ there is a bounded measurable function on $\mathbb{R}^m$, denoted by $u(\cdot,t)$, such that (6.5.16) holds.

We now strengthen the mode of convergence in (6.5.16). For any $y \in \mathbb{R}^m$, the functions u_h and $\bar{u}_h$, $\bar{u}_h(x,t) = u_h(x+y,t)$, are both solutions of (6.1.1) in every layer. Let us fix $t > 0$ and $r > 0$. Suppose $t \in [\ell h, \ell h + h)$. Applying repeatedly (6.2.9) and (6.5.5) (recalling (6.5.11)), we conclude

(6.5.21)

$$\begin{aligned}\int_{|x|<r} |u_h(x+y,t) - u_h(x,t)|dx &\leq \int_{|x|<r+s(t-\ell h)} |u_h(x+y,\ell h) - u_h(x,\ell h)|dx \\ &\leq \int_{|x|<r+s(t-\ell h)+\sqrt{m}ph} |u_h(x+y,\ell h-) - u_h(x,\ell h-)|dx \\ &\leq \dots \leq \int_{|x|<r+st+\sqrt{m}p(t+h)} |u_0(x+y) - u_0(x)|dx.\end{aligned}$$

It follows that the family $\{u_h(\cdot,t)\}$ is equicontinuous in the mean on every compact subset of $\mathbb{R}^m$. Therefore, the convergence in (6.5.16) is upgraded to strongly in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Thus, passing to a final subsequence we arrive at (6.5.12). This completes the proof.

6.6 Relaxation

Another interesting method for constructing admissible weak solutions of (6.1.1) is through *relaxation*. The point of departure is a semilinear system of $m+1$ equations,

$$\text{(6.6.1)} \quad \begin{cases} \partial_t v(x,t) + \displaystyle\sum_{\alpha=1}^{m} c_\alpha \partial_\alpha v(x,t) = \frac{1}{\mu} \sum_{\alpha=1}^{m} [F_\alpha(v(x,t)) - Z_\alpha(x,t)] \\ \partial_t Z_\alpha(x,t) - c_\alpha \partial_\alpha Z_\alpha(x,t) = \dfrac{1}{\mu}[F_\alpha(v(x,t)) - Z_\alpha(x,t)], \quad \alpha = 1,\cdots,m, \end{cases}$$

in the $m+1$ unknowns $(v, Z_1, \cdots, Z_m)$, where μ is a small positive parameter while, for $\alpha = 1,\cdots,m$, the c_α are given constants and the F_α are specified smooth functions such that

$$\text{(6.6.2)} \qquad F'_\alpha(v) < 0, \quad -\infty < v < \infty, \quad \alpha = 1,\cdots,m,$$

$$\text{(6.6.3)} \qquad F_\alpha(0) = 0, \quad F_\alpha(v) \to \pm\infty, \quad \text{as } v \to \mp\infty, \qquad \alpha = 1,\cdots,m.$$

Notice that solutions of (6.6.1) satisfy the conservation law

$$\partial_t[v(x,t) - \sum_{\alpha=1}^{m} Z_\alpha(x,t)] + \sum_{\alpha=1}^{m} c_\alpha \partial_\alpha [v(x,t) + Z_\alpha(x,t)] = 0. \tag{6.6.4}$$

Because of the form of the right-hand side of (6.6.1), one should expect that, as $\mu \downarrow 0$, the variables Z_α "relax" to their equilibrium states $F_\alpha(v)$, in which case (6.6.4) reduces to a scalar conservation law (6.1.1) with[1]

$$u = v - \sum_{\alpha=1}^{m} F_\alpha(v), \qquad G_\alpha(u) = c_\alpha[v + F_\alpha(v)], \quad \alpha = 1, \cdots, m. \tag{6.6.5}$$

The above considerations suggest a program for constructing solutions of (6.1.1) as asymptotic limits of solutions of (6.6.1).

The first step is to examine the Cauchy problem for (6.6.1), under assigned initial conditions

$$v(x,0) = v_0(x), \qquad Z_\alpha(x,0) = Z_{\alpha 0}(x), \quad \alpha = 1, \cdots, m, \quad x \in \mathbb{R}^m. \tag{6.6.6}$$

Since (6.6.1) is semilinear hyperbolic, when the initial data $(v_0, Z_{10}, \cdots, Z_{m0})$ are in $C_0^1(\mathbb{R}^m)$ there exists a unique classical solution $(v, Z_1, \cdots, Z_m)$ defined on a maximal time interval $[0,T)$, for some $0 < T \leq \infty$. For any $t \in [0,T)$, the functions $(v(\cdot,t), Z_1(\cdot,t), \cdots, Z_m(\cdot,t))$ are in $C_0^1(\mathbb{R}^m)$. Furthermore, if $T < \infty$,

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} + \sum_{\alpha=1}^{m} \|Z_\alpha(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \to \infty, \qquad \text{as } t \uparrow T. \tag{6.6.7}$$

Here we need (possibly weak) solutions, under a broader class of initial data, which exist globally in time. Such solutions do indeed exist because, under our assumptions (6.6.2), (6.6.3), the effect of the right-hand side in (6.6.1) is dissipative. This is manifested in the following

6.6.1 Theorem. *For any initial data $(v_0, Z_{10}, \cdots, Z_{m0})$ in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, there exists a unique weak solution $(v, Z_1, \cdots, Z_m)$ of* (6.6.1), (6.6.6) *on $\mathbb{R}^m \times [0,\infty)$ such that $(v(\cdot,t), Z_1(\cdot,t), \cdots, Z_m(\cdot,t))$ are in $C^0([0,\infty); L^1(\mathbb{R}^m))$. If*

$$a \leq v_0(x) \leq b, \quad F_\alpha(b) \leq Z_{\alpha 0}(x) \leq F_\alpha(a), \quad \alpha = 1, \cdots, m, \quad x \in \mathbb{R}^m, \tag{6.6.8}$$

then

$$a \leq v(x,t) \leq b\,, \; F_\alpha(b) \leq Z_\alpha(x,t) \leq F_\alpha(a)\,, \; \alpha = 1, \cdots, m\,, \; (x,t) \in \mathbb{R}^m \times [0,\infty). \tag{6.6.9}$$

Furthermore, if $(\bar{v}, \bar{Z}_1, \cdots, \bar{Z}_m)$ is another solution of (6.6.1), with initial data $(\bar{v}_0, \bar{Z}_{10}, \cdots, \bar{Z}_{m0})$ in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, then, for any $t \in [0,\infty)$,

[1] By virtue of (6.6.2), the transformation $(6.6.5)_1$ may be inverted to express v as a smooth, increasing function of u, and it is in that sense that G_α, defined by $(6.6.5)_2$, should be realized as a function of u.

$$\text{(6.6.10)} \qquad \int_{\mathbb{R}^m} \{[v(x,t) - \bar{v}(x,t)]^+ + \sum_{\alpha=1}^{m} [\bar{Z}_\alpha(x,t) - Z_\alpha(x,t)]^+\} dx$$

$$\leq \int_{\mathbb{R}^m} \{[v_0(x) - \bar{v}_0(x)]^+ + \sum_{\alpha=1}^{m} [\bar{Z}_{\alpha 0}(x) - Z_{\alpha 0}(x)]^+\} dx,$$

$$\text{(6.6.11)} \qquad \|v(\cdot,t) - \bar{v}(\cdot,t)\|_{L^1(\mathbb{R}^m)} + \sum_{\alpha=1}^{m} \|Z_\alpha(\cdot,t) - \bar{Z}_\alpha(\cdot,t)\|_{L^1(\mathbb{R}^m)}$$

$$\leq \|v_0(\cdot) - \bar{v}_0(\cdot)\|_{L^1(\mathbb{R}^m)} + \sum_{\alpha=1}^{m} \|Z_{\alpha 0}(\cdot) - \bar{Z}_{\alpha 0}(\cdot)\|_{L^1(\mathbb{R}^m)}.$$

In particular, if

$$\text{(6.6.12)} \qquad v_0(x) \leq \bar{v}_0(x), \quad Z_{\alpha 0}(x) \geq \bar{Z}_{\alpha 0}(x), \quad \alpha = 1, \cdots, m, \quad x \in \mathbb{R}^m,$$

then

(6.6.13)
$$v(x,t) \leq \bar{v}(x,t), \quad Z_\alpha(x,t) \geq \bar{Z}_\alpha(x,t), \quad \alpha = 1, \cdots, m, \quad (x,t) \in \mathbb{R}^m \times [0, \infty).$$

Proof. The first step is to establish (6.6.10) under the simplifying assumption that both solutions $(v, Z_1, \cdots, Z_m)$ and $(\bar{v}, \bar{Z}_1, \cdots, \bar{Z}_m)$ are classical, with initial data $(v_0, Z_{10}, \cdots, Z_{m0})$ and $(\bar{v}_0, \bar{Z}_{10}, \cdots, \bar{Z}_{m0})$ in $C_0^1(\mathbb{R}^m)$. For $\varepsilon > 0$, we recall the function η_ε defined through (6.3.8) and note that

$$\text{(6.6.14)} \quad \partial_t[\eta_\varepsilon(v - \bar{v}) + \sum_{\alpha=1}^{m} \eta_\varepsilon(\bar{Z}_\alpha - Z_\alpha)] + \sum_{\alpha=1}^{m} c_\alpha \partial_\alpha[\eta_\varepsilon(v - \bar{v}) - \eta_\varepsilon(\bar{Z}_\alpha - Z_\alpha)]$$

$$= \frac{1}{\mu} \sum_{\alpha=1}^{m} [\eta'_\varepsilon(v - \bar{v}) - \eta'_\varepsilon(\bar{Z}_\alpha - Z_\alpha)][F_\alpha(v) - F_\alpha(\bar{v}) + \bar{Z}_\alpha - Z_\alpha]$$

follows readily from (6.6.1). For fixed values of $v, \bar{v}, Z_\alpha, \bar{Z}_\alpha$, of any sign, the right-hand side of (6.6.14) has a nonpositive limit as $\varepsilon \downarrow 0$. Therefore, integrating (6.6.14) over $\mathbb{R}^m \times (0,t)$ and letting $\varepsilon \downarrow 0$ we arrive at (6.6.10).

When (6.6.12) holds, (6.6.10) immediately implies (6.6.13). Notice that, for any constants a and b, $(a, F_1(a), \cdots, F_m(a))$ and $(b, F_1(b), \cdots, F_m(b))$ are particular solutions of (6.6.1) and hence (6.6.8) implies (6.6.9). In particular, blow-up (6.6.7) cannot occur for any T and thus the solutions exist on $\mathbb{R}^m \times [0, \infty)$.

To get (6.6.11), it suffices to write (6.6.10) with the roles of $(v, Z_1, \cdots, Z_m)$ and $(\bar{v}, \bar{Z}_1, \cdots, \bar{Z}_m)$ reversed and then add the resulting inequality to the original (6.6.10).

We have now verified all the assertions of the theorem, albeit within the context of classical solutions, with initial data in $C_0^1(\mathbb{R}^m)$. Nevertheless, by virtue of the L^1-contraction estimate (6.6.11), weak solutions of (6.6.1), with any initial data in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, satisfying the asserted properties, may readily be constructed as L^1 limits of sequences of classical solutions. This completes the proof.

Our next task is to investigate the limiting behavior of solutions of (6.6.1) as $\mu \downarrow 0$. The mechanism that induces the Z_α to relax to their equilibrium values $F_\alpha(v)$ will be captured through an entropy-like inequality. We define the family

$$\Phi_\alpha(Z_\alpha) = -\int_0^{Z_\alpha} F_\alpha^{-1}(w)dw, \quad \alpha = 1, \cdots, m \tag{6.6.15}$$

of nonnegative, convex functions on $(-\infty, \infty)$. Assuming $(v, Z_1, \cdots, Z_m)$ is a classical solution of (6.6.1), with initial data $(v_0, Z_{10}, \cdots, Z_{m0})$ in $C_0^1(\mathbb{R}^m)$, we readily verify that

$$\begin{aligned} \partial_t \left[\tfrac{1}{2}v^2 + \sum_{\alpha=1}^m \Phi_\alpha(Z_\alpha)\right] &+ \sum_{\alpha=1}^m c_\alpha \partial_\alpha \left[\tfrac{1}{2}v^2 - \Phi_\alpha(Z_\alpha)\right] \\ &= \frac{1}{\mu} \sum_{\alpha=1}^m [v - F_\alpha^{-1}(Z_\alpha)][F_\alpha(v) - Z_\alpha]. \end{aligned} \tag{6.6.16}$$

Since $v - F_\alpha^{-1}(Z_\alpha) = F_\alpha^{-1}(F_\alpha(v)) - F_\alpha^{-1}(Z_\alpha)$, the mean value theorem implies

$$-[v - F_\alpha^{-1}(Z_\alpha)][F_\alpha(v) - Z_\alpha] \geq \frac{1}{k}[F_\alpha(v) - Z_\alpha]^2, \tag{6.6.17}$$

where k is any upper bound of $-F_\alpha'$ over the range of v. Therefore, upon integrating (6.6.16) over $\mathbb{R}^m \times [0, \infty)$ we deduce the inequality

$$\int_0^\infty \int_{\mathbb{R}^m} \sum_{\alpha=1}^m [F_\alpha(v) - Z_\alpha]^2 dxdt \leq k\mu \int_{\mathbb{R}^m} \left[\tfrac{1}{2}v_0^2 + \sum_{\alpha=1}^m \Phi_\alpha(Z_{\alpha_0})\right] dx. \tag{6.6.18}$$

As explained in the proof of Theorem 6.6.1, weak solutions of (6.6.1) are constructed as L^1 limits of sequences of classical solutions, and hence the inequality (6.6.18) will hold even for weak solutions with initial data in $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$.

6.6.2 Theorem. *Let* $(v^\mu, Z_1^\mu, \cdots, Z_m^\mu)$ *denote the family of solutions of* (6.6.1), (6.6.6), *with parameter* $\mu > 0$, *and initial data* $(v_0, F_1(v_0), \cdots, F_m(v_0))$, *where* v_0 *is in* $L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$. *Then there is a bounded measurable function* v *on* $\mathbb{R}^m \times [0, \infty)$ *such that, as* $\mu \downarrow 0$,

$$v^\mu(x,t) \longrightarrow v(x,t), \qquad Z_\alpha^\mu(x,t) \longrightarrow F_\alpha(v(x,t)), \quad \alpha = 1, \cdots, m, \tag{6.6.19}$$

almost everywhere on $\mathbb{R}^m \times [0, \infty)$. *The function*

$$u(x,t) = v(x,t) - \sum_{\alpha=1}^m F_\alpha(v(x,t)) \tag{6.6.20}$$

is the admissible weak solution of the conservation law (6.1.1), *with flux functions* G_α *defined through* (6.6.5), *and initial data*

$$u_0(x) = v_0(x) - \sum_{\alpha=1}^m F_\alpha(v_0(x)), \qquad x \in \mathbb{R}^m. \tag{6.6.21}$$

Proof. Let us set, for $(x,t) \in \mathbb{R}^m \times [0,\infty)$,

$$u^\mu(x,t) = v^\mu(x,t) - \sum_{\alpha=1}^m Z_\alpha^\mu(x,t), \tag{6.6.22}$$

$$G_\alpha^\mu(x,t) = c_\alpha[v^\mu(x,t) + Z_\alpha^\mu(x,t)]. \tag{6.6.23}$$

By virtue of (6.6.4),

$$\partial_t u^\mu(x,t) + \operatorname{div} G^\mu(x,t) = 0. \tag{6.6.24}$$

First we show that there is a bounded measurable function u on $\mathbb{R}^m \times [0,\infty)$ and some sequence $\{\mu_n\}$, with $\mu_n \downarrow 0$ as $n \to \infty$, such that

$$u^{\mu_n}(\cdot,t) \longrightarrow u(\cdot,t), \qquad n \to \infty, \tag{6.6.25}$$

in $L^\infty(\mathbb{R}^m)$ weak*, for all $t \in [0,\infty)$. To that end, let us fix any test function χ in $C_0^\infty(\mathbb{R}^m)$ and define the family of functions

$$w^\mu(t) = \int_{\mathbb{R}^m} \chi(x) u^\mu(x,t)dx, \qquad t \in [0,\infty), \tag{6.6.26}$$

which, on account of (6.6.24), are continuously differentiable with derivative

$$\frac{d}{dt} w^\mu(t) = \sum_{\alpha=1}^m \int_{\mathbb{R}^m} \partial_\alpha \chi(x)\, G_\alpha^\mu(x,t) \tag{6.6.27}$$

bounded, uniformly in $\mu > 0$. It then follows from Arzela's theorem that there is a sequence $\{\mu_n\}$, with $\mu_n \downarrow 0$ as $n \to \infty$, such that $\{w^{\mu_n}\}$ converges for all $t \in [0,\infty)$. By Cantor's diagonal process we may construct a subsequence of $\{\mu_n\}$, denoted again by $\{\mu_n\}$, such that the sequence

$$\left\{ \int_{\mathbb{R}^m} \chi(x) u^{\mu_n}(x,t)dx \right\} \tag{6.6.28}$$

is convergent for all $t \in [0,\infty)$ and every member χ of any given countable family of test functions. Consequently, (6.6.28) is convergent for any $\chi \in L^1(\mathbb{R}^m)$. Thus, for each $t \in [0,\infty)$ there is a bounded measurable function on $\mathbb{R}^m$, denoted by $u(\cdot,t)$, such that (6.6.25) holds in $L^\infty(\mathbb{R}^m)$ weak*. Next we note that, by the L^1 contraction estimate (6.6.11), for any fixed t in $[0,\infty)$ the family of functions $(v^\mu(\cdot,t), Z_1^\mu(\cdot,t), \cdots, Z_m^\mu(\cdot,t))$ is equicontinuous in the mean. Hence, the convergence in (6.6.25) is upgraded to strongly in $L^1(\mathbb{R}^m)$. In particular,

$$u^{\mu_n}(x,t) \longrightarrow u(x,t), \qquad n \to \infty, \tag{6.6.29}$$

almost everywhere on $\mathbb{R}^m \times [0,\infty)$.

We now apply (6.6.18) for our solutions $(v^{\mu_n}, Z_1^{\mu_n}, \cdots, Z_m^{\mu_n})$ and, passing if necessary to a subsequence, denoted again by $\{\mu_n\}$, we obtain

(6.6.30) $$F_\alpha(v^{\mu_n}(x,t)) - Z_\alpha^{\mu_n}(x,t) \to 0, \qquad n\to\infty, \qquad \alpha = 1,\cdots,m,$$

almost everywhere on $\mathbb{R}^m \times [0,\infty)$.

Combining (6.6.22), (6.6.29) and (6.6.30), we deduce

(6.6.31) $$v^{\mu_n}(x,t) - \sum_{\alpha=1}^m F_\alpha(v^{\mu_n}(x,t)) \to u(x,t), \qquad n\to\infty,$$

almost everywhere on $\mathbb{R}^m \times [0,\infty)$. Because of the monotonicity assumption (6.6.2), (6.6.31) implies that the sequence $\{v^{\mu_n}\}$ itself must be convergent, say

(6.6.32) $$v^{\mu_n}(x,t) \to v(x,t), \quad n\to\infty\,,$$

almost everywhere on $\mathbb{R}^m \times [0,\infty)$, where v is a function related to u through (6.6.20). Furthermore, (6.6.30) and (6.6.32) together imply

(6.6.33) $$Z_\alpha^{\mu_n}(x,t) \to F_\alpha(v(x,t)), \qquad n\to\infty\,, \qquad \alpha = 1,\cdots,m,$$

almost everywhere on $\mathbb{R}^m \times [0,\infty)$.

By virtue of (6.6.22), (6.6.23), (6.6.24), (6.6.32) and (6.6.33), u is a weak solution of (6.1.1), with fluxes G_α defined through (6.6.5). We proceed to show that this solution is admissible. We fix any constant $\bar v$ and write (6.6.14) for the two solutions $(v^{\mu_n}, Z_1^{\mu_n},\cdots,Z_m^{\mu_n})$ and $(\bar v, F_1(\bar v),\cdots,F_m(\bar v))$. We apply this (distributional) equation to any nonnegative Lipschitz continuous test function ψ, with compact support on $\mathbb{R}^m\times[0,\infty)$ and let $\varepsilon\downarrow 0$. Since the $\varepsilon\downarrow 0$ limit of the right-hand side of (6.6.14) is nonpositive, this calculation gives

(6.6.34) $$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^m} \partial_t\psi\,[(v^{\mu_n}-\bar v)^+ + \sum_{\alpha=1}^m (F_\alpha(\bar v) - Z_\alpha^{\mu_n})^+]\,dxdt\\ &+\int_0^\infty\int_{\mathbb{R}^m}\sum_{\alpha=1}^m c_\alpha\partial_\alpha\psi\,[(v^{\mu_n}-\bar v)^+ - (F_\alpha(\bar v) - Z_\alpha^{\mu_n})^+]\,dxdt\\ &+\int_{\mathbb{R}^m}\psi(x,0)[(v_0-\bar v)^+ + \sum_{\alpha=1}^m (F_\alpha(\bar v) - F_\alpha(v_0))^+]dx \geq 0.\end{aligned}$$

Letting $n\to\infty$ and using (6.6.32) and (6.6.33), (6.6.34) yields

(6.6.35) $$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^m} \partial_t\psi\,[(v-\bar v)^+ + \sum_{\alpha=1}^m (F_\alpha(\bar v) - F_\alpha(v))^+]\,dxdt\\ &+\int_0^\infty\int_{\mathbb{R}^m}\sum_{\alpha=1}^m c_\alpha\partial_\alpha\psi\,[(v-\bar v)^+ - (F_\alpha(\bar v) - F_\alpha(v))^+]\,dxdt\\ &+\int_{\mathbb{R}^m}\psi(x,0)[(v_0-\bar v)^+ + \sum_{\alpha=1}^m (F_\alpha(\bar v) - F_\alpha(v_0))^+]dx \geq 0.\end{aligned}$$

On account of (6.6.2), $v-\bar v$ and $F_\alpha(\bar v) - F_\alpha(v)$ have the same sign. Furthermore, if we set $\bar u = \bar v - \sum F_\alpha(\bar v)$, then $v-\bar v$ and $u-\bar u$ also have the same sign. Therefore, upon using (6.6.20), (6.6.21), and (6.6.5), we may rewrite (6.6.35) as

(6.6.36)

$$\int_0^\infty \int_{\mathbb{R}^m} [\partial_t \psi \eta(u;\bar{u}) + \sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u;\bar{u})] dxdt + \int_{\mathbb{R}^m} \psi(x,0)\eta(u_0;\bar{u})dx \geq 0,$$

where $(\eta(u;\bar{u}), Q(u;\bar{u}))$ is the entropy-entropy flux pair defined by (6.2.5). As noted in Section 6.2, the set of entropy-entropy flux pairs (6.2.5), with $\bar{u}$ arbitrary, is "complete" and hence (6.6.36) implies that (6.2.3) will hold for any entropy-entropy flux pair (η, Q) with η convex. This verifies that u is the admissible weak solution of (6.1.1), with initial data u_0 given by (6.6.21). Since u is unique, the convergence in (6.6.29), (6.6.32) and (6.6.33) applies not only along the particular sequence $\{\mu_n\}$ but also along the whole family $\{\mu\}$, as $\mu \downarrow 0$. This completes the proof.

Theorem 6.6.2 demonstrates how, starting out from a given system (6.6.1), one may construct, by relaxation, admissible solutions of a particular scalar conservation law induced by (6.6.1). Of course, we are interested in the reverse process, namely to determine the appropriate system (6.6.1) whose relaxed form is a given scalar conservation law (6.1.1). This may be accomplished when, given the fluxes $G_\alpha(u)$, it is possible to select constants c_α in such a way that the transformations (6.6.5) implicitly determine functions $F_\alpha(v)$ that satisfy the assumptions (6.6.2) and (6.6.3). Let us normalize the given fluxes by $G_\alpha(0) = 0$, $\alpha = 1, \cdots, m$. Since our solutions will be a priori bounded, let us assume, without loss of generality, that the $G'_\alpha(u)$ are uniformly bounded on $(-\infty, \infty)$. From (6.6.5),

$$(m+1)v = u + \sum_{\alpha=1}^m \frac{1}{c_\alpha} G_\alpha(u). \tag{6.6.37}$$

Therefore, the first constraint is to fix the $|c_\alpha|$ so large that

$$(m+1)\frac{dv}{du} = 1 + \sum_{\alpha=1}^m \frac{1}{c_\alpha} G'_\alpha(u) \geq \frac{1}{2}, \tag{6.6.38}$$

in order to secure that the map $v \mapsto u$ will possess a smooth inverse. Next we note

$$F'_\alpha(v) = -1 + \frac{1}{c_\alpha} G'_\alpha(u)\frac{du}{dv} = -1 + \frac{m+1}{c_\alpha}[1 + \sum_{\beta=1}^m \frac{1}{c_\beta} G'_\beta(u)]^{-1} G'_\alpha(u), \tag{6.6.39}$$

so that, by selecting the $|c_\alpha|$ sufficiently large, we can satisfy both assumptions (6.6.2) and (6.6.3). Restrictions on c_α that maintain that the convective characteristic speeds c_α should be high relative to the characteristic speeds G'_α of the relaxed conservation law are called *subcharacteristic conditions*.

6.7 A Kinetic Formulation

This section discusses an alternative, albeit equivalent, characterization of admissible weak solutions to (6.1.1), (6.1.2), which, as we shall see below, is motivated by the kinetic theory.

It has already been noted that the entropy production for any solution of (6.1.1) satisfying (6.2.2) is a nonpositive measure. In particular, if u is an admissible solution of (6.1.1), (6.1.2) in the sense of Definition 6.2.1, then for any $v \in (-\infty, \infty)$,

$$\partial_t\{|u-v|-|v|\} + \operatorname{div}\{\operatorname{sgn}(u-v)\,[G(u)-G(v)] - \operatorname{sgn} v\, G(v)\} = -2\nu_v\,, \tag{6.7.1}$$

where ν_v is a nonnegative measure on $\mathbb{R}^m \times \mathbb{R}^+$. For $|v| > \sup|u_0| = \sup|u|$, we have $2\nu_v = |\partial_t u + \operatorname{div} G(u)| = 0$.

We realize $\{\nu_v\}$ as a nonnegative measure ν on $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^+$ and differentiate (6.7.1), in the sense of distributions, with respect to v, to deduce

$$\partial_t \chi(v;u) + \sum_{\alpha=1}^{m} G'_\alpha(v)\partial_\alpha \chi(v;u) = \partial_v \nu, \tag{6.7.2}$$

where χ denotes the function

$$\chi(v;u) = \begin{cases} 1 & \text{if} \quad 0 < v < u \\ -1 & \text{if} \quad u < v < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.7.3}$$

The entropy production by any entropy-entropy flux pair (η, Q) is easily expressed in terms of ν. Indeed, let us multiply (6.7.2) by $\eta'(v)$ and integrate with respect to v over $(-\infty,\infty)$. Recalling (6.2.1) and after an integration by parts, we obtain

$$\partial_t \int_{-\infty}^{\infty} \eta'(v)\chi(v;u)dv + \operatorname{div} \int_{-\infty}^{\infty} Q'(v)\chi(v;u)dv = -\int_{-\infty}^{\infty} \eta''(v)d\nu(v;\cdot,\cdot). \tag{6.7.4}$$

One easily verifies that if $p(v)$ is any C^1 function, then

$$\int_{-\infty}^{\infty} p'(v)\chi(v;u)dv = p(u) - p(0), \tag{6.7.5}$$

and so (6.7.4) yields

$$\partial_t \eta(u) + \operatorname{div} Q(u) = -\int_{-\infty}^{\infty} \eta''(v)d\nu(v;\cdot,\cdot). \tag{6.7.6}$$

In particular, when $\eta(u)$ is convex the right-hand side of (6.7.6) is nonpositive. Furthermore, applying (6.7.6) for $\eta(u) = \frac{1}{2}u^2$ and integrating with respect to (x,t) over $\mathbb{R}^m \times [0,\infty)$, we deduce

$$\int_0^{\infty}\int_{\mathbb{R}^m}\int_{-\infty}^{\infty} d\nu(v;x,t) \le \frac{1}{2}\int_{\mathbb{R}^m} u_0^2(x)dx. \tag{6.7.7}$$

It is remarkable that (6.7.2) fully characterizes admissible weak solutions of (6.1.1), as shown in the following

6.7.1 Theorem. *A bounded measurable function $u(x,t)$ on $\mathbb{R}^m \times [0,\infty)$ is the admissible solution to* (6.1.1), (6.1.2) *if and only if the function $\chi(v;u(x,t))$, defined* *through* (6.7.3), *satisfies* (6.7.2) *on* $\mathbb{R} \times \mathbb{R}^m \times [0,\infty)$, *for some nonnegative measure* ν, *together with the initial condition*

$$\chi(v;u(x,0)) = \chi(v;u_0(x)), \qquad v \in (-\infty,\infty), \quad x \in \mathbb{R}^m. \tag{6.7.8}$$

Proof. Equation (6.7.2) admits solutions $\chi(\cdot\,;u(\cdot,t)) \in C^0([0,\infty);L^1(\mathbb{R}\times\mathbb{R}^m))$ and thus the initial condition (6.1.2) is attained strongly in $L^1(\mathbb{R}^m)$. Hence it remains to show that (6.2.2) holds for every entropy-entropy flux pair (η,Q) with η convex. Since u is bounded, it will suffice to establish (6.2.2) for entropies with linear growth, i.e., with $|\eta'(u)|$ bounded on $(-\infty,\infty)$.

Starting out from (6.7.2), one can show, as above, that (6.7.6) holds, albeit only for functions $\eta(v)$ whose derivative $\eta'(v)$ vanishes for $|v|$ large (in order to perform the integration by parts, as it is no longer known that ν vanishes for $|v| > \sup|u_0|$).

Fix any convex function η, with linear growth, and then for $k = 1,2,\cdots$, set $\eta_k(v) = \eta(v)\phi(v/k)$, where ϕ is a smooth even function on $(-\infty,\infty)$, with $\phi(v) = 1$ for $|v| \le 1$, $\phi(v) = 0$ for $|v| \ge 2$, and $\phi'(v) < 0$ for $v \in (1,2)$. We thus have

$$\partial_t \eta_k(u) + \operatorname{div} Q_k(u) = -\int_{-\infty}^{\infty} \eta_k''(v)d\nu(v;\cdot,\cdot) \tag{6.7.9}$$

$$= -\int_{-\infty}^{\infty} \left[\eta''(v)\phi\left(\frac{v}{k}\right) + \frac{2}{k}\eta'(v)\phi'\left(\frac{v}{k}\right) + \frac{1}{k^2}\eta(v)\phi''\left(\frac{v}{k}\right)\right] d\nu(v;\cdot,\cdot).$$

For k large, $\eta_k(u) = \eta(u)$ and $Q_k(u) = Q(u)$, on the range of the solution. Furthermore, $\eta''(v)\phi(v/k) \to \eta''(v)$ monotonically, as $k \to \infty$. Finally, it is clear that $\eta'(v)\phi'(v/k) = O(1)$ and $\eta(v)\phi''(v/k) = O(k)$, as $k \to \infty$. Thus, letting $k \to \infty$ in (6.7.9), we arrive at (6.7.6), and thereby at (6.2.2). This completes the proof.

The kinetic formulation (6.7.2), which may serve as an alternative, albeit equivalent, definition of admissible weak solutions of (6.1.1), provides a powerful instrument for discovering properties of these solutions. In particular, one obtains an alternative, direct proof of the L^1 contraction property (6.2.9), even under the more general assumption that the initial data are merely in $L^1(\mathbb{R}^m)$ and not necessarily in $L^\infty(\mathbb{R}^m)$; see references in Section 6.11.

Up to this point, we have been facing nonlinearity as an agent that provokes the development of discontinuities in solutions with smooth initial values. It turns out, however, that nonlinearity may also play the opposite role, of smoothing out solutions with rough initial data. In the course of the book, we shall encounter various manifestations of such behavior. The kinetic formulation provides valuable insight into the compactifying and smoothing effects of nonlinearity in scalar conservation laws. This will become evident in the next Section 6.8, but it is also seen in the following regularity theorem whose (hard and technical) proof is found in the references cited in Section 6.11.

6.7.2 Theorem. *Assume there are $r \in (0,1]$ and $C \geq 0$ such that*

$$\text{meas}\{v : |v| \leq \|u_0\|_{L^\infty},\ |p + G'(v)P| \leq \delta\} \leq C\delta^r, \tag{6.7.10}$$

for all $\delta \in (0,1)$, $p \in \mathbb{R}$, $P \in \mathbb{R}^m$ with $p^2 + |P|^2 = 1$. Then the admissible weak solution u of (6.1.1), (6.1.2) *satisfies*

$$u(\cdot,t) \in C^0((0,\infty); W^{s,1}_{\text{loc}}(\mathbb{R}^m)), \tag{6.7.11}$$

for any $s \in \left(0, \frac{r}{2r+1}\right)$.

It is condition (6.7.10) that encodes the aspect of nonlinearity of G responsible for the regularizing effect. For example, (6.7.10) fails, for any r, when G is linear, but it is satisfied, with $r = 1$, if the G_α are uniformly convex functions, $G''_\alpha(u) > 0$, $\alpha = 1, \cdots, m$.

The section closes with a discussion on how the kinetic formulation (6.7.2) of the scalar conservation law may be motivated by the kinetic theory of matter. As we saw in Chapter III, Example 3.3.7, in the classical kinetic theory of gases the state of the gas at the point x and time t is described by the molecular density function $f(\xi,x,t)$ of the molecular velocity ξ. The evolution of f is governed by the Boltzmann equation (3.3.51), which monitors the changes in the distribution of molecular velocities due to transport and collisions. The connection between the kinetic and the continuum approaches is established by identifying intensive quantities, such as density, velocity, pressure, temperature, etc., with appropriate moments of the molecular density function f, and then showing that these fields satisfy the balance laws of continuum physics. Thus, in principle one may construct solutions to systems of balance laws by treating the fields as moments of a molecular density in an underlying kinetic model with density function whose zero moment satisfies the scalar conservation law (6.1.1).

In the model, the "velocity" v is scalar-valued and the "molecular density" $f(v;x,t)$, at the point x and time t, is allowed to take positive and negative values. Then u is obtained from f by

$$u(x,t) = \int_{-\infty}^{\infty} f(v;x,t)dv. \tag{6.7.12}$$

In turn, f satisfies the transport equation

$$\partial_t f(v;x,t) + \sum_{\alpha=1}^{m} G'_\alpha(v)\partial_\alpha f(v;x,t) = \frac{1}{\mu}\left[\chi(v;u(x,t)) - f(v;x,t)\right], \tag{6.7.13}$$

where μ is a small positive parameter and $\chi(v;u)$ is the function defined by (6.7.3). Readers familiar with the kinetic theory will recognize in (6.7.13) a model of the BGK approximation to the classical Boltzmann equation. Hopefully, as $\mu \downarrow 0$, the stiff term on the right-hand side will force $f(v;x,t)$ to "relax" to $\chi(v;u(x,t))$ which satisfies (6.7.2). Before verifying that this expectation will be fulfilled, let us discuss properties of solutions of (6.7.13), (6.7.12).

6.7.3 Theorem. *Let* $u_0 \in L^\infty(\mathbb{R}^m) \bigcap L^1(\mathbb{R}^m)$. *For any* $\mu > 0$, *there exist bounded measurable functions*

$$f_\mu(\cdot\,;\cdot,t) \in C^0([0,\infty);L^1(\mathbb{R}\times\mathbb{R}^m)), \qquad u_\mu(\cdot,t) \in C^0([0,\infty);L^1(\mathbb{R}^m)) \tag{6.7.14}$$

which provide the unique solution of (6.7.13), (6.7.12) *with initial data*

$$f_\mu(v;x,0) = \chi(v;u_0(x)), \qquad v \in (-\infty,\infty),\ \ x \in \mathbb{R}^m, \tag{6.7.15}$$

induced by u_0. *For any* $(x,t) \in \mathbb{R}^m \times [0,\infty)$,

$$f_\mu(v;x,t) \in \begin{cases} [0,1] & \text{if } v > 0 \\ \\ [-1,0] & \text{if } v < 0. \end{cases} \tag{6.7.16}$$

If $(\bar{f}_\mu, \bar{u}_\mu)$ *is another solution of* (6.7.13), (6.7.12), *with initial data induced by* $\bar{u}_0$ *in* $L^\infty(\mathbb{R}^m)\bigcap L^1(\mathbb{R}^m)$, *then, for any* $t > 0$,

$$\|f_\mu(\cdot\,;\cdot,t) - \bar{f}_\mu(\cdot\,;\cdot,t)\|_{L^1(\mathbb{R}\times\mathbb{R}^m)} \leq \|f_\mu(\cdot\,;\cdot,0) - \bar{f}_\mu(\cdot\,;\cdot,0)\|_{L^1(\mathbb{R}\times\mathbb{R}^m)} \tag{6.7.17}$$

$$\|u_\mu(\cdot,t) - \bar{u}_\mu(\cdot,t)\|_{L^1(\mathbb{R}^m)} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R}^m)}. \tag{6.7.18}$$

Furthermore, if

$$u_0(x) \leq \bar{u}_0(x), \qquad x \in \mathbb{R}^m, \tag{6.7.19}$$

then

$$f_\mu(v;x,t) \leq \bar{f}_\mu(v;x,t), \qquad v \in (-\infty,\infty),\ \ x \in \mathbb{R}^m,\ \ t \in [0,\infty), \tag{6.7.20}$$

$$u_\mu(x,t) \leq \bar{u}_\mu(x,t), \qquad x \in \mathbb{R}^m,\ \ t \in [0,\infty). \tag{6.7.21}$$

Proof. Taking, for the time being, the existence of (f_μ, u_μ) and $(\bar{f}_\mu, \bar{u}_\mu)$ for granted, we integrate (6.7.13) along characteristics $dx/dt = G'(v)^\top$, $dv/dt = 0$ to deduce

(6.7.22)
$$f_\mu(v;x,t) = e^{-\frac{t}{\mu}} f_\mu(v;x - tG'(v)^\top, 0) + \frac{1}{\mu}\int_0^t e^{-\frac{t-\tau}{\mu}} \chi(v; u_\mu(x - (t-\tau)G'(v)^\top, \tau))d\tau.$$

Thus (6.7.16) readily follows from (6.7.22), (6.7.15) and the properties of the function χ.

We write the analog of (6.7.22) for the other solution $(\bar{f}_\mu, \bar{u}_\mu)$ and subtract the resulting equation from (6.7.22) to get

$$(6.7.23)\quad f_\mu(v;x,t)-\bar f_\mu(v;x,t)=e^{-\frac{t}{\mu}}[f_\mu(v;x-tG'(v)^\top,0)-\bar f_\mu(v;x-tG'(v)^\top,0)]$$
$$+\frac{1}{\mu}\int_0^t e^{-\frac{t-\tau}{\mu}}[\chi(v;u_\mu(x-(t-\tau)G'(v)^\top,\tau))-\ \chi(v;\bar u_\mu(x-(t-\tau)G'(v)^\top,\tau))]d\tau$$

whence, upon using

$$(6.7.24)\qquad \int_{-\infty}^{\infty}|\chi(v;u_\mu)-\chi(v;\bar u_\mu)|dv\le|u_\mu-\bar u_\mu|,$$

which follows from (6.7.3), and recalling (6.7.12),

$$(6.7.25)\quad \|f_\mu(\cdot\,;\cdot,t)-\bar f_\mu(\cdot\,;\cdot,t)\|_{L^1(\mathbb{R}\times\mathbb{R}^m)}\le e^{-\frac{t}{\mu}}\|f_\mu(\cdot\,;\cdot,0)-\bar f_\mu(\cdot\,;\cdot,0)\|_{L^1(\mathbb{R}\times\mathbb{R}^m)}$$
$$+\frac{1}{\mu}\int_0^t e^{-\frac{t-\tau}{\mu}}\|\chi(v;u_\mu(x-(t-\tau)G'(v)^\top,\tau))$$
$$-\chi(v;\bar u_\mu(x-(t-\tau)G'(v)^\top,\tau))\|_{L^1(\mathbb{R}\times\mathbb{R}^m)}d\tau$$
$$\le e^{-\frac{t}{\mu}}\|f_\mu(\cdot\,;\cdot,0)-\bar f_\mu(\cdot\,;\cdot,0)\|_{L^1(\mathbb{R}\times\mathbb{R}^m)}$$
$$+(1-e^{-\frac{t}{\mu}})\max_{0\le\tau\le t}\|f_\mu(\cdot\,;\cdot,\tau)-\bar f_\mu(\cdot\,;\cdot,\tau)\|_{L^1(\mathbb{R}\times\mathbb{R}^m)}\,.$$

Clearly, (6.7.25) implies (6.7.17) and this in turn yields (6.7.18). In particular, there is at most one solution to (6.7.13), (6.7.12), (6.7.15). Furthermore, this solution can be constructed from the integral equation (6.7.22) by Picard iteration.

Since $\chi(v;u)$ is increasing in u, (6.7.23) and (6.7.12) guarantee that (6.7.19) implies (6.7.20) and (6.7.21). This completes the proof.

We now turn to the limiting behavior of solutions as $\mu\downarrow 0$.

6.7.4 Theorem. *For $\mu>0$, let (f_μ,u_μ) denote the solution of* (6.7.13), (6.7.12), (6.7.15) *with $u_0\in L^\infty(\mathbb{R}^m)\bigcap L^1(\mathbb{R}^m)$. Then, as $\mu\downarrow 0$,*

$$(6.7.26)\qquad u_\mu(x,t)\to u(x,t),$$

$$(6.7.27)\qquad f_\mu(v;x,t)\to\chi(v;u(x,t)),$$

in L^1_{loc}, where $\chi(v;u)$ satisfies (6.7.2) *for some bounded, nonnegative measure ν, and hence u is the admissible weak solution of* (6.1.1.), (6.1.2).

Proof. The first step is to demonstrate that the family $\{(f_\mu,u_\mu):\mu>0\}$ is equicontinuous in the mean. This is clearly the case in the v and x directions by virtue of the contraction property (6.7.17), (6.7.18). For any $w\in\mathbb{R}$ and $y\in\mathbb{R}^m$, the functions $(\bar f_\mu,\bar u_\mu)$ defined by $\bar f_\mu(v;x,t)=f_\mu(v+w;x+y,t)$, $\bar u_\mu(x,t)=u_\mu(x+y,t)$ are solutions of (6.7.13), (6.7.12) with initial data $\bar f_\mu(v;x,0)=\chi(v+w;u_0(x+y))$, and so

(6.7.28)
$$\int_0^\infty \int_{\mathbb{R}^m} |f_\mu(v+w;x+y,t) - f_\mu(v;x,t)|dxdv \le \int_0^\infty \int_{\mathbb{R}^m} |\chi(v+w;u_0(x+y)) - \chi(v;u_0(x))|dxdv,$$

(6.7.29)
$$\int_{\mathbb{R}^m} |u_\mu(x+y,t) - u_\mu(x,t)|dx \le \int_{\mathbb{R}^m} |u_0(x+y) - u_0(x)|dx.$$

Equicontinuity in the t-direction easily follows from the above, in conjunction with the transport equation (6.7.13) itself; the details are omitted.

Next we consider the function

(6.7.30)
$$\omega_\mu(v;x,t) = \int_{-\infty}^{v} \left[\chi(w;u_\mu(x,t)) - f_\mu(w;x,t)\right] dw.$$

Let us fix (x,t), assuming for definiteness $u_\mu(x,t) > 0$ (the other cases being similarly treated). Clearly, $\omega_\mu(-\infty;x,t) = 0$. By virtue of (6.7.3) and (6.7.16), $\omega_\mu(\cdot\,;x,t)$ is nondecreasing on the interval $(-\infty, u_\mu(x,t))$ and nonincreasing on the interval $(u_\mu(x,t), \infty)$. Finally, on account of (6.7.12), $\omega_\mu(\infty;x,t) = 0$. Consequently, we may write

(6.7.31)
$$\frac{1}{\mu}\left[\chi(v;u_\mu(x,t)) - f_\mu(v;x,t)\right] = \partial_v \nu_\mu\,,$$

where ν_μ is a nonnegative measure which is bounded, uniformly in $\mu > 0$.

It follows that from any sequence $\{\mu_k\}$, $\mu_k \to 0$ as $k \to \infty$, we may extract a subsequence, denoted again by $\{\mu_k\}$, so that $\{(f_{\mu_k}, u_{\mu_k})\}$ converges in L^1_{loc} to functions (f,u), and $\{\nu_{\mu_k}\}$ converges weakly in the space of measures to a bounded nonnegative measure ν. Clearly, $f(v;x,t) = \chi(v;u(x,t))$ and (6.7.2) holds. By uniqueness, the whole family $\{(f_\mu, u_\mu)\}$ converges to $(\chi(\cdot\,;u),u)$, as $\mu \downarrow 0$. This completes the proof.

It is interesting that the transport equation (6.1.9), which, as we saw in Section 6.1, generates the classical solutions of (6.1.1), also arises in the theory of weak solutions, in the guise of (6.7.13) or (6.7.7). Remarkably, there is another connection between (6.1.10) and weak solutions of (6.1.1), emerging from the following considerations. Let $u(x,t)$ be the admissible weak solution of (6.1.1), (6.1.2), with initial datum u_0 that is 1-periodic in x_α and takes values in an interval $[a,b] \subset (0,1)$. Thus $u(\cdot,t)$ will also be 1-periodic in x_α and will take values in $[a,b]$. Let $\mathscr{C}$ denote the closed unit cube in $\mathbb{R}^m$, $\mathscr{C} = [0,1]^m$. The aim is to characterize $u(\cdot,t)$ through its level sets in $\mathscr{C}$. For that purpose, we introduce the function $f(v;x,t)$, defined for $v \in [0,1]$, $x \in \mathscr{C}$ and $t \ge 0$ by

(6.7.32)
$$\{x \in \mathscr{C} : f(v;x,t) \le 1\} = \{x \in \mathscr{C} : u(x,t) \ge v\}.$$

One may recover u from f through

(6.7.33)
$$u(x,t) = \int_0^1 h(1 - f(v;x,t))dv,$$

where h denotes the Heaviside function, $h(y) = 0$ for $y < 0$ and $h(y) = 1$ for $y \geq 0$.

As shown in the literature cited in Section 6.11, f satisfies the abstract differential equation

$$\partial_t f + \sum_{\alpha=1}^{m} Q'_\alpha(v)\partial_\alpha f + \partial\mathscr{K}(f) \ni 0, \tag{6.7.34}$$

on the Hilbert space $H = L^2([0,1]\times\mathscr{C})$, where $\partial\mathscr{K}$ denotes the subdifferential of the closed convex cone $\mathscr{K} = \{f \in H : f_v \geq 0\}$, i.e.,

$$\partial\mathscr{K}(f) = \{g \in H : \int_0^1 \int_{\mathscr{C}} (\bar{f} - f) g dx dv \leq 0, \text{ for all } \bar{f} \in \mathscr{K}\}. \tag{6.7.35}$$

The usefulness of the above observation lies in that the equation (6.7.34) involves a maximal monotone operator and thus generates a contraction semigroup on H. It is easily seen that the initial value $f(v;x,0)$ can be adjusted in such a way that $u(x,0)$, computed through (6.7.33), coincides with the given initial data $u_0(x)$. Therefore, the existence, uniqueness, stability and even numerical construction of f, and thereby of u, follow from the standard functional analytic theory of contraction semigroups in Hilbert space. The details on this approach, and its connection to the kinetic formulation, are found in the references listed in Section 6.11.

6.8 Fine Structure of L^∞ Solutions

According to Theorem 6.2.6, admissible solutions u to the scalar conservation law (6.1.1), with initial values u_0 of locally bounded variation on $\mathbb{R}^m$, have locally bounded variation on the upper half-space, and thereby inherit the fine structure of BV functions described in Sections 1.7 and 1.8. In particular, the points of approximate jump discontinuity of u assemble on the (at most) countable union of C^1 manifolds of codimension one. Furthermore, u has (generally distinct) traces on both sides of any oriented manifold of codimension one. However, when u_0 is merely in L^∞ the above structure is generally lost, as may be seen by considering the case where (6.1.1) is linear. On the other hand, we saw in Section 6.7 (Theorem 6.7.2) that nonlinearity in the flux function may exert a smoothing influence on L^∞ solutions. As another manifestation of this phenomenon, we shall see here that when the conservation law is linearly nondegenerate, in a sense to be made precise below, admissible solutions that are merely in L^∞ are nevertheless endowed with fine structure that closely resembles the structure of BV functions.

For the present purposes, the distinction between spatial and temporal variables is irrelevant, so it will be convenient to revert to the formulation and notations of Chapter I, by fusing the m-dimensional space and 1-dimensional time into k-dimensional space-time, $k = m+1$, and representing (x,t) by the vector X, with $X_\alpha = x_\alpha$, $\alpha = 1,\cdots,m$ and $X_k = t$. In what follows, div will denote the divergence operator in $\mathbb{R}^k$, acting on k-row vectors.

On some open subset $\mathscr{X}$ of $\mathbb{R}^k$, we consider scalar balance laws in the form

$$\operatorname{div} G(u(X)) = \nu_G\,, \tag{6.8.1}$$

where ν_G is a Radon measure. A function $u \in L^\infty(\mathscr{X})$ will be called an *admissible solution* of (6.8.1) if, for any companion Q of G,

$$\operatorname{div} Q(u(X)) = \nu_Q\,, \tag{6.8.2}$$

where ν_Q is a Radon measure on $\mathscr{X}$.

We recall, from Section 1.5, that companions Q are related to G by

$$Q'(u) = \eta'(u)G'(u), \tag{6.8.3}$$

where η is some scalar-valued function.

In the setting of Section 6.2, $G_k(u) = u$, $Q_k(u) = \eta(u)$ and $\nu_G = 0$. As noted in Section 6.2, any admissible solution u in the sense of Definition 6.2.1 renders the distribution $\operatorname{div} Q(u)$ a measure, for any companion Q (and in particular a nonpositive measure whenever Q_k is convex), so that it is also an admissible solution in the above sense.

In order to expunge linear systems, we introduce the following notion (compare with (6.7.10)):

6.8.1 Definition. The balance law (6.8.1) is called *linearly nondegenerate* if for each $N \in \mathbb{S}^{k-1}$

$$G'(u)N \neq 0, \quad \text{for almost all } u \in (-\infty,\infty). \tag{6.8.4}$$

The fine structure of admissible solutions of linearly nondegenerate scalar balance laws is described by the following

6.8.2 Theorem. *Assume* (6.8.1) *is linearly nondegenerate and let u be an admissible solution on $\mathscr{X}$. Then $\mathscr{X}$ is the union of three pairwise disjoint subsets $\mathscr{C}$, $\mathscr{J}$ and $\mathscr{I}$ with the following properties:*

(a) *$\mathscr{C}$ is the set of points of vanishing mean oscillation of u, i.e., for $\bar{X} \in \mathscr{C}$*

$$\lim_{r\downarrow 0} \frac{1}{r^k} \int_{\mathscr{B}_r(\bar{X})} |u(X) - \bar{u}_r(\bar{X})|dX = 0, \tag{6.8.5}$$

where $\bar{u}_r(\bar{X})$ denotes the average of u on the ball $\mathscr{B}_r(\bar{X})$.

(b) *$\mathscr{J}$ is rectifiable, namely it is essentially covered by the countable union of C^1 $(k-1)$-dimensional manifolds $\{\mathscr{F}_i\}$ embedded in $\mathbb{R}^k$: $\mathscr{H}^{k-1}(\mathscr{J}\backslash\bigcup\mathscr{F}_i) = 0$. When $\bar{X} \in \mathscr{J}\bigcap\mathscr{F}_i$, then the normal on $\mathscr{F}_i$ at $\bar{X}$ is interpreted as the normal on $\mathscr{J}$ at $\bar{X}$. The function u has distinct inward and outward traces u_- and u_+, in the sense of Definition* 1.7.7, *at any point $\bar{X} \in \mathscr{J}$.*

(c) *The $(k-1)$-dimensional Hausdorff measure of $\mathscr{I}$ is zero: $\mathscr{H}^{k-1}(\mathscr{I}) = 0$.*

A comparison between Theorems 1.7.4 and 6.8.2 reveals the striking similarity in the fine structure of admissible L^∞ solutions and BV functions. The reader should note, however, that there are some differences as well: points in the set $\mathscr{C}$ have merely vanishing mean oscillation in admissible L^∞ solutions, whereas they are Lebesgue points in the BV case. Furthermore, if u is a BV solution of (6.8.1), with $\nu_G = 0$, then, on account of Theorem 1.8.2, for any companion Q, ν_Q is concentrated on the set $\mathscr{J}$ of points of jump discontinuity. However, it is not known at the present time whether this important property carries over to L^∞ admissible solutions, except for $n = 1$; see Section 11.14.

The reader should consult the references in Section 6.11 for the proof of Theorem 6.8.2, which is lengthy and technical. Even so, a brief outline of some of the key ingredients is here in order.

Admissible L^∞ solutions u to (6.8.1) on $\mathscr{X}$ may be characterized by the kinetic formulation, discussed in Section 6.7. In the present setting, (6.7.2) takes the form

$$\sum_{\alpha=1}^{k} G'_\alpha(v)\partial_\alpha \chi(v;u) = \partial_v \nu, \tag{6.8.6}$$

where χ is the function defined by (6.7.3) and ν is a bounded measure on $\mathbb{R} \times \mathscr{X}$. Notice that here, in contrast to Section 6.7, the measure ν need not be nonnegative, as the notion of admissible solution adopted in this section is broader.

In analogy to (6.7.6), the measure ν_Q associated with any companion Q induced by some η through (6.8.3) is related to the measure ν by

$$\nu_Q = -\int_{-\infty}^{\infty} \eta''(v) d\nu(v;\cdot). \tag{6.8.7}$$

The measure ν also determines the "jump set" $\mathscr{J}$, in Theorem 6.8.2, by

$$\mathscr{J} = \{X \in \mathscr{X} : \limsup_{r \downarrow 0} \frac{|\nu|(\mathbb{R} \times \mathscr{B}_r(X))}{r^{k-1}} > 0\}, \tag{6.8.8}$$

where $|\nu|$ denotes the total variation measure of ν.

The resolution of the fine structure of u is achieved by "blowing up" the neighborhood of any point $X \in \mathscr{X}$, that is by rescaling u and ν in the vicinity of X in a manner that leaves (6.8.6) invariant. The linear nondegeneracy condition (6.8.4), in conjunction with velocity averaging estimates for the transport equation (6.8.6), induces the requisite compactness, so that the limits u_∞ and ν_∞ of u and ν under rescaling exist and satisfy (6.8.6). When $X \notin \mathscr{J}$, the measure ν_∞ vanishes. On the other hand, when $X \in \mathscr{J}$, ν_∞ is the tensor product of a measure on $\mathbb{R}$ and a measure on $\mathscr{X}$. It is by studying solutions of (6.8.6) with ν having this special tensor product structure that the assertion of Theorem 6.8.2 is established.

By the same techniques one verifies that admissible solutions of linearly nondegenerate scalar balance laws share another important property with BV functions, namely they have one-sided traces on manifolds of codimension one:

6.8.3 Theorem. *Let u be an admissible solution of the linearly nondegenerate balance law* (6.8.1) *on a Lipschitz subset $\mathscr{X}$ of $\mathbb{R}^k$ with boundary $\mathscr{B}$. Assume that for any companion Q the measure ν_Q in* (6.8.2) *is finite on $\mathscr{X}$. Then u has a strong trace $u_{\mathscr{B}} \in L^\infty(\mathscr{B})$ on $\mathscr{B}$.*

The strong trace is realized in L^1_{loc}, roughly as follows: Suppose that $\mathscr{B}$ contains a compact subset $\mathscr{P}$ of a $(k-1)$-dimensional hyperplane with outward unit normal N. Then the restriction of $u_{\mathscr{B}}$ to $\mathscr{P}$ is characterized by

$$\operatorname*{ess\,lim}_{\tau\downarrow 0} \int_{\mathscr{P}} |u(X-\tau N) - u_{\mathscr{B}}(X)| d\mathscr{H}^{k-1}(X) = 0. \tag{6.8.9}$$

In the general case, one employs Lipschitz transformations on $\mathbb{R}^k$ to map "pieces" of $\mathscr{B}$ into "pieces" $\mathscr{P}$ of a hyperplane, and then uses the above characterization.

Theorem 6.8.3 plays an important role in the theory of boundary value problems for scalar conservation laws, as we shall see in Section 6.9. Another important implication of Theorem 6.8.3 is the following

6.8.4 Corollary. *Assume that the scalar conservation law* (6.1.1) *is linearly nondegenerate, and let u be an L^∞ weak solution of the Cauchy problem* (6.1.1), (6.1.2), *on the upper half-space, which satisfies the inequalities* (6.2.2), *in the sense of distributions, for every convex entropy η. Then the map $t \mapsto u(\cdot,t)$ is strongly continuous in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$, for any $t \in [0,\infty)$.*

In particular, for linearly nondegenerate scalar conservation laws, admissible solutions to the Cauchy problem may be characterized merely by the set of inequalities (6.2.2), rather than by the stronger condition (6.2.3). Thus, referring back to the discussion on entropy admissibility, in Section 4.5, we conclude that for scalar, linearly nondegenerate conservation laws, the set $\mathscr{F}$ is empty.

6.9 Initial-Boundary Value Problems

Let $\mathscr{D}$ be an open bounded subset of $\mathbb{R}^m$, with smooth boundary $\partial\mathscr{D}$ and outward unit normal field ν. Here we consider the initial-boundary value problem

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = 0, \qquad (x,t) \in \mathscr{X}, \tag{6.9.1}$$

$$u(x,t) = 0, \qquad (x,t) \in \mathscr{B}, \tag{6.9.2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathscr{D}, \tag{6.9.3}$$

in the domain $\mathscr{X} = \mathscr{D} \times (0,\infty)$, with lateral boundary $\mathscr{B} = \partial\mathscr{D} \times (0,\infty)$.

The boundary condition (6.9.2) shall be interpreted in the context of the vanishing viscosity approach, as explained in Section 4.7. The inequality (4.7.5) motivates the following notion of admissible weak solution:

6.9.1 Definition. A bounded measurable function u on $\mathscr{X}$ is an *admissible weak solution* of (6.9.1), (6.9.2), (6.9.3), with initial data $u_0 \in L^\infty(\mathscr{D})$, if the inequality

(6.9.4)
$$\int_0^\infty \int_{\mathscr{D}} [\partial_t \psi \eta(u) + \sum_{\alpha=1}^m \partial_\alpha \psi Q_\alpha(u)] dxdt + \int_{\mathscr{D}} \psi(x,0)\eta(u_0(x))dx$$
$$\geq \int_0^\infty \int_{\partial\mathscr{D}} \psi \{Q^0_{\mathscr{B}} - \eta'(0)[G^0_{\mathscr{B}} - G_{\mathscr{B}}]\} d\mathscr{H}^{m-1}(x)dt$$

holds for every convex entropy η, with associated entropy flux Q determined by (6.2.1), and all nonnegative Lipschitz continuous test functions ψ with compact support in $\mathbb{R}^m \times [0,\infty)$. $G_{\mathscr{B}}$ denotes the trace of the normal component of G on $\mathscr{B}$, while $G^0_{\mathscr{B}}$ and $Q^0_{\mathscr{B}}$ stand for $G(0)\nu$ and $Q(0)\nu$, respectively.

Notice that (6.9.4) implies $\partial_t \eta + \operatorname{div} Q \leq 0$, and in particular $\partial_t u + \operatorname{div} G = 0$, so that the traces $Q_{\mathscr{B}}$ and $G_{\mathscr{B}}$ of the normal components of Q and G on $\mathscr{B}$ are well defined. Furthermore, (4.7.8) holds on $\mathscr{B}$, in the form

(6.9.5)
$$Q_{\mathscr{B}} - Q^0_{\mathscr{B}} - \eta'(0)[G_{\mathscr{B}} - G^0_{\mathscr{B}}] \geq 0.$$

At the price of technical complications, but without any essential difficulty, the special boundary condition $u = 0$ may be replaced with $u = \hat{u}(x,t)$, for any sufficiently smooth function $\hat{u}$.

The justification of Definition 6.9.1 is provided by

6.9.2 Theorem. *For each $u_0 \in L^\infty(\mathscr{D})$, there exists a unique admissible weak solution u of* (6.9.1), (6.9.2), (6.9.3), *and*

(6.9.6)
$$u(\cdot,t) \in C^0([0,\infty); L^1(\mathscr{D})).$$

Furthermore, if $u_0 \in BV(\mathscr{D})$, then $u \in BV_{\mathrm{loc}}(\mathscr{X})$.

Before establishing the existence of solutions by proving the above theorem, we demonstrate uniqueness and stability by means of the following analog of Theorem 6.2.3:

6.9.3 Theorem. *Let u and $\bar{u}$ be admissible weak solutions of* (6.9.1), (6.9.2) *with respective initial values u_0 and $\bar{u}_0$. Then, for any $t > 0$,*

(6.9.7)
$$\int_{\mathscr{D}} [u(x,t) - \bar{u}(x,t)]^+ dx \leq \int_{\mathscr{D}} [u_0(x) - \bar{u}_0(x)]^+ dx,$$

(6.9.8)
$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathscr{D})} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathscr{D})}.$$

Furthermore, if

(6.9.9)
$$u_0(x) \leq \bar{u}_0(x), \qquad a.e.\ on\ \mathscr{D},$$

then

(6.9.10)
$$u(x,t) \leq \bar{u}(x,t), \qquad a.e.\ on\ \mathscr{X}.$$

Proof. We sketch the proof under the simplifying assumption that both u and $\bar{u}$ attain strong traces $u_{\mathscr{B}}$ and $\bar{u}_{\mathscr{B}}$ on $\mathscr{B}$, in which case the traces of the normal components of G and Q on $\mathscr{B}$ are obtained via ordinary composition:

$$G_{\mathscr{B}} = G(u_{\mathscr{B}})\nu,\ Q_{\mathscr{B}} = Q(u_{\mathscr{B}})\nu,\ \bar{G}_{\mathscr{B}} = G(\bar{u}_{\mathscr{B}})\nu,\ \bar{Q}_{\mathscr{B}} = Q(\bar{u}_{\mathscr{B}})\nu. \tag{6.9.11}$$

The above assumption will hold when u and $\bar{u}$ are BV functions or when u and $\bar{u}$ are merely in L^∞ and G is linearly nondegenerate; see Theorem 6.8.3.

We retrace the steps in the proof of Theorem 6.2.3, employing the same entropy-entropy flux pair $(\eta(u;\bar{u}), Q(u;\bar{u}))$, defined by (6.2.5), and the same test function $\phi(x,t,\bar{x},\bar{t})$, given by (6.2.16). However, we now integrate over $\mathscr{D}\times[0,\infty)$, instead of $\mathbb{R}^m\times[0,\infty)$, and substitute (6.9.4) for (6.2.3). We thus obtain, in the place of (6.2.21),

(6.9.12)

$$\int_0^\infty\int_{\mathscr{D}}\{\partial_t\psi\,\eta(u;\bar{u})+\sum_{\alpha=1}^m\partial_\alpha\psi\, Q_\alpha(u;\bar{u})\}dxdt+\int_{\mathscr{D}}\psi(x,0)\eta(u_0(x);\bar{u}_0(x))dx$$
$$\geq\int_0^\infty\int_{\partial\mathscr{D}}\psi\,\mathrm{sgn}[u_{\mathscr{B}}-\bar{u}_{\mathscr{B}}]^+[G_{\mathscr{B}}-\bar{G}_{\mathscr{B}}]d\mathscr{H}^{m-1}(x)dt.$$

We verify that, as a consequence of the boundary condition (6.9.5), the integral on the right-hand side of (6.9.12) is nonnegative. Indeed, the integrand vanishes where $u_{\mathscr{B}}\leq\bar{u}_{\mathscr{B}}$, and has the sign of $G_{\mathscr{B}}-\bar{G}_{\mathscr{B}}$ where $u_{\mathscr{B}}>\bar{u}_{\mathscr{B}}$. In the latter case, we examine, separately, the following three subcases:

(a) $u_{\mathscr{B}}>\bar{u}_{\mathscr{B}}\geq 0$: (6.9.5), written for the solution u and the entropy-entropy flux pair $(\eta(u;\bar{u}_{\mathscr{B}}), Q(u;\bar{u}_{\mathscr{B}}))$, yields $G_{\mathscr{B}}\geq\bar{G}_{\mathscr{B}}$.
(b) $0\geq u_{\mathscr{B}}>\bar{u}_{\mathscr{B}}$: (6.9.5), written for the solution $\bar{u}$ and the entropy-entropy flux pair $(\eta(u_{\mathscr{B}};\bar{u}), Q(u_{\mathscr{B}};\bar{u}))$, again yields $G_{\mathscr{B}}\geq\bar{G}_{\mathscr{B}}$.
(c) $u_{\mathscr{B}}>0>\bar{u}_{\mathscr{B}}$: (6.9.5), written for the solution u and the entropy-entropy flux pair $(\eta(u;0), Q(u;0))$, yields $G_{\mathscr{B}}\geq G^0_{\mathscr{B}}$. Similarly, (6.9.5), written for the solution $\bar{u}$ and the entropy-entropy flux pair $(\eta(0;\bar{u}), Q(0;\bar{u}))$, yields $\bar{G}_{\mathscr{B}}\leq G^0_{\mathscr{B}}$. In particular, $G_{\mathscr{B}}\geq\bar{G}_{\mathscr{B}}$.

We apply (6.9.12) for the test function $\psi(x,\tau)=\chi(x)\omega(\tau)$, where $\chi(x)=1$ for $x\in\mathscr{D}$, and ω is defined by (5.3.11). Since the right-hand side of (6.9.12) is nonnegative, we deduce

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathscr{D}}[u(x,\tau)-\bar{u}(x,\tau)]^+\,dxd\tau\leq\int_{\mathscr{D}}[u_0(x)-\bar{u}_0(x)]^+\,dx. \tag{6.9.13}$$

Letting $\varepsilon\downarrow 0$, we arrive at (6.9.7). In turn, (6.9.7) readily implies the remaining assertions of the theorem. The proof is complete.

The next task is to construct the solution to (6.9.1), (6.9.2), (6.9.3) by the vanishing viscosity method. We thus consider the family of parabolic equations

$$\partial_t u(x,t) + \operatorname{div} G(u(x,t)) = \mu \Delta u(x,t), \qquad (x,t) \in \mathscr{X}, \tag{6.9.14}$$

with boundary condition (6.9.2) and initial condition (6.9.3). For any $\mu > 0$, (6.9.14), (6.9.2), (6.9.3) admits a unique solution u_μ which is smooth on $\bar{\mathscr{D}} \times (0,\infty)$. By the maximum principle,

$$\left|u_\mu(x,t)\right| \leq \sup|u_0(\cdot)|, \qquad x \in \mathscr{D}, \quad t \in (0,\infty). \tag{6.9.15}$$

Upon retracing the steps in the proof of Theorem 6.3.2, except that now (6.3.10) should be integrated over $\mathscr{D} \times (s,t)$ instead of $\mathbb{R}^m \times (s,t)$, one readily obtains

6.9.4 Theorem. *Let u_μ and $\bar{u}_\mu$ be solutions of* (6.9.14), (6.9.2) *with respective initial data u_0 and $\bar{u}_0$. Then, for any $t > 0$,*

$$\int_{\mathscr{D}} \left[u_\mu(x,t) - \bar{u}_\mu(x,t)\right]^+ dx \leq \int_{\mathscr{D}} [u_0(x) - \bar{u}_0(x)]^+ dx, \tag{6.9.16}$$

$$\|u_\mu(\cdot,t) - \bar{u}_\mu(\cdot,t)\|_{L^1(\mathscr{D})} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathscr{D})}. \tag{6.9.17}$$

Furthermore, if

$$u_0(x) \leq \bar{u}_0(x), \qquad a.e. \text{ on } \mathscr{D}, \tag{6.9.18}$$

then

$$u_\mu(x,t) \leq \bar{u}_\mu(x,t), \qquad (x,t) \in \mathscr{D} \times (0,\infty). \tag{6.9.19}$$

We proceed to show that the family $\{u_\mu : \mu > 0\}$ of solutions to (6.9.14), (6.9.2), (6.9.3) is relatively compact in L^1.

6.9.5 Lemma. *Let u_μ be the solution of* (6.9.14), (6.9.2), (6.9.3) *with initial data $u_0 \in L^\infty(\mathscr{D}) \bigcap W^{2,1}(\mathscr{D})$. Then, for any $t > 0$,*

$$\left\|\partial_t u_\mu(\cdot,t)\right\|_{L^1(\mathscr{D})} \leq c_0 \|u_0(\cdot)\|_{W^{1,1}(\mathscr{D})} + \mu \|u_0(\cdot)\|_{W^{2,1}(\mathscr{D})}, \tag{6.9.20}$$

$$\sum_{\beta=1}^{m} \left\|\partial_\beta u_\mu(\cdot,t)\right\|_{L^1(\mathscr{D})} \leq a(t)\|u_0(\cdot)\|_{W^{1,1}(\mathscr{D})} + \mu b(t)\|u_0(\cdot)\|_{W^{2,1}(\mathscr{D})}, \tag{6.9.21}$$

where c_0 and the continuous functions $a(t)$, $b(t)$ do not depend on μ.

Proof. For $h > 0$, we apply (6.9.17) for the two solutions, $u_\mu(x,t)$, with initial value $u_0(x)$, and $\bar{u}_\mu(x,t) = u_\mu(x,t+h)$, with initial value $\bar{u}_0(x) = u_\mu(x,h)$. Upon dividing by h, and then letting $h \downarrow 0$, we deduce $\left\|\partial_t u_\mu(\cdot,t)\right\|_{L^1(\mathscr{D})} \leq \left\|\partial_t u_\mu(\cdot,0)\right\|_{L^1(\mathscr{D})}$, whence (6.9.20) follows with the help of (6.9.14).

One cannot use the same procedure for estimating spatial derivatives, because shifting in the spatial direction no longer carries solutions into solutions. We thus have to employ a different argument.

For $\varepsilon > 0$, we define the function

$$(6.9.22) \qquad \eta_\varepsilon(w) = \begin{cases} -w-\varepsilon & -\infty < w \le -2\varepsilon \\[2ex] \dfrac{w^2}{4\varepsilon} & -2\varepsilon < w \le 2\varepsilon \\[2ex] w-\varepsilon & 2\varepsilon < w < \infty. \end{cases}$$

We set $w = \partial_\beta u_\mu$, differentiate (6.9.14) with respect to x_β, multiply the resulting equation by $\eta'_\varepsilon(w)$ and integrate over $\mathscr{D}$. After an integration by parts, this yields

$$(6.9.23) \quad \frac{d}{dt}\int_{\mathscr{D}} \eta_\varepsilon(w)dx = \int_{\mathscr{D}} \left[\eta_\varepsilon(w) - \eta'_\varepsilon(w)w\right] \operatorname{div} G'(u_\mu)\,dx$$
$$-\mu \int_{\mathscr{D}} \eta''_\varepsilon(w)\,|\nabla w|^2\,dx + \int_{\partial\mathscr{D}} [\mu\eta'_\varepsilon(w)\frac{\partial w}{\partial \nu} - \eta_\varepsilon(w)G'(0)\nu]d\mathscr{H}^{m-1}(x).$$

As $\varepsilon \downarrow 0$, the integrand on the left-hand side of (6.9.23) tends to $|w|$. On the right-hand side, the first integral is $O(\varepsilon)$ and the second integral is nonnegative. To estimate the integral over $\partial\mathscr{D}$, we note that since u_μ vanishes on the boundary, $\partial_\alpha u_\mu = \frac{\partial u_\mu}{\partial \nu}\nu_\alpha$, $\alpha = 1, \cdots, m$. In particular, $w = \frac{\partial u_\mu}{\partial \nu}\nu_\beta$. Then (6.9.14) implies $\frac{\partial u_\mu}{\partial \nu}G'(0)\nu = \mu\Delta u_\mu$. Finally, it is clear that $\frac{\partial w}{\partial \nu} = \frac{\partial^2 u_\mu}{\partial \nu^2}\nu_\beta + O(1)\frac{\partial u_\mu}{\partial \nu}$ and $\Delta u_\mu = \frac{\partial^2 u_\mu}{\partial \nu^2} + O(1)\frac{\partial u_\mu}{\partial \nu}$. We thus have

$$(6.9.24) \quad \mu\eta'_\varepsilon(w)\frac{\partial w}{\partial \nu} - \eta_\varepsilon(w)G'(0)\nu = \mu[\eta'_\varepsilon(w) - \frac{\eta_\varepsilon(w)}{w}]\frac{\partial^2 u_\mu}{\partial \nu^2}\nu_\beta + O(1)\mu\frac{\partial u_\mu}{\partial \nu},$$

which tends to $O(1)\mu\frac{\partial u_\mu}{\partial \nu}$, as $\varepsilon \downarrow 0$. Therefore, in the limit, as $\varepsilon \downarrow 0$, (6.9.23) yields

$$(6.9.25) \qquad \frac{d}{dt}\int_{\mathscr{D}} |\partial_\beta u_\mu|\,dx \le c\int_{\partial\mathscr{D}} \mu\left|\frac{\partial u_\mu}{\partial \nu}\right| d\mathscr{H}^{m-1}(x) \le c'\int_{\mathscr{D}} \mu\,|\Delta u_\mu|\,dx.$$

We sum (6.9.25) over $\beta = 1, \cdots, m$, and also substitute $\mu\Delta u_\mu$ by $\partial_t u_\mu + \operatorname{div} G(u_\mu)$. Using (6.9.15), (6.9.20) and applying Gronwall's inequality, we arrive at (6.9.21). The proof is complete.

Proof of Theorem 6.9.2. Assume first $u_0 \in L^\infty(\mathscr{D})\bigcap W^{2,1}(\mathscr{D})$. By virtue of Lemma 6.9.5, the family $\{u_\mu : \mu > 0\}$ of solutions to (6.9.14), (6.9.2), (6.9.3) is relatively compact in $L^1(\mathscr{D} \times (0,T))$, for any $T > 0$. Therefore, recalling (6.9.15), we may extract a sequence $\{u_{\mu_k}\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, which converges boundedly almost everywhere on $\mathscr{D} \times (0,\infty)$ to some function u. As shown in Section 4.7, u satisfies (6.9.4) and hence is the unique solution of (6.9.1), (6.9.2), (6.9.3). In particular, the entire family $\{u_\mu\colon \mu > 0\}$ converges to u, as $\mu \downarrow 0$. Moreover, it follows from (6.9.20), (6.9.21) that u is in $BV_{\mathrm{loc}}(\mathscr{D} \times (0,\infty))$ and, for any $T > 0$,

$$TV_{\mathscr{D}\times(0,T)}u \le c(T)\|u_0\|_{W^{1,1}(\mathscr{D})}\,. \tag{6.9.26}$$

In addition, u inherits from (6.9.15) the maximum principle: $|u(x,t)| \le \sup|u_0(\cdot)|$.

Assume now $u_0 \in L^\infty(\mathscr{D})$. We construct a sequence of functions $\{u_{0n}\}$ in $L^\infty(\mathscr{D})\bigcap W^{2,1}(\mathscr{D})$ with $\|u_{0n}\|_{L^\infty(\mathscr{D})} \le \|u_0\|_{L^\infty(\mathscr{D})}$ and $u_{0n} \to u_0$ in $L^1(\mathscr{D})$. By virtue of (6.9.8), the sequence $\{u_n\}$ of admissible solutions to (6.9.1), (6.9.2), with initial data u_{0n}, converges in L^1 to a function u which satisfies (6.9.4) and hence is the admissible solution of (6.9.1), (6.9.2), (6.9.3). Moreover, when u_0 is in $BV(\mathscr{D})$, the sequence $\{u_{0n}\}$ may be constructed with the additional requirement that $\|u_{0n}\|_{W^{1,1}(\mathscr{D})} \le C\left[TV_{\mathscr{D}}u_0 + \|u_0\|_{L^\infty(\mathscr{D})}\right]$, in which case (6.9.26) implies that u is in $BV(\mathscr{D}\times(0,T))$, for any $T>0$. This completes the proof.

6.10 The L^1 Theory for Systems of Conservation Laws

The successful treatment of the scalar conservation law, based on L^1 and L^∞ estimates, which we witnessed in the previous sections, naturally raises the expectation that a similar approach may also be effective for systems of conservation laws. Unfortunately, this does not seem to be the case. In order to gain some insight into the difficulty, let us consider the Cauchy problem for a symmetrizable system of conservation laws:

$$\partial_t U + \sum_{\alpha=1}^m \partial_\alpha G_\alpha(U) = 0, \qquad x \in \mathbb{R}^m,\ t>0, \tag{6.10.1}$$

$$U(x,0) = U_0(x), \qquad x \in \mathbb{R}^m. \tag{6.10.2}$$

In analogy to Definition 6.2.1, for the scalar case, we shall require that admissible solutions of (6.10.1), (6.10.2) satisfy (4.5.3), for any entropy-entropy flux pair (η, Q) with η convex. The first test of this should be whether the trivial, constant solutions $\bar{U}$ of (6.10.1) are L^p-stable in the class of admissible solutions:

$$\|U(\cdot,t)-\bar{U}\|_{L^p(\mathscr{B}_r)} \le c_p\|U_0(\cdot)-\bar{U}\|_{L^p(\mathscr{B}_{r+st})}\,. \tag{6.10.3}$$

Since the system is symmetrizable, and thereby endowed with a convex entropy of quadratic growth, (6.10.3) will be satisfied at least for $p=2$, by virtue of Theorem 5.3.1. The question then arises whether such an estimate may also hold for $p \ne 2$, with the cases $p=1$ and $p=\infty$ being of particular interest.

For the linear system

$$\partial_t V + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(\bar{U})\partial_\alpha V = 0, \tag{6.10.4}$$

resulting from linearizing (6.10.1) about a constant state $\bar{U}$, it is known (references in Section 6.11) that the following three statements are equivalent: (a) the zero solution is L^p-stable for some $p \neq 2$; (b) the zero solution is L^p-stable for all $1 \leq p \leq \infty$; (c) the Jacobian matrices $\mathrm{D}G_\alpha(\bar{U})$ commute:

$$\mathrm{D}G_\alpha(\bar{U})\mathrm{D}G_\beta(\bar{U}) = \mathrm{D}G_\beta(\bar{U})\mathrm{D}G_\alpha(\bar{U}), \qquad \alpha, \beta = 1, \cdots, m. \tag{6.10.5}$$

The system (6.10.1) inherits (6.10.5) as a necessary condition for L^p-stability:

6.10.1 Theorem. *Assume that the constant state $\bar{U}$ is L^p-stable,* (6.10.3) *for some $p \neq 2$, within the class of classical solutions. Then* (6.10.5) *holds.*

Sketch of Proof. For ε small, let $U_\varepsilon(x,t)$ denote the solution of (6.10.1) with initial values $U_\varepsilon(x,0) = \bar{U} + \varepsilon V_0(x)$, where $V_0 \in H_\ell$ for $\ell > \frac{m}{2} + 1$. By Theorem 5.1.1, U_ε exists, as a classical solution, on a time interval with length $O(\varepsilon^{-1})$. Furthermore,

$$U_\varepsilon(x,t) = \bar{U} + \varepsilon V(x,t) + O(\varepsilon^2), \tag{6.10.6}$$

where $V(x,t)$ is the solution of (6.10.4) with initial value $V_0(x)$. Now if (6.10.3) is satisfied by the solutions U_ε, for any $\varepsilon > 0$, it follows that the zero solution of (6.10.4) is L^p-stable and hence (6.10.5) must hold. This completes the proof.

A similar argument shows that (6.10.3) is also necessary for stability of solutions of (6.10.1), (6.10.2) in the space BV:

$$TV_{\mathscr{B}_r} U(\cdot,t) \leq c\, TV_{\mathscr{B}_{r+st}} U_0(\cdot). \tag{6.10.7}$$

The above results douse any hope that the elegant L^1 and BV theory of the scalar conservation law may be readily extended to general systems of conservation laws for which (6.10.5) is violated. A question of some relevance is whether (6.10.3) may at least hold in the special class of systems that satisfy (6.10.5). This is indeed the case, at least for systems of just two conservation laws:

6.10.2 Theorem. *Let* (6.10.1) *be a symmetrizable system of two conservation laws* $(n = 2)$ *with the property that* (6.10.5) *holds for all $\bar{U}$. Then, for any fixed $\bar{U}$ and $1 \leq p \leq 2$, there are $\delta > 0$ and $c_p > 0$ such that* (6.10.3) *holds for any admissible solution U of* (6.10.1), (6.10.2), *taking values in the ball $\mathscr{B}_\delta(\bar{U})$.*

The proof, which is found in the references cited in Section 6.11, employs a convex entropy η for (6.10.1) such that

$$c|U - \bar{U}|^p \leq \eta(U) \leq C|U - \bar{U}|^p, \qquad U \in \mathscr{B}_\delta(\bar{U}). \tag{6.10.8}$$

Recall that in order to construct an entropy for a system of n conservation laws in m spatial variables, one has to solve the generally overdetermined system (3.2.4) of $\frac{1}{2}n(n-1)m$ equations for the single scalar η. However, as noted in Section 3.2, when (6.10.5) holds, the number of independent equations is reduced to $\frac{1}{2}n(n-1)$, and in the special case $n = 2$ to just one. It thus becomes possible to construct a convex entropy with the requisite property (6.10.8), for $1 \leq p \leq 2$, by solving a Goursat problem on $\mathscr{B}_\delta(\bar{U})$. In fact, under additional assumptions on the system, it is even possible to construct convex entropies that satisfy (6.10.8) for all p, and for such systems constant solutions are L^p-stable over the full range $1 \leq p \leq \infty$.

The class of systems that satisfy (6.10.5) includes, in particular, the scalar conservation laws $(n = 1)$, in any spatial dimension m, as well as the systems of arbitrary size n, in a single spatial dimension $(m = 1)$; but beyond that it contains very few representatives of (even modest) physical interest. An example is the system

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha [F_\alpha(|U|)U] = 0, \tag{6.10.9}$$

which governs the flow of a fluid in an anisotropic porous medium. The special features of this system make it analytically tractable, so that it may serve as a vehicle for exhibiting some of the issues facing the study of hyperbolic systems of conservation laws in several space dimensions.

If U is a classical solution of (6.10.9), it is easy to see that its "density" $\rho = |U|$ satisfies the scalar conservation law

$$\partial_t \rho + \sum_{\alpha=1}^{m} \partial_\alpha [\rho F_\alpha(\rho)] = 0, \tag{6.10.10}$$

while its directional unit vector field $\Theta = \rho^{-1} U$ satisfies the transport equation

$$\partial_t \Theta + \sum_{\alpha=1}^{m} F_\alpha(\rho) \partial_\alpha \Theta = 0. \tag{6.10.11}$$

Thus, classical solutions to the Cauchy problem (6.10.9), (6.10.2) can be constructed by first solving (6.10.10), with initial data $\rho(\cdot,0) = |U_0(\cdot)|$, say by the method of characteristics expounded in Section 6.1, and then determining Θ by its property of staying constant along the trajectories of the ordinary differential equation

$$\frac{dx}{dt} = F(\rho(x,t)). \tag{6.10.12}$$

It is not obvious how to adapt the above procedure to weak solutions. It is of course still possible to determine ρ as the admissible weak solution of (6.10.10) with initial data $|U_0|$ merely in L^∞, but it is by no means clear how one should interpret (6.10.12) when $F(\rho(x,t))$ is just an L^∞ function. In fact, it has been shown (references in Section 6.11) that the Cauchy problem for (6.10.9) is generally ill-posed in L^∞. A relevant, powerful theory of ordinary differential equations $\dot{X} = P(X)$ exists, but it requires that P be a divergence-free vector field in BV. In order to use that theory, we restrict the initial data so that $|U_0|$ is a positive function of locally bounded variation on $\mathbb{R}^m$. This will guarantee, by virtue of Theorems 6.2.3 and 6.2.6, that ρ is a positive function of locally bounded variation on the upper half-space. Next, we rescale the time variable and rewrite (6.10.12) in the implicit form

$$
\text{(6.10.13)} \qquad \begin{cases} \dfrac{dt}{d\tau} = \rho(x,t) \\[2ex] \dfrac{dx}{d\tau} = \rho(x,t)F(\rho(x,t)), \end{cases}
$$

which has the desired feature that the vector field $(\rho, \rho F(\rho))$ is divergence-free on the upper half-space, by virtue of (6.10.10).

By eliminating τ in the family of solutions $(t(\tau), x(\tau))$ of (6.10.13), one obtains the family of curves $x = x(t)$, namely the formal trajectories of (6.10.12), along which Θ stays constant. Thus Θ can be determined from its initial data, which may merely be in L^∞. Finally, it can be shown (references in Section 6.11) that $U = \rho\Theta$ is a weak solution of (6.10.9), (6.10.2):

6.10.3 Theorem. *For any $U_0 \in L^\infty(\mathbb{R}^m; \mathbb{R}^n)$ with $|U_0| \in BV_{\mathrm{loc}}(\mathbb{R}^m)$, there exists a unique L^∞ weak solution U of* (6.10.9), (6.10.2) *on $[0,\infty)$, such that $\rho = |U|$ is the admissible weak solution of* (6.10.10) *with initial data $\rho(\cdot, 0) = |U_0(\cdot)|$.*

It has been shown, further, that the above solution depends continuously on its initial value in L^1_{loc}, and it satisfies the entropy admissibility condition, for any convex entropy of (6.10.9), at least when the set of critical points of the function $\rho F(\rho)$ on $(0,\infty)$ has measure zero.

On the other hand, the Cauchy problem for (6.10.9) is ill-posed in BV, even when the initial data have small total variation:

6.10.4 Theorem. *Let $m \geq 3$ and $n \geq 2$. For any nonzero $\bar{U} \in \mathbb{R}^n$, such that $|\bar{U}|$ is not a critical point of $\rho F(\rho)$, and any $\delta > 0$, there exist initial data U_0 that take values in $\mathscr{B}_\delta(\bar{U})$, are equal to $\bar{U}$ for $|x| > 1$, have total variation on $\mathbb{R}^m$ that is less than δ, and have the property that if U is any admissible L^∞ weak solution of* (6.10.9), (6.10.2) *on some time interval $[0,T)$, then the total variation of U on $\mathbb{R}^m \times [0,T)$ is infinite.*

The reader should bear in mind that (6.10.9) is so special that the above should not necessarily be interpreted as representative of the behavior of generic systems. The theory of hyperbolic systems of conservation laws in several spatial variables is still in its infancy.

6.11 Notes

More extensive discussion on the breakdown of classical solutions of scalar conservation laws can be found in Majda [4]. Theorem 6.1.1 is due to Conway [1]. For a systematic study of the geometric features of shock formation and propagation, see Izumiya and Kossioris [1], and Danilov and Mitrovic [3]. The reduction of (6.1.1) to the linear transport equation (6.1.10) is classical; see Courant-Hilbert [1,§I.5].

There is voluminous literature on weak solutions of the scalar conservation law. The investigation was initiated in the 1950's, in the framework of the single space dimension, stimulated by the seminal paper of Hopf [1], already cited in the historical introduction. References to this early work will be provided, as they become relevant, in Section 11.12.

The first existence proof in several space dimensions is due to Conway and Smoller [1], who recognized the relevance of the space BV and constructed solutions with bounded variation through the Lax-Friedrichs difference scheme. The definitive treatment in the space BV was later given by Volpert [1], who was apparently the first to realize the L^1 contraction property in several space dimensions. Building on Volpert's work, Kruzkov [1] proposed the characterization of admissible weak solutions recorded in Section 6.2, derived the L^1 contraction estimate, and established the convergence of the method of vanishing viscosity along the lines of our discussion in Section 6.3. More delicate treatment is needed when the flux is merely continuous in u; see Kruzhkov and Panov [1], Bénilan and Kruzkov [1], and Andreianov, Bénilan and Kruzhkov [1]. Moreover, when the flux is discontinuous the notion of an admissible weak solution must be clarified and redefined; see Panov [8,9,10], Andreianov, Karlsen and Risebro [1], and Audusse and Perthame [1]. On the other hand, the analysis extends routinely to inhomogeneous scalar balance laws (3.3.1), though solutions may blow up in finite time when the production grows superlinearly with u; see Natalini, Sinestrari and Tesei [1]. In particular, the inhomogeneous conservation law of "transport type," with flux $G(u,x) = f(u)V(x)$, has interesting structure, especially when $\operatorname{div} V = 0$; see Caginalp [1] and Otto [2].

New ideas, with geometric flavor, are needed in order to treat scalar conservation laws on a manifold, because the notion of entropy does not extend to that setting in a straightforward manner; see Amorim, Ben-Artzi and LeFloch [1], Ben-Artzi and LeFloch [1], Ben-Artzi, Falcovitz and LeFloch [1], LeFloch and Okutmustur [1], Dziuk, Kröner and Müller [1], Kröner, Müller and Strehlau [1], and Panov [3].

The theory of nonlinear contraction semigroups in general, not necessarily reflexive, Banach space is due to Crandall and Liggett [1]. The application to the scalar conservation law presented in Section 6.4 is taken from Crandall [1]. For an alternative functional analytic characterization of admissible solutions, see Portilheiro [1].

The construction of solutions by the layering method, discussed in Section 6.5, was suggested by Roždestvenskii [1] and was carried out by Kuznetsov [1] and Douglis [2].

There is an active research program aiming at treating hyperbolic conservation laws as the "relaxed" form of larger, but simpler, systems that may govern, or model, relaxation phenomena in physics. Further discussion and references are found in Chapter XVII. The presentation in Section 6.6 follows Katsoulakis and Tzavaras [1]. Though artificially constructed for the purposes of the analysis, (6.6.1) may be interpreted a posteriori as a system governing the evolution of an ensemble of interacting particles, at the mesoscopic scale. See Katsoulakis and Tzavaras [2], and Jin, Katsoulakis and Xin [1]. An alternative construction of solutions to multidimensional scalar conservation laws by a relaxation scheme is discussed in Natalini [2].

The kinetic formulation described in Section 6.7 is due to Perthame and Tadmor [1], and Lions, Perthame and Tadmor [2]. Theorem 6.7.2 was first established by Lions, Perthame and Tadmor [2], with $s < r/(r+2)$. The improved range $s < r/(2r+1)$

was derived by Tadmor and Tao [1], with the help of a sharper velocity averaging estimate. A detailed discussion, with extensions, applications and an extensive bibliography, is found in the recent monograph and survey article by Perthame [2,3]. For related results, see Giga and Miyakawa [1], Bäcker and Dressler [1], Brenier [1], James, Peng and Perthame [1], Natalini [2], Perthame [1], Perthame and Pulvirenti [1], Hwang [1,2], Vasseur [2,3,5], Westdickenberg and Noelle [1], Kissling and LeFloch [1], and Dalibard [1]. The mechanism that induces the regularizing effect stated in Theorem 6.7.2 plays a prominent role in the theory of nonlinear transport equations in general, including the classical Boltzmann equation (cf. DiPerna and Lions [1]).

The surprising association of the level sets of admissible solutions with contraction semigroups in Hilbert space, outlined in Section 6.7, was discovered by Brenier [3]. See also Bolley, Brenier and Loeper [1].

For another interesting example of a kinetic model that relaxes, in the hydrodynamic limit, to the scalar conservation law, see Portilheiro and Tzavaras [1].

There are several other methods for constructing solutions, most notably by fractional stepping, spectral viscosity approximation, or through various difference schemes that may also be employed for efficient computation. See, for example, Bouchut and Perthame [1], Chen, Du and Tadmor [1], Cockburn, Coquel and LeFloch [1], and Crandall and Majda [1]. For references on the numerics the reader should consult LeVeque [1], Godlewski and Raviart [1,2], and Kröner [1].

In addition to L^1 and BV, other function spaces are relevant to the theory. DeVore and Lucier [1] show that solutions of (6.1.1) reside in Besov spaces.

Perthame and Westdickenberg [1] establish a total oscillation diminishing property for solutions.

To get a feel for the limiting behavior of solutions when the conservation law is singularly perturbed, the reader may consult Botchorishvilli, Perthame and Vasseur [1], for the effect of stiff sources, Hwang [3], for diffusive-dispersive limits, Aggarwal, Colombo and Goatin [1], for nonlocal effects, and Dalibard [2], for the consequences of homogenization.

The fine structure of L^∞ solutions, and in particular Theorem 6.8.2, is discussed in De Lellis, Otto and Westdickenberg [1]. See also De Lellis and Rivière [1], De Lellis and Golse [1], and Crippa, Otto and Westdickenburg [1]. Theorem 6.8.3 is due to Vasseur [4]. See also Chen and Rascle [1], and, for more recent developments, Panov [5,6], and Kwon and Vasseur [1].

The construction of BV solutions to the initial-boundary value problem by the method of vanishing viscosity, expounded in Section 6.9, is taken from Bardos, Leroux and Nédélec [1]. For a proof of Theorem 6.9.3 when u and $\bar{u}$ are merely in L^∞, see Otto [1] and Málek, Nečas, Rokyta and Růžička [1]. For recent results in that direction, see Kwon [1], and Coclite, Karlsen and Kwon [1]. For the initial-boundary value problem in L^∞, with the flux vanishing at the boundary, see Bürger, Frid and Karlsen [1]. The case of discontinuous flux is discussed by Carrillo [1]. Solutions in L^∞ have been constructed via the kinetic formulation by Nouri, Omrane and Vila [1], and Tidriri [1]. For measure-valued solutions, see Szepessy [1], and Kondo and LeFloch [1].

The large time behavior of solutions of (6.1.1), (6.1.2) is discussed in Conway [1], Engquist and E [1], Bauman and Phillips, and Feireisl and Petzeltová [1]. Chen and Frid [1,3,4,6] set a framework for investigating, in general systems of conservation laws, decay of solutions induced by scale invariance and compactness. In particular, this theory establishes the long time behavior of solutions of (6.1.1), (6.1.2) when u_0 is either periodic or of the form $u_0(x) = v(|x|^{-1}x) + w(x)$, with $w \in L^1(\mathbb{R}^m)$. This framework has been refined by Panov [11,13] and extended to the almost periodic case by Frid [5], and Panov [12]. The approach, via contraction semigroups, leading to Theorem 6.4.9, is taken from Dafermos [35]. In that connection, see also Dafermos and Slemrod [1].

For stochastic effects see Lions, Perthame and Souganidis [2,3], Gess and Souganidis [1,2], and the references in these interesting papers.

The proof that (6.10.5) is necessary and sufficient for L^p-stability in symmetrizable linear systems, is due to Brenner [1]. Rauch [1] demonstrated Theorem 6.10.1, and Dafermos [22] proved Theorem 6.10.2. See also Frid and LeFloch [1], for a uniqueness result. Theorem 6.10.3 is due to Ambrosio and De Lellis [1]. See also Ambrosio, Bouchut and De Lellis [1]. Finally, Bressan [11,13] and De Lellis [1,2] explain why the Cauchy problem for the system (6.10.9) is not generally well-posed in L^∞ or in BV, when $m > 1$ (Theorem 6.10.4). By contrast, when $m = 1$ the Cauchy problem for this system is well-posed and has an interesting theory; see Temple [2], Isaacson and Temple [1], Liu and Wang [1], Tveito and Winther [1], Freistühler [7], and Panov [4].

VII

Hyperbolic Systems of Balance Laws in One-Space Dimension

Chapters VII-XVI will be devoted to the study of systems of balance laws in one space dimension. This narrowing of focus is principally dictated by necessity: At the present time the theory of multidimensional systems is terra incognita, replete with fascinating problems. In any event, the reader should bear in mind that certain multidimensional phenomena, with special symmetry, such as wave focussing, may be studied in the context of the one-space-dimensional theory. We will return to several space dimensions in Chapter XVII.

This chapter introduces many of the concepts that serve as foundation of the theory of hyperbolic systems of balance laws in one space dimension: strict hyperbolicity; Riemann invariants and their relation to entropy; simple waves; genuine nonlinearity and its role in the breakdown of classical solutions.

In order to set the stage, the chapter opens with the presentation of a number of illustrative examples of hyperbolic systems of balance laws in one space dimension, arising in physics or other branches of science and technology.

7.1 Balance Laws in One-Space Dimension

When $m = 1$, the general system of balance laws (3.1.1) reduces to

$$\partial_t H(U(x,t),x,t) + \partial_x F(U(x,t),x,t) = \Pi(U(x,t),x,t). \tag{7.1.1}$$

Systems (7.1.1) naturally arise in the study of gas flow in ducts, vibration of elastic bars or strings, etc., in which the medium itself is modeled as one-dimensional. The simplest examples are homogeneous systems of conservation laws, beginning with the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{7.1.2}$$

Despite its apparent simplicity, the scalar conservation law provides valuable insight into complex processes, in physics and elsewhere. The simple hydrodynamic theory of traffic flow in a stretch of highway is a case in point.

C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*,
Grundlehren der mathematischen Wissenschaften 325,
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The state of the traffic at location x and time t is described by the traffic density $\rho(x,t)$ (measured, say, in vehicles per mile) and the traffic speed $v(x,t)$ (in miles per hour). The fields ρ and v are related by the law of conservation of vehicles, which is identical to mass conservation (2.3.2), for rectilinear motion:

$$\partial_t \rho + \partial_x(\rho v) = 0. \tag{7.1.3}$$

This equation is then closed by the behavioral assumption that drivers set their vehicles' speed according to the local density, $v = g(\rho)$. In order to account for the congestion effect, g must be decreasing with ρ, for instance $g(\rho) = v_0(1-\rho/\rho_0)$, where v_0 is the speed limit and ρ_0 is the saturation density beyond which traffic crawls to a standstill. For that $g(\rho)$, (7.1.3) becomes

$$\partial_t \rho + \partial_x \left[v_0 \rho \left(1 - \frac{\rho}{\rho_0} \right) \right] = 0. \tag{7.1.4}$$

This simplistic model manages, nevertheless, to capture some of the qualitative features of traffic flow in congested highways, and serves as the springboard for more sophisticated models, developed in the references cited in Section 7.10.

Another important example of a scalar conservation law is the *Buckley-Leverett equation*

$$\partial_t u + \partial_x \left[\frac{u^2}{u^2 + a(1-u)^2} \right] = 0, \tag{7.1.5}$$

where a is some positive parameter. It provides a simple model for the rectilinear flow of two immiscible fluids (phases), such as oil and water, through a porous medium, and thus finds applications in enhanced oil recovery operations by the petroleum industry. The variable u, taking values in $[0,1]$, measures the saturation (i.e., volume fraction) of one of the fluid phases, and the flux measures the fractional flow rate of that phase. Thus, the equation expresses mass conservation. If one neglects the effects of inertia and capillarity, the fractional flow rate is determined through Darcy's law, and it depends on the ratio of viscosities as well as on the ratio of relative permeabilities of the two phases. In turn, the relative permeabilities depend on the saturation u, and hence the equation closes. The empirical flux function employed in (7.1.5) captures the salient traits of the fractional flow rate. It should be noted that, in contrast to (7.1.4), the second derivative of the flux in (7.1.5) changes sign on $[0,1]$. As we shall see later, this renders the structure of solutions substantially more complex.

Still another instructive example of a scalar conservation law arises in chromatography in a single solute. The concentration c of a dilute solute of a chemical species moving, with speed v, through the interstices of a finely divided solid bed of particles, and absorbed on the solid surfaces, satisfies the conservation law

$$\partial_x c + \partial_t [v^{-1} g(c)] = 0, \tag{7.1.6}$$

where x is the space variable along the bed, t is time, and $g(c)$ is the *column isotherm function*. The reader should notice that space and time exchange roles in this case.

An interesting example of a scalar conservation law (7.1.2) in which both variables are spatial arises in the theory of composite materials consisting of an incompressible matrix, such as rubber, reinforced with inextensible fibers. In the mathematical modeling, it is assumed that one fiber passes from any particle of the matrix, so the material is inextensible in the direction of the tangent to that fiber. In the equilibrium state of a body made of this material, the fibers are assumed parallel straight lines, tangential to some fixed vector $A \in \mathbb{R}^3$. Taking this equilibrium state as reference configuration, we consider a placement (bilipschitz homeomorphism) $X^* = X^*(X)$, with deformation gradient $F = \partial X^*/\partial X$. Each fiber becomes a Lipschitz curve with tangent vector field $A^* = FA$. As the fibers are inextensible, $|A^*| = 1$. Furthermore, since the material is incompressible, $\det F = 1$. Hence $A^{*\top} = (\det F)^{-1} A^\top F^\top$, so that $\operatorname{div} A^{*\top} = 0$, by virtue of Theorem 1.3.1. Assume now that $A^\top = (1,0,0)$ and that the deformation is planar, with F_{13}, F_{23}, F_{31} and F_{32} all zero. Thus $A^{*\top} = (F_{11}, F_{21}, 0)$, where F_{11} and F_{21} depend solely on the first two coordinates (x,y) of X^*. Assuming $F_{11} > 0$, setting $F_{21} = u(x,y)$ and noting that $|A^*| = 1$ implies $F_{11} = \sqrt{1-u^2}$, we conclude that $\operatorname{div} A^{*\top} = 0$ reduces to the scalar conservation law

$$\partial_y u + \partial_x \sqrt{1-u^2} = 0. \tag{7.1.7}$$

Thermoelasticity is a rich source of interesting examples of systems. In Lagrangian coordinates, the rectilinear adiabatic flow of a thermoelastic fluid (gas) in a duct is governed by the one-dimensional version of (3.3.4), in the form

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x p(u,s) = 0 \\ \partial_t \left[\varepsilon(u,s) + \frac{1}{2} v^2\right] + \partial_x [v p(u,s)] = 0, \end{cases} \tag{7.1.8}$$

where u is the *specific volume* ($u = 1/\rho$, on account of (2.3.3)), v denotes the velocity, ε is the internal energy and p stands for the *pressure* (-p is the stress). Note that $\rho > 0$ restricts u to positive values.

The thermodynamic relations (3.3.5) here read

$$p(u,s) = -\varepsilon_u(u,s), \qquad \theta(u,s) = \varepsilon_s(u,s). \tag{7.1.9}$$

The system (7.1.8) is hyperbolic if

$$\varepsilon_s(u,s) > 0, \qquad \varepsilon_{uu}(u,s) > 0, \tag{7.1.10}$$

that is, the absolute temperature θ is positive and the internal energy ε is convex in u, or equivalently, $p_u(u,s) < 0$.

The one-dimensional version of (3.3.19),

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u) = 0, \end{cases} \tag{7.1.11}$$

with monotone increasing σ, $\sigma'(u) > 0$, is the hyperbolic system governing the rectilinear isentropic flow of a gas, as well as the isentropic longitudinal oscillation of an elastic solid bar and the isentropic shearing motion of an elastic layer. For the case of a gas, where u is the specific volume, (7.1.11) written with $\sigma = -p$, is commonly known as the *p-system*. In the case of the bar, u is the *strain*. Specific volume and strain are both restricted to positive values, whereas in the case of the shearing motion u may take values of either sign. In what follows, we shall often use (7.1.11) as a mathematical paradigm under the assumption that σ is a smooth monotone increasing function on $(-\infty,\infty)$.

In Eulerian coordinates, rectilinear isentropic flow of a gas is governed by the one-dimensional version of (3.3.36), namely

$$(7.1.12) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x[\rho v^2 + p(\rho)] = 0. \end{cases}$$

This system is hyperbolic when $p'(\rho) > 0$. In particular, when the fluid is an ideal gas (2.5.31), (7.1.12) becomes

$$(7.1.13) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x[\rho v^2 + \kappa\rho^\gamma] = 0. \end{cases}$$

For $\gamma > 1$, hyperbolicity breaks down at the vacuum state $\rho = 0$.

The so called system of *pressureless gas dynamics*

$$(7.1.14) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x\left(\rho v^2\right) = 0, \end{cases}$$

which is not hyperbolic, governs the flow of an aggregate of "sticky" particles: colliding particles fuse into a single particle that combines their masses and moves with velocity that conserves the total linear momentum. The propensity of solutions of (7.1.14) to develop mass concentrations may serve as an explanation for the formation of large-scale structures in the universe.

Next we derive the system that governs isentropic, planar oscillations of a three-dimensional, homogeneous thermoelastic medium, with reference density $\rho_0 = 1$. In the terminology and notation of Chapter II, we consider motions in the particular form $\chi = x + \phi(x\cdot\nu, t)$, where ν is the (constant) unit vector pointing in the direction of the oscillation. For consistency with the notation of this chapter, we shall denote the scalar variable $x\cdot\nu$ by x, so that $\partial_x = \sum_{\alpha=1}^3 \nu_\alpha \partial_\alpha$. The velocity in the ν-direction is $v(x,t) = \partial_t\phi(x,t)$. We also set $u(x,t) = \partial_x\phi(x,t)$, in which case the deformation gradient is $F = I + u\otimes\nu$. The stress vector, per unit area, on planes perpendicular to ν is $\sigma(u) = S(I + u\nu^\top)\nu$, where $S(F)$ is the Piola-Kirchhoff stress. We thus end up with a system of six conservation laws

$$
(7.1.15) \qquad \begin{cases} \partial_t u - \partial_x v = 0 \\ \\ \partial_t v - \partial_x \sigma(u) = 0, \end{cases}
$$

which looks identical to (7.1.11), except that here u, v and σ are no longer scalars but 3-vectors.

The internal energy $\varepsilon(F)$ also becomes a function of u: $\varepsilon(I + u\nu^\top) = e(u)$. Then, (2.5.30) yields $\sigma(u) = \partial e(u)/\partial u$. Thus the Jacobian matrix of $\sigma(u)$ is the Hessian matrix of $e(u)$, which in turn is the acoustic tensor (3.3.8) evaluated at $F = I + u\nu^\top$. The system (7.1.15) is hyperbolic when the function $e(u)$ is convex.

As explained in Section 2.5 (recall (2.5.25)), when the medium is an isotropic solid, the internal energy depends on F solely through the invariants $|F|$, $|F^*|$ and $\det F$. Here $F = I + u\nu^\top$ and so $|F|^2 = 3 + 2u\cdot\nu + |u|^2$, $|F^*|^2 = (u\cdot\nu)^2 + 4u\cdot\nu + |u|^2$ and $\det F = 1 + u\cdot\nu$. Thus, the internal energy depends on just two variables, $|u|$ and $u\cdot\nu$. If, in addition, the material is incompressible, the kinematic constraint (2.7.1) becomes $u\cdot\nu = 0$, in which case the internal energy depends solely on $|u|$, $e(u) = h(|u|)$. The stress tensor is now given by (2.7.2), where p is the hydrostatic pressure. After a short calculation, recalling that $\sigma = S\nu$, we deduce that (7.1.15) takes the form

$$
(7.1.16) \qquad \begin{cases} \partial_t u - \partial_x v = 0 \\ \\ \partial_t v + \partial_x p\,\nu - \partial_x \left(\dfrac{h'(|u|)}{|u|} u \right) = 0. \end{cases}
$$

However, the incompressibility condition $u\cdot\nu = 0$ implies $\partial_x v\cdot\nu = 0$; let us take $v\cdot\nu = 0$ so as to eliminate a trivial rigid motion in the direction ν. Then $(7.1.16)_2$ yields $\partial_x p = 0$, and thus (7.1.16) reduces to

$$
(7.1.17) \qquad \begin{cases} \partial_t u - \partial_x v = 0 \\ \\ \partial_t v - \partial_x \left(\dfrac{h'(|u|)}{|u|} u \right) = 0. \end{cases}
$$

The special symmetry encoded in the flux function of (7.1.17) induces rich geometric structure which is a gift to the geometer at the expense of the analyst who has to deal with particular analytical difficulties, a taste of which will emerge later. The next example indicates that the same symmetry structure arises in other contexts as well.We derive the system that governs the oscillation of a flexible, extensible elastic string. The reference configuration of the string lies along the x-axis, and is assumed to be a natural state of (linear) density one. The motion $\chi = \chi(x,t)$ is monitored through the *velocity* $v = \partial_t \chi$ and the *stretching* $u = \partial_x \chi$ which take values in $\mathbb{R}^3$ or in $\mathbb{R}^2$, depending on whether the string is free to move in 3-dimensional space or is constrained to undergo planar oscillations. The tension τ of the string is assumed to depend solely on $|u|$, $\tau = \tau(|u|)$, which measures the stretch of the string. Since the string cannot sustain any compression, the natural range of $|u|$ is $[1,\infty)$, and τ

is assumed to satisfy $\tau(r) > 0$, $[\tau(r)/r]' > 0$, for $r > 1$. The compatibility relation between u and v together with balance of momentum, in Lagrangian coordinates, yields the hyperbolic system

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \left(\dfrac{\tau(|u|)}{|u|} u \right) = 0, \end{cases} \tag{7.1.18}$$

which is identical to (7.1.17).

Our next example is the classical system of conservation laws that governs the propagation of long gravity waves in shallow water. It may be derived either by asymptotic analysis of the Euler equations or ab initio, by appealing to gross balance of mass and momentum. We follow here the latter approach.

An incompressible, inviscid fluid of density one flows isentropically in an open channel with horizontally level bottom and unit width. The atmospheric pressure on the free surface is taken to be zero. The flow is driven by the hydrostatic pressure gradient induced by variations in the height of the free surface. Assume the channel lies along the x-axis, the y-axis is vertical, pointing upwards, and the bottom rests on the x-z plane. It is assumed that the height of the free surface is constant in the z-direction and thus is described by a function h of (x,t) alone. Moreover, the velocity vector points in the x-direction and is constant on any cross section of the channel, so its length is likewise described by a function v of (x,t).

As explained in Section 2.7, the stress tensor for an incompressible, inviscid fluid is just a hydrostatic pressure $-pI$. The balance of linear momentum in the y and the z-direction yields $\partial_y p = -g$ and $\partial_z p = 0$, respectively, where g is the acceleration of gravity. Thus, $p = g[h(x,t) - y]$, for $0 \le y \le h(x,t)$. Integrating with respect to y and z, we find that the total pressure force exerted on the x-cross section at time t is $P(x,t) = \frac{1}{2} g h^2(x,t)$.

We treat the flow in the channel as a rectilinear motion of a continuum governed by conservation of mass and linear momentum, exactly as in (7.1.12), where now the role of density is naturally played by the cross sectional area h and the role of pressure is played by the pressure force P. We thus arrive at the system of shallow water waves:

$$\begin{cases} \partial_t h + \partial_x (hv) = 0 \\ \partial_t (hv) + \partial_x \left(hv^2 + \frac{1}{2} g h^2 \right) = 0. \end{cases} \tag{7.1.19}$$

Notice that (7.1.19) is identical to (7.1.13), with $\gamma = 2$.

As we saw earlier, the flow of two phases through a porous medium, with volume fractions u and $1 - u$, is governed by the Buckley-Leverett equation (7.1.5). In the case of the flow of n phases, with respective volume fractions $U_1, \cdots, U_n$,

$$U_1 + \cdots + U_n = 1, \tag{7.1.20}$$

the conservation of mass equations reduce to a system of the following form:

$$\partial_t U_i + \partial_x \left[\frac{c_i U_i^2}{\sum_{j=1}^n c_j U_j^2} \right] = 0\,, \qquad i = 1, \cdots, n. \tag{7.1.21}$$

Notice that (7.1.20) and (7.1.21) are compatible. Of course, one may eliminate one of the unknowns, with the help of (7.1.20), thus reducing the size of the system (7.1.21) by one. For instance, when $n = 3$ and $c_1 = c_2 = c_3$,

$$\begin{cases} \partial_t u + \partial_x \left[\dfrac{u^2}{u^2 + v^2 + (1-u-v)^2} \right] = 0 \\ \\ \partial_t v + \partial_x \left[\dfrac{v^2}{u^2 + v^2 + (1-u-v)^2} \right] = 0. \end{cases} \tag{7.1.22}$$

Systems of this type are employed by the petroleum industry in oil recovery operations.

Systems with interesting features govern the propagation of planar electromagnetic waves through special isotropic dielectrics in which the electromagnetic energy depends on the magnetic induction B and the electric displacement D solely through the scalar $r = (B \cdot B + D \cdot D)^{\frac{1}{2}}$; i.e., in the notation of Section 3.3.8, $\eta(B,D) = \psi(r)$, with $\psi'(0) = 0$, $\psi''(0) > 0$, and $\psi'(r) > 0, \psi''(r) > 0$ for $r > 0$. Waves propagating in the direction of the 3-axis are represented by solutions of Maxwell's equations (3.3.66), with $J = 0$, in which the fields B, D, E and H depend solely on the single spatial variable $x = x_3$ and on time t. In particular, (3.3.66) imply $B_3 = 0$ and $D_3 = 0$ so that B and D should be regarded as vectors in $\mathbb{R}^2$ satisfying the hyperbolic system

$$\begin{cases} \partial_t B - \partial_x \left[\dfrac{\psi'(r)}{r} A D \right] = 0 \\ \\ \partial_t D + \partial_x \left[\dfrac{\psi'(r)}{r} A B \right] = 0, \end{cases} \tag{7.1.23}$$

where A is the alternating 2×2 matrix, with $A_{11} = A_{22} = 0, A_{12} = -A_{21} = 1$.

Returning to the general balance law (7.1.1), we note that H and/or F may depend explicitly on x, to account for inhomogeneity of the medium. For example, isentropic gas flow through a duct of (slowly) varying cross section $a(x)$ is governed by the system

$$\begin{cases} \partial_t [a(x)\rho] + \partial_x [a(x)\rho v] = 0 \\ \\ \partial_t [a(x)\rho v] + \partial_x [a(x)\rho v^2 + a(x) p(\rho)] = a'(x) p(\rho), \end{cases} \tag{7.1.24}$$

which reduces to (7.1.12) in the homogeneous case a =constant. On the other hand, explicit dependence of H or F on t, indicating "ageing" of the medium, is fairly rare. By contrast, dependence of Π on t is not uncommon, because external forcing is generally time-dependent.

The source Π may depend on the state vector U, to account for relaxation or reaction effects. A simple example of the latter case is provided by the system

$$
(7.1.25) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x[\rho v^2 + (\gamma - 1)c\rho\theta] = 0 \\ \partial_t\left[c\rho\theta + \beta\rho z + \frac{1}{2}\rho v^2\right] + \partial_x\left[\left(\gamma c\rho\theta + \beta\rho z + \frac{1}{2}\rho v^2\right)v\right] = 0 \\ \partial_t(\rho z) + \partial_x(\rho z v) = -\delta h(\theta - \theta_i)\rho z, \end{cases}
$$

which governs the flow of a combustible ideal gas in a duct. In addition to density ρ, velocity v and temperature θ, the state vector here comprises the *mass fraction* z of the unburnt gas, which takes values in $[0,1]$. The first three equations in (7.1.25) express the balance of mass, momentum and energy. As in (2.5.18), the equation of state for the pressure is $p = R\rho\theta = (\gamma - 1)c\rho\theta$, where γ is the adiabatic exponent and c is the specific heat. On the other hand, unlike (2.5.19), the internal energy here depends also on z, $\varepsilon = c\theta + \beta z$, where $\beta > 0$ is the *heat of reaction* (assumed exothermic). In the fourth equation of (7.1.25), which governs the reaction, h is the standard Heaviside function ($h(\zeta) = 0$ for $\zeta < 0$ and $h(\zeta) = 1$ for $\zeta \geq 0$), θ_i is the *ignition temperature* and $\delta > 0$ is the *reaction rate*.

A simple model system that captures the principal features of (7.1.25) is

$$
(7.1.26) \qquad \begin{cases} \partial_t(u + \beta z) + \partial_x f(u) = 0 \\ \partial_t z = -\delta h(u) z, \end{cases}
$$

where both u and z are scalar variables, and $f(u)$ is a strictly increasing convex function.

As an example of a source that manifests relaxation, consider the isothermal flow of a binary mixture of ideal gases in a duct. Both constituents of the mixture satisfy partial balance laws of mass and momentum: For $\alpha = 1,2$,

$$
(7.1.27)_\alpha \qquad \begin{cases} \partial_t \rho_\alpha + \partial_x(\rho_\alpha v_\alpha) = 0 \\ \partial_t(\rho_\alpha v_\alpha) + \partial_x\left[\rho_\alpha v_\alpha^2 + \nu_\alpha \rho_\alpha\right] = \chi_\alpha \,. \end{cases}
$$

The coupling is induced by the source term χ_α, which accounts for the momentum transfer to the α-constituent by the other constituent, as a result of the disparity between v_1 and v_2. In particular, $\chi_1 + \chi_2 = 0$. In nonisothermal flow, the coupling is enhanced by the balance law of energy. In more sophisticating modeling of mixtures, the density gradient appears, along with the density, as a state variable (Fick's law), in which case second-order spatial derivatives of the concentrations emerge in the field equations. Such terms induce diffusion, similar to the effect of heat conduction or viscosity. Here, however, we shall deal with the simple system $(7.1.27)_1$-$(7.1.27)_2$, which is hyperbolic.

So as to realize the mixture as a single continuous medium, it is expedient to replace the original state vector $(\rho_1, \rho_2, v_1, v_2)$ with new state variables (ρ, c, v, m), where ρ and v are the density and mean velocity of the mixture, that is, $\rho = \rho_1 + \rho_2$,

$\rho v = \rho_1 v_1 + \rho_2 v_2$, c is the concentration of the first constituent, i.e., $c = \rho_1/\rho$, and $m = (-1)^\alpha \rho_\alpha (v - v_\alpha)$. It is assumed that $\chi_\alpha = \beta \rho_\alpha (v_\alpha - v) = (-1)^\alpha \beta m$, where β is a positive constant. One may then rewrite the system (7.1.27) in the form

$$\text{(7.1.28)} \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho c) + \partial_x(\rho c v + m) = 0 \\ \partial_t(\rho v) + \partial_x \left[\rho v^2 + (\nu_2 + (\nu_1 - \nu_2)c)\rho + \dfrac{m^2}{\rho c(1-c)} \right] = 0 \\ \partial_t(\rho c v + m) + \partial_x \left[\rho c v^2 + 2mv + \dfrac{m^2}{\rho c} + \nu_1 c \rho \right] = -\beta m. \end{cases}$$

Indeed, the second and fourth equations in the above system are just $(7.1.27)_1$, rewritten in terms of the new state variables, while the first and the third equations are obtained by adding the corresponding equations of $(7.1.27)_1$ and $(7.1.27)_2$.

Single-space-dimensional systems (7.1.1) also derive from multispace-dimensional systems (3.1.1), in the presence of symmetry (planar, cylindrical, radial, etc.) that reduces spatial dependence to a single parameter. In that process, parent multidimensional homogeneous systems of conservation laws may yield one-dimensional inhomogeneous systems of balance laws, as a reflection of multidimensional geometric effects. For example, the single-space-dimensional system governing radial, isentropic gas flow, which results from the homogeneous Euler equations (3.3.36), is inhomogeneous:

$$\text{(7.1.29)} \qquad \begin{cases} \partial_t \rho + \partial_r(\rho v) + \dfrac{2\rho v}{r} = 0 \\ \partial_t(\rho v) + \partial_r[\rho v^2 + p(\rho)] + \dfrac{2\rho v^2}{r} = 0. \end{cases}$$

In particular, certain multidimensional phenomena, such as wave focusing, may be investigated in the framework of one space dimension.

7.2 Hyperbolicity and Strict Hyperbolicity

As in earlier chapters, to avoid inessential technical complications, the theory will be developed in the context of homogeneous systems of conservation laws in canonical form:

$$\text{(7.2.1)} \qquad \partial_t U(x,t) + \partial_x F(U(x,t)) = 0.$$

F is a C^3 map from an open convex subset $\mathscr{O}$ of $\mathbb{R}^n$ to $\mathbb{R}^n$.

Often in the applications, systems (7.2.1) govern planar front solutions, namely, $U = U(\nu \cdot x, t)$, in the spatial direction $\nu \in \mathbb{S}^{m-1}$, of multispace-dimensional systems of conservation laws (4.1.1). In that connection,

$$F(U) = \sum_{\alpha=1}^{m} \nu_\alpha G_\alpha(U), \qquad U \in \mathscr{O}. \tag{7.2.2}$$

Referring to the examples introduced in Section 7.1, in order to cast the system (7.1.8) of thermoelasticity in canonical form, we have to switch from (u,v,s) to new state variables (u,v,E), where $E = \varepsilon + \frac{1}{2}v^2$ is the total energy. Similarly, the system (7.1.12) of isentropic gas flow is written in canonical form in terms of the state variables (ρ,m), where $m = \rho v$ is the momentum.

By Definition 3.1.1, the system (7.2.1) is hyperbolic if for every $U \in \mathscr{O}$ the $n \times n$ Jacobian matrix $\mathrm{D}F(U)$ has real eigenvalues $\lambda_1(U) \le \cdots \le \lambda_n(U)$ and n lineary independent eigenvectors $R_1(U), \cdots, R_n(U)$. For future use, we also introduce left (row) eigenvectors $L_1(U), \cdots, L_n(U)$ of $\mathrm{D}F(U)$, normalized by

$$L_i(U)R_j(U) = \begin{cases} 0 & \text{if } \ i \neq j \\ \\ 1 & \text{if } \ i = j. \end{cases} \tag{7.2.3}$$

Henceforth, the symbols λ_i, R_i and L_i will be reserved to denote these objects.

Clearly, the multispace-dimensional system (4.1.1) is hyperbolic if and only if all one-space-dimensional systems (7.2.1) resulting from it through (7.2.2), for arbitrary $\nu \in \mathbb{S}^{m-1}$, are hyperbolic. Thus hyperbolicity is essentially a one-space-dimensional notion.

For the system (7.1.11) of one-dimensional isentropic elasticity, in Lagrangian coordinates, which will serve throughout as a vehicle for illustrating the general concepts, we have

$$\lambda_1 = -\sigma'(u)^{1/2}, \qquad \lambda_2 = \sigma'(u)^{1/2}, \tag{7.2.4}$$

$$R_1 = \tfrac{1}{2}\begin{pmatrix} -\sigma'(u)^{-1/2} \\ \\ -1 \end{pmatrix}, \qquad R_2 = \tfrac{1}{2}\begin{pmatrix} -\sigma'(u)^{-1/2} \\ \\ 1 \end{pmatrix}, \tag{7.2.5}$$

$$L_1 = (-\sigma'(u)^{1/2}, -1), \qquad L_2 = (-\sigma'(u)^{1/2}\ , \ 1). \tag{7.2.6}$$

The eigenvalue λ_i of $\mathrm{D}F$, $i = 1, \cdots, n$, is called the *i-characteristic speed* of the system (7.2.1). The term derives from the following

7.2.1 Definition. An *i-characteristic*, $i = 1, \cdots, n$, of the system (7.2.1), associated with a classical solution U, is a C^1 function $x = x(t)$, with graph contained in the domain of U, which is an integral curve of the ordinary differential equation

$$\frac{dx}{dt} = \lambda_i(U(x,t)). \tag{7.2.7}$$

The standard existence-uniqueness theory for ordinary differential equations (7.2.7) implies that through any point $(\bar{x},\bar{t})$ in the domain of a classical solution of (7.2.1) passes precisely one characteristic of each characteristic family.

Characteristics are carriers of waves of various types. For example, Eq. (1.6.1), for the general system (1.4.3) of balance laws, specialized to (7.2.1), implies that weak fronts propagate along characteristics. As a result, the presence of multiple eigenvalues of DF may induce severe complexity in the behavior of solutions, because of resonance. It is thus natural to single out systems that are free from such complication:

7.2.2 Definition. The system (7.2.1) is *strictly hyperbolic* if for any $U \in \mathscr{O}$ the Jacobian D$F(U)$ has real, distinct eigenvalues

$$\lambda_1(U) < \cdots < \lambda_n(U). \tag{7.2.8}$$

By virtue of (7.2.4), the system (7.1.11) of isentropic elasticity in Lagrangian coordinates is strictly hyperbolic. The same is true for the system (7.1.8) of adiabatic thermoelasticity, for which the characteristic speeds are

$$\lambda_1 = -\sqrt{-p_u(u,s)}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \sqrt{-p_u(u,s)}. \tag{7.2.9}$$

The system (7.1.13) for the ideal gas has characteristic speeds

$$\lambda_1 = v - (\kappa\gamma)^{1/2}\rho^{\frac{\gamma-1}{2}}, \qquad \lambda_2 = v + (\kappa\gamma)^{1/2}\rho^{\frac{\gamma-1}{2}}, \tag{7.2.10}$$

and so it is strictly hyperbolic on the part of the state space with $\rho > 0$.

Furthermore, any one-dimensional system resulting, through (7.2.2), from the Euler equations for two-dimensional isentropic flow is strictly hyperbolic.

In view of the above examples, the reader may form the impression that strict hyperbolicity is the norm in systems arising in continuum physics. However, this is not the case. For example, the system (7.1.15) of planar elastic oscillations fails to be strictly hyperbolic in those directions ν for which the acoustic tensor (3.3.8) has multiple eigenvalues. Indeed, it has been shown that in one-space-dimensional systems (7.2.1), of size $n = \pm 2, \pm 3, \pm 4 \pmod 8$, which result from parent three-space-dimensional systems (4.1.1) through (7.2.2), strict hyperbolicity necessarily fails, at least in some spatial direction $\nu \in \mathbb{S}^2$. In particular, one-dimensional systems resulting from the Euler equations for two-dimensional non-isentropic flow $(n = 4)$, or for three-dimensional isentropic or non-isentropic flow ($n = 4$ or $n = 5$) are not strictly hyperbolic. Actually, failure of strict hyperbolicity is often a byproduct of symmetry. For instance, the systems (7.1.17) and (7.1.18) are not strictly hyperbolic.

In systems of size $n = 2$, strict hyperbolicity typically fails at isolated *umbilic points*, at which DF reduces to a multiple of the identity matrix. Even the presence of a single umbilic point is sufficient to create havoc in the behavior of solutions. This will be demonstrated in following chapters by means of the simple system

$$\begin{cases} \partial_t u + \partial_x[(u^2+v^2)u] = 0 \\ \partial_t v + \partial_x[(u^2+v^2)v] = 0, \end{cases} \tag{7.2.11}$$

which is a caricature of (7.1.17) and (7.1.18). The characteristic speeds of (7.2.11) are

$$\lambda_1 = u^2 + v^2\,, \qquad \lambda_2 = 3(u^2+v^2), \tag{7.2.12}$$

with corresponding eigenvectors

$$R_1 = \begin{pmatrix} v \\ -u \end{pmatrix}, \qquad R_2 = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{7.2.13}$$

so this system is strictly hyperbolic, except at the origin $(0,0)$ which is an umbilic point.

We close this section with the derivation of a useful identity. We apply D to both sides of the equation $\mathrm{D}FR_j = \lambda_j R_j$ and then multiply, from the left, by $R_k^\top$; we also apply D to $\mathrm{D}FR_k = \lambda_k R_k$ and then multiply, from the left, by $R_j^\top$. Upon combining the resulting two equations, we deduce

$$(\mathrm{D}\lambda_j R_k)R_j - (\mathrm{D}\lambda_k R_j)R_k = \mathrm{D}F[R_j,R_k] - \lambda_j \mathrm{D}R_j R_k + \lambda_k \mathrm{D}R_k R_j\,, \quad j,k = 1,\cdots,n, \tag{7.2.14}$$

where $[R_j,R_k]$ denotes the Lie bracket:

$$[R_j,R_k] = \mathrm{D}R_j R_k - \mathrm{D}R_k R_j\,. \tag{7.2.15}$$

In particular, at a point $U \in \mathscr{O}$ where strict hyperbolicity fails, say $\lambda_j(U) = \lambda_k(U)$, (7.2.14) yields

$$(\mathrm{D}\lambda_j R_k)R_j - (\mathrm{D}\lambda_k R_j)R_k = (\mathrm{D}F - \lambda_j I)[R_j,R_k]. \tag{7.2.16}$$

Upon multiplying (7.2.16), from the left, by $L_j(U)$ and by $L_k(U)$, we conclude from (7.2.3):

$$\mathrm{D}\lambda_j(U)R_k(U) = \mathrm{D}\lambda_k(U)R_j(U) = 0. \tag{7.2.17}$$

7.3 Riemann Invariants

Consider a hyperbolic system (7.2.1) of conservation laws on $\mathscr{O} \subset \mathbb{R}^n$. A very important concept is introduced by the following

7.3.1 Definition. An *i-Riemann invariant* of (7.2.1) is a smooth scalar-valued function w on $\mathscr{O}$ such that

$$\mathrm{D}w(U)R_i(U) = 0, \qquad U \in \mathscr{O}. \tag{7.3.1}$$

For example, recalling (7.2.5), one readily verifies that the functions

$$w = -\int^{u} \sqrt{\sigma'(\omega)}\,d\omega + v, \qquad z = -\int^{u} \sqrt{\sigma'(\omega)}\,d\omega - v \tag{7.3.2}$$

are, respectively, 1- and 2-Riemann invariants of the system (7.1.11). Similarly, it can be shown that

$$w = v + \frac{2(\kappa\gamma)^{1/2}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}, \qquad z = v - \frac{2(\kappa\gamma)^{1/2}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \tag{7.3.3}$$

are 1- and 2-Riemann invariants of the system (7.1.13) of isentropic flow of an ideal gas.[1]

By solving the first-order linear differential equation (7.3.1) for w, one may construct in the vicinity of any point $U \in \mathscr{O}$ $n-1$ i-Riemann invariants whose gradients are linearly independent and span the orthogonal complement of R_i. For example, the reader may verify as an exercise that the three pairs of functions

$$\begin{cases} s\,, \quad -\displaystyle\int^{u} \sqrt{-p_{\omega}(\omega,s)}\,d\omega + v \\ v\,, \quad -p(u,s) \\ s\,, \quad -\displaystyle\int^{u} \sqrt{-p_{\omega}(\omega,s)}\,d\omega - v \end{cases} \tag{7.3.4}$$

are, respectively, 1-, 2-, and 3-Riemann invariants of the system (7.1.8) of adiabatic thermoelasticity.

Riemann invariants are particularly useful in systems with the following special structure:

7.3.2 Definition. The system (7.2.1) is endowed with a *coordinate system of Riemann invariants* if there exist n scalar-valued functions $(w_1, \cdots, w_n)$ on $\mathscr{O}$ such that, for any $i, j = 1, \cdots, n$, with $i \neq j$, w_j is an i-Riemann invariant of (7.2.1).

An immediate consequence of Definitions 7.3.1 and 7.3.2 is

7.3.3 Theorem. *The functions* $(w_1, \cdots, w_n)$ *form a coordinate system of Riemann invariants for* (7.2.1) *if and only if*

$$\mathrm{D}w_i(U)R_j(U) \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases} \tag{7.3.5}$$

i.e., if and only if, for $i = 1, \cdots, n$, $\mathrm{D}w_i(U)$ *is a left eigenvector of the matrix* $\mathrm{D}F(U)$, *associated with the characteristic speed* $\lambda_i(U)$. *Equivalently, the tangent hyperplane to the level surface of* w_i *at any point* U, *is spanned by the vectors* $R_1(U), \dots, R_{i-1}(U), R_{i+1}(U), \dots, R_n(U)$.

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[1] In the isothermal case, $\gamma = 1$, $w = v + \kappa^{1/2}\log\rho$, $z = v - \kappa^{1/2}\log\rho$.

Assuming (7.2.1) is endowed with a coordinate system $(w_1,\cdots,w_n)$ of Riemann invariants and multiplying from the left by $\mathrm{D}w_i, i=1,\cdots,n$, we reduce this system to diagonal form:

$$\partial_t w_i + \lambda_i \partial_x w_i = 0, \qquad i=1,\cdots,n, \tag{7.3.6}$$

which is equivalent to the original form (7.2.1), albeit only in the context of classical solutions. The left-hand side of (7.3.6) is just the derivative of w_i in the i-characteristic direction. Therefore,

7.3.4 Theorem. *Assume* $(w_1,\cdots,w_n)$ *form a coordinate system of Riemann invariants for* (7.2.1). *For* $i=1,\cdots,n$, w_i *stays constant along every i-characteristic associated with any classical solution U of* (7.2.1).

Clearly, any hyperbolic system of two conservation laws is endowed with a coordinate system of Riemann invariants. By contrast, in systems of size $n\geq 3$, coordinate systems of Riemann invariants will exist only in the exceptional case where the formally overdetermined system (7.3.5), with $n(n-1)$ equations for the n unknown $(w_1,\cdots,w_n)$, has a solution. By the Frobenius theorem, the hyperplane to the level surface of w_i will be spanned by $R_1,\ldots,R_{i-1}$, $R_{i+1},\ldots,R_n$ if and only if, for $i\neq j\neq k\neq i$, the Lie bracket $[R_j,R_k]$ (cf. (7.2.15)) lies in the span of $\{R_1,\cdots,R_{i-1},R_{i+1},\cdots,R_n\}$. Consequently, the system (7.2.1) is endowed with a coordinate system of Riemann invariants if and only if

$$[R_j,R_k] = \alpha_j^k R_j - \alpha_k^j R_k\,, \qquad j,k=1,\cdots,n, \tag{7.3.7}$$

where the α_i^ℓ are scalar fields.

When a coordinate system $(w_1,\cdots,w_n)$ of Riemann invariants exists for (7.2.1), it is convenient to normalize the eigenvectors $R_1,\cdots,R_n$ so that

$$\mathrm{D}w_i(U)R_j(U) = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j. \end{cases} \tag{7.3.8}$$

In that case we note the identity

$$\mathrm{D}w_i\mathrm{D}R_jR_k = \mathrm{D}(\mathrm{D}w_iR_j)R_k - R_j^\top\mathrm{D}^2w_iR_k = -R_j^\top\mathrm{D}^2w_iR_k\,, \quad i,j,k=1,\cdots,n, \tag{7.3.9}$$

which implies, in particular, $\mathrm{D}w_i[R_j,R_k]=0,\ i=1,\cdots,n$, i.e.,

$$[R_j,R_k]=0, \qquad j,k=1,\cdots,n. \tag{7.3.10}$$

Recalling the identity (7.2.14) and using (7.2.15), (7.3.10), we deduce that whenever $\lambda_j(U)\neq\lambda_k(U)$, $\mathrm{D}R_j(U)R_k(U)$ lies in the span of $\{R_j(U),R_k(U)\}$. This, together with (7.3.8) and (7.3.9), yields

$$R_j^\top\mathrm{D}^2w_iR_k = -\mathrm{D}w_i\mathrm{D}R_jR_k = 0, \qquad i\neq j\neq k\neq i. \tag{7.3.11}$$

When (7.2.1) possesses a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, the map that carries U to $W = (w_1, \cdots, w_n)^\top$ is locally a diffeomorphism. It is often convenient to regard W rather than U as the state vector. To avoid proliferation of symbols, when there is no danger of confusion we shall be using the same symbol to denote fields as functions of either U or W. By virtue of (7.3.8), $\partial U/\partial w_i = R_i$ and so the chain rule yields, for the typical function ϕ,

$$\frac{\partial \phi}{\partial w_i} = \mathrm{D}\phi R_i\,, \qquad i = 1, \cdots, n. \tag{7.3.12}$$

For example, (7.3.10) reduces to $\partial R_j/\partial w_k = \partial R_k/\partial w_j = \partial^2 U/\partial w_j \partial w_k$.

We proceed to derive certain identities that will help us later to establish other remarkable properties of systems endowed with a coordinate system of Riemann invariants. Upon combining (7.2.14), (7.2.15), (7.3.10) and (7.3.12), we deduce

$$-\frac{\partial R_j}{\partial w_k} = g_{jk}R_j + g_{kj}R_k\,, \qquad j,k = 1, \cdots, n;\ \ j \neq k, \tag{7.3.13}$$

where we have set

$$g_{jk} = \frac{1}{\lambda_j - \lambda_k}\,\frac{\partial \lambda_j}{\partial w_k}\,, \qquad j,k = 1, \cdots, n\ ;\ \ j \neq k. \tag{7.3.14}$$

Notice that g_{jk} may be defined even when $\lambda_j = \lambda_k$, because at such points $\partial \lambda_j/\partial w_k$ vanishes by virtue of (7.2.17) and (7.3.12). From (7.3.13),

$$-\frac{\partial^2 R_j}{\partial w_i \partial w_k} = \frac{\partial g_{jk}}{\partial w_i} R_j - g_{jk}(g_{ji}R_j + g_{ij}R_i) + \frac{\partial g_{kj}}{\partial w_i} R_k - g_{kj}(g_{ki}R_k + g_{ik}R_i). \tag{7.3.15}$$

Since R_i, R_j, R_k are linearly independent for $i \neq j \neq k \neq i$, and the right-hand side of (7.3.15) has to be symmetric in (i,k), we deduce

$$\frac{\partial g_{jk}}{\partial w_i} = \frac{\partial g_{ji}}{\partial w_k}\,, \qquad i \neq j \neq k \neq i, \tag{7.3.16}$$

$$\frac{\partial g_{ij}}{\partial w_k} + g_{ij}g_{jk} - g_{ij}g_{ik} + g_{ik}g_{kj} = 0, \qquad i \neq j \neq k \neq i. \tag{7.3.17}$$

Of the hyperbolic systems of conservation laws of size $n \geq 3$ that arise in the applications, few possess coordinate systems of Riemann invariants. A noteworthy example is the system of *electrophoresis*:

$$\partial_t U_i + \partial_x \left[\frac{c_i U_i}{\sum_{j=1}^n U_j} \right] = 0, \qquad i = 1, \cdots, n, \tag{7.3.18}$$

where $c_1 < c_2 < \cdots < c_n$ are positive constants. This system governs the process used to separate n ionized chemical compounds in solution by applying an electric field.

In that context, U_i denotes the concentration and c_i measures the electrophoretic mobility of the i-th species. In particular, $U_i \geq 0$. As an exercise, the reader may verify that the characteristic speeds of (7.3.18) are given by

$$\lambda_i = \mu_i \sum_{j=1}^{n} U_j\,, \qquad i = 1, \cdots, n, \tag{7.3.19}$$

where for $i = 1, \cdots, n-1$ the value of μ_i at U is the solution of the equation

$$\sum_{j=1}^{n} \frac{c_j U_j}{c_j - \mu} = \sum_{j=1}^{n} U_j \tag{7.3.20}$$

lying in the interval (c_i, c_{i+1}); and $\mu_n = 0$. Moreover, (7.3.18) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, where, for $i = 1, \cdots, n-1$, the value of w_i at U is the solution of the equation

$$\sum_{j=1}^{n} \frac{U_j}{c_j - w} = 0 \tag{7.3.21}$$

that lies in the interval (c_i, c_{i+1}); and

$$w_n = \sum_{j=1}^{n} \frac{1}{c_j} U_j\,. \tag{7.3.22}$$

Later we shall see that the system (7.3.18) has very special structure and a host of interesting properties.

Another interesting system endowed with coordinate systems of Riemann invariants is (7.1.23), which, as we recall, governs the propagation of planar electromagnetic waves through special isotropic dielectrics. This is seen by passing from (B_1, B_2, D_1, D_2) to the new state vector (p, q, a, b) defined through

$$\begin{cases} \sqrt{2} p \exp(ia) = B_2 + D_1 - i(B_1 - D_2) \\ \\ \sqrt{2} q \exp(ib) = -B_2 + D_1 + i(B_1 + D_2). \end{cases} \tag{7.3.23}$$

In particular, $p^2 + q^2 = r^2$. A simple calculation shows that, at least in the context of classical solutions, (7.1.23) reduces to

$$\begin{cases} \partial_t p + \partial_x \left[\dfrac{\psi'(r)}{r} p \right] = 0 \\ \\ \partial_t q - \partial_x \left[\dfrac{\psi'(r)}{r} q \right] = 0, \end{cases} \tag{7.3.24}$$

$$\begin{cases} \partial_t a + \dfrac{\psi'(r)}{r} \partial_x a = 0 \\ \\ \partial_t b - \dfrac{\psi'(r)}{r} \partial_x b = 0. \end{cases} \tag{7.3.25}$$

Notice that (7.3.24) constitutes a closed system of two conservation laws, from which p,q, and thereby r, may be determined. Subsequently (7.3.25) may be solved, as two independent nonhomogeneous scalar conservation laws, to determine a and b. In particular, a and b together with any pair of Riemann invariants of (7.3.24) will constitute a coordinate system of Riemann invariants for (7.1.23).

7.4 Entropy-Entropy Flux Pairs

Entropies play a central role in the theory of hyperbolic systems of conservation laws in one space dimension. Adapting the discussion of Section 3.2 to the present setting, we infer that functions η and q on $\mathscr{O}$ constitute an entropy-entropy flux pair for the system (7.2.1) if

$$\mathrm{D}q(U) = \mathrm{D}\eta(U)\mathrm{D}F(U), \qquad U \in \mathscr{O}. \tag{7.4.1}$$

Furthermore, the integrability condition (3.2.4) here reduces to

$$\mathrm{D}^2\eta(U)\mathrm{D}F(U) = \mathrm{D}F(U)^\top \mathrm{D}^2\eta(U), \qquad U \in \mathscr{O}. \tag{7.4.2}$$

Upon multiplying (7.4.2) from the left by $R_j(U)^\top$ and from the right by $R_k(U)$, $j \neq k$, we deduce that (7.4.2) is equivalent to

$$R_j(U)^\top \mathrm{D}^2\eta(U)R_k(U) = 0, \qquad j,k = 1,\cdots,n;\ \ j \neq k, \tag{7.4.3}$$

with the understanding that (7.4.3) holds automatically when $\lambda_j(U) \neq \lambda_k(U)$ but may require renormalization of eigenvectors R_i associated with multiple characteristic speeds. (Compare with (3.2.5).) Note that the requirement that some entropy η is convex may now be conveniently expressed as

$$R_j(U)^\top \mathrm{D}^2\eta(U)R_j(U) > 0, \qquad j = 1,\cdots,n. \tag{7.4.4}$$

When the system (7.2.1) is symmetric,

$$\mathrm{D}F(U)^\top = \mathrm{D}F(U), \qquad U \in \mathscr{O}, \tag{7.4.5}$$

it admits two interesting entropy-entropy flux pairs:

$$\eta = \tfrac{1}{2}|U|^2, \qquad q = U \cdot F(U) - h(U), \tag{7.4.6}$$

$$\eta = h(U), \qquad q = \tfrac{1}{2}|F(U)|^2, \tag{7.4.7}$$

where h is defined by the condition

$$\mathrm{D}h(U) = F(U)^\top. \tag{7.4.8}$$

As explained in Chapter III, the systems (7.1.8), (7.1.11), (7.1.13) are endowed with entropy-entropy flux pairs, respectively,

$$\eta = -s, \qquad q = 0, \tag{7.4.9}$$

$$\eta = \tfrac{1}{2}v^2 + e(u), \qquad q = -v\sigma(u), \qquad e(u) = \int^u \sigma(\omega)d\omega, \tag{7.4.10}$$

$$\eta = \frac{1}{2}\rho v^2 + \frac{\kappa}{\gamma-1}\rho^\gamma, \qquad q = \frac{1}{2}\rho v^3 + \frac{\kappa\gamma}{\gamma-1}\rho^\gamma v, \tag{7.4.11}$$

induced by the Second Law of thermodynamics.[2] In fact, (7.4.10), with $v\sigma$ and $\sigma d\omega$ interpreted as $v\cdot\sigma$ and $\sigma\cdot d\omega$, constitutes an entropy-entropy flux pair even for the system (7.1.15). When expressed as functions of the canonical state variables, that is (u,v,E) for (7.4.9), (u,v) for (7.4.10), and (ρ,m) for (7.4.11), the above entropies are convex.

In developing the theory of systems (7.2.1), it will be useful to construct entropies with given specifications. These must be solutions of (7.4.2), which is a linear, second-order system of $\frac{1}{2}n(n-1)$ partial differential equations in a single unknown η. Thus, when $n=2$, (7.4.2) reduces to a single linear hyperbolic equation which may be solved to produce an abundance of entropies. By contrast, for $n \geq 3$, (7.4.2) is formally overdetermined. Notwithstanding the presence of special solutions such as (7.4.6) and (7.4.7), one should not expect an abundance of entropies, unless (7.2.1) is special. It is remarkable that the overdeterminacy of (7.4.2) vanishes when (7.2.1) is endowed with a coordinate system $(w_1,\cdots,w_n)$ of Riemann invariants. In that case it is convenient to seek η and q as functions of the state vector $W=(w_1,\cdots,w_n)^\top$. Upon multiplying (7.4.1), from the right, by $R_j(U)$ and by using (7.3.12), we deduce that (7.4.1) is now equivalent to

$$\frac{\partial q}{\partial w_j} = \lambda_j \frac{\partial \eta}{\partial w_j}, \qquad j=1,\cdots,n. \tag{7.4.12}$$

The integrability condition associated with (7.4.12) takes the form

$$\frac{\partial^2 \eta}{\partial w_j \partial w_k} + g_{jk}\frac{\partial \eta}{\partial w_j} + g_{kj}\frac{\partial \eta}{\partial w_k} = 0, \qquad j,k=1,\cdots,n;\ j\neq k, \tag{7.4.13}$$

where g_{jk}, g_{kj} are the functions defined through (7.3.14). An alternative, useful expression for g_{jk} arises if one derives (7.4.13) directly from (7.4.3). Indeed, for $j,k=1,\cdots,n$,

(7.4.14)
$$R_j^\top \mathrm{D}^2\eta R_k = \mathrm{D}(\mathrm{D}\eta R_j)R_k - \mathrm{D}\eta \mathrm{D}R_j R_k = \mathrm{D}(\mathrm{D}\eta R_j)R_k - \sum_{i=1}^n \frac{\partial \eta}{\partial w_i}\mathrm{D}w_i \mathrm{D}R_j R_k\,.$$

Combining (7.4.3), (7.3.12), (7.3.10) and (7.3.9), we arrive at an equation of the form (7.4.13) with

[2] In the isothermal case, $\gamma=1$, the entropy-entropy flux pair of (7.1.13) takes the following form: $\eta = \frac{1}{2}\rho v^2 + \kappa\rho\log\rho$, $q = \frac{1}{2}\rho v^3 + \kappa\rho v\log\rho + \kappa\rho v$.

(7.4.15) $$g_{jk} = R_j^\top \mathrm{D}^2 w_j R_k\,, \qquad j,k = 1,\cdots,n;\ j \neq k.$$

The reader may verify directly, as an exercise, with the help of (7.2.14), (7.3.8), (7.3.11), (7.3.10), (7.3.9) and (7.3.12), that (7.3.14) and (7.4.15) are equivalent.

Applying (7.4.14) with $k = j$, using (7.3.12), (7.3.9) and recalling (7.4.4), we deduce that, in terms of Riemann invariants, the convexity condition on η is expressed by the set of inequalities

(7.4.16) $$\frac{\partial^2\eta}{\partial w_j^2} + \sum_{i=1}^n a_{ij}\frac{\partial\eta}{\partial w_i} \geq 0, \qquad j = 1,\cdots,n,$$

where

(7.4.17) $$a_{ij} = R_j^\top \mathrm{D}^2 w_i R_j\,, \qquad i,j = 1,\cdots,n.$$

The system (7.4.13) contains $\frac{1}{2}n(n-1)$ equations in the single unknown η and thus seems overdetermined when $n \geq 3$. It turns out, however, that this set of equations is internally consistent. To see this, differentiate (7.4.13) with respect to w_i to get

(7.4.18) $$\begin{aligned}\frac{\partial^3\eta}{\partial w_i \partial w_j \partial w_k} = &-\frac{\partial g_{jk}}{\partial w_i}\,\frac{\partial\eta}{\partial w_j} + g_{jk}\left(g_{ji}\frac{\partial\eta}{\partial w_j} + g_{ij}\frac{\partial\eta}{\partial w_i}\right)\\ &-\frac{\partial g_{kj}}{\partial w_i}\,\frac{\partial\eta}{\partial w_k} + g_{kj}\left(g_{ki}\frac{\partial\eta}{\partial w_k} + g_{ik}\frac{\partial\eta}{\partial w_i}\right).\end{aligned}$$

The system (7.4.13) will be integrable if and only if, for $i \neq j \neq k \neq i$, the right-hand side of (7.4.18) is symmetric in (i,j,k). But this is always the case, on account of the identities (7.3.16) and (7.3.17). Consequently, it is possible to construct, in a neighborhood of any given state $\bar{W} = (\bar{w}_1,\cdots,\bar{w}_n)^\top$, entropies η with prescribed values $\{\eta(w_1,\bar{w}_2,\cdots,\bar{w}_n), \eta(\bar{w}_1,w_2,\cdots,\bar{w}_n),\cdots,\eta(\bar{w}_1,\cdots,\bar{w}_{n-1},w_n)\}$ along straight lines parallel to the coordinate axes. When $n = 2$, this amounts to solving a classical Goursat problem.

We have thus shown that systems endowed with coordinate systems of Riemann invariants are also endowed with an abundance of entropies. For this reason, such systems are called *rich*. In particular, the system (7.3.18) of electrophoresis and the system (7.1.23) of electromagnetic waves are rich. The reader will find how to construct the family of entropies of these systems in the references cited in Section 7.10.

7.5 Genuine Nonlinearity and Linear Degeneracy

The feature distinguishing the behavior of linear and nonlinear hyperbolic systems of conservation laws is that in the former, characteristic speeds being constant, all waves of the same family propagate with fixed speed; while in the latter, wave speeds vary with wave-amplitude. As we proceed with our study, we will encounter various manifestations of nonlinearity, and in every case we shall notice that its effects will be particularly pronounced when the characteristic speeds λ_i vary in the direction of the corresponding eigenvectors R_i. This motivates the following

7.5.1 Definition. For the hyperbolic system (7.2.1) of conservation laws on $\mathscr{O}$, U in $\mathscr{O}$ is called a *state of genuine nonlinearity of the i-characteristic family* if

$$\mathrm{D}\lambda_i(U)R_i(U) \neq 0, \tag{7.5.1}$$

or a *state of linear degeneracy of the i-characteristic family* if

$$\mathrm{D}\lambda_i(U)R_i(U) = 0. \tag{7.5.2}$$

When (7.5.1) holds for all $U \in \mathscr{O}$, i is a *genuinely nonlinear characteristic family*, while if (7.5.2) is satisfied for all $U \in \mathscr{O}$, then i is a *linearly degenerate characteristic family*. When every characteristic family is genuinely nonlinear, (7.2.1) is a *genuinely nonlinear system.*

It is clear that the i-characteristic family is linearly degenerate if and only if the i-characteristic speed λ_i is constant along the integral curves of the vector field R_i.

The scalar conservation law (7.1.2), with characteristic speed $\lambda = f'(u)$, is genuinely nonlinear when f has no inflection points: $f''(u) \neq 0$. In particular, the Burgers equation (4.2.1) is genuinely nonlinear.

Using (7.2.4) and (7.2.5), one readily checks that the system (7.1.11) is genuinely nonlinear when $\sigma''(u) \neq 0$. As an exercise, the reader may verify that the system (7.1.12) is genuinely nonlinear if $2p'(\rho) + \rho p''(\rho) > 0$ so, in particular, the system (7.1.13) for the ideal gas is genuinely nonlinear. The system (7.1.19) of waves in shallow water is likewise genuinely nonlinear.

On account of (7.2.9), the 2-characteristic family of the system (7.1.8) of thermoelasticity is linearly degenerate. It turns out that the other two characteristic families are genuinely nonlinear, provided $\sigma_{uu}(u,s) \neq 0$.

Consider next the system (7.1.15) of planar elastic oscillations in the direction ν, recalling that $\sigma(u) = \partial e(u)/\partial u$, with $e(u)$ convex. The six characteristic speeds are the square roots $\pm\sqrt{\mu_1}$, $\pm\sqrt{\mu_2}$, $\pm\sqrt{\mu_3}$ of the eigenvalues $\mu_1(u)$, $\mu_2(u)$, $\mu_3(u)$ of the Hessian matrix of $e(u)$, namely the eigenvalues of the acoustic tensor (3.3.8) evaluated at $F = I + u\nu^\top$. A simple calculation shows that the characteristic families associated with the characteristic speeds $\pm\sqrt{\mu_\ell}$ are genuinely nonlinear at $u = F\nu$ if

$$\sum_{i,j,k=1}^{3} \frac{\partial^3 e(u)}{\partial u_i \partial u_j \partial u_k} \xi_i \xi_j \xi_k = \sum_{i,j,k=1}^{3} \sum_{\alpha,\beta,\gamma=1}^{3} \frac{\partial^3 \varepsilon(F)}{\partial F_{i\alpha} \partial F_{j\beta} \partial F_{k\gamma}} \xi_i \xi_j \xi_k \nu_\alpha \nu_\beta \nu_\gamma \neq 0, \tag{7.5.3}$$

where ξ is the eigenvector of the acoustic tensor associated with the eigenvalue μ_ℓ.

Applying the above to the special system (7.1.17), one finds that $\mu_1 = h''(|u|)$ is a simple eigenvalue, with eigenvector u, and $\mu_2 = \mu_3 = h'(|u|)/|u|$ is a double eigenvalue, with eigenspace the orthogonal complement of u. Thus, the characteristic speeds $\pm[h''(|u|)]^{1/2}$ are associated with longitudinal oscillations, while

$\pm[h'(|u|)/|u|]^{1/2}$ are associated with transverse oscillations. However, only transverse oscillations that are also orthogonal to ν are compatible with incompressibility. The characteristic families associated with $\pm[h''(|u|)]^{1/2}$ are genuinely nonlinear at u if $h'''(|u|) \neq 0$, while the characteristic families associated with $\pm[h'(|u|)/|u|]^{1/2}$ are linearly degenerate. Clearly, the same conclusions apply to the system of elastic string oscillations (7.1.18), with $\tau(|u|)$ replacing $h'(|u|)$. For this system, all transverse oscillations are physically meaningful, as the incompressibility constraint is no longer relevant. The model system (7.2.11) exhibits similar behavior, as its 1-characteristic family is linearly degenerate, while its 2-characteristic family is genuinely nonlinear, except at the origin.

In the system (7.1.22) of three-phase flow through a porous medium, genuine nonlinearity breaks down along the three lines of symmetry $u = v$, $u = 1 - u - v$ and $v = 1 - u - v$, as well as along a closed curve surrounding the point $u = v = 1/3$.

Finally, in the system (7.3.18) of electrophoresis the n-characteristic family is linearly degenerate while the rest are genuinely nonlinear.

The system of Maxwell's equations (3.3.66) for the Born-Infeld medium (3.3.73) has the remarkable property that planar oscillations in any spatial direction $\nu \in S^2$ are governed by a system whose characteristic families are all linearly degenerate.

Quite often, linear degeneracy results from the loss of strict hyperbolicity. Indeed, an immediate consequence of (7.2.17) is

7.5.2 Theorem. *In the hyperbolic system* (7.2.1) *of conservation laws, assume that the j- and k-characteristic speeds coincide*: $\lambda_j(U) = \lambda_k(U)$, $U \in \mathscr{O}$. *Then both the j- and the k-characteristic families are linearly degenerate.*

When the system (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants and one uses $W = (w_1, \cdots, w_n)^\top$ as state vector, the conditions of genuine nonlinearity and linear degeneracy assume an elegant and suggestive form. Indeed, upon using (7.3.12), we deduce that (7.5.1) and (7.5.2) are respectively equivalent to

$$\frac{\partial \lambda_i}{\partial w_i} \neq 0 \tag{7.5.4}$$

and

$$\frac{\partial \lambda_i}{\partial w_i} = 0. \tag{7.5.5}$$

7.6 Simple Waves

In the context of classical solutions, the scalar conservation law (7.1.2), with characteristic speed $\lambda = f'(u)$, takes the form

$$\partial_t u(x,t) + \lambda(u(x,t))\partial_x u(x,t) = 0. \tag{7.6.1}$$

As noted already in Section 6.1, by virtue of (7.6.1) u stays constant along characteristics and this, in turn, implies that each characteristic propagates with constant speed, i.e., it is a straight line. It turns out that general hyperbolic systems (7.2.1) of conservation laws admit special solutions with the same features:

7.6.1 Definition. A classical, C^1 solution U of the hyperbolic system (7.2.1) of conservation laws is called an *i-simple wave* if U stays constant along any i-characteristic associated with it.

Thus a C^1 function U, defined on an open subset of $\mathbb{R}^2$ and taking values in $\mathscr{O}$, is an i-simple wave if it satisfies (7.2.1) together with

$$\partial_t U(x,t) + \lambda_i(U(x,t))\partial_x U(x,t) = 0. \tag{7.6.2}$$

In particular, in an i-simple wave each i-characteristic propagates with constant speed and so it is a straight line.

If U is an i-simple wave, combining (7.2.1) with (7.6.2) we deduce

$$\begin{cases} \partial_x U(x,t) = a(x,t)R_i(U(x,t)) \\ \\ \partial_t U(x,t) = -a(x,t)\lambda_i(U(x,t))R_i(U(x,t)), \end{cases} \tag{7.6.3}$$

where a is a scalar field. Conversely, any C^1 function U that satisfies (7.6.3) is necessarily an i-simple wave.

It is possible to give still another characterization of simple waves, in terms of Riemann invariants:

7.6.2 Theorem. *A classical, C^1 solution U of* (7.2.1) *is an i-simple wave if and only if every i-Riemann invariant is constant on each connected component of the domain of U.*

Proof. For any i-Riemann invariant w, $\partial_x w = \mathrm{D}w\,\partial_x U$ and $\partial_t w = \mathrm{D}w\,\partial_t U$. If U is an i-simple wave, $\partial_x w$ and $\partial_t w$ vanish identically, by virtue of (7.6.3) and (7.3.1), so that w is constant on any connected component of the domain of U.

Conversely, recalling that the gradients of i-Riemann invariants span the orthogonal complement of R_i, we infer that when $\partial_x w = \mathrm{D}w\,\partial_x U$ vanishes identically for all i-Riemann invariants w, $\partial_x U$ must satisfy $(7.6.3)_1$. Substituting $(7.6.3)_1$ into (7.2.1) we conclude that $(7.6.3)_2$ holds as well, i.e., U is an i-simple wave. This completes the proof.

Any constant function $U = \bar{U}$ qualifies, according to Definition 7.6.1, to be viewed as an i-simple wave, for every $i = 1, \cdots, n$. It is expedient, however, to refer to such trivial solutions as *constant states* and reserve the term simple wave for solutions that are not constant on any open subset of their domain. The following proposition, which demonstrates that simple waves are the natural neighbors of constant states, is stated informally, in physical rather than mathematical terminology.

The precise meaning of assumptions and conclusions may be extracted from the proof.

7.6.3 Theorem. *Any weak front moving into a constant state propagates with constant characteristic speed of some family i. Furthermore, the wake of this front is necessarily an i-simple wave.*

Proof. The setting is as follows: The system (7.2.1) is assumed strictly hyperbolic. U is a classical, Lipschitz solution which is C^1 on its domain, except along the graph of a C^1 curve $x = \chi(t)$. U is constant, $\bar{U}$, at any point of its domain lying on one side, say to the right, of the graph of χ. By contrast, $\partial_x U$ and $\partial_t U$ attain nonzero limits from the left along the graph of χ. Thus, according to the terminology of Section 1.6, χ is a weak front propagating with speed $\dot{\chi} = d\chi/dt$. In particular, (1.6.1) here reduces to

$$[\mathrm{D}F(\bar{U}) - \dot{\chi}I][\![\partial U/\partial N]\!] = 0, \tag{7.6.4}$$

which shows that $\dot{\chi}$ is constant and equal to $\lambda_i(\bar{U})$ for some i.

Next we show that to the left of, and sufficiently close to, the graph of χ the solution U is an i-simple wave. By virtue of Theorem 7.6.2, it suffices to prove that $n-1$ independent i-Riemann invariants, which will be denoted by $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$, are constant.

For U near $\bar{U}$, the vectors $\{\mathrm{D}w_1(U), \cdots, \mathrm{D}w_{i-1}(U), \mathrm{D}w_{i+1}(U), \cdots, \mathrm{D}w_n(U)\}$ span the orthogonal complement of $R_i(U)$ and this is also the case for the vectors $\{L_1(U), \cdots, L_{i-1}(U), L_{i+1}(U), \cdots, L_n(U)\}$. Consequently, there is a nonsingular $(n-1)\times(n-1)$ matrix $B(U)$ such that

$$L_j(U) = \sum_{k\neq i} B_{jk}(U)\mathrm{D}w_k(U), \qquad j = 1, \cdots, i-1, i+1, \cdots, n. \tag{7.6.5}$$

Multiplying (7.2.1), from the left, by $L_j(U)$ yields

$$L_j(U)\partial_t U + \lambda_j(U)L_j(U)\partial_x U = 0, \qquad j = 1, \cdots, n. \tag{7.6.6}$$

Combining (7.6.5) with (7.6.6), we conclude

$$\sum_{k\neq i} B_{jk}\partial_t w_k + \sum_{k\neq i} \lambda_j B_{jk}\partial_x w_k = 0, \qquad j = 1, \cdots, i-1, i+1, \cdots, n. \tag{7.6.7}$$

We regard (7.6.7) as a first-order linear inhomogeneous system of $n-1$ equations in the $n-1$ unknowns $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$. In that sense, (7.6.7) is strictly hyperbolic, with characteristic speeds $\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n$. Along the graph of χ, the $n-1$ Riemann invariants are constant, namely, equal to their values at $\bar{U}$: $w_1(\bar{U}), \cdots, w_{i-1}(\bar{U}), w_{i+1}(\bar{U}), \cdots, w_n(\bar{U})$. Also, the graph of χ is non-characteristic for the system (7.6.7). Consequently, the standard uniqueness theorem for the Cauchy problem for linear hyperbolic systems implies that (7.6.7) may admit only one solution compatible with the Cauchy data, namely the trivial one:

$w_1 = w_1(\bar{U}), \cdots, w_{i-1} = w_{i-1}(\bar{U}), w_{i+1} = w_{i+1}(\bar{U}), \cdots, w_n = w_n(\bar{U})$. This completes the proof.

At any point (x,t) in the domain of an i-simple wave U of (7.2.1), we let $\xi(x,t)$ denote the slope at (x,t) of the i-characteristic associated with U, i.e.,

$$\xi(x,t) = \lambda_i(U(x,t)). \tag{7.6.8}$$

The derivative of ξ in the direction of the line with slope ξ is zero, that is

$$\partial_t \xi + \xi \partial_x \xi = 0. \tag{7.6.9}$$

Thus ξ satisfies the Burgers equation (4.2.1).

In the vicinity of any point $(\bar{x},\bar{t})$ in the domain of U, we shall say that the i-simple wave is an *i-rarefaction wave* if $\partial_x \xi(\bar{x},\bar{t}) > 0$, i.e., if the i-characteristics diverge, or an *i-compression wave* if $\partial_x \xi(\bar{x},\bar{t}) < 0$, i.e., if the i-characteristics converge. This terminology originated in the context of gas dynamics.

Since in an i-simple wave U stays constant along i-characteristics, on a small neighborhood $\mathscr{X}$ of any point $(\bar{x},\bar{t})$ where $\partial_x \xi(\bar{x},\bar{t}) \neq 0$ we may use the single variable ξ to label U, i.e., there is a function V_i, defined on an interval $(\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon)$, with $\bar{\xi} = \lambda_i(U(\bar{x},\bar{t}))$, taking values in $\mathscr{O}$ and such that

$$U(x,t) = V_i(\xi(x,t)), \quad (x,t) \in \mathscr{X}. \tag{7.6.10}$$

Furthermore, by virtue of (7.6.3) and (7.6.8), V_i satisfies

$$\dot{V}_i(\xi) = b(\xi) R_i(V_i(\xi)), \qquad \xi \in (\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon), \tag{7.6.11}$$

$$\lambda_i(V_i(\xi)) = \xi, \qquad \xi \in (\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon), \tag{7.6.12}$$

where b is a scalar function and an overdot denotes derivative with respect to ξ.

Conversely, if V_i satisfies (7.6.11), (7.6.12) and ξ is any C^1 solution of (7.6.9) taking values in the interval $(\bar{\xi} - \varepsilon, \bar{\xi} + \varepsilon)$, then $U = V_i(\xi(x,t))$ is an i-simple wave. The above considerations motivate the following

7.6.4 Definition. An *i-rarefaction wave curve* in the state space $\mathbb{R}^n$, for the hyperbolic system (7.2.1), is a curve $U = V_i(\cdot)$, where the function V_i satisfies (7.6.11) and (7.6.12).

Rarefaction wave curves will provide one of the principal tools for solving the Riemann problem in Chapter IX. The construction of these curves is particularly simple in the neighborhood of states of genuine nonlinearity:

7.6.5 Theorem. *Assume $\bar{U} \in \mathscr{O}$ is a state of genuine nonlinearity of the i-characteristic family of the hyperbolic system* (7.2.1) *of conservation laws. Then there exists a unique i-rarefaction wave curve V_i through $\bar{U}$. If R_i is normalized on a neighborhood of $\bar{U}$ through*

(7.6.13) $$\mathrm{D}\lambda_i(U)R_i(U) = 1,$$

and V_i is reparametrized by $\tau = \xi - \bar{\xi}$, where $\bar{\xi} = \lambda_i(\bar{U})$, then V_i is the solution of the ordinary differential equation

(7.6.14) $$\dot{V}_i = R_i(V_i)$$

with initial condition $V_i(0) = \bar{U}$. In particular, V_i is C^3. The more explicit notation $V_i(\tau;\bar{U})$ shall be employed when it becomes necessary to display the point of origin of this rarefaction wave curve.

Proof. Any solution V_i of (7.6.14) clearly satisfies (7.6.11) with $b = 1$. At $\xi = \bar{\xi}$, i.e., $\tau = 0$, $\lambda_i(V_i) = \lambda_i(\bar{U}) = \bar{\xi}$. Furthermore, $\dot{\lambda}_i(V_i) = \mathrm{D}\lambda_i(V_i)\dot{V}_i = 1$, by virtue of (7.6.14) and (7.6.13). This establishes (7.6.12) and completes the proof.

By contrast, when the i-characteristic family is linearly degenerate, differentiating (7.6.12) with respect to ξ and combining the resulting equation with (7.6.11) yields a contradiction: $0 = 1$. In that case, i-characteristics in any i-simple wave are necessarily parallel straight lines. It is still true, however, that any i-simple wave takes values along some integral curve of the differential equation (7.6.14).

Motivated by Theorem 7.6.2, we may characterize rarefaction wave curves in terms of Riemann invariants:

7.6.6 Theorem. *Every i-Riemann invariant is constant along any i-rarefaction wave curve of the system* (7.2.1). *Conversely, if $\bar{U}$ is any state of genuine nonlinearity of the i-characteristic family of* (7.2.1) *and $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$ are independent i-Riemann invariants on some neighborhood of $\bar{U}$, then the i-rarefaction curve through $\bar{U}$ is determined implicitly by the system of equations $w_j(U) = w_j(\bar{U})$, for $j = 1, \cdots, i-1, i+1, \cdots, n$.*

Proof. Any i-rarefaction curve V_i satisfies (7.6.11). If w is an i-Riemann invariant of (7.2.1), multiplying (7.6.11), from the left, by $\mathrm{D}w(V_i(\xi))$ and using (7.3.1) yields $\dot{w}(V_i(\xi)) = 0$, i.e., w stays constant along V_i.

Assume now $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$ are i-Riemann invariants such that $\mathrm{D}w_1, \cdots, \mathrm{D}w_{i-1}, \mathrm{D}w_{i+1}, \cdots, \mathrm{D}w_n$ are linearly independent. Then the $n-1$ surfaces $w_j(U) = w_j(\bar{U})$, $j = 1, \cdots, i-1, i+1, \cdots, n$, intersect transversely to form a C^1 curve V_i through $\bar{U}$, parametrized by arclength s, whose tangent $V_i^{'}$ must satisfy, on account of Definition 7.3.1, $V_i^{'}(s) = c(s)R_i(V(s))$, for some nonzero scalar function c. For as long as V_i is a state of genuine nonlinearity of the i-characteristic field, $\lambda_i^{'}(V_i) = \mathrm{D}\lambda_i V_i^{'} = c\mathrm{D}\lambda_i R_i \neq 0$. We may thus find the proper parametrization $s = s(\xi)$ so that V_i satisfies both (7.6.11) and (7.6.12). This completes the proof.

As an application of Theorem 7.6.6, we infer that the 1- and 2-rarefaction wave curves of the system (7.1.11) through a point $(\bar{u}, \bar{v})$, with $\sigma''(\bar{u}) \neq 0$, are determined, in terms of the Riemann invariants (7.3.2), by the equations

(7.6.15) $$v = \bar{v} \pm \int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}\, d\omega.$$

Similarly, the 1- and 3-rarefaction wave curves of the system (7.1.8) through a point $(\bar{u}, \bar{v}, \bar{s})$, with $p_{uu}(\bar{u}, \bar{s}) \neq 0$, are described in terms of the Riemann invariants (7.3.4), by the equations

(7.6.16) $$v = \bar{v} \pm \int_{\bar{u}}^{u} \sqrt{-p_\omega(\omega, \bar{s})}\, d\omega\,, \qquad s = \bar{s}.$$

When the system (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants and we use $W = (w_1, \cdots, w_n)^\top$, instead of U, as our state variable, the rarefaction wave curves assume a very simple form. Indeed, by virtue of Theorem 7.6.4, the i-rarefaction wave curve through the point $\bar{W} = (\bar{w}_1, \cdots, \bar{w}_n)^\top$ is the straight line $w_j = \bar{w}_j\ ,\ j \neq i$, parallel to the i-axis.

7.7 Explosion of Weak Fronts

The aim here is to expose the decisive role played by genuine nonlinearity in the amplification and eventual explosion of weak fronts.

We consider a Lipschitz continuous solution U of the strictly hyperbolic system (7.2.1), defined on a strip $(-\infty, \infty) \times [0, T)$ and having the following structure: A C^1 curve $x = \chi(t)$ issues from the origin, and $U(x,t) = \bar{U} =$ constant on the set $\{(x,t) : 0 \leq t < T,\ x > \chi(t)\}$, while on the set $\{(x,t) : 0 \leq t < T,\ x < \chi(t)\}$ U is C^2 and its first and second partial derivatives attain non-zero limits, as $x \uparrow \chi(t)$. Thus, $\chi(\cdot)$ is a weak front moving into a constant state.

On the set $\{(x,t) : 0 \leq t < T,\ x < \chi(t)\}$,

(7.7.1) $$\partial_t U(x,t) + \mathrm{D}F(U(x,t))\partial_x U(x,t) = 0.$$

Since $U(\chi(t)-, t) = \bar{U}$,

(7.7.2) $$\partial_t U(\chi(t)-, t) + \dot{\chi}(t)\partial_x U(\chi(t)-, t) = 0.$$

By combining (7.7.1) with (7.7.2),

(7.7.3) $$\left[\mathrm{D}F(\bar{U}) - \dot{\chi}(t)I\right]\partial_x U(\chi(t)-, t) = 0.$$

Therefore, $\dot{\chi}(t)$ is constant, equal to $\lambda_i(\bar{U})$, for some characteristic family i, and

(7.7.4) $$\partial_x U(\chi(t)-, t) = a(t)R_i(\bar{U}).$$

The function $a(t)$ measures the strength of the weak front.

We multiply (7.7.4), from the left, by $L_i(\bar{U})$, use (7.2.3) and differentiate with respect to t to get

(7.7.5) $$\frac{da(t)}{dt} = L_i(\bar{U})\left[\partial_x\partial_t U(\chi(t))-, t) + \lambda_i(\bar{U})\partial_x\partial_x U(\chi(t)-, t)\right].$$

Next, we multiply (7.7.1), from the left, by $L_i(U(x,t))$,

$$L_i(U(x,t))\,[\partial_t U(x,t) + \lambda_i(U(x,t))\partial_x U(x,t)] = 0, \tag{7.7.6}$$

then differentiate with respect to x and let $x \uparrow \chi(t)$. Upon combining $\dot{\chi}(t) = \lambda_i(\bar{U})$, (7.7.2), (7.7.5), (7.7.4) and (7.2.3), we conclude that $a(t)$ satisfies an ordinary differential equation of Bernoulli type:

$$\frac{da}{dt} + \mathrm{D}\lambda_i(\bar{U})R_i(\bar{U})a^2 = 0. \tag{7.7.7}$$

Thus, if $\bar{U}$ is a state of genuine nonlinearity for the i-characteristic family and $\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U})a(0) < 0$, then the strength of the weak wave increases with time and eventually explodes as $t \uparrow \left[-\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U})a(0)\right]^{-1}$. The issue of breakdown of classical solutions will be discussed from a broader perspective in the following section.

7.8 Existence and Breakdown of Classical Solutions

When the system (7.2.1) is equipped with a convex entropy, Theorem 5.1.1 guarantees the existence of a unique, locally defined, classical solution, with initial data U_0 in the Sobolev space H_2. In one space dimension, however, there is a sharper existence theory which applies to quasilinear hyperbolic systems in general, that may or may not be conservation laws, and does not rely on the existence of entropies:

7.8.1 Theorem. *Assume* (7.2.1) *is strictly hyperbolic on $\mathscr{O}$. For any initial data U_0 in $C^1(-\infty,\infty)$, with values in a compact subset of $\mathscr{O}$ and bounded derivative, there exists a unique C^1 solution of the Cauchy problem on a strip $(-\infty,\infty)\times[0,T_\infty)$, for some $0<T_\infty\leq\infty$, and values in $\mathscr{O}$. Moreover, if $T_\infty<\infty$, then, as $t\uparrow T_\infty$, $\|\partial_x U(\cdot,t)\|_{L^\infty}\to\infty$ and/or the range of $U(\cdot,t)$ escapes from every compact subset of $\mathscr{O}$.*

The proof of the above theorem, which may be found in the references cited in Section 7.10, relies on pointwise bounds for U and $\partial_x U$ obtained by monitoring the evolution of U and its derivatives along characteristics. Estimates of this nature will be established below but they will be employed for establishing the breakdown of classical solutions in finite time.

We have already encountered a number of examples of breakdown of classical solutions, notably for scalar conservation laws, in Section 6.1, and for weak fronts, in Section 7.7. Breakdown also occurs in the presence of compressive simple waves. Indeed, as shown in Section 7.6, an i-simple wave solution U is obtained by taking the composition (7.6.10) of a (smooth) solution V_i to the ordinary differential equation (7.6.11) with a classical solution ξ to the Burgers equation (7.6.9). When that solution of (7.6.9) breaks down, so does the i-simple wave. The above examples involve a single characteristic family. The aim here is to demonstrate that, in the presence of genuine nonlinearity, the interaction of waves from different characteristic families cannot prevent the breakdown of smooth solutions.

Any classical, C^2 solution U of (7.2.1) on $(-\infty,\infty)\times[0,T)$ may be written as

$$\begin{cases} \partial_x U = \sum_{j=1}^{n} a_j R_j(U) \\ \\ \partial_t U = -\sum_{j=1}^{n} a_j \lambda_j(U) R_j(U) \end{cases} \tag{7.8.1}$$

with

$$a_j = L_j(U)\partial_x U, \qquad j = 1,\cdots,n. \tag{7.8.2}$$

In view of (7.6.3), one may interpret (7.8.1) as a decomposition of U into simple waves, one for each characteristic family, with respective strengths $a_1,\cdots,a_n$. Our aim is to study the evolution of a_i along the i-characteristics associated with U. We let

$$\frac{d}{dt} = \partial_t + \lambda_i \partial_x \tag{7.8.3}$$

denote differentiation in the i-characteristic direction. Combining (7.8.2) with (7.8.1) yields

$$\begin{aligned} \partial_t a_i &= L_i \partial_t \partial_x U + \partial_x U^\top \mathrm{D}L_i^\top \partial_t U \\ &= \partial_x(L_i \partial_t U) - \partial_t U^\top \mathrm{D}L_i^\top \partial_x U + \partial_x U^\top \mathrm{D}L_i^\top \partial_t U \\ &= \partial_x(L_i \partial_t U) + \sum_{j,k=1}^{n} (\lambda_j - \lambda_k) R_j^\top \mathrm{D}L_i^\top R_k a_j a_k \,, \end{aligned} \tag{7.8.4}$$

$$\begin{aligned} \lambda_i \partial_x a_i &= \partial_x(\lambda_i L_i \partial_x U) - (\mathrm{D}\lambda_i \partial_x U)(L_i \partial_x U) \\ &= \partial_x(\lambda_i L_i \partial_x U) - \sum_{j,k=1}^{n} (\mathrm{D}\lambda_i R_j)\delta_{ik} a_j a_k \,, \end{aligned} \tag{7.8.5}$$

where δ_{ik} is the Kronecker delta. From (7.2.1), $L_i\partial_t U + \lambda_i L_i \partial_x U = 0$. Also, by virtue of (7.2.3), $R_j^\top \mathrm{D}L_i^\top R_k = -L_i \mathrm{D}R_j R_k$. Therefore, combining (7.8.3), (7.8.4), (7.8.5) and symmetrizing we conclude

$$\frac{da_i}{dt} = \sum_{j,k=1}^{n} \gamma_{ijk} a_j a_k \tag{7.8.6}$$

with

$$\gamma_{ijk} = -\tfrac{1}{2}(\lambda_j - \lambda_k) L_i [R_j, R_k] - (\mathrm{D}\lambda_i R_j)\delta_{ik} \,, \tag{7.8.7}$$

where $[R_j, R_k]$ denotes the Lie bracket (7.2.15). Note, in particular, that

$$\gamma_{iii} = -\mathrm{D}\lambda_i R_i \,, \tag{7.8.8}$$

$$\gamma_{ijj} = 0, \qquad j \neq i. \tag{7.8.9}$$

It is clear that in any argument showing blow-up of a_i through (7.8.6), the coefficient γ_{iii} will play a pivotal role. By virtue of (7.8.8), γ_{iii} never vanishes when the i-characteristic family is genuinely nonlinear, and vanishes identically when the i-characteristic family is linearly degenerate.

To gain insight, let us consider first the case where U is just an i-simple wave, i.e., $a_i \neq 0$ and $a_j = 0$ for $j \neq i$. In that case, (7.8.6) reduces to

$$\frac{da_i}{dt} = \gamma_{iii} a_i^2 \,. \tag{7.8.10}$$

Furthermore, since U is constant along characteristics, γ_{iii} in (7.8.10) is a constant. When $\gamma_{iii} \neq 0$ and a_i has the same sign as γ_{iii}, (7.8.10) induces blow-up of a_i in a finite time.

Next we consider the noteworthy special case where the system (7.2.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants. In that case $L_j = \mathrm{D}w_j$ and so, by (7.8.2),

$$a_j = \partial_x w_j \,. \tag{7.8.11}$$

Moreover, in virtue of (7.8.7), (7.3.10) and (7.3.12), (7.8.6) reduces to

$$\frac{da_i}{dt} = -\sum_{j=1}^{n} \frac{\partial \lambda_i}{\partial w_j} a_i a_j \,. \tag{7.8.12}$$

We seek an integrating factor for (7.8.12). If ϕ is any smooth scalar function of U, we get from (7.8.1):

$$\frac{d\phi}{dt} = \mathrm{D}\phi(\partial_t U + \lambda_i \partial_x U) = \sum_{j\neq i} (\lambda_i - \lambda_j)(\mathrm{D}\phi R_j) a_j = \sum_{j\neq i} (\lambda_i - \lambda_j) \frac{\partial \phi}{\partial w_j} a_j \,. \tag{7.8.13}$$

Combining (7.8.12) with (7.8.13) yields

$$\frac{d}{dt}(e^{\phi} a_i) = -e^{\phi} \frac{\partial \lambda_i}{\partial w_i} a_i^2 - \sum_{j\neq i} e^{\phi} [\frac{\partial \lambda_i}{\partial w_j} - (\lambda_i - \lambda_j) \frac{\partial \phi}{\partial w_j}] a_i a_j \,. \tag{7.8.14}$$

From (7.3.14) and (7.3.16), it follows that there exists ϕ that satisfies

$$\frac{\partial \phi}{\partial w_j} = \frac{1}{\lambda_i - \lambda_j} \frac{\partial \lambda_i}{\partial w_j} \,, \qquad j = 1, \cdots, i-1, i+1, \cdots, n. \tag{7.8.15}$$

For that ϕ, (7.8.14) reduces to

$$\frac{d}{dt}(e^{\phi}a_i) = -e^{-\phi}\frac{\partial\lambda_i}{\partial w_i}(e^{\phi}a_i)^2. \tag{7.8.16}$$

When the i-characteristic family is genuinely nonlinear, $\partial\lambda_i/\partial w_i \neq 0$. Whenever $e^{-\phi}\partial\lambda_i/\partial w_i$ is bounded away from zero, uniformly on the range of the solution, (7.8.16) will induce blowup of a_i, in finite time, along any characteristic emanating from a point $\bar{x}$ of the x-axis where a_i has the opposite sign of $\partial\lambda_i/\partial w_i$. Uniform boundedness of $e^{-\phi}\partial\lambda_i/\partial w_i$ is maintained, because, by Theorem 7.3.4, the range of any classical solution in the state space of Riemann invariants coincides with the range of its initial values. In the opposite case where the i-characteristic family is linearly degenerate, so that $\partial\lambda_i/\partial w_i$ vanishes identically, (7.8.16) implies that $\partial_x w_i$ stays bounded for as long as the solution exists. We have thus established

7.8.2 Theorem. *Assume* (7.2.1) *is endowed with a coordinate system of Riemann invariants* $(w_1,\cdots,w_n)$. *Suppose the i-characteristic family is genuinely nonlinear. Then any classical solution U with bounded initial values U_0, such that $dw_i(U_0)/dx$ has the opposite sign from $\partial\lambda_i/\partial w_i$ at some point $\bar{x}\in(-\infty,\infty)$, breaks down in finite time.*

7.8.3 Theorem. *Assume the strictly hyperbolic system* (7.2.1) *is linearly degenerate and is endowed with a coordinate system of Riemann invariants* $(w_1,\cdots,w_n)$, *on a domain $\mathscr{O} = \{U\in\mathbb{R}^n : a_i < w_i < b_i,\ i = 1,\cdots,n\}$. For any initial data $U_0\in C^1(-\infty,\infty)$ with values in $\mathscr{O}$, there exists a unique C^1 solution U to the Cauchy problem on the upper half-plane $(-\infty,\infty)\times[0,\infty)$.*

We now return to the general situation. When the i-characteristic field is genuinely nonlinear, and thus, by (7.8.8), $\gamma_{iii}\neq 0$, the term $\gamma_{iii}a_i^2$ in (7.8.6) will have a destabilizing effect. Any expectation that this may be offset by the remaining terms in (7.8.6), which account for the interaction effects with the other characteristic fields, is not likely to be fulfilled, at least when the initial data are constant outside a bounded interval, for the following reason. Equation (7.8.9) rules out the possibility of self-interactions of the remaining characteristic fields: all interactions, other than $\gamma_{iii}a_i^2$, involve two distinct characteristic families. Now, when the initial data are constant outside a bounded interval, mutual interactions eventually become insignificant, because waves of distinct characteristic families propagate with different speeds and thus eventually separate. Consequently, in the long run the term $\gamma_{iii}a_i^2$ becomes the dominant factor and drives a_i to infinity in finite time. The above heuristic arguments can be formalized and lead to the following

7.8.4 Theorem. *Assume* (7.2.1) *is a genuinely nonlinear strictly hyperbolic system of conservation laws. Consider initial data $U_0\in C^2(-\infty,\infty)$ such that $U_0(x)=\bar{U}$, a constant state, for $x\le a$ and $x\ge b$. If $(b-a)^2\max|U_0''(x)|$ is a sufficiently small positive number, then the classical solution of the initial value problem breaks down in finite time.*

In the literature cited in Section 7.9, the reader will find the (long and technical) proof of the above theorem as well as analogous results on the breakdown of classical solutions under weaker hypotheses, namely when the requirement of strict hyperbolicity of the system is relaxed and only some of the characteristic families are genuinely nonlinear. There are also extensions of Theorem 7.8.3, in which global existence of C^1 solutions is established when the characteristic families are merely *weakly linearly degenerate* relative to some constant state $\bar{U}$, in that, for $i = 1, \cdots, n$, $\mathrm{D}\lambda_i R_i$ need only vanish along the i-rarefaction wave curve emanating from $\bar{U}$, and the initial data $U_0 \in C^1(-\infty, \infty)$ stay close to $\bar{U}$, in the sense that

$$\sup\{(1+|x|)^{1+\mu}(|U_0(x) - \bar{U}| + |U_0'(x)|)\} \tag{7.8.17}$$

is sufficiently small, for some $\mu > 0$.

There is also a substantial body of research on the initial-boundary value problem. For comparison with the discussion in Section 5.6, let us consider a strictly hyperbolic system (7.2.1) on the quadrant $\{(x,t) : x > 0, t > 0\}$, under the assumption that the boundary $x = 0$ is noncharacteristic, i.e., $\lambda_k(U) < 0 < \lambda_{k+1}(U)$, for some $k = 0, \cdots, n$, with $\lambda_0(U) = -\infty$, $\lambda_{n+1}(U) = \infty$. We impose boundary conditions of the form (5.6.4), namely $BF(U(0,t)) = 0$, where B is a $n \times n$ matrix such that, for any U in the manifold $\mathscr{M} = \{U : BF(U) = 0\}$, $\mathbb{R}^n$ is the direct sum of the kernel of $B\mathrm{D}F(U)$ and the subspace spanned by the eigenvectors $\{R_{k+1}(U), \cdots, R_n(U)\}$ associated with the incoming characteristic families. We also assign initial data $U_0 \in C^1([0,\infty); \mathbb{R}^n)$ that are compatible with the boundary conditions, in that $BF(U_0(0)) = 0$ and $B[\mathrm{D}F(U_0(0))]^2 U_0'(0) = 0$. Under the above hypotheses, one can show (a) local existence of classical solutions; (b) breakdown, in finite time, of classical solutions, when (at least some of) the incoming characteristic families are genuinely nonlinear; and (c) global existence of classical solutions, when all of the incoming characteristic families are (at least) weakly linearly degenerate and the supremum (7.8.17) of the initial data is sufficiently small.

7.9 Weak Solutions

In view of the breakdown of classical solutions, demonstrated in the previous section, in order to solve the initial value problem in the large, for nonlinear hyperbolic systems of conservation laws, one has to resort to weak solutions. As explained in Chapter IV, the issue of the admissibility of weak solutions will have to be addressed.

In earlier chapters, we mainly considered weak solutions that are merely bounded measurable functions. Existence in that function class will indeed be established, for certain systems, in Chapter XVII, through the functional analytic method of compensated compactness. On the other hand, there are systems of three conservation laws for which the Cauchy problem is not well-posed in L^1. Apparently, the function class of choice for hyperbolic systems of conservation laws in one spatial dimension is BV, which provides the natural framework for envisioning the most important features of weak solutions, namely shocks and their interactions.

The finite domain of dependence property for solutions of hyperbolic systems, combined with the fact that our system (7.2.1) is invariant under uniform stretching of coordinates: $x = \bar{x} + ay$, $t = \bar{t} + a\tau$, $a > 0$, suggests that the admissibility of BV weak solutions may be decided locally, through examination of shocks and wave fans. These issues will be discussed thoroughly in the following two chapters.

7.10 Notes

The general mathematical framework of the theory of hyperbolic systems of conservation laws in one space dimension was set in the seminal paper of Lax [2], which distills the material collected over the years in the context of special systems. The notions of Riemann invariants, genuine nonlinearity, simple waves and simple wave curves, at the level of generality considered here, were introduced in that paper. The books by Smoller [3] and Serre [11] contain expositions of these topics, illustrated by interesting examples.

The simple hydrodynamic model (7.1.4) for traffic flow was introduced by Lighthill and Whitham [1]. Its elaborations and extensions have provided the vehicle for exhibiting and exploring a variety of features of hyperbolic systems of conservation laws. Extensions address traffic flow on road networks, under proper modeling of interactions at the junctions. In particular, when users plan their itinerary so as to minimize their personal travel cost, the network is expected to operate at a state of Nash equilibrium. Systems of conservation laws with the same flavor model pedestrian flow and gas flow in a network of pipes. A comprehensive treatment is found in the monograph by Garavello and Piccoli [2]. Other references include Holden and Risebro [3], Aw and Rascle [1], Tong Li [1,2,3,4], Greenberg [4,5], Colombo [1], Greenberg, Klar and Rascle [1], Bagnerini and Rascle [1], Benzoni-Gavage and Colombo [1], Tong Li and Hailiang Liu [1,2], Coclite, Garavello and Piccoli [1], Herty and Rascle [1], Garavello and Piccoli [1,2,3,4,5,6,7,8], Benzoni-Gavage, Colombo and Gwiazda [1], Colombo, Goatin and Priuli [1], Colombo, Goatin and Piccoli [1], Colombo, Goatin and Rosini [1], Berthelin, Degond, Delitala and Rascle [1], D'Apice, Manzo and Piccoli [1], Marigo and Piccoli [1], Godvik and Hanche-Olsen [1], Coclite and Garavello [1], Colombo and Marcellini [4], Colombo, Marcellini and Rascle [1], Colombo and Garavello [1,2] Colombo and Mauri [1], Colombo and Marcellini [1,2,3], Chalons, Goatin and Seguin [1], Amadori, Goatin and Rosini [1], Goatin [2], Garavello [1], Garavello and Goatin [1,2], Lee and Liu [1], Bressan and Han [1,2], and Bressan, Liu, Shen and Yu [1].

The derivation of the chromatography equation (7.16), the Buckley-Leverett equation (7.1.5) for two-phase flow, and the systems (7.1.21) and (7.1.22) for multiphase flow is found in the book by Rhee, Aris and Amundson [1], which also provides extensive discussions and a generous list of references. See also Bourdarias, Gisclon and Junca [1].

For the equation (7.1.7) of the rubber sheet reinforced with inextensible fibers, see Choksi [1].

The connection of the system (7.1.14) of pressureless gas dynamics with astrophysics is discussed in Shandarin and Zeldovich [1]. For a different application of the system of pressureless gas dynamics, see Ha, Huang and Wang [1].

A systematic, rigorous exposition of the theory of one-dimensional elastic continua (strings, rods, etc.) is found in the book by Antman [1]. See also Antman [2], and Antman and Jian-Guo Liu [1]. The system (7.1.17) was studied by Freistühler.

The shallow water wave system (7.1.19), originally derived (in a somewhat different form) by Lagrange [1], has been used extensively in hydraulic theory to model flood and tidal waves and bores. A few relevant references, out of an immense bibliography, are Airy [1], Saint Venant [1], Stoker [1], Whitham [1], Gerbeau and Perthame [1], and Holden and Risebro [5].

The system (7.1.23) for planar electromagnetic waves was studied thoroughly by Serre [4].

The system (7.1.25) represents the (inviscid) *Zeldovich-von Neumann-Döring* combustion theory, which reduces, as the reaction rate tends to infinity, to the simpler *Chapman-Jouguet* combustion theory. See the book by Williams [1]. The system (7.1.26) was proposed by Majda [1], as a model for the Zeldovich-von Neumann-Döring theory. As $\delta \to \infty$, it yields a model for the Chapman-Jouguet theory.

For a general thermodynamic theory of mixtures, see I. Müller [2] and Müller and Ruggeri [1]. A thorough treatment of the mathematical properties of the non-isothermal version of the system (7.1.28) is given in Ruggeri and Simić [1].

There are many other interesting examples of hyperbolic systems of conservation laws, for example the equations governing sedimentation and suspension flows (Bürger and Wendland [1], Bürger [1]), the system of flood waves (Whitham [2]), the system of polymer flooding (Holden, Risebro and Tveito [1]), the system of granular flow and its slow erosion limit (Amadori and Shen [1,2,3,4], Bressan and Shen [3], Cattani, Colombo and Graziano [1], Colombo, Guerra and Shen [1], Shen and Zhang [1], May, Shearer and Daniels [1], Shearer and Giffen [1], Shearer, Gray and Thornton [1]), and a system modeling the advance of avalanches (Shen [1]).

The possibility of recovering the flux F from the eigenvectors $R_1, \cdots, R_n$ of $\mathrm{D}F$ is discussed by Dafermos [22], when $n = 2$, and by Jenssen and Kogan [1], for any n.

The failure of strict hyperbolicity in one space-dimensional systems deriving from three-space-dimensional parent systems is discussed by Lax [6]. The system (7.2.11) has been used extensively as a vehicle for demonstrating the features of non-strictly hyperbolic systems of conservation laws, beginning with the work of Keyfitz and Kranzer [2].

As we saw in the historical introduction, Riemann invariants were first considered by Earnshaw [1] and by Riemann [1], in the context of the systems (7.1.11) and (7.1.12) of isentropic gas dynamics. Conditions for existence of coordinate systems of Riemann invariants and its implications on the existence of entropies were investigated by Conlon and Liu [1] and by Sévennec [1]. The calculation of the characteristic speeds and Riemann invariants of the system (7.3.18) of electrophoresis is due to Fife and Geng [1]. A detailed exposition of the noteworthy properties of this system is contained in Serre [11]. Serre [4] shows that the system (7.1.23) is equivalent to (7.3.24), (7.3.25) even within the realm of weak solutions.

As already mentioned in Section 1.10, the special entropy-entropy flux pair (7.4.6), for symmetric systems, was noted by Godunov [1,2,3] and by Friedrichs and Lax [1]. Over the years, a great number of entropy-entropy flux pairs with special properties have been constructed, mainly for systems of two conservation laws, beginning with the pioneering paper of Lax [4]. We shall see some of that work in later chapters. The characterization of systems of size $n \geq 3$ endowed with an abundance of entropies is due to Tsarev [1], who calls them *semi-Hamiltonian*, and Serre [6], who named them *rich*. A comprehensive exposition of their theory is contained in Serre [11]. For more recent developments in that direction, see Jenssen and Kogan [2]. For related discussions, see Sever [5,6].

Theorem 7.5.2 is due to Boillat [2].

The earliest examples of simple waves appear in the works of Poisson [1], Airy [2], and Earnshaw [1]; see the historical introduction. Theorem 7.6.3 is taken from Lax [2], who attributes the proof to Friedrichs.

A thorough discussion on the explosion of weak waves in continuum physics, together with extensive bibliography, are found in the encyclopedic article by Peter Chen [1]. Danilov and Mitrovic [1] describe the process of shock generation by the collision of two weak waves, in a scalar conservation law.

Local existence of C^1 solutions to the initial value problem in one space dimension was established by Friedrichs [1], Douglis [1], and Hartman and Winter [1]. For a comprehensive treatment of the initial as well as the initial-boundary value problem see the monograph by Ta-tsien Li and Wen-ci Yu [1]. Under certain conditions on the initial data, smooth solutions may exist globally in time.

It was pointed out in the historical introduction that the process of wave-breaking was first described by Stokes [1]. The earliest result on generic breakdown of classical solutions to systems of conservation laws caused by wave-breaking is due to Lax [3], who proved directly the case $n = 2$ of Theorem 7.8.2. This work was extended in several directions: Klainerman and Majda [1] established breakdown in the case $n = 2$ so long as neither of the two characteristic families is linearly degenerate. John [1] derived[3] (7.8.6) and used it to prove Theorem 7.8.3. A detailed discussion is found in Hörmander [1,2]. Tai-Ping Liu [13] gives an extension of Theorem 7.8.3 covering the case where some of the characteristic families are linearly degenerate. Ta-tsien Li, Zhou Yi and De-xing Kong [1] consider the case of weakly linearly degenerate characteristic families. See also Ta-tsien Li and De-xing Kong [1], Kong and Yang [1], and Ta-tsien Li and Libin Wang [1,3]. A direct proof of Theorem 7.8.2, for any n, is found in Serre [11]. Precise estimates for the equations of (nonisentropic) gas flow and inhomogeneous nonlinear wave equations are found in Geng Chen [1] and Chen and Young [1]. In particular, for the system of nonisentropic gas flow, see Chen, Young and Zhang [1], and Hualin Zheng [1]. For systems with relaxation, see Li and Liu [2,3,4].

Corli and Guès [1] consider systems with a linearly degenerate characteristic field and establish the local existence of "stratified" weak solutions, which have Lipschitz continuous components along the Riemann invariants of the degenerate field, but the remaining component may have infinite variation. See also Heibig [1]. On the other hand, the plausible conjecture that solutions to the Cauchy problem for totally

[3] John's formula for γ_{ijk} is different from (7.8.7) but, of course, the two expressions are equivalent.

degenerate systems, under smooth initial data, are globally smooth is generally false; see Neves and Serre [2].

Examples of systems for which the Cauchy problem is not well-posed in L^1 are found in Bressan and Shen [1]. See also Lewicka [1].

VIII

Admissible Shocks

Shock fronts were introduced in Section 1.6, for general systems of balance laws, and were placed in the context of *BV* solutions in Section 1.8. They were encountered again, briefly, in Section 3.1, where the governing Rankine-Hugoniot condition was recorded.

Since shock fronts have codimension one, important aspects of their local behavior may be investigated, without loss of generality, within the framework of systems in one space dimension. This will be the object of the present chapter. The discussion will begin with an exploration of the geometric features of the Rankine-Hugoniot condition, leading to the introduction of the Hugoniot locus.

The necessity of imposing admissibility conditions on weak solutions was pointed out in Chapter IV. These in turn induce, or at least motivate, admissibility conditions on shocks. Indeed, the prevailing view is that the issue of admissibility of general *BV* weak solutions should be resolved through a test applied to every point of the shock set. In particular, the shock admissibility conditions associated with the entropy condition of Section 4.5 and the vanishing viscosity approach of Section 4.6 will be introduced, and they will be compared with each other as well as with other important shock admissibility conditions proposed by Lax and by Liu.

8.1 Strong Shocks, Weak Shocks, and Shocks of Moderate Strength

For the hyperbolic system

$$\partial_t U + \partial_x F(U) = 0, \tag{8.1.1}$$

in one space dimension, the Rankine-Hugoniot jump condition (3.1.3) reduces to

$$F(U_+) - F(U_-) = s(U_+ - U_-). \tag{8.1.2}$$

Actually, (8.1.2) is as general as the multi-space-dimensional version (3.1.3), once the direction ν of propagation of the shock has been fixed and F has been defined through (7.2.2).

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When (8.1.2) holds, we say that *the state* U_-, *on the left, is joined to the state* U_+, *on the right, by a shock of speed* s. Note that "left" and "right" may be interchanged in (8.1.2), by the invariance of (8.1.1) under the transformation $(x,t) \mapsto (-x,-t)$. Nevertheless, later on we shall introduce admissibility conditions inducing irreversibility, as a result of which the roles of U_- and U_+ cannot be interchanged.

The jump $U_+ - U_-$ is the *amplitude* and its size $|U_+ - U_-|$ is the *strength* of the shock. Properties established without restriction on the strength are said to hold even for *strong shocks*. Quite often, however, we shall have to impose limitations on the strength of shocks: $|U_+ - U_-| < \delta$, with δ depending on DF through parameters such as the size of the gaps between characteristic speeds of distinct families, which induce the separation of waves of different families, and the size of derivatives of the functions λ_i and R_i, which manifest the nonlinearity of the system. In particular, when δ depends solely on the size of the first derivatives of the λ_i and R_i, the shock is of *moderate strength*; while if δ also depends on the size of second derivatives, the shock is termed *weak*. Of course, the size of these parameters may be changed by rescaling the variables x,t and U, so the relevant factor is the relative rather than the absolute size of δ.

Notice that (8.1.2) may be written as

$$[A(U_-,U_+) - sI](U_+ - U_-) = 0, \tag{8.1.3}$$

where we are using the notation

$$A(V,U) = \int_0^1 \mathrm{D}F(\tau U + (1-\tau)V)d\tau. \tag{8.1.4}$$

Thus s must be a real eigenvalue of $A(U_-,U_+)$, with associated eigenvector $U_+ - U_-$. If for some $\bar{U} \in \mathscr{O}$ the characteristic speed $\lambda_i(\bar{U})$ is a simple eigenvalue of D$F(\bar{U})$, then for V and U near $\bar{U}$, $A(V,U)$ will have a simple real eigenvalue $\mu_i(V,U)$ with associated eigenvector $S_i(V,U)$. In particular, $A(U,U) = \mathrm{D}F(U)$ whence we get $\mu_i(U,U) = \lambda_i(U)$, $S_i(U,U) = R_i(U)$. Notice that $A(V,U)$, and thereby also $\mu_i(V,U)$ and $S_i(V,U)$, are symmetric in (V,U). Therefore, (finite) Taylor expansion of these functions about the midpoint $\frac{1}{2}(V+U)$ yields

$$\mu_i(V,U) = \lambda_i(\tfrac{1}{2}(V+U)) + O(|V-U|^2), \tag{8.1.5}$$

$$S_i(V,U) = R_i(\tfrac{1}{2}(V+U)) + O(|V-U|^2). \tag{8.1.6}$$

Suppose then that for a shock of moderate strength

$$s = \mu_i(U_-,U_+), \tag{8.1.7}$$

$$U_+ - U_- = \zeta S_i(U_-,U_+). \tag{8.1.8}$$

Thus s will be close to the characteristic speed λ_i. Such a shock is then called an *i-shock*.

An interesting implication of (8.1.5), (8.1.7) is the useful identity

$$s = \tfrac{1}{2}[\lambda_i(U_-) + \lambda_i(U_+)] + O(|U_- - U_+|^2). \tag{8.1.9}$$

In special systems it is possible to associate even strong shocks with a particular characteristic family. For example, the Rankine-Hugoniot condition

$$\begin{cases} v_+ - v_- + s(u_+ - u_-) = 0 \\ \sigma(u_+) - \sigma(u_-) + s(v_+ - v_-) = 0 \end{cases} \tag{8.1.10}$$

for the system (7.1.11) of isentropic elasticity implies

$$s = \pm\sqrt{\frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-}}\ . \tag{8.1.11}$$

Recalling the characteristic speeds (7.2.4) of this system, it is natural to call shocks propagating to the left $(s < 0)$ 1-shocks and shocks propagating to the right $(s > 0)$ 2-shocks.

Another important example is the system (7.1.8), which governs rectilinear adiabatic flow of inviscid gases. The Rankine-Hugoniot jump conditions read [1]

$$\begin{cases} v_+ - v_- + r(u_+ - u_-) = 0 \\ p(u_+, s_+) - p(u_-, s_-) - r(v_+ - v_-) = 0 \\ v_+ p(u_+, s_+) - v_- p(u_-, s_-) - r[\varepsilon(u_+, s_+) + \tfrac{1}{2}v_+^2 - \varepsilon(u_-, s_-) - \tfrac{1}{2}v_-^2] = 0. \end{cases} \tag{8.1.12}$$

The 2-shocks, associated with the characteristic speed $\lambda_2 = 0$, are stationary, $r = 0$, in which case (8.1.12) reduces to $v_- = v_+$ and $p_- = p_+$. On the other hand, 1-shocks and 3-shocks propagate with negative and positive speed

$$r = \pm\sqrt{-\frac{p(u_+, s_+) - p(u_-, s_-)}{u_+ - u_-}}\ . \tag{8.1.13}$$

Furthermore, when $r \neq 0$, we may combine the three equations in (8.1.12) to deduce the celebrated *Hugoniot equation*

$$\varepsilon(u_+, s_+) - \varepsilon(u_-, s_-) = -\frac{1}{2}[p(u_+, s_+) + p(u_-, s_-)](u_+ - u_-) \tag{8.1.14}$$

(see the historical introduction), which does not involve velocity or the shock speed, but relates only the thermodynamic state variables u and s.

[1] Here r stands for shock speed, as the symbol s is retained to denote specific entropy.

8.2 The Hugoniot Locus

The set of points U in state space that may be joined to a fixed point $\bar{U}$ by a shock is called the *Hugoniot locus* of $\bar{U}$. It has a simple geometric structure in the vicinity of any point U of strict hyperbolicity of the system.

8.2.1 Theorem. *For a given state $\bar{U} \in \mathscr{O}$, assume that the characteristic speed $\lambda_i(\bar{U})$ is a simple eigenvalue of $\mathrm{D}F(\bar{U})$. Then there is a C^3 curve $U = W_i(\tau)$ in state space, called the i-shock curve through $\bar{U}$, and a C^2 function $s = s_i(\tau)$, both defined for τ in some neighborhood of 0, with the following property: A state U can be joined to $\bar{U}$ by an i-shock of moderate strength and speed s if and only if $U = W_i(\tau)$, $s = s_i(\tau)$, for some τ. Furthermore, $W_i(0) = \bar{U}$ and*

$$s_i(0) = \lambda_i(\bar{U}), \tag{8.2.1}$$

$$\dot{s}_i(0) = \tfrac{1}{2}\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U}), \tag{8.2.2}$$

$$\dot{W}_i(0) = R_i(\bar{U}), \tag{8.2.3}$$

$$\ddot{W}_i(0) = \mathrm{D}R_i(\bar{U})R_i(\bar{U}). \tag{8.2.4}$$

The more explicit notation $W_i(\tau;\bar{U})$, $s_i(\tau;\bar{U})$ shall be employed when one needs to identify the point of origin of this shock curve.

Proof. Recall the notation developed in Section 8.1 and, in particular, Equations (8.1.7), (8.1.8). A state U may be joined to $\bar{U}$ by an i-shock of speed s if and only if

$$U = \bar{U} + \tau S_i(\bar{U}, U), \tag{8.2.5}$$

$$s = \mu_i(\bar{U}, U). \tag{8.2.6}$$

Accordingly, we consider the function

$$H(U,\tau) = U - \bar{U} - \tau S_i(\bar{U}, U), \tag{8.2.7}$$

defined on $\mathscr{O} \times \mathbb{R}$, and note that $H(\bar{U}, 0) = 0$, $\mathrm{D}H(\bar{U}, 0) = I$. Consequently, by the implicit function theorem, there is a curve $U = W_i(\tau)$ in state space, with $W_i(0) = \bar{U}$, such that $H(U,\tau) = 0$ for τ near 0 if and only if $U = W_i(\tau)$. We then define

$$s_i(\tau) = \mu_i(\bar{U}, W_i(\tau)). \tag{8.2.8}$$

In particular, $s_i(0) = \mu_i(\bar{U},\bar{U}) = \lambda_i(\bar{U})$. Furthermore, differentiating (8.2.5) with respect to τ and setting $\tau = 0$, we deduce $\dot{W}_i(0) = S_i(\bar{U},\bar{U}) = R_i(\bar{U})$. To establish the remaining equations (8.2.2) and (8.2.4), we appeal to (8.1.5) and (8.1.6) to get

$$s_i(\tau) = \lambda_i(\tfrac{1}{2}(\bar{U} + W_i(\tau))) + O(\tau^2) \tag{8.2.9}$$
$$= \lambda_i(\bar{U}) + \tfrac{1}{2}\tau \mathrm{D}\lambda_i(\bar{U})R_i(\bar{U}) + O(\tau^2),$$

$$W_i(\tau) = \bar{U} + \tau R_i(\tfrac{1}{2}(\bar{U} + W_i(\tau))) + O(\tau^3) \tag{8.2.10}$$
$$= \bar{U} + \tau R_i(\bar{U}) + \tfrac{1}{2}\tau^2 \mathrm{D}R_i(\bar{U})R_i(\bar{U}) + O(\tau^3).$$

This completes the proof.

In particular, if $\bar{U}$ is a point of strict hyperbolicity of the system (8.1.1), Theorem 8.2.1 implies that the Hugoniot locus of $\bar{U}$ is the union of n shock curves, one for each characteristic family.

The shock curve constructed above is generally confined to the regime of shocks of moderate strength, because of the use of the implicit function theorem, which applies only when the strength of the shock, measured by $|\tau|$, is sufficiently small: $|\tau| < \delta$ with δ depending on the C^1 norm of S_i, which in turn can be estimated in terms of the C^1 norm of $\mathrm{D}F$ and the inverse of the gap between λ_i and the other characteristic speeds. Nevertheless, in special systems one may often use more delicate analytical or topological arguments or explicit calculation to extend shock curves to the range of strong shocks. For example, in the case of the system (7.1.11), combining (8.1.10) with (8.1.11) we deduce that the Hugoniot locus of any point $(\bar{u}, \bar{v})$ in state space consists of two curves

$$v = \bar{v} \pm \sqrt{[\sigma(u) - \sigma(\bar{u})](u - \bar{u})}\,, \tag{8.2.11}$$

defined on the whole range of u.

Another noteworthy case of a system in which the shock curves may be extended to the realm of strong shocks is (7.1.8). Recalling the discussion at the end of Section 8.1, we infer that the 2-shock curve through the state $(\bar{u}, \bar{v}, \bar{s})$ is determined by the Rankine-Hugoniot jump conditions

$$v = \bar{v}, \qquad p(u,s) = p(\bar{u}, \bar{s}). \tag{8.2.12}$$

Since $p_u < 0$, (8.2.12) describes a simple curve parametrized by s. As regards the other two shock curves, the Hugoniot equation (8.1.14), with $(u_-, s_-) = (\bar{u}, \bar{s})$ and $(u_+, s_+) = (u, s)$, determines the projection of the Hugoniot locus on the u-s plane. Realizing this projection as a curve $s = s(u)$, we differentiate with respect to u and use (7.1.9) to get

$$[2\theta - \theta_u(u - \bar{u})]\frac{ds}{du} = p - \bar{p} - p_u(u - \bar{u}). \tag{8.2.13}$$

Since $\theta > 0$, (8.2.13) induces a simple curve $s = s(u)$ on some neighborhood of $\bar{u}$. When the equations of state satisfy $2\theta - u\theta_u > 0$ on their domain of definition, as is the case with ideal gases (2.5.20), the curve $s(u)$ is extended to the regime of strong shocks. However, if $2\theta - u\theta_u$ is allowed to change signs, upon encountering a singular point at which $2\theta - \theta_u(u-\bar{u})$ and $p - \bar{p} - p_u(u-\bar{u})$ vanish simultaneously, $s(u)$ may split into infinitely many branches. The velocity components of the 1- and the 3-shock curves follow from (8.1.12):

$$v(u) = \bar{v} \pm \sqrt{-[p(u,s(u)) - p(\bar{u},\bar{s})](u-\bar{u})}. \tag{8.2.14}$$

As we shall see in Section 8.5, parametrizing the shock curves by u will elucidate the admissibility of strong shocks. In order to prepare the ground for that investigation, we differentiate (8.2.13) to get

(8.2.15)
$$[2\theta - \theta_u(u-\bar{u})]\frac{d^2s}{du^2} + [2\theta_s - \theta_{us}(u-\bar{u})]\left(\frac{ds}{du}\right)^2 + 2[\theta_u - \theta_{uu}(u-\bar{u})]\frac{ds}{du} = -p_{uu}(u-\bar{u}).$$

The shock speed, $r = r(u)$, parametrized by u, is determined by (8.1.13), with $(u_-, s_-) = (\bar{u}, \bar{s})$ and $(u_+, s_+) = (u, s(u))$. Upon using (8.2.13), we deduce

$$r\frac{dr}{du} = \frac{\theta}{(u-\bar{u})^2}\frac{ds}{du}\,. \tag{8.2.16}$$

Returning to the general system (8.1.1), we note that the i-shock curves introduced above have common features with the i-rarefaction wave curves defined in Section 7.6. Indeed, recalling Theorems 7.6.5 and 8.2.1, and, in particular, comparing (7.6.14) with (8.2.3), (8.2.4), we deduce

8.2.2 Theorem. *Assume $\bar{U} \in \mathscr{O}$ is a point of genuine nonlinearity of the i-characteristic family of the hyperbolic system* (8.1.1) *of conservation laws, and $\lambda_i(\bar{U})$ is a simple eigenvalue of* $\mathrm{D}F(\bar{U})$. *Normalize R_i so that* (7.6.13) *holds on some neighborhood of $\bar{U}$. Then the i-rarefaction wave curve V_i, defined through Theorem* 7.6.5, *and the i-shock curve W_i, defined through Theorem* 8.2.1, *have a second order contact at $\bar{U}$.*

Recall that, by Theorem 7.6.6, i-Riemann invariants are constant along i-rarefaction wave curves. At the same time, as shown above, i-shock curves are very close to i-rarefaction wave curves. It is then to be expected that i-Riemann invariants vary very slowly along i-shock curves. Indeed,

8.2.3 Theorem. *The jump of any i-Riemann invariant across a weak i-shock is of third order in the strength of the shock.*

Proof. Assume $\lambda_i(\bar{U})$ is a simple eigenvalue of $\mathrm{D}F(\bar{U})$ and consider the i-shock curve W_i through $\bar{U}$. For any i-Riemann invariant w, differentiating along the curve $W_i(\cdot)$,

$$\dot{w} = \mathrm{D}w\dot{W}_i\,, \tag{8.2.17}$$

$$\ddot{w} = \dot{W}_i^{\top}\mathrm{D}^2w\dot{W}_i + \mathrm{D}w\ddot{W}_i\,. \tag{8.2.18}$$

By virtue of (8.2.3) and (7.3.1), $\dot{w} = 0$ at $\tau = 0$.

We now apply D to (7.3.1) and then multiply the resulting equation from the right by R_i to deduce the identity

$$R_i^\top \mathrm{D}^2 w R_i + \mathrm{D}w \mathrm{D}R_i R_i = 0. \tag{8.2.19}$$

Combining (8.2.18), (8.2.3), (8.2.4) and (8.2.19), we conclude that $\ddot{w} = 0$ at $\tau = 0$. This completes the proof.

In the special case where the system (8.1.1) is endowed with a coordinate system $(w_1, \cdots, w_n)$ of Riemann invariants, we may calculate the leading term in the jump of w_j across a weak i-shock, $i \neq j$, as follows. The Rankine-Hugoniot condition reads

$$F(W_i(\tau)) - F(\bar{U}) = s_i(\tau)[W_i(\tau) - \bar{U}]. \tag{8.2.20}$$

Differentiating with respect to τ yields

$$[\mathrm{D}F(W_i(\tau)) - s_i(\tau)I]\dot{W}_i(\tau) = \dot{s}_i(\tau)[W_i(\tau) - \bar{U}]. \tag{8.2.21}$$

Multiplying (8.2.21), from the left, by $\mathrm{D}w_j(W_i)$ gives

$$(\lambda_j - s_i)\dot{w}_j = \dot{s}_i \mathrm{D}w_j[W_i - \bar{U}]. \tag{8.2.22}$$

Next we differentiate (8.2.22), with respect to τ, thus obtaining

$$(\lambda_j - s_i)\ddot{w}_j + (\dot{\lambda}_j - 2\dot{s}_i)\dot{w}_j = \ddot{s}_i \mathrm{D}w_j[W_i - \bar{U}] + \dot{s}_i \dot{W}_i^\top \mathrm{D}^2 w_j [W_i - \bar{U}]. \tag{8.2.23}$$

We differentiate (8.2.23), with respect to τ, and then set $\tau = 0$. We use (8.2.1), (8.2.2), (8.2.3), (7.3.12) and that both $\dot{w}_j$ and $\ddot{w}_j$ vanish at 0, by virtue of Theorem 8.2.3, to conclude

$$\dddot{w}_j = \frac{1}{2} \frac{1}{\lambda_j - \lambda_i} \frac{\partial \lambda_i}{\partial w_i} R_i^\top \mathrm{D}^2 w_j R_i, \tag{8.2.24}$$

where $\dddot{w}_j$ is evaluated at 0 and the right-hand side is evaluated at $\bar{U}$.

Returning to the general case, we next investigate how the shock speed function $s_i(\tau)$ evolves along the i-shock curve. We multiply (8.2.21), from the left, by $L_i(W_i(\tau))$ to get

$$[\lambda_i(W_i(\tau)) - s_i(\tau)]L_i(W_i(\tau))\dot{W}_i(\tau) = \dot{s}_i(\tau)L_i(W_i(\tau))[W_i(\tau) - \bar{U}]. \tag{8.2.25}$$

For τ sufficiently close to 0, but $\tau \neq 0$,

$$L_i(W_i(\tau))\dot{W}_i(\tau) > 0, \qquad \tau L_i(W_i(\tau))[W_i(\tau) - \bar{U}] > 0, \tag{8.2.26}$$

by virtue of (8.2.3). In the applications it turns out that (8.2.26) continue to hold for a broad range of τ, often extending to the regime of strong shocks. In that case, (8.2.25) and (8.2.21) immediately yield the following

8.2.4 Lemma. *Assume* (8.2.26) *hold. Then*

$$\dot{s}_i(\tau) > 0 \quad \textit{if and only if } \tau[\lambda_i(W_i(\tau)) - s_i(\tau)] > 0, \tag{8.2.27}$$

$$\dot{s}_i(\tau) = 0 \quad \textit{if and only if } \lambda_i(W_i(\tau)) = s_i(\tau). \tag{8.2.28}$$

Moreover, $\dot{s}_i(\tau) = 0$ *implies that* $\dot{W}_i(\tau)$ *is collinear to* $R_i(W_i(\tau))$*.*

In order to see how s_i varies across points where $\dot{s}_i$ vanishes, we differentiate (8.2.25) with respect to τ and then evaluate the resulting expression at any τ where $\dot{s}_i(\tau) = 0$. Since $s_i(\tau) = \lambda_i(W_i(\tau))$ and $\dot{W}_i(\tau) = aR_i(W_i(\tau))$, upon recalling (7.2.3) we deduce

$$\ddot{s}_i(\tau) L_i\big(W_i(\tau)\big)\big[W_i(\tau) - \bar{U}\big] = a^2 \mathrm{D}\lambda_i\big(W_i(\tau)\big) R_i\big(W_i(\tau)\big), \tag{8.2.29}$$

whence it follows that at points where $\dot{s}_i = 0$, $\ddot{s}_i$ has the same sign as $\tau \mathrm{D}\lambda_i R_i$.

By Lemma 8.2.4, s_i constant implies that the i-shock curve is an integral curve of the vector field R_i, along which λ_i is constant. Consequently, all points along such a shock curve are states of linear degeneracy of the i-characteristic family. The converse of this statement is also valid:

8.2.5 Theorem. *Assume the i-characteristic family of the hyperbolic system* (8.1.1) *of conservation laws is linearly degenerate and* $\lambda_i(\bar{U})$ *is a simple eigenvalue of* $\mathrm{D}F(\bar{U})$*. Then the i-shock curve* W_i *through* $\bar{U}$ *is the integral curve of* R_i *through* $\bar{U}$*. In fact, under the proper parametrization,* W_i *is the solution of the differential equation*

$$\dot{W}_i = R_i(W_i) \tag{8.2.30}$$

with initial condition $W_i(0) = \bar{U}$*. Along* W_i*, the characteristic speed* λ_i *and all i-Riemann invariants are constant. The shock speed function* s_i *is also constant:*

$$s_i(\tau) = \lambda_i(W_i(\tau)) = \lambda_i(\bar{U}). \tag{8.2.31}$$

Proof. Let W_i denote the solution of (8.2.30) with initial condition $W_i(0) = \bar{U}$. Then

$$[\mathrm{D}F(W_i(\tau)) - \lambda_i(W_i(\tau))I]\dot{W}_i(\tau) = 0. \tag{8.2.32}$$

Since $\mathrm{D}\lambda_i(U)R_i(U) = 0$, $\dot{\lambda}_i(W_i(\tau)) = 0$ and so $\lambda_i(W_i(\tau)) = \lambda_i(\bar{U})$. Integrating (8.2.32) from 0 to τ yields

$$F(W_i(\tau)) - F(\bar{U}) = \lambda_i(\bar{U})[W_i(\tau) - \bar{U}], \tag{8.2.33}$$

which establishes that W_i is the i-shock curve through $\bar{U}$, with corresponding shock speed function s_i given by (8.2.31). This completes the proof.

The following important implication of Theorem 8.2.5 provides an alternative characterization of linear degeneracy:

8.2.6 Corollary. *When the i-characteristic family of the hyperbolic system* (8.1.1) *is linearly degenerate, there exist traveling wave solutions*

$$U(x,t) = V(x - \sigma t), \tag{8.2.34}$$

for any σ in the range of the i-characteristic speed λ_i.

Proof. Let $\sigma = \lambda_i(\bar{U})$, for some state $\bar{U}$. Consider the i-shock curve W_i through $\bar{U}$, which satisfies (8.2.30). Take any C^1 function $\tau = \tau(\xi)$ and define U by (8.2.34), with $V(\xi) = W_i(\tau(\xi))$. On account of (8.2.30) and (8.2.31),

$$\partial_t U + \partial_x F(U) = \frac{d\tau}{d\xi}\left[\mathrm{D}F\big(W_i(\tau)\big) - \lambda_i\big(W_i(\tau)\big)I\right] R_i\big(W_i(\tau)\big) = 0. \tag{8.2.35}$$

The proof is complete.

It is natural to inquire whether an i-shock curve may be an integral curve of the vector field R_i in the absence of linear degeneracy. It turns out that this may only occur under very special circumstances:

8.2.7 Theorem. *For the hyperbolic system* (8.1.1), *assume $\bar{U}$ is a state of genuine nonlinearity for the i-characteristic family and $\lambda_i(\bar{U})$ is a simple eigenvalue of $\mathrm{D}F(\bar{U})$. The i-shock curve through $\bar{U}$ coincides with the integral curve of the field R_i (i.e., the i-rarefaction wave curve) through $\bar{U}$ if and only if the latter is a straight line in state space.*

Proof. If the i-shock curve W_i through $\bar{U}$ coincides with the integral curve of R_i through $\bar{U}$, then $\dot{W}_i(\tau)$ must be collinear to $R_i(W_i(\tau))$. In that case, (8.2.21) implies

$$[\lambda_i(W_i(\tau)) - s_i(\tau)]\dot{W}_i(\tau) = \dot{s}_i(\tau)[W_i(\tau) - \bar{U}]. \tag{8.2.36}$$

For τ near 0, but $\tau \neq 0$, it is $\lambda_i(W_i(\tau)) \neq s_i(\tau)$, by genuine nonlinearity. Therefore, (8.2.36) implies that the graph of W_i is a straight line through $\bar{U}$.

Conversely, assume the integral curve of R_i through $\bar{U}$ is a straight line, which may be parametrized as $U = W_i(\tau)$, where W_i is some smooth function satisfying $W_i(0) = \bar{U}$, as well as (8.2.3) and (8.2.4) (note that $\mathrm{D}R_i(\bar{U})R_i(\bar{U})$ is necessarily collinear to $R_i(\bar{U})$). We may then determine a scalar-valued function $s_i(\tau)$ such that

$$\begin{aligned} F(W_i(\tau)) - F(\bar{U}) &= \int_0^\tau \mathrm{D}F(W_i(\zeta))\dot{W}_i(\zeta)d\zeta \\ &= \int_0^\tau \lambda_i(W_i(\zeta))\dot{W}_i(\zeta)d\zeta = s_i(\tau)[W_i(\tau) - \bar{U}]. \end{aligned} \tag{8.2.37}$$

Thus W_i is the i-shock curve through $\bar{U}$. This completes the proof.

Special as it may be, the class of hyperbolic systems of conservation laws with coinciding shock and rarefaction wave curves of each characteristic family includes some noteworthy examples. Consider, for instance, the system (7.3.18) of electrophoresis. Notice that, for $i = 1, \cdots, n$, the level surfaces of the i-Riemann invariant W_i, determined through (7.3.21) or (7.3.22), are hyperplanes. In particular, for $i = 1, \cdots, n$, the integral curves of the vector field R_i are the straight lines produced by the intersection of the level hyperplanes of the $n-1$ Riemann invariants $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$. Consequently, the conditions of Theorem 8.2.7 apply to the system (7.3.18).

In the presence of multiple characteristic speeds, the Hugoniot locus may contain multi-dimensional varieties, in the place of shock curves. In that connection it is instructive to consider the model system (7.2.11), for which the origin is an umbilic point. When a state $(\bar{u}, \bar{v})$ is joined to a state (u, v) by a shock of speed s, the Rankine-Hugoniot condition reads

$$
\begin{cases} (u^2 + v^2)u - (\bar{u}^2 + \bar{v}^2)\bar{u} = s(u - \bar{u}) \\ \\ (u^2 + v^2)v - (\bar{u}^2 + \bar{v}^2)\bar{v} = s(v - \bar{v}). \end{cases} \tag{8.2.38}
$$

Notice that when $(\bar{u}, \bar{v}) \neq (0,0)$, the Hugoniot locus of $(\bar{u}, \bar{v})$ consists of the circle $u^2 + v^2 = \bar{u}^2 + \bar{v}^2$, along which the shock speed is constant, $s = \bar{u}^2 + \bar{v}^2$, and the straight line $\bar{v}u = \bar{u}v$, which connects $(\bar{u}, \bar{v})$ to the origin. Thus, the 1-characteristic family provides a case in which Theorem 8.2.5 applies, while, at the same time, the 2-characteristic family satisfies the assumptions of Theorem 8.2.7. On the other hand, the Hugoniot locus of the umbilic point $(0,0)$ is the entire plane, because any point (u, v) can be joined to $(0,0)$ by a shock of speed $s = u^2 + v^2$.

Not all systems in which strict hyperbolicity fails exhibit the same behavior. For instance, the Hugoniot locus of $(0,0)$ for the system

$$
\begin{cases} \partial_t u + \partial_x[2(u^2 + v^2)u] = 0 \\ \\ \partial_t v + \partial_x[(u^2 + v^2)v] = 0, \end{cases} \tag{8.2.39}
$$

in which strict hyperbolicity also fails at the origin, consists of two lines, namely the u-axis and the v-axis.

8.3 The Lax Shock Admissibility Criterion; Compressive, Overcompressive and Undercompressive Shocks

The object of this section is to introduce conditions under which the interaction of a shock with its surrounding "smoother" part of the solution is stable. A manifestation of stability would be that smooth waves of small amplitude colliding with the shock are absorbed, transmitted and/or reflected as waves with small amplitude, without affecting the integrity of the shock itself, by changing substantially its strength or its speed of propagation. With this in mind, let us consider a solution of (8.1.1), on the upper half-plane, consisting of a constant state U_-, on the left, joined to a constant state U_+, on the right, by a shock $x = st$. In particular, the Rankine-Hugoniot jump condition (8.1.2) holds. Assume that the speed s of the shock satisfies

$$\begin{cases} \lambda_1(U_-) \leq \cdots \leq \lambda_{i-1}(U_-) < s < \lambda_i(U_-) \leq \cdots \leq \lambda_n(U_-) \\ \lambda_1(U_+) \leq \cdots \leq \lambda_j(U_+) < s < \lambda_{j+1}(U_+) \leq \cdots \leq \lambda_n(U_+)\,, \end{cases} \tag{8.3.1}$$

for some $i = 1,\cdots,n$ and $j = 1,\cdots,n$, with the understanding that if $i = 1$ then $\lambda_{i-1}(U_-) = -\infty$ and if $j = n$ then $\lambda_{j+1}(U_+) = \infty$. The plan is to construct a family of solutions by perturbing slightly the constant states U_- and U_+. Classical solutions of (8.1.1) with small oscillation are closely approximated by solutions of the linearized system. Thus, in order to reach the desired conclusion without facing laborious technical details, we make the simplifying assumption that $\mathrm{D}F(U)$ is constant, equal to $\mathrm{D}F(U_-)$, for $|U - U_-| < \varepsilon$, and also constant, equal to $\mathrm{D}F(U_+)$, for $|U - U_+| < \varepsilon$. We then seek solutions of (8.1.1), on the upper half-plane, in the form

$$U(x,t) = \begin{cases} U_- + \sum_{k=1}^n \omega_k^-(x,t)R_k(U_-)\,, & x < st + \sigma(t) \\ U_+ + \sum_{k=1}^n \omega_k^+(x,t)R_k(U_+)\,, & x > st + \sigma(t)\,, \end{cases} \tag{8.3.2}$$

where the $\omega_k^{\pm}(x,t)$ and $\sigma(t)$ are C^1 functions such that $|\omega_k^{\pm}| < a\varepsilon$ and $|\dot{\sigma}| < a\varepsilon$, for some $a << 1$. Since $F(U) = F(U_{\pm}) + \mathrm{D}F(U_{\pm})(U - U_{\pm})$ in the vicinity of $U_{\pm}$, and $R_k(U_{\pm})$ are eigenvectors of $\mathrm{D}F(U_{\pm})$ with eigenvalue $\lambda_k(U_{\pm})$, U from (8.3.2) will satisfy (8.1.1), for $x \neq st + \sigma(t)$, if, for $k = 1,\cdots,n$,

$$\begin{cases} \partial_t \omega_k^-(x,t) + \lambda_k(U_-)\partial_x \omega_k^-(x,t) = 0\,, & x < st + \sigma(t) \\ \partial_t \omega_k^+(x,t) + \lambda_k(U_+)\partial_x \omega_k^+(x,t) = 0\,, & x > st + \sigma(t) \end{cases} \tag{8.3.3}$$

and it will also satisfy the Rankine-Hugoniot jump conditions across the perturbed shock $x = st + \sigma(t)$ if

$$\sum_{k=1}^n [\lambda_k(U_+) - s - \dot{\sigma}]\omega_k^+ R_k(U_+) - \sum_{k=1}^n [\lambda_k(U_-) - s - \dot{\sigma}]\omega_k^- R_k(U_-) = \dot{\sigma}(U_+ - U_-). \tag{8.3.4}$$

By virtue of (8.3.3), $\omega_k^{\pm}$ are constant along k-characteristics, which are straight lines with slope $\lambda_k(U_{\pm})$. In particular, on account of (8.3.1), $\omega_i^-,\cdots,\omega_n^-$ and $\omega_1^+,\cdots,\omega_j^+$ are everywhere determined by their prescribed initial data, at $t = 0$. By contrast, $\omega_1^-,\cdots,\omega_{i-1}^-$ and $\omega_{j+1}^+,\cdots,\omega_n^+$ are determined partly by their prescribed initial data, at $t = 0$, and partly by their values along the shock. The latter are constrained by the Rankine-Hugoniot condition (8.3.4), which shall be regarded as a system of n equations in the $n + i - j$ unknowns $\omega_1^-,\cdots,\omega_{i-1}^-,\omega_{j+1}^+,\cdots,\omega_n^+$ and $\dot{\sigma}$.

Commonly, shocks satisfy (8.3.1) with $i = j$ so that

$$\begin{cases} \lambda_{i-1}(U_-) < s < \lambda_i(U_-) \\ \lambda_i(U_+) < s < \lambda_{i+1}(U_+). \end{cases} \tag{8.3.5}$$

In that case, the shock is called *compressive*, a term borrowed from gas dynamics (see below). For compressive shocks, the number of unknowns in (8.3.4) is n, matching the number of equations. If in addition

$$(8.3.6) \qquad \det\left[R_1(U_-),\cdots,R_{i-1}(U_-),U_+-U_-,R_{i+1}(U_+),\cdots,R_n(U_+)\right]\neq 0\,,$$

one readily shows, by the implicit function theorem, that (8.3.4) determines uniquely $\omega_1^-,\cdots,\omega_{i-1}^-,\omega_{j+1}^+,\cdots,\omega_n^+$ and $\dot{\sigma}$. Under this condition, a solution U of the form (8.3.2) is uniquely determined upon prescribing initial data $\omega_k^\pm(x,0)$, within the allowable range, and the shock is termed *evolutionary*.

Shocks satisfying (8.3.1) with $i<j$ are called *overcompressive*. In that case, the number of equations in (8.3.4) exceeds the number of unknowns, so that existence of solutions U of the form (8.3.2) is not guaranteed. Nevertheless, shocks of this type do arise in certain applications. A simple example is provided by System (7.2.11): recalling the form of its Hugoniot locus, described in Section 8.2, we consider a shock of speed s, joining, on the left, a state (u_-,v_-), lying on the unit circle, to a state $(u_+,v_+)=a(u_-,v_-)$, on the right, where a is some constant. From (7.2.12), $\lambda_1(u_-,v_-)=1$, $\lambda_2(u_-,v_-)=3$, $\lambda_1(u_+,v_+)=a^2$, $\lambda_2(u_+,v_+)=3a^2$. Furthermore, the Rankine-Hugoniot condition (8.2.38) yields $s=a^2+a+1$. Therefore, if $a\in(-\frac{1}{2},0)$, then $\lambda_2(u_-,v_-)>\lambda_1(u_-,v_-)>s>\lambda_2(u_+,v_+)>\lambda_1(u_+,v_+)$, i.e., the shock is overcompressive.

In the opposite case, where (8.3.1) holds with $i>j$, the shock is termed *undercompressive*.

In that situation, (8.3.4) is underdetermined and generally admits multiple solutions. However, when shocks of this type arise in the applications, the Rankine-Hugoniot jump conditions are usually supplemented with equations of the form $G(U_-,U_+,s)=0$, dubbed *kinetic relations*, which render the solution unique.

The above will be clarified further in Section 8.6.

The conditions (8.3.1) exclude shocks traveling with characteristic speed but such shocks do exist. In particular, any shock joining U_-, on the left, with U_+, on the right, and traveling with speed s will be called a *left i-contact discontinuity* if $s=\lambda_i(U_-)$, a *right i-contact discontinuity* if $s=\lambda_i(U_+)$, and simply an *i-contact discontinuity* if $s=\lambda_i(U_-)=\lambda_i(U_+)$. By virtue of Theorem 8.2.5, any weak shock associated with a linearly degenerate characteristic family is necessarily a contact discontinuity.

In what follows, we assume λ_i is a simple eigenvalue of DF and focus attention on compressive i-shocks of moderate strength. In that case, $|s-\lambda_i|$ is small compared to both $\lambda_{i+1}-\lambda_i$ and $\lambda_i-\lambda_{i-1}$, so that the first and the fourth inequalities in (8.3.5) always hold. The remaining two inequalities, slightly relaxed to allow for left and/or right i-contact discontinuities, combine into

$$(8.3.7) \qquad \lambda_i(U_+)\leq s\leq\lambda_i(U_-),$$

which is the celebrated *Lax E-condition*. Furthermore, as U_+-U_- is nearly collinear to $R_i(U_\pm)$, (8.3.6) is implied by hyperbolicity.

For orientation, let us consider a few examples, beginning with the scalar conservation law (7.1.2). The characteristic speed is $\lambda(u) = f'(u)$ and so (8.3.7) takes the form

$$f'(u_+) \le s \le f'(u_-)\,, \tag{8.3.8}$$

where s is the shock speed computed through the Rankine-Hugoniot jump condition:

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}\,. \tag{8.3.9}$$

The reader will immediately realize the geometric interpretation of (8.3.8) upon noticing that $f'(u_-)$ and $f'(u_+)$ are the slopes of the graph of f at the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$, while s is the slope of the chord that connects $(u_-, f(u_-))$ with $(u_+, f(u_+))$. In particular, when (7.1.2) is genuinely nonlinear, i.e., $f''(u) \neq 0$ for all u, then (8.3.8) reduces to $u_- < u_+$ if $f''(u) < 0$, and $u_- > u_+$ if $f''(u) > 0$.

Next we consider the system (7.1.11) of isentropic elasticity. The characteristic speeds are recorded in (7.2.4) and the shock speeds in (8.1.11), so that (8.3.7) assumes the form

(8.3.10)
$$\sigma'(u_+) \ge \frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-} \ge \sigma'(u_-) \quad \text{or} \quad \sigma'(u_+) \le \frac{\sigma(u_+) - \sigma(u_-)}{u_+ - u_-} \le \sigma'(u_-),$$

for 1-shocks or 2-shocks, respectively. The geometric interpretation of (8.3.10) is again clear. When (7.1.11) is genuinely nonlinear, i.e., $\sigma''(u) \neq 0$ for all u, (8.3.10) reduces to $u_- < u_+$ or $u_- > u_+$ if $\sigma''(u) > 0$, and to $u_- > u_+$ or $u_- < u_+$ if $\sigma''(u) < 0$. Equivalently, in terms of velocity, by virtue of (8.1.10): $v_- < v_+$ if $\sigma''(u) > 0$ and $v_- > v_+$ if $\sigma''(u) < 0$, for both shock families.

A similar analysis applies to the system (7.1.13) of isentropic flow of an ideal gas, with characteristic speeds given by (7.2.10), and yields that a 1-shock (or 2-shock) that joins the state (ρ_-, v_-), on the left, to the state (ρ_+, v_+), on the right, satisfies the Lax E-condition if and only if $\rho_- < \rho_+$ (or $\rho_- > \rho_+$). In other words, the passing of an admissible shock front compresses the gas.

Returning now to the general system (8.1.1), let us consider a state U_-, on the left, which is joined to a state U_+, on the right, by an i-shock of moderate strength, with speed s. Assuming $\lambda_i(U_-)$ is a simple eigenvalue of $\mathrm{D}F(U_-)$, let W_i denote the i-shock curve through U_- (cf. Theorem 8.2.1), so that $U_- = W_i(0)$ and $U_+ = W_i(\tau)$. Furthermore, $\lambda_i(U_-) = s_i(0)$ and $s = s_i(\tau)$. We show that if $\tau < 0$ and $\dot{s}_i(\cdot) \ge 0$ on $(\tau, 0)$, then the shock satisfies the Lax E-condition. Indeed, $\dot{s}_i(\cdot) \ge 0$ implies $s = s_i(\tau) \le s_i(0) = \lambda_i(U_-)$, which is the right half of (8.3.7). At the same time, so long as (8.2.27) and (8.2.28) hold at τ, $\dot{s}_i(\cdot) \ge 0$ implies, by virtue of Lemma 8.2.4, that $s = s_i(\tau) \ge \lambda_i(W_i(\tau)) = \lambda_i(U_+)$, namely, the left half of (8.3.7). A similar argument demonstrates that the Lax E-condition also holds when $\tau > 0$ and $\dot{s}_i(\cdot) \le 0$ on $(0, \tau)$, but it is violated if either $\tau < 0$ and $\dot{s}_i(\cdot) < 0$ on $(\tau, 0)$ or $\tau > 0$ and $\dot{s}_i(\cdot) > 0$ on $(0, \tau)$. The implications of the above statements to the genuine nonlinear case, in which, by virtue of (8.2.2), $\dot{s}_i(\cdot)$ does not change sign across 0, are recorded in the following

8.3.1 Theorem. *Assume U_- is a point of genuine nonlinearity of the i-characteristic family of the system* (8.1.1), *with* $\mathrm{D}\lambda_i(U_-)R_i(U_-) > 0$ (or < 0). *Suppose $\lambda_i(U_-)$ is a simple eigenvalue of* $\mathrm{D}F(U_-)$ *and let W_i denote the i-shock curve through U_-, with $U_- = W_i(0)$. Then a weak i-shock that joins U_- to a state $U_+ = W_i(\tau)$ satisfies the Lax E-condition if and only if $\tau < 0$ (or $\tau > 0$).*

Thus, in the genuinely nonlinear case, one half of the shock curve is compatible with the Lax E-condition (8.3.7), as strict inequalities, and the other half is incompatible with it. When U_- is a point of linear degeneracy of the i-characteristic field, so that $\dot{s}_i(0) = 0$, the situation is more delicate: If $\ddot{s}_i(0) < 0$, $\dot{s}_i(\tau)$ is positive for $\tau < 0$ and negative for $\tau > 0$, so that weak i-shocks that join U_- to $U_+ = W_i(\tau)$ are admissible, regardless of the sign of τ. On the other hand, if $\ddot{s}_i(0) > 0$, $\dot{s}_i(\tau)$ is negative for $\tau < 0$ and positive for $\tau > 0$, in which case all (sufficiently) weak i-shocks violate the Lax E-condition. As noted above, when the i-characteristic family itself is linearly degenerate, i-shocks are i-contact discontinuities satisfying (8.3.7) as equalities.

The relation of (8.3.5), (8.3.6) to stability, hinted by the heuristic analysis in the beginning of this section, is established by

8.3.2 Theorem. *Assume the system* (8.1.1) *is strictly hyperbolic. Consider initial data U_0 such that $U_0(x) = U_L(x)$ for x in $(-\infty,0)$ and $U_0(x) = U_R(x)$ for x in $(0,\infty)$, where U_L and U_R are smooth functions that are bounded, together with their first derivatives, on $(-\infty,\infty)$. Assume, further, that the state $U_- = U_L(0)$, on the left, is joined to the state $U_+ = U_R(0)$, on the right, by a compressive shock satisfying* (8.3.5) *and* (8.3.6). *Then there exist: $T > 0$; a smooth function $x = \chi(t)$ on $[0,T)$, with $\chi(0) = 0$; and a function U on $(-\infty,\infty)\times[0,T)$ with initial values U_0 and the following properties. U is smooth and satisfies* (8.1.1), *in the classical sense, for any (x,t), with $t \in [0,T)$ and $x \neq \chi(t)$. Furthermore, for $t \in [0,T)$ one-sided limits $U(\chi(t)-,t)$ and $U(\chi(t)+,t)$ exist and are joined by a compressive shock of speed $\dot{\chi}(t)$.*

The proof, which is found in the references cited in Section 8.8, employs pointwise bounds on U and its derivatives, obtained by monitoring the evolution of these functions along characteristics; i.e., it is of the same genre as the proof of Theorem 7.8.1. The role of (8.3.5) and (8.3.6) is to secure that characteristics of each family, originating either at the x-axis or at the graph of the shock, will reach every point of the upper half-plane. Compare with the discussion at the opening of this section. One may gain some insight from the very simple special case $n = 1$.

We thus consider the scalar conservation law (7.1.2) and assign initial data u_0 such that $u_0(x) = u_L(x)$ for $x \in (-\infty,0)$ and $u_0(x) = u_R(x)$ for $x \in (0,\infty)$, where u_L and u_R are bounded and uniformly Lipschitz continuous functions on $(-\infty,\infty)$. Furthermore, $u_- = u_L(0)$ and $u_+ = u_R(0)$ satisfy

$$f'(u_+) < \frac{f(u_+) - f(u_-)}{u_+ - u_-} < f'(u_-). \tag{8.3.11}$$

Let $u_-(x,t)$ and $u_+(x,t)$ be the classical solutions of (7.1.2) with initial data $u_L(x)$ and $u_R(x)$, respectively, which exist on $(-\infty,\infty) \times [0,T)$, for some $T > 0$, by virtue of Theorem 6.1.1. On $[0,T)$ we define the function χ as the solution of the ordinary differential equation

$$\frac{dx}{dt} = \frac{f(u_+(x,t)) - f(u_-(x,t))}{u_+(x,t) - u_-(x,t)} \tag{8.3.12}$$

with $\chi(0) = 0$. Finally, we define the function u on $(-\infty,\infty) \times [0,T)$ by

$$u(x,t) = \begin{cases} u_-(x,t), & t \in [0,T),\ x < \chi(t) \\ u_+(x,t), & t \in [0,T),\ x > \chi(t). \end{cases} \tag{8.3.13}$$

Clearly, u satisfies (7.1.2), in the classical sense, for any (x,t) with $t \in [0,T)$ and $x \neq \chi(t)$. Furthermore, $u(\chi(t)-,t)$ and $u(\chi(t)+,t)$ are joined by a shock of speed $\dot{\chi}(t)$. Finally, for T sufficiently small, the Lax E-condition

$$f'(u(\chi(t)+,t)) < \dot{\chi}(t) < f'(u(\chi(t)-,t)), \qquad t \in [0,T), \tag{8.3.14}$$

holds by continuity, since it is satisfied at $t = 0$. Notice that it is because of the Lax E-condition that the solution u solely depends on the initial data, i.e., it is independent of the “extraneous” information carried by $u_L(x)$ for $x > 0$ and $u_R(x)$ for $x < 0$.

As noted in Section 7.1, one-dimensional systems of conservation laws arise either in connection to media that are inherently one-dimensional or in the context of multispace-dimensional media wherein the fields stay constant in all but one spatial dimension. In the latter situation, Theorem 8.3.2 establishes the stability of planar shock fronts, albeit only for perturbations that likewise vary solely in the normal spatial direction. Naturally, it is important to investigate the stability of multispace-dimensional planar shocks under a broader class of perturbations and, more generally, the stability of non-planar shock fronts in $\mathbb{R}^m$.

The type of problem addressed by Theorem 8.3.2 may be formulated for hyperbolic systems (4.1.1) of conservation laws in $\mathbb{R}^m$ as follows. Let U_L, U_R be smooth functions on $\mathbb{R}^m$, and $\mathscr{F}$ a smooth $(m-1)$-dimensional hypersurface embedded in $\mathbb{R}^m$ and oriented by means of its unit normal vector field ν. Assume that the traces U_- and U_+ of U_L and U_R on $\mathscr{F}$ satisfy the Rankine-Hugoniot jump condition (4.3.5). Denote by U_0 the function on $\mathbb{R}^m$ that coincides with U_L on the negative side of $\mathscr{F}$ and with U_R on the positive side of $\mathscr{F}$. One has to construct an m-dimensional hypersurface $\mathscr{S}$ embedded in $\mathbb{R}^m \times [0,T)$, with trace $\mathscr{F}$ at $t = 0$, together with a piecewise smooth solution U of (4.1.1) on $\mathbb{R}^m \times [0,T)$, with initial values U_0, such that U is smooth for $(x,t) \notin \mathscr{S}$. Thus $\mathscr{S}$ will be a shock evolving out of $\mathscr{F}$.

The above problem has been solved under the following assumptions. The system (4.1.1) is endowed with a uniformly convex entropy and satisfies a certain structural condition, valid in particular for the Euler equations of isentropic or nonisentropic gas dynamics. At each point of $\mathscr{F}$, where the unit normal is ν, (8.3.5) and (8.3.6) hold, with $\lambda_j(\nu;U)$ and $R_j(\nu;U)$ in the place of $\lambda_j(U)$ and $R_j(U)$. Finally, a complicated set of compatibility conditions, involving the normal derivatives $\partial^p U_L/\partial\nu^p$ and $\partial^p U_R/\partial\nu^p$ of U_L and U_R, up to a certain order depending on m, is satisfied on $\mathscr{F}$. These are needed in order to avert the emission of spurious waves from $\mathscr{F}$.

The construction of $\mathscr{S}$ and U is performed within the framework of Sobolev spaces and involves quite sophisticated tools (pseudodifferential operators, paradifferential calculus, etc.). The relevant references are listed in Section 8.8.

Another serious issue of concern is the internal stability of shocks. It turns out that the Lax E-condition is effective in that direction as well, so long as the system is genuinely nonlinear and the shocks are weak; however, it is insufficient in more general situations. For that purpose, we have to consider additional, more selective shock admissibility criteria, which will be introduced in the following sections.

8.4 The Liu Shock Admissibility Criterion

The Liu shock admissibility test is more discriminating than the Lax E-condition and strives to capture the internal stability of shocks. By its very design, it makes sense only in the context of shocks joining states that may be connected by shock curves. Thus, its applicability to general systems is limited to shocks of moderate strength. Nevertheless, in special systems it also applies to strong shocks.

For a given state U_-, assume $\lambda_i(U_-)$ is a simple eigenvalue of $\mathrm{D}F(U_-)$ so that the i-shock curve $W_i(\tau;U_-)$ through U_- is well defined, by Theorem 8.2.1. An i-shock that joins U_-, on the left, to a state $U_+ = W_i(\tau_+;U_-)$, on the right, of speed s, satisfies the *Liu E-condition* if

$$(8.4.1) \qquad s = s_i(\tau_+;U_-) \leq s_i(\tau;U_-), \qquad \text{for all } \tau \text{ between } 0 \text{ and } \tau_+ .$$

The justification of the above admissibility criterion will be established a posteriori, through its connection to other, physically motivated, shock admissibility criteria, as well as by its role in the construction of stable solutions to the Riemann problem, in Chapter IX.

As U_- and U_+ are joined by an i-shock, U_- must also lie on the i-shock curve emanating from U_+, say $U_- = W_i(\tau_-;U_+)$. So long as (8.2.26) holds along the above shock curves, (8.4.1) is equivalent to the dual statement

$$(8.4.2) \qquad s = s_i(\tau_-;U_+) \geq s_i(\tau;U_+), \qquad \text{for all } \tau \text{ between } 0 \text{ and } \tau_- .$$

We proceed to verify that (8.4.1) implies (8.4.2) under the hypothesis that all minima of the function $s_i(\tau;U_-)$ are nondegenerate. (The general case may be reduced to the above by a perturbation argument, and the proof of the converse statement is similar.) For definiteness, we assume that $\tau_+ > 0$, in which case $\tau_- < 0$. From (8.4.1) it follows that either $\dot{s}_i(\tau_+;U_-) < 0$ or $\dot{s}_i(\tau_+;U_-) = 0$ and $\ddot{s}_i(\tau_+;U_-) > 0$. Thus, by virtue of (8.2.27), (8.2.28) and (8.2.29), either $\lambda_i(U_+) < s$ or $\lambda_i(U_+) = s$ and $\mathrm{D}\lambda_i(U_+)R_i(U_+) > 0$. In either case, recalling (8.2.1) and (8.2.2), we deduce that $s_i(\tau;U_+) < s$ for $\tau < 0$, near zero. If (8.4.2) is violated for some $\tau \in (\tau_-,0)$, $s_i(\tau;U_+) - s$ must be changing sign across some $\tau_0 \in (\tau_-,0)$. Let $U_0 = W_i(\tau_0;U_+)$. Since both U_- and U_0 are joined to U_+ by shocks of speed s, U_- and U_0 can also be joined to each other by a shock of speed s, i.e., $U_0 = W_i(\tau_1;U_-)$, for some $\tau_1 \in (0,\tau_+)$, and $s_i(\tau_1;U_-) = s$. Thus, (8.4.1) implies $\dot{s}_i(\tau_1;U_-) = 0$ and $\ddot{s}_i(\tau_1;U_-) > 0$, and so, on account of (8.2.28) and (8.2.29), $\lambda_i(U_0) = s$ and $\mathrm{D}\lambda_i(U_0)R_i(U_0) > 0$. But then (8.2.28) and (8.2.29) again yield $\dot{s}_i(\tau_0;U_+) = 0$ and $\ddot{s}_i(\tau_0;U_+) < 0$ so that, contrary to our hypothesis, $s_i(\tau;U_+) - s$ cannot change sign across τ_0.

In particular, applying (8.4.1) and (8.4.2) for $\tau = 0$ and recalling (8.2.1), we arrive at (8.3.1). We have thus established

8.4.1 Theorem. *Within the range where* (8.2.26) *holds, any shock satisfying the Liu E-condition also satisfies the Lax E-condition.*

When the system is genuinely nonlinear, these two criteria coincide, at least in the realm of weak shocks:

8.4.2 Theorem. *Assume the i-characteristic family is genuinely nonlinear and λ_i is a simple characteristic speed. Then weak i-shocks satisfy the Liu E-condition if and only if they satisfy the Lax E-condition.*

Proof. The Liu E-condition implies the Lax E-condition by Theorem 8.4.1. To show the converse, assume the state U_-, on the left, is joined to the state $U_+ = W_i(\tau_+;U_-)$, on the right, by a weak i-shock of speed s, which satisfies the Lax E-condition (8.3.1). Suppose, for definiteness, $\mathrm{D}\lambda_i(U_-)R_i(U_-) > 0$ (the case of the opposite sign is similar). By virtue of Theorem 8.3.1, $\tau_+ < 0$. Since the shock is weak, by Theorem 8.2.1, $\dot{s}_i(\tau;U_-) > 0$ on the interval $(\tau_+,0)$. Then $s = s_i(\tau_+;U_-) < s_i(\tau;U_-)$ for $\tau \in (\tau_+,0)$, i.e., the Liu E-condition holds. This completes the proof.

When the system is not genuinely nonlinear and/or the shocks are not weak, the Liu E-condition is stricter than the Lax E-condition. This will be demonstrated by means of the following examples.

Let us first consider the scalar conservation law (7.1.2). The shock curve is the u-axis and we may use u as the parameter ξ. The shock speed is given by (8.3.9). It is then clear that a shock joining the states u_- and u_+ will satisfy the Liu E-condition (8.4.1), (8.4.2) if and only if

$$\frac{f(u_+) - f(u_0)}{u_+ - u_0} \leq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \leq \frac{f(u_0) - f(u_-)}{u_0 - u_-} \tag{8.4.3}$$

holds for every u_0 between u_- and u_+. This is the celebrated *Oleinik E-condition*. It is easily memorized as a geometric statement: When $u_- < u_+$ (or $u_- > u_+$) the shock that joins u_-, on the left, to u_+, on the right, is admissible if the arc of the graph of f with endpoints $(u_-, f(u_-))$ and $(u_+, f(u_+))$ lies above (or below) the chord that connects the points $(u_-, f(u_-))$ and $(u_+, f(u_+))$. Letting u_0 converge to u_- and to u_+, we deduce that (8.4.3) implies (8.3.8). The converse, of course, is generally false, unless f is convex or concave. We have thus demonstrated that in the scalar conservation law the Liu E-condition is stricter than the Lax E-condition when f contains inflection points. In the genuinely nonlinear case, the Liu and Lax E-conditions are equivalent.

We now turn to the system (7.1.11) of isentropic elasticity. The shock curves are determined by (8.2.11) so we may use u as parameter instead of ξ. The shock speed is given by (8.1.11). Therefore, a shock joining the states (u_-, v_-) and (u_+, v_+) will satisfy the Liu E-condition (8.4.1), (8.4.2) if and only if

$$\frac{\sigma(u_0)-\sigma(u_-)}{u_0-u_-} \lesseqgtr \frac{\sigma(u_+)-\sigma(u_-)}{u_+-u_-} \lesseqgtr \frac{\sigma(u_+)-\sigma(u_0)}{u_+-u_0} \tag{8.4.4}$$

holds for all u_0 between u_- and u_+, where "$\leq$" applies for 1-shocks and "$\geq$" applies for 2-shocks. This is called the *Wendroff E-condition*. In geometric terms, it may be stated as follows: When $s(u_+ - u_-) < 0$ (or > 0) the shock that joins (u_-, v_-), on the left, to (u_+, v_+), on the right, is admissible if the arc of the graph of σ with endpoints $(u_-, \sigma(u_-))$ and $(u_+, \sigma(u_+))$ lies below (or above) the chord that connects the points $(u_-, \sigma(u_-))$ and $(u_+, \sigma(u_+))$. Clearly, there is close analogy with the Oleinik E-condition. Letting u_0 in (8.4.4) converge to u_- and to u_+, we deduce that the Wendroff E-condition implies the Lax E-condition (8.3.4). The converse is true when σ is convex or concave, but false otherwise. Thus, for the system (7.1.11) the Liu E-condition is stricter than the Lax E-condition when σ contains inflection points. In the genuinely nonlinear case, the Liu and Lax E-conditions are equivalent.

As we shall see, the Oleinik E-condition and the Wendroff E-condition follow naturally from other admissibility criteria. To a great extent these special E-conditions provided the motivation for postulating the general Liu E-condition.

8.5 The Entropy Shock Admissibility Criterion

The idea of employing entropy inequalities to weed out spurious weak solutions of general hyperbolic systems of conservation laws was introduced in Section 4.5 and was used repeatedly in Chapters IV, V, and VI. It was observed that in the context of BV weak solutions, the entropy condition reduces to the set of inequalities (4.5.9), to be tested at every point of the shock set. For the system (8.1.1), in one space dimension, (4.5.9) assumes the form

$$-s[\eta(U_+)-\eta(U_-)]+q(U_+)-q(U_-) \leq 0, \tag{8.5.1}$$

where (η, q) is an entropy-entropy flux pair satisfying (7.4.1), $\mathrm{D}q = \mathrm{D}\eta\mathrm{D}F$. The quantity on the left-hand side of (8.5.1) will be called henceforth the *entropy production across the shock*.

The fact that the entropy condition reduces to a pointwise test on shocks has played a dominant role in shaping the prevailing view that admissibility need be tested only at the level of shocks, i.e., that a general BV weak solution will be admissible if and only if each one of its shocks is admissible.

In setting up an entropy admissibility condition (8.5.1), the first task is to designate the appropriate entropy-entropy flux pair (η, q). Whenever (8.1.1) arises in connection to physics, the physically appropriate entropy should always be designated. In particular, the pairs (7.4.9), (7.4.10) and (7.4.11) must be designated for the systems (7.1.8), (7.1.11) and (7.1.13), respectively.[2]

In the absence of guidelines from physics, or when the entropy-entropy flux pair supplied by physics is inadequate to rule out all spurious shocks, additional entropy-entropy flux pairs must be designated (whenever available), motivated by other admissibility criteria, such as viscosity. In that connection, we should bear in mind that, as demonstrated in earlier chapters, convexity of the entropy function is a desirable feature.

Let us begin the investigation with the scalar conservation law (7.1.2). The shock speed s is given by (8.3.3). In accordance with the discussion in Chapter VI, admissible shocks must satisfy (8.5.1) for all convex functions η. However, as explained in Section 6.2, (8.5.1) need be tested only for the family (6.2.5) of entropy-entropy flux pairs, namely

$$\eta(u;\bar{u}) = (u-\bar{u})^+ , \qquad q(u;\bar{u}) = \mathrm{sgn}(u-\bar{u})^+[f(u)-f(\bar{u})]. \tag{8.5.2}$$

It is immediately seen that (8.5.1) will be satisfied for every (η, q) in the family (8.5.2) if and only if (8.4.3) holds for all u_0 between u_- and u_+. We have thus rederived the Oleinik E-condition encountered in Section 8.4. This implies that, for the scalar conservation law, the entropy admissibility condition, applied for all convex entropies, is equivalent to the Liu E-condition.

It is generally impossible to recover the Oleinik E-condition from the entropy condition (8.5.1) for a single entropy-entropy flux pair. Take for example

$$\eta(u) = \tfrac{1}{2}u^2 , \qquad q(u) = \int_0^u \omega f'(\omega) d\omega. \tag{8.5.3}$$

By virtue of (8.3.3) and after a short calculation, (8.5.1) takes the form

$$\tfrac{1}{2}[f(u_+)+f(u_-)](u_+-u_-) - \int_{u_-}^{u_+} f(\omega)d\omega \leq 0. \tag{8.5.4}$$

Notice that the entropy production across the shock is here measured by the signed area of the domain bordered by the arc of the graph of f with endpoints $(u_-, f(u_-))$, $(u_+, f(u_+))$, and the chord that connects $(u_-, f(u_-))$ with $(u_+, f(u_+))$. Clearly, the

[2] In applying (8.5.1) to the system (7.1.8), with entropy-entropy flux pair (7.4.9), one should not confuse s in (7.4.9), namely the physical entropy, with s in (8.5.1), the shock speed. Since $q = 0$, (8.5.1) here states that "after a shock passes, the physical entropy must increase." The reader is warned that this statement is occasionally misinterpreted as a general physical principle and is applied even when it is no longer appropriate.

Oleinik E-condition (8.4.3) implies (8.5.4) but the converse is generally false. Moreover, neither does (8.5.4) generally imply the Lax E-condition (8.3.8) nor is the converse true. However, when f is convex or concave, (8.5.4), (8.4.3) and (8.3.2) are all equivalent.

Next we turn to the system (7.1.11) of isentropic elasticity. We employ the entropy-entropy flux pair (η, q) given by (7.4.10). An interesting, rather lengthy, calculation, which involves the Rankine-Hugoniot condition (8.1.10), shows that (8.5.1) here reduces to

$$s\{\tfrac{1}{2}[\sigma(u_+)+\sigma(u_-)](u_+-u_-)-\int_{u_-}^{u_+}\sigma(\omega)d\omega\}\le 0. \tag{8.5.5}$$

The quantity in braces on the left-hand side of (8.5.5) measures the signed area of the set bordered by the arc of the graph of σ with endpoints $(u_-,\sigma(u_-))$, $(u_+,\sigma(u_+))$ and the chord that connects $(u_-,\sigma(u_-))$ with $(u_+,\sigma(u_+))$. Hence, the Wendroff E-condition (8.4.4) implies (8.5.5) but the converse is generally false. Condition (8.5.5) does not necessarily imply the Lax E-condition (8.3.10) nor is the converse valid. However, when σ is convex or concave, (8.5.5), (8.4.4) and (8.3.10) are all equivalent. Of course, the system (7.1.11) is endowed with a rich collection of entropies, so one may employ additional entropy-entropy flux pairs to recover the Wendroff E-condition from the entropy condition, but this shall not be attempted here.

We now consider the entropy shock admissibility condition (8.5.1) for a general system (8.1.1), under the assumption that U_- and U_+ are connected by a shock curve. In particular, this will encompass the case of shocks of moderate strength. We thus assume $\lambda_i(U_-)$ is a simple characteristic speed, consider the i-shock curve $W_i(\tau;U_-)$ through U_-, and let $U_+=W_i(\tau_+;U_-)$, $s=s_i(\tau_+;U_-)$. The entropy production along the i-shock curve is given by

$$E(\tau)=-s_i(\tau)[\eta(W_i(\tau))-\eta(U_-)]+q(W_i(\tau))-q(U_-)\,. \tag{8.5.6}$$

Differentiating (8.5.6) and using (7.4.1) yields

$$\dot E=-\dot s_i[\eta(W_i)-\eta(U_-)]-s_i\mathrm{D}\eta(W_i)\dot W_i+\mathrm{D}\eta(W_i)\mathrm{D}F(W_i)\dot W_i\,. \tag{8.5.7}$$

Combining (8.5.7) with (8.2.21) (for $\bar U=U_-$), we deduce

$$\dot E=-\dot s_i\{\eta(W_i)-\eta(U_-)-\mathrm{D}\eta(W_i)[W_i-U_-]\}\,. \tag{8.5.8}$$

Notice that the right-hand side of (8.5.8) is of quadratic order in the strength of the shock. Therefore, the entropy production $E(\tau_+)$ across the shock, namely the integral of $\dot E(\tau)$ from 0 to τ_+, is of cubic order in τ_+. We have thus established the following

8.5.1 Theorem. *The entropy production across a weak shock is of third order in the strength of the shock.*

When U_- is a point of linear degeneracy of the i-characteristic family, $\dot s_i(0;U_-)$ vanishes and so the entropy production across the shock will be of (at most) fourth order in the strength of the shock. In particular, when the i-characteristic family is linearly degenerate, $\dot{s}_i$ vanishes identically, by Theorem 8.2.5, and so

8.5.2 Theorem. *When the i-characteristic family is linearly degenerate, the entropy production across any i-shock (i-contact discontinuity) is zero.*

Turning now to the issue of admissibility of the shock, we observe that when η is a convex function, the expression in braces on the right-hand side of (8.5.8) is nonpositive. Thus $\dot{E}$ and $\dot{s}_i$ have the same sign. Consequently, the entropy admissibility condition $E(\tau_+) \leq 0$ will hold if $\tau_+ < 0$ and $\dot{s}_i \geq 0$ on $(\tau_+, 0)$, or if $\tau_+ > 0$ and $\dot{s}_i \leq 0$ on $(0, \tau_+)$; while it will be violated when either $\tau_+ < 0$ and $\dot{s}_i < 0$ on $(\tau_+, 0)$ or $\tau_+ > 0$ and $\dot{s}_i > 0$ on $(0, \tau_+)$. Recalling our discussion in Section 8.3, we conclude that the entropy admissibility condition and the Lax E-condition are equivalent in the range of τ, on either side of 0, where $\dot{s}_i(\tau)$ does not change sign. In particular, this will be the case when the characteristic family is genuinely nonlinear and the shocks are weak:

8.5.3 Theorem. *When the i-characteristic family is genuinely nonlinear and λ_i is a simple characteristic speed, the entropy admissibility condition and the Lax E-condition for weak i-shocks are equivalent.*

In order to escape from the realm of genuine nonlinearity and weak shocks, let us consider the condition

$$\tau \dot{W}_i^\top(\tau; U_-)\, \mathrm{D}^2\eta(W_i(\tau; U_-))\, [W_i(\tau; U_-) - U_-] \geq 0. \tag{8.5.9}$$

Recalling (7.4.3), (7.4.4) and Theorem 8.2.1, we conclude that when the entropy η is convex, (8.5.9) will always hold for weak i-shocks; it will also be satisfied for shocks of moderate strength when the i-shock curves extend into that regime; and may even hold for strong shocks, so long as $\dot{W}_i$ and $W_i - U_-$ keep pointing nearly in the direction of R_i.

8.5.4 Theorem. *Assume that the i-shock curve $W_i(\tau; U_-)$ through U_-, and corresponding shock speed function $s_i(\tau; U_-)$, are defined on an interval (α, β) containing* 0, *and satisfy* (8.5.9) *for $\tau \in (\alpha, \beta)$, where η is a convex entropy of the system. Then any i-shock joining U_-, on the left, to $U_+ = W_i(\tau_+; U_-)$, on the right, with speed s, that satisfies the Liu E-condition* (8.4.1), *also satisfies the entropy admissibility condition* (8.5.1).

Proof. We set

$$Q(\tau) = \eta(W_i(\tau; U_-)) - \eta(U_-) - \mathrm{D}\eta(W_i(\tau; U_-))\, [W_i(\tau; U_-) - U_-]. \tag{8.5.10}$$

By virtue of (8.5.9),

$$\tau \dot{Q}(\tau) \leq 0. \tag{8.5.11}$$

Integrating (8.5.8) from 0 to τ_+, integrating by parts, and using (8.5.10), (8.5.11) and (8.4.1), we obtain

$$
\begin{aligned}
(8.5.12)\qquad E(\tau_+) &= -\int_0^{\tau_+} \dot{s}_i(\tau;U_-)Q(\tau)d\tau = -sQ(\tau_+) + \int_0^{\tau_+} s_i(\tau;U_-)\dot{Q}(\tau)d\tau \\
&\le -sQ(\tau_+) + s\int_0^{\tau_+} \dot{Q}(\tau)d\tau = 0,
\end{aligned}
$$

which shows that the shock satisfies (8.5.1). This completes the proof.

In the realm of strong shocks, the above hierarchy of the various admissibility criteria may be violated. We have seen that for the system (7.1.11) of isentropic thermoelasticity, under the condition $\sigma''(u) < 0$ of genuine nonlinearity, compressive shocks of arbitrary strength satisfy the entropy admissibility criterion (8.5.5) induced by the Second Law of thermodynamics, as well as the Liu E-condition (8.4.4). We proceed to test whether this also applies to the system (7.1.8) of rectilinear adiabatic gas flow, under the assumption $p_{uu} > 0$ of genuine nonlinearity for the first and the third characteristic families.

Assume first that the constitutive equations satisfy $2\theta - u\theta_u > 0$ on their domain, as is the case with ideal gases (2.5.20). Then the initial value problem for the differential equation (8.2.13), which governs the shock curves, is well-posed. Consider states (u_-, v_-, s_-) and (u_+, v_+, s_+) joined by a 3-shock, propagating with speed $r_+ > 0$, that compresses the medium, i.e., $u_+ > u_-$. Thus $s_+ = s(u_+)$, where $s(u)$ denotes the solution of (8.2.13), for $\bar{u} = u_-$, with initial condition $s(u_-) = s_-$. From (8.2.13) and (8.2.15) we deduce that both ds/du and d^2s/du^2 vanish at u_-. Moreover, differentiating (8.2.15) with respect to u yields $d^3s/du^3 < 0$ at u_-. Therefore, $ds/du < 0$, at least for $u - u_-$ positive and small. The above are in accordance with Theorems 8.5.1, 8.5.3 and 8.5.4. We now note that, by virtue of (8.2.15), any critical point of $s(u)$, for $u > u_-$, must be a strict maximum. Hence, $ds/du < 0$ for all $u \in (u_-, u_+)$. In particular, this implies that the entropy shock admissibility criterion $s_+ < s_-$, induced by the entropy-entropy flux pair $(-s, 0)$ and expressing the Second Law of thermodynamics, is satisfied by the shock. Furthermore, it follows from (8.2.16) that the shock speed $r(u)$ is decreasing on (u_-, u_+), whence

$$
(8.5.13)\qquad \sqrt{-p_u(u_-, s_-)} = r(u_-) > r(u) > r(u_+) = r_+, \qquad u_- < u < u_+.
$$

Similarly, $s_- = s(u_-)$, where now $s(u)$ denotes the solution of (8.2.13), for $\bar{u} = u_+$, with initial condition $s(u_+) = s_+$. Similar arguments yield $\sqrt{-p_u(u_+, s_+)} < r_+$. We conclude that, under the assumptions $p_{uu} > 0$ and $2\theta - u\theta_u > 0$, 3-shocks of arbitrary strength that compress the medium satisfy the Lax E-condition, the Liu E-condition, and the entropy shock admissibility criterion. This also holds for compressive 1-shocks.

By contrast, it is possible to construct constitutive equations satisfying the thermodynamic relations (7.1.9) together with the hyperbolicity conditions (7.1.10), but allowing $2\theta - u\theta_u$ to change sign, for which (7.1.8) supports compressive shocks that satisfy the Liu E-condition but violate the Second Law of thermodynamics. Such conditions may arise when the gas undergoes phase transitions.

8.6 Viscous Shock Profiles

The idea of using the vanishing viscosity approach for identifying admissible weak solutions of hyperbolic systems of conservation laws was introduced in Section 4.6. In the present setting of one space dimension, for the system (8.1.1), Equation (4.6.1) reduces to

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu \partial_x [B(U(x,t)) \partial_x U(x,t)]. \tag{8.6.1}$$

As already explained in Section 4.6, the selection of the $n \times n$ matrix-valued function B may be suggested by the physical context of the system or it may just be an artifact of the analysis. Consider for example the dissipative systems

$$\partial_t u + \partial_x f(u) = \mu \partial_x^2 u, \tag{8.6.2}$$

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u) = \mu \partial_x (u^{-1} \partial_x v), \end{cases} \tag{8.6.3}$$

$$\begin{cases} \partial_t u + \partial_x [(u^2 + v^2) u] = \mu \partial_x^2 u \\ \partial_t v + \partial_x [(u^2 + v^2) v] = \mu \partial_x^2 v, \end{cases} \tag{8.6.4}$$

associated with the hyperbolic systems (7.1.2), (7.1.11), and (7.2.11). In so far as (7.1.11) is interpreted as the system of isentropic gas dynamics, the selection of viscosity in (8.6.3) is dictated by physics[3]. On the other hand, in (8.6.2) and (8.6.4) the viscosity is artificial.

In contrast to the entropy criterion, it is not at all clear that admissibility of weak solutions by means of the vanishing viscosity criterion is decided solely at the level of the shock set. However, taking that premise for granted, it will suffice to test admissibility in the context of solutions in the simple form

$$U(x,t) = \begin{cases} U_- , & x < st \\ U_+ , & x > st, \end{cases} \tag{8.6.5}$$

namely a shock of constant speed s joining the constant state U_-, on the left, to the constant state U_+, on the right. Presumably, functions (8.6.5) may be approximated, as $\mu \downarrow 0$, by a family of solutions U_μ of (8.6.1) in the form of *traveling waves*, namely functions of the single variable $x - st$. Taking advantage of the scaling in (8.6.1), we seek a family of solutions in the form

[3] Compare with (4.6.2). The variable viscosity coefficient μu^{-1} is adopted so that in the spatial setting, where measurements are usually performed, viscosity will be constant μ. Of course this will make sense only when $u > 0$.

(8.6.6)
$$U_\mu(x,t) = V\left(\frac{x-st}{\mu}\right).$$

Substituting in (8.6.1), we deduce that V should satisfy the ordinary differential equation

(8.6.7)
$$[B(V(\tau))\dot{V}(\tau)]^{\cdot} = \dot{F}(V(\tau)) - s\dot{V}(\tau),$$

where the overdot denotes differentiation with respect to $\tau = (x-ct)/\mu$. We are interested in solutions in which $\dot{V}$ vanishes at $V = U_-$ and so, upon integrating (8.6.7) once with respect to τ,

(8.6.8)
$$B(V)\dot{V} = F(V) - F(U_-) - s[V - U_-].$$

Notice that the right-hand side of (8.6.8) vanishes on the set of V that may be joined to U_- by a shock of speed s. This set includes, in particular, the state U_+. Thus both U_- and U_+ are equilibrium points of (8.6.8).

A *viscous shock profile* connecting the left state U_- and the right state U_+ of a shock is a smooth arc with endpoints U_- and U_+ that is an invariant set of the differential equation (8.6.8) and, in addition, at any nonequilibrium point on the arc the flow is directed from U_- to U_+.

The shock that joins U_-, on the left, to U_+, on the right, is said to satisfy the *viscous shock admissibility criterion* if U_- can be connected to U_+ by a viscous shock profile.

Determining viscous shock profiles is important not only because they shed light on the issue of admissibility but also because they provide information (at least when the matrix B is physically motivated) on the nature of the sharp transition modeled by the shock, the so-called *structure of the shock*. Indeed, the stretching of coordinates involved in (8.6.6), as $\mu \downarrow 0$, allows one, as it were, to observe the shock under the microscope.

In general, the viscous shock profile may contain a (finite or infinite) number of equilibrium points, with any two consecutive ones connected by orbits of (8.6.8). As an extreme case, notice that when the shock joining U_- and U_+ is a contact discontinuity associated with a linearly degenerate characteristic family, then, by virtue of Theorem 8.2.5, every point of the shock curve connecting U_- and U_+ is an equilibrium point of (8.6.8) and hence the shock curve itself serves as the viscous shock profile. We now consider the more typical situation in which U_- and U_+ are the only equilibrium points on the viscous shock profile, so that U_- is the α-limit set and U_+ is the ω-limit set of an orbit of (8.6.8).

For orientation, and in order to establish a connection with the discussion in Section 8.3, let us assume that the speed s of a shock joining U_-, on the left, with U_+, on the right, satisfies (8.3.1). For definiteness, let us choose the identity as viscosity matrix B, so that viscous shock profiles shall be orbits of the system

(8.6.9)
$$\dot{V} = F(V) - F(U_-) - s[V - U_-].$$

Recalling (8.1.2) and linearizing (8.6.9) about $U_\pm$ yields

(8.6.10) $$\dot{W} = [\mathrm{D}F(U_\pm) - sI]W.$$

The matrix $\mathrm{D}F(U_-) - sI$ has eigenvalues $\lambda_1(U_-) - s, \cdots, \lambda_n(U_-) - s$ of which, by (8.3.1), the first $i-1$ are negative and the remaining $n-i+1$ positive. Similarly, of the eigenvalues $\lambda_1(U_+) - s, \cdots, \lambda_n(U_+) - s$ of the matrix $\mathrm{D}F(U_+) - sI$ the first j are negative and the remaining $n-j$ are positive. It follows that U_- and U_+ are locally isolated equilibrium points of (8.6.9). Furthermore, on some neighborhood of U_-, any orbit of (8.6.9) whose α-limit set is U_- dwells on the unstable manifold $\mathscr{U}$, which has dimension $n-i+1$ and is tangent at U_- to the hyperplane spanned by the eigenvectors $R_i(U_-), \cdots, R_n(U_-)$. Similarly, on some neighborhood of U_+, any orbit of (8.6.9) whose ω-limit set is U_+ dwells on the stable manifold $\mathscr{S}$, which has dimension j and is tangent at U_+ to the hyperplane spanned by the eigenvectors $R_1(U_+), \cdots, R_j(U_+)$. We assume that the strength of the shock is commensurate to this local structure, so that viscous shock profiles are orbits lying on the intersection of $\mathscr{U}$ and $\mathscr{S}$.

For compressive shocks, i.e., $i = j$, $\dim \mathscr{U} + \dim \mathscr{S} = n+1$, so that $\mathscr{U}$ and $\mathscr{S}$ intersect transversely to form a curve which is the unique viscous shock profile connecting U_- and U_+. Furthermore, this profile is *structurally stable*, i.e., stable under small perturbations of U_-, U_+ and the flux function F. Figure 8.6.1(a) depicts a pattern of this type, for $n = 2$ and $i = j = 1$, in which the unstable node U_- is connected with the saddle U_+ by a unique, structurally stable viscous shock profile. Figure 8.6.2 depicts a configuration for $n = 3$ and $i = j = 2$, in which the two-dimensional unstable manifold $\mathscr{U}$ and the two-dimensional stable manifold $\mathscr{S}$ intersect transversely to form the unique, structurally stable viscous shock profile connecting U_- with U_+.

For overcompressive shocks, i.e., $i < j$, $\dim \mathscr{U} + \dim \mathscr{S} = n+1+j-i$, so that $\mathscr{U} \bigcap \mathscr{S}$ is a manifold of dimension $j-i+1 \geq 2$. Therefore, U_- and U_+ are connected by infinitely many viscous shock profiles. This configuration is also structurally stable. Figure 8.6.1(b) depicts a pattern of this type, for $n = 2$, $i = 1$ and $j = 2$, in which the unstable node U_- is connected with the stable node U_+ by infinitely many viscous shock profiles.

For undercompressive shocks, i.e., $i > j$, $\dim \mathscr{U} + \dim \mathscr{S} < n+1$, so that, generically, $\mathscr{U}$ and $\mathscr{S}$ do not intersect. Thus, generically, no viscous shock profile connecting U_- with U_+ exists; and in the nongeneric situation where it exists, it is structurally unstable. This is seen in Figure 8.6.1(c), depicting a pattern of this type, for $n = 2$, $i = 2$ and $j = 1$, in which a structurally unstable viscous shock profile connects the saddles U_- and U_+.

The situation described above, in which the existence of viscous shock profiles hinges on compressibility alone, arises when the shock is weak or when the system has special structure. In general, the existence of viscous shock profiles also depends on the internal structure of the shock. In order to gain insight on this issue, we first consider two simple examples, beginning with the case of the scalar conservation law (7.1.2) with corresponding dissipative equation (8.6.2). The system (8.6.8) now reduces to the scalar equation

(8.6.11) $$\dot{u} = f(u) - f(u_-) - s(u - u_-).$$

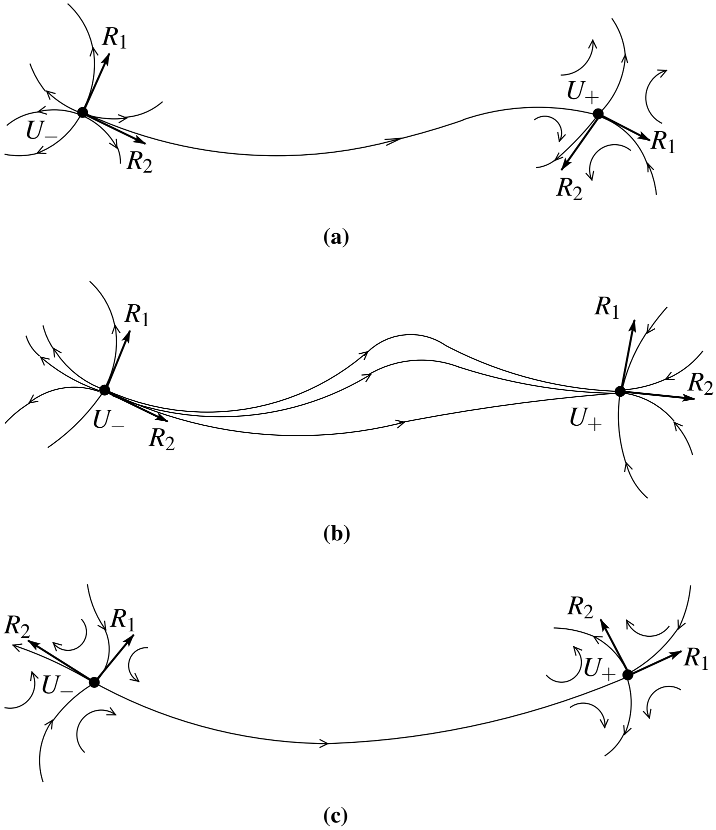

Fig. 8.6.1

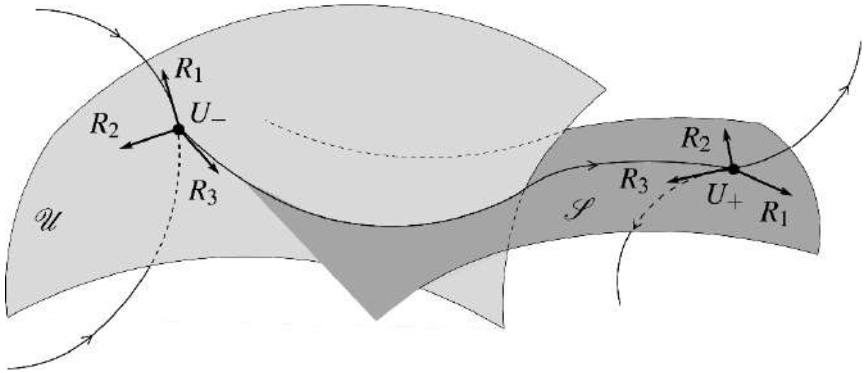

Fig. 8.6.2

It is clear that u_- will be connected to u_+ by a viscous shock profile if and only if the right-hand side of (8.6.11) does not change sign between u_- and u_+, and indeed it is nonnegative when $u_- < u_+$ and nonpositive when $u_- > u_+$. Recalling (8.3.3), we conclude that in the scalar conservation law (7.1.2), a shock satisfies the viscous shock admissibility criterion if and only if the Oleinik E-condition (8.4.3) holds. When (8.4.3) holds as a strict inequality for any u_0 (strictly) between u_- and u_+, then u_- is connected to u_+ with a single orbit. By contrast, when (8.4.3) becomes equality for a set of intermediate u_0, we need more than one orbit and perhaps even a number of contact discontinuities in order to build the viscous shock profile. In that case one may prefer to visualize the shock as a composite of several shocks and/or contact discontinuities, all traveling with the same speed.

Next we turn to the system (7.1.11) and the corresponding dissipative system (8.6.3). In that case (8.6.8) reads

$$\text{(8.6.12)} \qquad \begin{cases} 0 = -v + v_- - s(u - u_-) \\ \\ u^{-1}\dot{v} = -\sigma(u) + \sigma(u_-) - s(v - v_-). \end{cases}$$

The reason we end up here with a combination of algebraic and differential equations, rather than just differential equations, is that B is a singular matrix. In any event, upon eliminating v between the two equations in (8.6.12), we deduce

$$\text{(8.6.13)} \qquad su^{-1}\dot{u} = \sigma(u) - \sigma(u_-) - s^2(u - u_-).$$

Since $u > 0$, (u_-, v_-) will be connected to (u_+, v_+) by a viscous shock profile if and only if the right-hand side of (8.6.13) does not change sign between u_- and u_+ and is in fact nonnegative when $s(u_+ - u_-) > 0$ and nonpositive when $s(u_+ - u_-) < 0$. In view of (8.1.13), we conclude that in the system (7.1.11) of isentropic elasticity a shock satisfies the viscous shock admissibility criterion if and only if the Wendroff E-condition (8.4.4) holds.

It was the Oleinik E-condition and the Wendroff E-condition, originally derived through the above argument, that motivated the general Liu E-condition. We now proceed to show that the viscous shock admissibility criterion is generally equivalent to the Liu E-condition, at least in the range of shocks of moderate strength. For simplicity, only the special case $B = I$ will be discussed here; the case of more general B is treated in the references cited in Section 8.8.

8.6.1 Theorem. *Assume λ_i is a simple eigenvalue of $\mathrm{D}F$. Then an i-shock of moderate strength satisfies the viscous shock admissibility criterion, with $B = I$, if and only if it satisfies the Liu E-condition.*

Proof. Assume the state U_-, on the left, is joined to the state U_+, on the right, by an i-shock of moderate strength and speed s. In order to apply the viscous shock admissibility test, the first task is to construct a curve in state space that connects U_+ with U_- and is invariant under the flow generated by (8.6.9). To that end, we embed (8.6.9) into a larger, autonomous, system by introducing a new (scalar) variable r:

$$
\begin{cases} \dot{V} = F(V) - F(U_-) - r[V - U_-] \\ \\ \dot{r} = 0. \end{cases} \tag{8.6.14}
$$

Notice that the Jacobian matrix of the right-hand side of (8.6.14), evaluated at the equilibrium point $V = U_-$, $r = \lambda_i(U_-)$, is

$$
J = \left(\begin{array}{c|c} \mathrm{D}F(U_-) - \lambda_i(U_-)I & 0 \\ \hline 0 & 0 \end{array} \right), \tag{8.6.15}
$$

with eigenvalues $\lambda_j(U_-) - \lambda_i(U_-)$, $j = 1, \cdots, n$, and 0; the corresponding eigenvectors being

$$
\begin{pmatrix} R_j(U_-) \\ 0 \end{pmatrix}, \quad j = 1, \cdots, n, \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.6.16}
$$

We see that J has two zero eigenvalues, associated with a two-dimensional eigenspace, while the remaining eigenvalues are nonzero real numbers. The center manifold theorem then implies that any trajectory of (8.6.14) that is confined in a small neighborhood of the point $(U_-, \lambda_i(U_-))$ must lie on a two-dimensional manifold $\mathscr{M}$, which is invariant under the flow generated by (8.6.14), and may be parametrized by

$$
V = \Phi(\zeta, r) = U_- + \zeta R_i(U_-) + S(\zeta, r), \qquad r = r, \tag{8.6.17}
$$

with

$$
S(0, \lambda_i(U_-)) = 0, \quad S_\zeta(0, \lambda_i(U_-)) = 0, \quad S_r(0, \lambda_i(U_-)) = 0. \tag{8.6.18}
$$

In particular, the equilibrium point (U_+, s) of (8.6.14) must lie on $\mathscr{M}$, in which case $U_+ = \Phi(\rho, s)$, for some ρ near zero. Thus U_- and U_+ are connected by the smooth curve $V = \Phi(\zeta, s)$, for ζ between 0 and ρ, and this curve is invariant under the flow generated by (8.6.9).

The flow induced by $(8.6.14)_1$ along the invariant curve $V = \Phi(\cdot, r)$ is represented by a function $\zeta = \zeta(\cdot)$ which satisfies the scalar ordinary differential equation

$$
\dot{\zeta} = g(\zeta, r), \tag{8.6.19}
$$

with g defined through

$$
g(\zeta, r)\Phi_\zeta(\zeta, r) = F(\Phi(\zeta, r)) - F(U_-) - r[\Phi(\zeta, r) - U_-]. \tag{8.6.20}
$$

In particular, recalling (8.6.17) and (8.6.18),

$$
g(0, r) = 0, \qquad g_\zeta(0, r) = \lambda_i(U_-) - r. \tag{8.6.21}
$$

Clearly, the viscous shock admissibility criterion will be satisfied if and only if $\rho g(\zeta, s) \geq 0$ for all ζ between 0 and ρ.

Suppose now the shock satisfies the Liu E-condition. Thus, if W_i denotes the i-shock curve through U_- and s_i is the corresponding shock speed function, so that $U_- = W_i(0)$, $U_+ = W_i(\tau_+)$, $s = s_i(\tau_+)$, we must have $s_i(\tau) \geq s$ for τ between 0 and τ_+. For definiteness, let us assume $U_+ - U_-$ points nearly in the direction of $R_i(U_-)$, in which case both ρ and τ_+ are positive.

We fix $r < s$, with $s - r$ very small, consider the curve $\Phi(\cdot, r)$ and identify $\kappa > 0$ such that $[\Phi(\kappa, r) - U_+]^\top R_i(U_-) = 0$. We show that $g(\zeta, r) > 0$, $0 < \zeta < \kappa$. Indeed, if $g(\zeta, r) = 0$ for some ζ, $0 < \zeta < \kappa$, then, by virtue of (8.6.20), the state $\Phi(\zeta, r)$ may be joined to the state U_- by a shock of speed r. Thus, $\Phi(\zeta, r)$ lies on the shock curve W_i, say $\Phi(\zeta, r) = W_i(\tau)$, for some τ. By the construction of κ, since $0 < \zeta < \kappa$, it is necessarily $0 < \tau < \tau_+$. However, in that case it is $r = s_i(\tau) \geq s$, namely, a contradiction to our assumption $r < s$. This establishes that $g(\zeta, r)$ does not change sign on $(0, \kappa)$. At the same time, recalling (8.6.21), $g_\zeta(0, r) = s_i(0) - r \geq s - r > 0$, which shows that $g(\zeta, r) > 0$, $0 < \zeta < \kappa$. Finally, we let $r \uparrow s$, in which case $\kappa \to \rho$. Hence $g(\zeta, s) \geq 0$ for $\zeta \in (0, \rho)$.

By a similar argument one shows the converse, namely that $\rho g(\zeta, s) \geq 0$, for ζ between 0 and ρ, implies $s_i(\tau) \geq s$, for τ between 0 and τ_+. This completes the proof.

Combining Theorems 8.4.1, 8.4.2 and 8.6.1, we conclude that the viscous shock admissibility criterion generally implies the Lax E-condition but the converse is generally false, unless the system is genuinely nonlinear and the shocks are weak.

It should be emphasized that the equivalence between the viscous shock admissibility criterion and the Liu E-condition generally hinges on the choice of the viscosity matrix B. To see this, consider the simple system

$$\partial_t (u, v, w)^\top + \partial_x (u^2, v^2, w^2)^\top = 0, \tag{8.6.22}$$

which consists of three uncoupled copies of the Burgers equation. The undercompressive shock joining the state $(-3, 7, -1)^\top$, on the left, with the state $(5, -5, 3)^\top$, on the right, and propagating with speed $s = 2$, violates the Lax E-condition, the Liu E-condition and also the viscous shock admissibility criterion when the viscosity matrix is $B = I$. However, this shock does satisfy the viscous shock admissibility condition for the symmetric positive definite viscosity matrix

$$B = \begin{pmatrix} 9 & 8 & 2 \\ 8 & 9 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \tag{8.6.23}$$

Indeed, the corresponding viscous shock profile is given by

$$V(\tau) = (1, 1, 1)^\top + \tanh(2\tau)(4, -6, 2)^\top, \qquad -\infty < \tau < \infty. \tag{8.6.24}$$

Our next task is to compare the viscous shock admissibility criterion with the entropy shock admissibility criterion. We thus assume that the system (8.1.1) is equipped with an entropy-entropy flux pair (η, q), satisfying (7.4.1), $\mathrm{D}q = \mathrm{D}\eta\mathrm{D}F$. The natural compatibility condition between the entropy and the viscosity matrix B was already discussed in Section 4.6. We write (a weaker form of) the condition (4.6.7) in the present, one-dimensional setting:

$$H^\top \mathrm{D}^2\eta(U)B(U)H \geq 0, \quad H \in \mathbb{R}^n, \quad U \in \mathscr{O}. \tag{8.6.25}$$

As already noted in Section 4.6, when $B = I$, (8.6.25) will hold if and only if η is convex.

8.6.2 Theorem. *When* (8.6.25) *holds, any shock that satisfies the viscous shock admissibility criterion also satisfies the entropy shock admissibility criterion.*

Proof. Consider a shock of speed s that joins the state U_-, on the left, with the state U_+, on the right, and satisfies the viscous shock admissibility condition.

Assume first U_- is connected to U_+ with a single orbit of (8.6.8), i.e., there is a function V which satisfies (8.6.8), and thereby also (8.6.7), on $(-\infty,\infty)$, together with the conditions $V(\tau) \to U_\pm$, as $\tau \to \pm\infty$. We multiply (8.6.7), from the left, by $\mathrm{D}\eta(V(\tau))$ and use (7.4.1) to get

$$[\mathrm{D}\eta(V)B(V)\dot{V}]^{\cdot} - \dot{V}^\top \mathrm{D}^2\eta(V)B(V)\dot{V} = \dot{q}(V) - s\dot{\eta}(V). \tag{8.6.26}$$

Integrating (8.6.26) over $(-\infty,\infty)$ and using (8.6.25), we arrive at (8.5.1). We have thus proved that the shock satisfies the entropy condition.

In the general case where the viscous shock profile contains intermediate equilibrium points, we realize the shock as a composite of a (finite or infinite) number of simple shocks of the above type and/or contact discontinuities, all propagating with the same speed s. As shown above, the entropy production across each simple shock is nonpositive. On the other hand, by Theorem 8.5.2, the entropy production across any contact discontinuity will be zero. Therefore, combining the partial entropy productions, we conclude that the total entropy production (8.5.1) is nonpositive. This completes the proof.

The converse of Theorem 8.6.2 is generally false. Consider for example the system (7.1.11) of isentropic elasticity, with corresponding dissipative system (8.6.3) and entropy-entropy flux pair (7.4.10), which satisfy the compatibility condition (8.6.25). As shown in Section 8.5, the entropy shock admissibility criterion is tested through the inequality (8.5.5), which follows from, but does not generally imply, the Wendroff E-condition (8.4.4).

To be robust, the viscosity approach should generate the shock (8.6.5) as the $\mu \downarrow 0$ limit of solutions of (8.6.1) that are perturbations of the associated viscous shock profile (8.6.6). Passing to the $\mu \downarrow 0$ limit, for fixed t, is equivalent, by rescaling coordinates, to passing to the $t \to \infty$ limit, for fixed μ. One may thus plausibly argue that viscous shock profiles employed to test the admissibility of shocks must derive from traveling wave solutions of the system (8.6.1) that are asymptotically stable. This issue has been investigated thoroughly in recent years and a complete theory has emerged, warranting the writing of a monograph on the subject. A detailed presentation would lie beyond the scope of the present book, so only the highlights shall be reported here. For details and proofs the reader may consult the references cited in Section 8.8.

For simplicity, we limit our discussion to viscosity matrix $B = I$ and normalize (8.6.1) by setting $\mu = 1$. We consider a weak i-shock, joining the states U_-, on the left, and U_+, on the right, which admits a viscous shock profile V. A change of variable $x \mapsto x + st$ renders the shock stationary. The viscous shock profile V is called *asymptotically stable* if the solution $U(x,t)$ of (8.6.1) with initial values $U(x,0) = V(x) + U_0(x)$, where U_0 is a "small" perturbation decaying at $\pm\infty$, satisfies

$$U(x,t) \to V(x+h), \quad \text{as } t \to \infty, \tag{8.6.27}$$

for some appropriate phase shift $h \in \mathbb{R}$.

Motivated by the observation that the total mass of solutions of (8.6.1) is conserved, it seems natural to require that the convergence in (8.6.27) be in $L^1(-\infty,\infty)$. In particular, this would imply that $V(x+h)$ carries the excess mass introduced by the perturbation:

$$\int_{-\infty}^{\infty} U_0(x)dx = \int_{-\infty}^{\infty} [V(x+h) - V(x)]dx = h[U_+ - U_-]. \tag{8.6.28}$$

In the scalar case, $n = 1$, any viscous shock profile is asymptotically stable in $L^1(-\infty,\infty)$, under arbitrary perturbations $U_0 \in L^1(-\infty,\infty)$, with h determined through (8.6.28).

For systems, $n \geq 2$, the single scalar parameter h is generally inadequate to balance the vectorial equation (8.6.28), in which case (8.6.27) cannot hold in $L^1(-\infty,\infty)$, as no h-translate of V alone may carry the excess mass. Insightful analysis of the asymptotics of (8.6.1) suggests that, for large t, the solution U should develop a viscous shock profile accompanied by a family of so-called diffusion waves, which share the burden of carrying the mass:

$$U(x,t) \sim V(x+h) + W(x,t) + \sum_{j<i} \theta_j(x,t) R_j(U_-) + \sum_{j>i} \theta_j(x,t) R_j(U_+). \tag{8.6.29}$$

The j-term in the summation on the right-hand side of (8.6.29) represents a *decoupled j-diffusion wave*. The scalar function θ_j is a self-similar solution,

$$\theta_j(x,t) = \frac{1}{\sqrt{t}} \phi_j\left(\frac{x - \lambda_j t}{\sqrt{t}}\right), \tag{8.6.30}$$

of the nonlinear diffusion equation

$$\partial_t \theta_j + \partial_x \left[\lambda_j \theta_j + \tfrac{1}{2}(\mathrm{D}\lambda_j R_j)\theta_j^2\right] = \partial_x^2 \theta_j. \tag{8.6.31}$$

In (8.6.30) and (8.6.31), λ_j, $\mathrm{D}\lambda_j$ and R_j are evaluated at U_-, for $j = 1, \cdots, i-1$, or at U_+, for $j = i+1, \cdots, n$. Thus the j-diffusion wave has a bell-shaped profile which propagates at characteristic speed λ_j; its peak decays like $O(t^{-\frac{1}{2}})$, while its mass stays constant, say m_jR_j. The remaining term W on the right-hand side of (8.6.29) represents the *coupled diffusion wave*, which satisfies a complicated linear diffusion equation and decays at the same rate as the uncoupled diffusion waves, but carries no mass. Therefore, mass conservation as $t \to \infty$ yields, in lieu of (8.6.28), the equation

$$\int_{-\infty}^{\infty} U_0(x)dx = \sum_{j<i} m_jR_j(U_-) + h[U_+ - U_-] + \sum_{j>i} m_jR_j(U_+), \tag{8.6.32}$$

which dictates how the excess mass is distributed among the viscous shock profile and the decoupled diffusion waves. Since $U_+ - U_-$ and $R_i(U_\pm)$ are nearly collinear, (8.6.32) determines explicitly and uniquely the phase shift h of the viscous shock profile as well as the masses m_j of the j-diffusion waves.

It has been established that the viscous shock profile V is asymptotically stable (8.6.27) in $L^\infty(-\infty,\infty)$, for the h determined through (8.6.32), under any perturbation $U_0 \in H^1(-\infty,\infty)$ of V with

$$\int_{-\infty}^{\infty} (1+x^2)|U_0(x)|^2 dx << 1, \tag{8.6.33}$$

provided only that the eigenvalue λ_i is simple and the shock satisfies the strict form of the Lax E-condition. It should be noted that this assertion holds even when the i-characteristic family fails to be genuinely nonlinear.

The orderly structure depicted above disintegrates when dealing with overcompressive or undercompressive shocks, and occasionally even with strong compressive shocks. In order to catch a glimpse of the geometric complexity that may arise in such cases, let us discuss the construction of viscous shock profiles for 2-shocks of the simple system (7.2.11), with dissipative form (8.6.4). The properties of shocks were already discussed in Section 8.3. Taking advantage of symmetry under rotations and scaling properties of the system, we may fix, without loss of generality, the left state (u_-,v_-) at the point $(1,0)$. The right state (u_+,v_+) will be located at a point $(a,0)$, with $a \in (-\frac{1}{2},0)$. In that case, as shown in Section 8.3, the shock speed is $s = a^2+a+1$ and the shock is overcompressive (8.3.7). Notice that the state $(b,0)$, where $b=-1-a$, is also joined to $(1,0)$ by a 2-shock of the same speed s, which satisfies the Lax E-condition and is not overcompressive, but does not satisfy the Liu E-condition.

The system (8.6.8) associated with (8.6.4) reads:

$$\begin{cases} \dot{u} = -s(u-1) + u(u^2+v^2) - 1 \\ \dot{v} = -sv + v(u^2+v^2); \end{cases} \tag{8.6.34}$$

or, equivalently, in polar coordinates (ρ,θ), $u = \rho\cos\theta$, $v = \rho\sin\theta$:

$$\begin{cases} \dot{\rho} = \rho(\rho^2 - s) + (s-1)\cos\theta \\ \rho\dot{\theta} = -(s-1)\sin\theta. \end{cases} \tag{8.6.35}$$

Notice that (8.6.34) possesses three equilibrium points: (a) $(1,0)$ which is an unstable node; (b) $(a,0)$ which is a stable node; and (c) $(b,0)$ which is a saddle. The phase portrait, which may be easily determined through elementary analysis of (8.6.34) and (8.6.35), is depicted in Fig. 8.6.3.

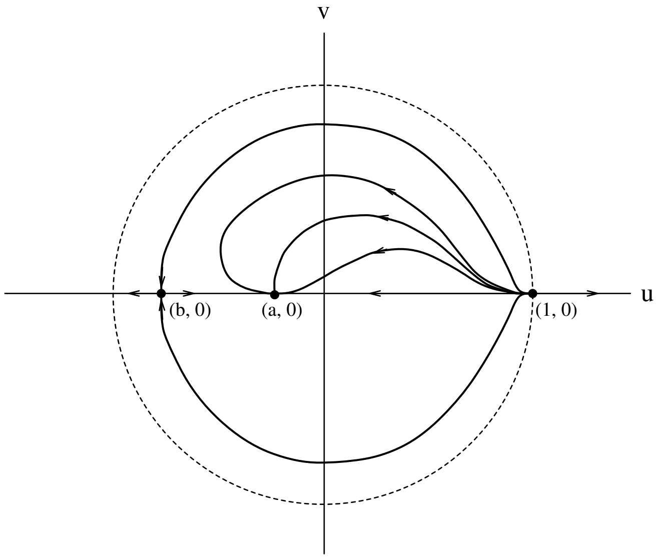

Fig. 8.6.3

Even though the shock joining $(1,0)$ to $(b,0)$ violates the Liu E-condition, these states are connected by two viscous shock profiles, symmetric with respect to the u-axis. By contrast, the states $(1,0)$ and $(a,0)$ are connected by infinitely many viscous shock profiles. To test the asymptotic stability of any one of these viscous shock profiles, say $(\bar{u}(\tau),\bar{v}(\tau))$, in the light of our discussion above, we introduce a small perturbation $(u_0(x),v_0(x))$ and inquire whether the solution $(u,v)(x,t)$ of (8.6.4) with initial values

$$(u,v)(x,0) = \left(\bar{u}(\frac{x}{\mu}) + u_0(x),\ \bar{v}(\frac{x}{\mu}) + v_0(x)\right) \tag{8.6.36}$$

satisfies

$$(u,v)(x,t) \to \left(\hat{u}\left(\frac{x-st}{\mu}\right),\ \hat{v}\left(\frac{x-st}{\mu}\right)\right), \qquad \text{as } t \to \infty, \tag{8.6.37}$$

where $(\hat{u}(\tau),\hat{v}(\tau))$ is a (generally different) viscous shock profile. Because no diffusion waves are possible here, the convergence in (8.6.37) must be in $L^1(-\infty,\infty)$. In particular, the v-component of the excess mass conservation yields

(8.6.38)
$$\int_{-\infty}^{\infty} v_0(x)dx = \int_{-\infty}^{\infty}\left[\hat{v}\left(\frac{x-st}{\mu}\right) - \bar{v}\left(\frac{x-st}{\mu}\right)\right]dx = \mu\int_{-\infty}^{\infty}[\hat{v}(\tau)-\bar{v}(\tau)]\,d\tau.$$

It can be shown that the integral on the right-hand side of (8.6.38) is uniformly bounded, independently of the choice of $\bar{v}$ and $\hat{v}$. Consequently, when v_0 is fixed so that $\int v_0 dx \neq 0$, (8.6.38) cannot hold when μ is sufficiently small. Thus, insofar as shock admissibility hinges on stability of the connecting shock profiles, the overcompressive shocks of the system (7.2.11) should be termed inadmissible.

A very technical theory of linear stability for multi-space-dimensional viscous shock profiles has emerged in recent years, paralleling the corresponding theory for multi-space-dimensional shocks, briefly outlined at the end of Section 8.3. Expositions are found in the references cited in Section 8.8.

As pointed out in Section 4.6, in certain cases the physically relevant admissibility condition is provided not by the viscosity criterion but by the viscosity-capillarity criterion, in which (8.6.1) is replaced by

(8.6.39)
$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu\partial_x[B(U(x,t))\partial_x U(x,t)] + \nu\partial_x[H(U(x,t))\partial_x^2 U(x,t)].$$

In general, diffusion is dominant when $\nu = o(\mu^2)$, while dispersion prevails if $\mu = o(\sqrt{\nu})$. The two effects are balanced when $\nu = \mu^2$. In that case, shock profiles are governed by the ordinary differential equation

$$H(V)\ddot{V} + B(V)\dot{V} = F(V) - F(U_-) - s[V - U_-], \tag{8.6.40}$$

replacing (8.6.8). A theory of these profiles is gradually emerging in the literature.

8.7 Nonconservative Shocks

In continuum physics one occasionally encounters quasilinear hyperbolic systems

$$\partial_t U(x,t) + A(U(x,t))\partial_x U(x,t) = 0 \tag{8.7.1}$$

that are not in conservative form. In that case, it is not possible to characterize weak solutions within the setting of the theory of distributions. It is still possible, however, to introduce a notion of weak solution within the class BV of functions of bounded variation by postulating jump conditions that play the role of the Rankine-Hugoniot jump condition (8.1.2) at the points of approximate jump discontinuity.

Appropriate jump conditions can be motivated by prior information on shock profiles, deriving from the vanishing viscosity approach, the vanishing viscosity-capillarity argument, or from (so-called) *kinetic relations* in the theory of phase transitions. The formulation of this theory proceeds along the following lines.

To (8.7.1) one links a function $V(\tau; U_-, U_+)$, defined on $(-\infty,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$ and taking values in $\mathbb{R}^n$, which has the following properties:

$$(8.7.2) \qquad V(-\infty;U_-,U_+) = U_-\,, \qquad V(\infty;U_-,U_+) = U_+\,,$$

$$(8.7.3) \qquad V(\tau;U,U) = U,$$

$$(8.7.4) \qquad |\partial_\tau V(\tau;U_-,U_+) - \partial_\tau V(\tau;\bar{U}_-,\bar{U}_+)| \leq a|(U_+ - \bar{U}_+) - (U_- - \bar{U}_-)|,$$

for all U, U_-, U_+, $\bar{U}_-$, $\bar{U}_+$ in $\mathbb{R}^n$, any $\tau \in (-\infty,\infty)$, and some $a > 0$. One then requires that a shock joining the state U_-, on the left, with the state U_+, on the right, and propagating with speed s must satisfy the jump condition

$$(8.7.5) \qquad \int_{-\infty}^{\infty} A(V(\tau;U_-,U_+))\partial_\tau V(\tau;U_-,U_+)d\tau = s[U_+ - U_-].$$

In the above setting, $V(\cdot;U_-,U_+)$ represents the shock profile. Notice that in the conservative case, $A(U) = \mathrm{D}F(U)$, (8.7.5) reduces to the Rankine-Hugoniot jump condition (8.1.2), regardless of the particular choice of V.

The literature cited in Section 8.8 explains how the above device naturally leads to a notion of weak solution to (8.7.1), within the framework of BV functions.

8.8 Notes

The study of shock waves originated in the context of gas dynamics. The book by Courant and Friedrichs [1], already cited in Chapter III, presented a coherent, mathematical exposition of material from the physical and engineering literature, accumulated over the past 150 years, paving the way for the development of a general theory by Lax [2].

For as long as gas dynamics remained the prototypical paradigm, the focus of the research effort was set on strictly hyperbolic, genuinely nonlinear systems. The intricacy of shock patterns in nonstrictly hyperbolic systems was not recognized until recently, and this subject is currently undergoing active development.

Expositions of many of the topics covered in this chapter are also contained in the books of Smoller [3] and Serre [11].

The notion of Hugoniot locus in gas dynamics may be traced back to the work of Riemann [1]; but the definition of shock curves in the general setting is due to Lax [2], who first established the properties stated in Theorems 8.2.1, 8.2.2 and 8.2.3. The elegant proof of Theorem 8.2.1 is here taken from Serre [11]. The significance of systems with coinciding shock and rarefaction wave curves was first recognized by Temple [3], who conducted a thorough study of their noteworthy properties. A detailed discussion is also contained in Serre [11].

For gas dynamics, the statement that admissible shocks should be subsonic relative to their front state and supersonic relative to their back state is found in the pioneering paper of Riemann [1]. This principle was postulated as a general shock admissibility criterion, namely the Lax E-condition, by Lax [2], who also proved

Theorem 8.3.1. A proof of Theorem 8.3.2 is given in Ta-tsien Li and Wen-ci Yu [1]. See also Hsiao and Chang [1]. A different connection between the Lax E-condition and stability is established in Smoller, Temple and Xin [1].

Shock admissibility in the absence of genuine nonlinearity was first discussed by Bethe [1] and Weyl [1], for the system of gas dynamics. The Liu E-condition and related Theorems 8.4.1, 8.4.2 and 8.5.4 are due to Tai-Ping Liu [2]. The motivation was provided by the Oleinik E-condition, derived in Oleinik [4], and the Wendroff E-condition, established in Wendroff [1]. This admissibility criterion was apparently anticipated in the 1960's by Chang and Hsiao [1,2] (see also Hsiao and Zhang [1]) but their work was not published until much later.

The entropy shock admissibility condition has been part of the basic theory of continuum thermomechanics for over a century (see the historical introduction). The form (8.5.1), for general systems (8.1.1), was postulated by Lax [4], who established Theorems 8.5.1, 8.5.3 and 8.6.2. The proofs of Theorems 8.5.1, 8.5.2, 8.5.3 and 8.5.4, based on Equation (8.5.8), are here taken from Dafermos [11]. Stricter versions of the entropy admissibility criterion that are equivalent to the Liu E-condition have been proposed by Dafermos (see Section 9.7) and by Liu and Ruggeri [1].

The relevance of the condition $2\theta - u\theta_u > 0$ to the equations of state in the system (7.1.8) of gas dynamics was originally recognized by Bethe [1]. Dafermos [30] shows that when Bethe's condition fails, as may occur in physical gases undergoing phase transitions, the Liu E-condition no longer guarantees that strong shocks will satisfy the Second Law of thermodynamics.

As we saw in the historical introduction, the notion of viscous shock profile was introduced to gas dynamics by Rayleigh [4] and G.I. Taylor [1], and then developed by Becker [1], Weyl [1], and Gilbarg [1]. The general form (8.6.8), for systems (8.1.1), was first written down by Gelfand [1].

Theorem 8.6.1 is due to Majda and Pego [1]. See also Conlon [2]. An earlier paper by Foy [1] had established the result in the special case where the system is genuinely nonlinear and the shocks are weak. Also Mock [1] has proved a similar result under the assumption that the system is genuinely nonlinear and it is endowed with a convex entropy. The issue of characterizing appropriate viscosity matrices B has been discussed by several authors, including Conley and Smoller [1,3], Majda and Pego [1], and Serre [11]. See also Bianchini and Spinolo [1].

For a detailed study of viscous shock profiles in isentropic (or isothermal) elastodynamics, under physically natural assumptions, see Antman and Malek-Madani [1]. The case of general, nonisentropic, gas dynamics, with nonconvex equations of state, was investigated by Pego [2], who established that strong shocks satisfying the Liu E-condition do not necessarily admit viscous shock profiles when the viscosity is dominated by the heat conductivity. The existence of viscous shock profiles with large amplitude for a general class of systems of two conservation laws that are not necessarily strictly hyperbolic is discussed by Yang, Zhang and Zhu [1]. The system (8.6.22), with viscosity matrix (8.6.23), is treated in detail by Mailybaev and Marchesin [1].

Smooth shock profiles are also induced by alternative dissipative or dispersive mechanisms, such as: capillarity (for an early study, see Conley and Smoller [2]; see also Bertozzi and Shearer [1], as well as numerous references below); and relaxation (see Tai-Ping Liu [21], Yong and Zumbrun [1], Dressel and Yong [1], and numerous references below). The study of shock profiles associated with solutions to the Boltzmann equation is still in progress; see Caflisch and Nicolaenko [1], and Liu and Yu [4].

To keep matters relatively simple, the discussion in this chapter has been biased in favor of shocks with moderate strength for strictly hyperbolic systems, as just only hints have been provided on the complexity associated with strong shocks for systems in which strict hyperbolicity, or even hyperbolicity proper, fails. To a great extent, the bewildering variety of shock structure encountered in the equations of mathematical physics is a consequence of symmetry, which is a cause of delight for the geometer and frustration for the analyst. In particular, the existence and stability of shocks in systems with rotational invariance, with application to elasticity and magnetohydrodynamics, has been discussed by Brio and Hunter [1], Freistühler [1,2,3,4,5,6], Freistühler and Liu [1], and Freistühler and Szmolyan [1]. A thorough investigation has been conducted on the admissibility of overcompressive and undercompressive shocks in the realm of nonstrictly hyperbolic systems of two conservation laws with quadratic or cubic flux functions. Viscosity or viscosity-capillarity conditions, as well as kinetic relations, have been employed for that purpose as admissibility criteria. The ultimate test of success of this endeavor is the well-posedness of the Riemann problem, to be discussed in Chapter IX. Out of an extensive body of literature, a sample is M. Shearer [2], Schaeffer and Shearer [1], Schecter and Shearer [1], Schaeffer, Schecter and Shearer [1], Jacobs, MacKinney and Shearer [1], Schulze and Shearer [1], Čanić and Plohr [1], Asakura and Yamazaki [1], Marchesin and Mailybaev [1], and Mailybaev and Marchesin [2,3]. For additional related references, see Section 9.12. The discussion of the stability of overcompressive shocks for the system (7.2.11) was borrowed from Tai-Ping Liu [27]; see also Tai-Ping Liu [27].

Another class of jump discontinuities that has not been discussed here are phase boundaries and transonic shocks, which arise in systems of conservation laws of mixed, elliptic-hyperbolic type, governing phase transitions or transonic gas flow. A prototypical example is the system (7.1.11) with nonmonotone $\sigma(u)$, and in particular the classical (isentropic) van der Waals fluid, whose shock curves are described in LeFloch and Thanh [3]. Entropy, viscosity and viscosity-capillarity admissibility criteria have been tried in that context, together with a new criterion based on kinetic relations, motivated by considerations at the microscale. A relevant, comprehensive reference on the physical background is the interesting monograph by Abeyaratne and Knowles [3]. Shocks induced by such kinetic relations are typically undercompressive. A sample of references in that area are Abeyaratne and Knowles [1,2], Asakura [2], Bedjaoui and LeFloch [1], Benzoni-Gavage [4], Fan [3,4], Hagan and Slemrod [1], Hayes and LeFloch [2,3], Hayes and Shearer [1], R.D. James [1], Keyfitz [2], Keyfitz and Warnecke [1], Pego [3], Pence [1], Rosakis [1], M. Shearer [2,5], Slemrod [2,3,4], and Truskinovsky [1,2]. The area is amply surveyed in the article by Fan and Slemrod [1] and in the paper and monograph by LeFloch

[4,5]. A thoughtful discussion of the admissibility issue for solutions to the Euler equations is found in Slemrod [8].

The admissibility of detonation waves in the theory of combustion also poses interesting and difficult problems. It has been discussed by Tai-Ping Liu and Tong Zhang [1] in the context of the highly simplified model of Chapman-Jouguet type that results from letting $\delta \to \infty$ in (7.1.26).

The asymptotic stability theory of viscous shock profiles, which was just sketched in Section 8.6, is presented in detail in the text by Serre [11]. The seminal papers in that direction include Ilin and Oleinik [1], on the scalar case; Goodman [1], on systems for perturbations with zero excess mass; Tai-Ping Liu [19], where the decoupled diffusion waves were introduced; and Szepessy and Xin [1], where the coupled diffusion waves first appeared. The definitive treatment of the scalar case is found in the survey by Serre [21], which presents, among other topics, the relevant contributions by Freistüler and Serre [2,3]. For a comprehensive recent treatment, with precise estimates on the rate of decay established by the use of the Green's function of the system, the reader should consult the memoir by Liu and Zeng [4], as well as the earlier paper [3], by the same authors, together with Liu and Yu [5]. For earlier works and for extensions of the theory, dealing with contact discontinuities, undercompressive shocks, boundary effects on stationary shocks, nonstrictly hyperbolic systems, and various types of viscosity, and based on either energy estimates or on Green's function, see Matsumura and Nishihara [1,4], Tai-Ping Liu [20,26], Liu and Nishihara [1], Liu and Xin [2,3], Liu and Yu [1], Tai-Ping Liu and Yanni Zeng [1,2], Liu and Zumbrun [1], Chern and Liu [1], Lan, Liu and Yu [1], Goodman, Szepessy and Zumbrun[1], Kawashima and Matsumura [1], Xin [3,5], Luo and Xin [1], Huang, Matsumura and Xin [1], Huang Xin and Yang [1], Fries [1,2], and Yanni Zeng [1,2,3]. The competition between viscosity and dispersion in controlling the asymptotic behavior of shock profiles is investigated by Perthame and Ryzhik [1]. The asymptotic stabilityof weak detonation waves, in the context of the simple model (7.1.26) equipped with viscosity, was established by Szepessy [3]. For the effect of damping, see Liao, Wang and Yang [1].

The asymptotic stability of viscous rarefaction wave profiles has also been investigated along similar lines; see Liu and Yu [6], Matsumura and Nishihara [2,3,5], Liu, Matsumura and Nishihara [1], Liu and Xin [1], Xin [1,2], Nishihara, Yang and Zhao [1], and Yang and Zhao [1]. Finally, for the asymptotic stability of composite viscous waves, shocks and/or rarefaction, see Huang, Li and Matsumura [1], and Huang and Matsumura [1].

A parallel theory has emerged on the asymptotic stability of relaxation shock profiles (including contact discontinuities) and relaxation rarefaction wave profiles; see Tai-Ping Liu [21], Hsiao and Pan [1], Tao Luo [1], Luo and Xin [1], Hailiang Liu [1], Mascia and Natalini [1], Mascia and Zumbrun [5], Yang and Zhu [1], Huijiang Zhao [2], Ueda and Kawashima [1], and Feimin Huang, Ronghua Pan and Yi Wang [1]. In the same vein, Kawashima and Nishibata [1,2], and Kawashima and Tanaka [1] establish the asymptotic stability of shock and rarefaction profiles for a scalar balance law with dissipative source induced by coupling to an elliptic equation (a radiating gas model). See also Lattanzio, Mascia and Serre [1].

The topic of shock stability, which was barely touched upon in Section 8.3, had its origins in the physics literature (e.g., Erpenbeck [1]) but has evolved, over the past thirty years, into an active field of mathematical research. By a suitable change of coordinates, the shock may be reduced to a stationary surface and thus may be treated by the machinery developed for the study of boundary value problems, outlined in Section 5.6. One may employ various notions of stability, spectral, linear or nonlinear, independently or interconnectedly. The construction and stability theory of multi-dimensional shocks was pioneered by Majda [2,3,4]. This seminal work has been extended in various directions and now encompasses compressive, undercompressive, and overcompressive shocks for hyperbolic systems, as well as phase boundaries for systems that change type. A small sample of relevant references, out of a voluminous literature, includes Benzoni-Gavage [2,3], Corli and Sablé-Tougeron [1], Franchéteau and Métivier [1], Freistühler [8], Freistühler and Rohde [1], Freistühler and Szmolyan [2], Freistühler, Szmolyan and Wächtler [1], Freistühler and Trakhinin [1], Godin [1], Métivier [2], Serre [18], Coulombel [1,2], Freistühler and Plaza [1], Lewicka and Zumbrun [1], Howard [1,2], and Barker, Freistühler and Zumbrun [1]. Detailed expositions are found in the monograph by Benzoni-Gavage and Serre [2], and the survey article by Métivier [3]. The same methodology has been used in the investigation of existence and stability of multi-dimensional detonation fronts; see Costanzino, Jenssen, Lyng and Williams [1].

Closely related to the above, in spirit as well as in technique, is the stabiliity theory for one-dimensional viscous, capillary, radiative, and relaxation shock profiles, phase boundaries and detonation fronts. This has been an area of active research over the past several years, and it is still undergoing rapid development. The survey papers by Zumbrun [5.6,10] provide detailed expositions, together with an exhaustive list of references. A sample of relevant contributions is Zumbrun [3,4,7,8,9], Gardner and Zumbrun [1], Howard and Zumbrun [1,2], Benzoni-Gavage, Serre and Zumbrun [1,2], Benzoni-Gavage [8], Guès, Métivier, Williams and Zumbrun [1,2,3,4,7], Hoff and Zumbrun [1,2], Lyng and Zumbrun [1], Plaza and Zumbrun [1], Mascia and Zumbrun [1,2,3,4,5,6], Texier and Zumbrun [1], Barker, Humpherys, Rudd and Zumbrun [1], Howard, Raoofi and Zumbrun [1], Métivier, Texier and Zumbrun [1], Howard [1], Jenssen, Lyng and Williams [1], Barker, Lewicka and Zumbrun [1], Nguyen and Zumbrun [1], Ha and Yu [1], Härterich [2], Kreiss, Kreiss and Lorenz [1], and Luo, Rauch, Xie and Xin [1]. Similar ideas and techniques apply in the investigation of stability of periodic solutions to systems of conservation laws with viscous dissipation; see Oh and Zumbrun [1,2], Beck, Sandstede and Zumbrun [1], Freistühler and Szmolyan [3], and Serre [23].

A parallel theory is emerging on the stability of shock profiles associated with finite difference systems that result from discretization of hyperbolic systems of conservation laws; see Jian-Guo Liu and Zhou Ping Xin [1,2], Jiang and Yu [1], Liu and Yu [2], and Benzoni-Gavage [6].

The notion of weak solution for quasilinear hyperbolic systems that are not in conservation form, outlined in Section 8.7, was introduced by LeFloch [2], and Dal Maso, LeFloch and Murat [1]. For developments and applications of these ideas, see Amadori, Baiti, LeFloch and Piccoli [1], Hayes and LeFloch [2,3], LeFloch and

Tzavaras [1,2], and references in Section 14.13. For a survey and list of diverse applications, see LeFloch [4], and Berthon, Coquel and LeFloch [1]. For stability issues, see Xiao-Biao Lin [1].

IX

Admissible Wave Fans and the Riemann Problem

The property of systems of conservation laws to be invariant under uniform stretching of the space-time coordinates induces the existence of self-similar solutions, which stay constant along straight-line rays emanating from some focal point in space-time. Such solutions depict a collection of waves converging to the focal point and interacting there to produce a jump discontinuity which is in turn resolved into an outgoing wave fan.

This chapter investigates the celebrated Riemann problem, whose object is the resolution of jump discontinuities into wave fans. A solution will be constructed in three different ways, namely: (a) by the classical method of piecing together elementary centered solutions encountered in earlier chapters, i.e., constant states, shocks joining constant states, and centered rarefaction waves bordered by constant states or contact discontinuities; (b) by minimizing the total entropy production of the outgoing wave fan; and (c) by a vanishing viscosity approach which employs time-dependent viscosity so that the resulting dissipative system is invariant under stretching of coordinates, just like the original hyperbolic system. A new type of discontinuity, called a delta shock, will emerge in the process.

The issue of admissibility of wave fans will be raised. In particular, it will be examined whether shocks contained in solutions constructed by any one of the above methods are necessarily admissible.

Next, the wave fan that best approximates the complex wave pattern generated by the interaction of two wave fans will be determined.

A system will be exhibited in which bounded initial data generate a resonating wave pattern that drives the solution amplitude to infinity, in finite time.

9.1 Self-Similar Solutions and the Riemann Problem

The hyperbolic system of conservation laws

$$\partial_t U + \partial_x F(U) = 0 \tag{9.1.1}$$

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is invariant under uniform stretching of coordinates: $(x,t) \mapsto (\alpha x, \alpha t)$; hence it admits *self-similar solutions*, defined on the space-time plane and constant along straight-line rays emanating from the origin. Since (9.1.1) is also invariant under translations of coordinates: $(x,t) \mapsto (x+\bar{x}, t+\bar{t})$, the focal point of self-similar solutions may be translated from the origin to any fixed point $(\bar{x}, \bar{t})$ in space-time.

A (generally weak) self-similar solution U of (9.1.1), defined on the upper or on the lower half-plane and focussed at the origin admits the representation

$$U(x,t) = V\left(\frac{x}{t}\right), \qquad -\infty < x < \infty, \text{ and } 0 < t < \infty \text{ or } -\infty < t < 0, \tag{9.1.2}$$

where V is a bounded measurable function on $(-\infty, \infty)$, which satisfies the ordinary differential equation

$$[F(V(\xi)) - \xi V(\xi)]^{\cdot} + V(\xi) = 0, \tag{9.1.3}$$

in the sense of distributions. To verify this statement for U defined on the upper half-plane in the form (9.1.2), we fix any test function ϕ with compact support on $(-\infty,\infty) \times (0,\infty)$ and notice that

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty [\partial_t \phi(x,t) U(x,t) + \partial_x \phi(x,t) F(U(x,t))] dxdt \\ &= \int_{-\infty}^\infty \{\dot{\psi}(\xi)[F(V(\xi)) - \xi V(\xi)] - \psi(\xi) V(\xi)\} d\xi, \end{aligned} \tag{9.1.4}$$

where

$$\psi(\xi) = \int_0^\infty \phi(\xi t, t) dt, \qquad -\infty < \xi < \infty. \tag{9.1.5}$$

The case of U defined on the lower half-plane in the form (9.1.2) is treated in a similar manner, yielding again (9.1.3).

From (9.1.3) we infer that $F(V) - \xi V$ is Lipschitz and

$$F(V(\xi)) - \xi V(\xi) - F(V(\zeta)) + \zeta V(\zeta) = -\int_\zeta^\xi V(\theta) d\theta \tag{9.1.6}$$

holds for all ζ and ξ in $(-\infty, \infty)$.

The domain $(-\infty,\infty)$ of any bounded measurable solution V of (9.1.3) is partitioned into the set $\mathscr{S}$ of points of discontinuity of V, the set $\mathscr{W}$ of points ξ of continuity of V with the property

$$\lambda_i(V(\xi)) = \xi, \tag{9.1.7}$$

for some $i \in \{1, \dots, n\}$, and the set $\mathscr{C}$ of points ξ of continuity of V such that $\lambda_i(V(\xi)) \neq \xi$, for $i = 1, \dots, n$.

Any increasing sequence $\{\zeta_k\}$ and decreasing sequence $\{\xi_k\}$ converging to a point ξ of $\mathscr{S}$ contain subsequences, again denoted by $\{\zeta_k\}$ and $\{\xi_k\}$, such that $\{V(\zeta_k)\}$ and $\{V(\xi_k)\}$ converge to respective limits V_- and V_+. By virtue of (9.1.6),

$$F(V_+) - F(V_-) = \xi[V_+ - V_-]. \tag{9.1.8}$$

In particular, when one-sided limits $V(\xi\pm)$ exist, ξ marks a shock and (9.1.8) expresses the Rankine-Hugoniot jump condition. However, at the present level of generality, we cannot preclude the possibility that different sequences converging to ξ may generate distinct limiting values V_- and/or V_+. In any case, from (9.1.8) follows that whenever $|V_+ - V_-|$ is sufficiently small ξ must be close to $\lambda_i(V_\pm)$ for some $i \in \{1,\dots,n\}$. This in turn implies that any point of continuity of V in the closure of $\mathscr{S}$ must belong to $\mathscr{W}$. At the same time, any point of continuity of V in the closure of $\mathscr{W}$ also belongs to $\mathscr{W}$. Thus $\mathscr{S}\bigcup\mathscr{W}$ is closed and $\mathscr{C}$ is open.

As an open set, $\mathscr{C}$ is the (at most) countable union of disjoint open intervals. Let us fix ξ and ζ in any one of these open intervals, say $\mathscr{I}$. We rewrite (9.1.6) in the form

$$(A - \xi I)[V(\xi) - V(\zeta)] = -\int_\zeta^\xi [V(\theta) - V(\zeta)]d\theta, \tag{9.1.9}$$

where A is the $n \times n$ matrix

$$A = \int_0^1 [s\mathrm{D}F(V(\zeta)) + (1-s)\mathrm{D}F(V(\xi))]ds. \tag{9.1.10}$$

As $\zeta \to \xi$, ξ is bounded away from the eigenvalues of A and the right-hand side of (9.1.9) behaves like $o(1)|\xi - \zeta|$. Thus (9.1.9) implies that V is Lipschitz on $\mathscr{I}$ and its derivative vanishes almost everywhere, i.e., V is constant on $\mathscr{I}$.

The above constraints still leave ample room for complexity in the structure of V, depending on the properties of the system (9.1.1). However, a simple and orderly configuration emerges when (9.1.1) is strictly hyperbolic and the range of V is confined in a ball $\mathscr{B}_\delta(\bar{U})$, with small radius δ, centered at some state $\bar{U}$. The size of δ is subjected to the same type of constraints characterizing shocks of moderate strength in Section 8.1. Under these assumptions the sets $\mathscr{S}$ and $\mathscr{W}$ are confined in the union of n intervals of length $O(\delta)$ centered at the points $\lambda_1(\bar{U}),\dots,\lambda_n(\bar{U})$. We let $\mathscr{S}_i$ and $\mathscr{W}_i$ denote the parts of $\mathscr{S}$ and $\mathscr{W}$ lying in the vicinity of $\lambda_i(\bar{U})$, and identify the convex hull $[\zeta_i\,,\,\xi_i]$ of the closed set $\mathscr{S}_i\bigcup\mathscr{W}_i$. Thus V takes constant values $U_0\,,U_1\,,\dots,U_n$ on the intervals $(-\infty,\,\zeta_1)$, $(\xi_1\,,\,\zeta_2)$, $\dots$, $(\xi_n\,,\infty)$. Each interval $[\zeta_i\,,\,\xi_i]$ may also contain an open subset of $\mathscr{C}$ denoted by $\mathscr{C}_i$. We call $[\zeta_i\,,\,\xi_i]$ an i-wave. We conclude that under the above assumptions V comprises a fan of n waves, one of each characteristic family, propagating at nearly characteristic speed and separated by constant states. The wave fan is outgoing from the origin when the self-similar solution is defined on the upper half-plane, or incoming into the origin when the self-similar solution is defined on the lower half-plane.

The structure of the i-wave becomes particularly clear when the function V has bounded variation. In that case $\mathscr{S}_i$ is countable and one-sided limits $V_\pm = V(\xi\pm)$ exist at every $\xi \in \mathscr{S}_i$, satisfying the jump condition (9.1.8). Thus $\mathscr{S}_i$ consists of i-shocks. Furthermore, V is differentiable almost everywhere. In particular, if V is differentiable at a point $\xi \in \mathscr{W}_i$, (9.1.3) and (9.1.7) yield

$$[\mathrm{D}F(V(\xi)) - \xi I]\dot{V}(\xi) = 0, \tag{9.1.11}$$

$$\mathrm{D}\lambda_i(V(\xi))\dot{V}(\xi) = 1. \tag{9.1.12}$$

Thus, $\dot{V}(\xi)$ is a nonzero vector, collinear to the i-th right eigenvector $R_i(V(\xi))$ of $\mathrm{D}F(V(\xi))$. Furthermore, $V(\xi)$ is necessarily a state of genuine nonlinearity of the i-th characteristic family. Accordingly, we realize $\mathscr{W}_i$ as the centered i-simple wave component of the i-wave, which will be a rarefaction wave in the case of an outgoing wave fan or a compression wave in the case of an incoming wave fan.

We now demonstrate that V has bounded variation on $[\zeta_i, \xi_i]$, at least when the i-wave is *unidirectional*, in the sense that the left i-th eigenvector $L_i(\bar{U})$ of $\mathrm{D}F(\bar{U})$ may be oriented in such a way that

$$L_i(\bar{U})[V(\xi) - V(\zeta)] \geq 0 \tag{9.1.13}$$

holds for any pair (ζ, ξ) of points of continuity of V with[1] $\xi_{i-1} < \zeta < \xi < \zeta_{i+1}$.

Fixing points of continuity ζ and ξ, as above, with $\xi_{i-1} < \zeta < \xi < \zeta_{i+1}$, we pass from (9.1.6), via (9.1.9) to

$$\begin{aligned}[\mathrm{D}F(\bar{U}) - \lambda_i(\bar{U})I][V(\xi) - V(\zeta)]& \\ = O(\delta)[V(\xi) - V(\zeta)] - \int_\zeta^\xi [V(\theta) - V(\zeta)]d\theta.&\end{aligned} \tag{9.1.14}$$

We write

$$V(\xi) - V(\zeta) = \sum_{k=1}^{n} a_k R_k(\bar{U}). \tag{9.1.15}$$

Multiplying (9.1.15) from the left by $L_i(\bar{U})$ and recalling (7.2.3) and (9.1.13) we deduce $a_i \geq 0$. On the other hand, combining (9.1.14) with (9.1.15) yields

$$\sum_{k \neq i} |a_k| \leq c\delta a_i + c\delta(\zeta - \xi). \tag{9.1.16}$$

Hence

$$TV_{[\zeta_i, \xi_i]} V(\cdot) \leq L_i(\bar{U})[U_i - U_{i-1}] + c\delta^2. \tag{9.1.17}$$

In the following sections, we shall impose admissibility conditions on wave fans, requiring, for instance, that any jump discontinuity must meet some or all of the shock admissibility criteria discussed in Chapter VIII, and as a minimum must satisfy the Lax E-condition. Notice that the Lax E-condition induces the monotonicity of the function $\lambda_i(V(\xi))$ on any incoming or outgoing i-wave $[\zeta_i, \xi_i]$. As we shall see in the sequel, in any strictly hyperbolic system, admissible outgoing i-waves of moderate strength end up being unidirectional, though this is not necessarily the case for

[1] For $i = 1$, $\xi_{i-1} = -\infty$; for $i = n$, $\zeta_{i+1} = \infty$.

incoming i-waves. However, when the i-characteristic field is genuinely nonlinear, then under the Lax E-condition even incoming i-waves must be unidirectional. The reason is that, as we saw in Chapter VIII, when the i characteristic field is genuinely nonlinear, then λ_i varies monotonically along the i wave curve, at least near the point of origin.

From now on, we confine our discussion to wave fans with bounded variation. As a function of bounded variation, V is the sum of a saltus function, representing the shocks, and a continuous function. In turn, the continuous part is the sum of an absolutely continuous function and a purely singular function, the so-called "Cantor part". It has been shown (references cited in Section 9.11) that when the system is genuinely nonlinear, the continuous part is Lipschitz, so in particular V is an *SBV* function. This is an indication of the role of genuine nonlinearity in fostering discontinuity or Lipschitz continuity, while impeding the "middle ground" of mere continuity. Other manifestations of that feature of genuine nonlinearity will be encountered later, most notably in Chapter XI. Of central importance will be to understand how a jump discontinuity at the origin, introduced by the initial data, is resolved into an outgoing wave fan. This is the object of the

9.1.1 Riemann Problem: Determine a self-similar (generally weak) solution U of (9.1.1) on $(-\infty,\infty)\times(0,\infty)$, with initial condition

$$U(x,0) = \begin{cases} U_L\,, & \text{for } x<0 \\ U_R\,, & \text{for } x>0, \end{cases} \tag{9.1.18}$$

where U_L and U_R are given states in $\mathscr{O}$.

Following our discussion, above, we shall seek a solution of the Riemann problem in the form (9.1.2), where V satisfies the ordinary differential equation (9.1.3), on $(-\infty,\infty)$, together with boundary conditions

$$V(-\infty) = U_L\,, \qquad V(\infty) = U_R\,. \tag{9.1.19}$$

The specter of nonuniqueness again raises the issue of admissibility, which will be the subject of discussion in the following sections.

9.2 Wave Fan Admissibility Criteria

Various aspects of admissibility have already been discussed, for general weak solutions, in Chapter IV, and for single shocks, in Chapter VIII. We have thus encountered a number of admissibility criteria and we have seen that they are strongly interrelated but not quite equivalent. As we shall see later, the most discriminating among these criteria, namely viscous shock profiles and the Liu E-condition, are sufficiently powerful to weed out all spurious solutions, so long as we are confined to strictly hyperbolic systems and shocks of moderate strength. However, once one moves to systems that are not strictly hyperbolic and/or to solutions with strong shocks, the situation becomes murky. The question of admissibility is still open.

Any rational new admissibility criterion should adhere to certain basic principles, the fruits of accumulated experience. They include:

9.2.1 Localization: The test of admissibility of a solution should apply individually to each point $(\bar{x},\bar{t})$ in the domain and should thus involve only the restriction of the solution to an arbitrarily small neighborhood of $(\bar{x},\bar{t})$, say the circle $\{(x,t) : |x-\bar{x}|^2 + |t-\bar{t}|^2 < r^2\}$ where r is fixed but arbitrarily small. This is compatible with the general principle that solutions of hyperbolic systems should have the local dependence property.

9.2.2 Evolutionarity: The test of admissibility should be forward-looking, without regard for the past. Thus, admissibility of a solution at the point $(\bar{x},\bar{t})$ should depend solely on its restriction to the semicircle $\{(x,t) : |x-\bar{x}|^2 + |t-\bar{t}|^2 < r^2, t \geq \bar{t}\}$. This is in line with the principle of time irreversibility, which pervades the admissibility criteria we have encountered thus far, such as entropy, viscosity, etc.

9.2.3 Invariance Under Translations: A solution U will be admissible at $(\bar{x},\bar{t})$ if and only if the translated solution $\bar{U}$, $\bar{U}(x,t) = U(x+\bar{x}, t+\bar{t})$, is admissible at the origin $(0,0)$.

9.2.4 Invariance Under Dilations: A solution U will be admissible at $(0,0)$ if and only if, for each $\alpha > 0$, the dilated solution $\bar{U}_\alpha$, $\bar{U}_\alpha(x,t) = U(\alpha x, \alpha t)$, is admissible at $(0,0)$.

Let us focus attention on weak solutions U with the property that, for each fixed point $(\bar{x},\bar{t})$ in the domain, the limit

$$\bar{U}(x,t) = \lim_{\alpha \downarrow 0} U(\bar{x} + \alpha x, \bar{t} + \alpha t) \tag{9.2.1}$$

exists in $L^1_{loc}((-\infty,\infty) \times (0,\infty))$. Notice that in that case $\bar{U}$ is necessarily a self-similar solution of (9.1.1). In the spirit of the principles listed above, one may use the admissibility of $\bar{U}$ at the origin as a test for the admissibility of U at the point $(\bar{x},\bar{t})$. Since $\bar{U}$ depicts a fan of waves radiating from the origin, such tests constitute *wave fan admissibility criteria*.

Passing to the limit in (9.2.1) is tantamount to observing the solution U under a microscope focused at the point $(\bar{x},\bar{t})$. When $U \in BV_{loc}$, the limit exists, by virtue of Theorem 1.7.4, except possibly on the set of irregular points $(\bar{x},\bar{t})$, which has one-dimensional Hausdorff measure zero. In particular, if $(\bar{x},\bar{t})$ is a point of approximate continuity of U, then $\bar{U}(x,t)$ will be a constant state U_0, while if $(\bar{x},\bar{t})$ is a point of approximate jump discontinuity, then $\bar{U}(x,t) = U_-$, for $x < st$, and $\bar{U}(x,t) = U_+$, for $x > st$, where $U_\pm$ are the approximate one-sided limits of U, and s is the slope of the jump discontinuity at $(\bar{x},\bar{t})$. Whether the limit will also exist at the irregular points

$(\bar{x},\bar{t})$ of U, where the resulting wave fan $\bar{U}$ should be more complex, will be investigated in Chapter XI, for genuinely nonlinear scalar conservation laws; in Chapter XII, for genuinely nonlinear systems of two conservation laws; and in Chapter XIV, for general genuinely nonlinear systems of conservation laws.

As we saw in Section 9.1, the wave fan $\bar{U}$ is generally a composite of constant states, shocks, and centered rarefaction waves. The simplest wave fan admissibility criterion postulates that the fan is admissible if each one of its shocks, individually, satisfies the shock admissibility conditions discussed in Chapter VIII. As we shall see in the following section, this turns out to be adequate in many cases. Other fan admissibility criteria, which regard the wave fan as an entity rather than as a collection of individual waves, include the entropy rate condition and the viscous fan profile test. These will be discussed later.

9.3 Solution of the Riemann Problem via Wave Curves

The aim here is to construct a solution of the Riemann problem by piecing together constant states, centered rarefaction waves, and admissible shocks. We limit our investigation to the case where wave speeds of different characteristic families are strictly separated. This will cover waves of small amplitude in general strictly hyperbolic systems as well as waves of any amplitude in special systems such as (7.1.11), in which all 1-waves travel to the left and all 2-waves travel to the right.

Let us then consider an outgoing wave fan (9.1.2), of bounded variation. Following the discussion in Section 9.1, $(-\infty,\infty)$ is decomposed into the union of the shock set $\mathscr{S}$, the rarefaction wave set $\mathscr{W}$, and the constant state set $\mathscr{C}$. Since the wave speeds of distinct characteristic families are strictly separated, $\mathscr{S} = \bigcup_{i=1}^{n} \mathscr{S}_i$ and $\mathscr{W} = \bigcup_{i=1}^{n} \mathscr{W}_i$, where $\mathscr{S}_i$ is the (at most countable) set of points of jump discontinuity of V that are i-shocks, and $\mathscr{W}_i$ is the (possibly empty) set of points of continuity of V that satisfy (9.1.7). The set $\mathscr{S}_i \bigcup \mathscr{W}_i$ is closed and contains points in the range of wave speeds of the i-characteristic family.

We now assume that the shocks satisfy the Lax E-condition, i.e., for all $\xi \in \mathscr{S}_i$,

(9.3.1) $$\lambda_i(V(\xi+)) \leq \xi \leq \lambda_i(V(\xi-)).$$

Then $\mathscr{S}_i \bigcup \mathscr{W}_i$ is necessarily a closed interval $[\zeta_i, \xi_i]$. Indeed, suppose $\mathscr{S}_i \bigcup \mathscr{W}_i$ is disconnected. Then there is an open interval $(\zeta, \xi) \subset \mathscr{C}$ with endpoints ζ and ξ contained in $\mathscr{S}_i \bigcup \mathscr{W}_i$. In particular, $V(\zeta+) = V(\xi-)$. On the other hand, by virtue of (9.1.7) and (9.3.1), $\zeta \geq \lambda_i(V(\zeta+))$, $\xi \leq \lambda_i(V(\xi-))$, which is a contradiction to $\zeta < \xi$. Notice further that any $\xi \in \mathscr{S}_i$ with $\xi > \zeta_i$ (or $\xi < \xi_i$) is the limit of an increasing (or decreasing) sequence of points of $\mathscr{W}_i$ and so $\lambda_i(V(\xi-)) = \xi$ (or $\lambda_i(V(\xi+)) = \xi$). We have thus established the following

9.3.1 Theorem. *Assume the wave speeds of distinct characteristic families are strictly separated. Any self-similar solution* (9.1.2) *of the Riemann Problem* (9.1.1), (9.1.18), *with shocks satisfying the Lax E-condition, comprises* $n+1$ *constant states* $U_L = U_0, U_1, \cdots, U_{n-1}, U_n = U_R$. *For* $i = 1, \cdots, n$, U_{i-1} *is joined to* U_i *by an i-wave,* *namely a sequence of centered i-rarefaction waves and/or i-shocks with the property that i-shocks bordered from the left (and/or the right) by i-rarefaction waves are left (and/or right) i-contact discontinuities* (*Fig.* 9.3.1).

Single i-shocks and single i-rarefaction waves are *elementary i-waves*. The term *composite i-wave* will be employed when it is necessary to recall or emphasize that the i-wave may contain more than one elementary i-wave.

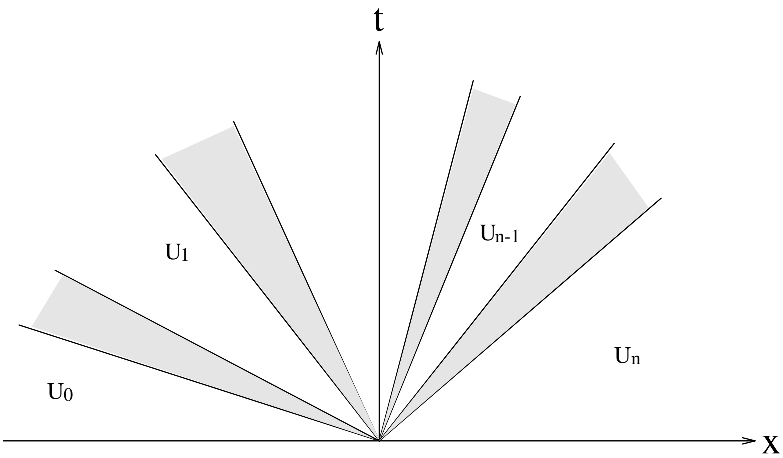

Fig. 9.3.1

It will be shown in the following two sections that the locus of states that may be joined on the right (or left) of a fixed state $\bar{U} \in \mathscr{O}$ by an admissible i-wave, composed of i-rarefaction waves and admissible i-shocks, is a Lipschitz curve $\Phi_i(\tau;\bar{U})$ (or $\Psi_i(\tau;\bar{U})$), called the *forward* (or *backward*) *i-wave curve through* $\bar{U}$, which may be parametrized so that

(9.3.2) $$\Phi_i(\tau;\bar{U}) = \bar{U} + \tau R_i(\bar{U}) + P_i(\tau;\bar{U}),$$

(9.3.3) $$\Psi_i(\tau;\bar{U}) = \bar{U} + \tau R_i(\bar{U}) + Q_i(\tau;\bar{U}),$$

where P_i and Q_i are Lipschitz continuous functions of (τ, U) that vanish at $\tau = 0$, and their Lipschitz constant becomes arbitrarily small if τ is restricted to a sufficiently small neighborhood of the origin.

Taking, for the time being, the existence of wave curves with the above properties for granted, we note that to solve the Riemann problem we have to determine an n-tuple $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, realized as a point in $\mathbb{R}^n$, such that, starting out from $U_0 = U_L$ and computing successively $U_i = \Phi_i(\varepsilon_i; U_{i-1})$, $i = 1, \cdots, n$, we end up with $U_n = U_R$. Accordingly, we define the function

(9.3.4)
$$\Omega(\varepsilon;\bar{U}) = \Phi_n(\varepsilon_n;\Phi_{n-1}(\varepsilon_{n-1};\cdots\Phi_1(\varepsilon_1;\bar{U})\cdots)).$$

Clearly,

(9.3.5)
$$\Omega(\varepsilon;\bar{U}) = \bar{U} + \sum_{i=1}^{n} \varepsilon_i R_i(\bar{U}) + G(\varepsilon;\bar{U}),$$

where G is a Lipschitz function that vanishes at $\varepsilon = 0$ and whose Lipschitz constant becomes arbitrarily small when ε is confined to a sufficiently small neighborhood of the origin. When U_R is sufficiently close to U_L, there exists a unique ε near 0 such that $\Omega(\varepsilon;U_L) = U_R$. Indeed, this ε may be constructed through the iteration scheme: $\varepsilon^{(0)} = 0$ and for $m = 1,2,\ldots$

(9.3.6)
$$\varepsilon_i^{(m)} = L_i(U_L)\big[U_R - U_L\big] - L_i(U_L)G\big(\varepsilon^{(m-1)};U_L\big), \qquad i = 1,\ldots,n,$$

which converges by an obvious contraction argument. This generates a solution to the Riemann problem that is unique within the class of self-similar solutions with waves of moderate strength. The wave fan joining U_L with U_R is conveniently identified by its left state U_L and the n-tuple $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$. The value of ε_i determines the *i-wave amplitude* and $|\varepsilon_i|$ measures the *i-wave strength.*

In the special case where the Φ_i are $C^{2,1}$, we shall see that

(9.3.7)
$$\dot{\Phi}_i(0;\bar{U}) = R_i(\bar{U}), \qquad \ddot{\Phi}_i(0;\bar{U}) = \mathrm{D}R_i(\bar{U})R_i(\bar{U}).$$

Then Ω is also $C^{2,1}$. Since $\Omega(0,\cdots,0,\varepsilon_i,0,\cdots,0;\bar{U}) = \Phi_i(\varepsilon_i;\bar{U})$,

(9.3.8)
$$\frac{\partial\Omega}{\partial\varepsilon_i}(0;\bar{U}) = R_i(\bar{U}), \qquad 1 \le i \le n,$$

(9.3.9)
$$\frac{\partial^2\Omega}{\partial\varepsilon_i^2}(0;\bar{U}) = \mathrm{D}R_i(\bar{U})R_i(\bar{U}), \qquad 1 \le i \le n.$$

Moreover, for $j<k$, $\Omega(0,\cdots,0,\varepsilon_j,0,\cdots,0,\varepsilon_k,0,\cdots,0;\bar{U}) = \Phi_k\big(\varepsilon_k;\Phi_j\big(\varepsilon_j;\bar{U}\big)\big)$ and so

(9.3.10)
$$\frac{\partial^2\Omega}{\partial\varepsilon_j\partial\varepsilon_k}(0;\bar{U}) = \mathrm{D}R_k(\bar{U})R_j(\bar{U}), \qquad 1 \le j < k \le n.$$

By virtue of (9.3.8), (9.3.9) and (9.3.10),

(9.3.11)
$$\begin{aligned} U_R = U_L &+ \textstyle\sum_{i=1}^n \varepsilon_i R_i(U_L) + \frac{1}{2}\sum_{i=1}^n \varepsilon_i^2 \mathrm{D}R_i(U_L)R_i(U_L) \\ &+ \textstyle\sum_{j=1}^n \sum_{k=j+1}^n \varepsilon_j\varepsilon_k \mathrm{D}R_k(U_L)R_j(U_L) + O\big(|\varepsilon|^3\big). \end{aligned}$$

Clearly, we may also synthesize the solution of the Riemann problem in the reverse order, starting out from $U_n = U_R$ and computing successively the states $U_{i-1} = \Psi_i(\varepsilon_i;U_i)$, $i = n,\cdots,1$, until we reach $U_0 = U_L$. Under certain circumstances, a mixed strategy may be advantageous. For example, the most efficient procedure for solving the Riemann problem for a system of two conservation laws, $n = 2$, is to draw the forward 1-wave curve $\Phi_1(\varepsilon_1;U_L)$ through the left state U_L and the backward 2-wave curve $\Psi_2(\varepsilon_2;U_R)$ through the right state U_R. The intersection of these two curves will determine the intermediate constant state: $U_M = \Phi_1(\varepsilon_1;U_L) = \Psi_2(\varepsilon_2;U_R)$.

9.4 Systems with Genuinely Nonlinear or Linearly Degenerate Characteristic Families

Our project here is to construct the wave curves for systems in which waves are particularly simple. When the i-characteristic family is linearly degenerate, no centered i-rarefaction waves exist and hence, by Theorem 8.2.5, any i-wave is necessarily an i-contact discontinuity. In that case the forward and backward i-wave curves coincide with the shock curve W_i in Theorem 8.2.5, namely, $\Phi_i(\tau;\bar{U}) = \Psi_i(\tau;\bar{U}) = W_i(\tau;\bar{U})$.

When the i-characteristic family is genuinely nonlinear, i-contact discontinuities are ruled out by Theorem 8.2.1, and so any i-wave of small amplitude must be either a single centered i-rarefaction wave or a single compressive i-shock. Let us normalize the field R_i so that (7.6.13) holds, $\mathrm{D}\lambda_i R_i = 1$. The states that may be joined to $\bar{U}$ by a weak i-shock lie on the i-shock curve $W_i(\tau;\bar{U})$ described by Theorem 8.2.1. On account of Theorem 8.3.1, the shock that joins $\bar{U}$, on the left, with $W_i(\tau;\bar{U})$, on the right, is compressive if and only if $\tau < 0$. On the other hand, by Theorem 7.6.5, the state $\bar{U}$ may be joined on the right (or left) by centered i-rarefaction waves to states $V_i(\tau;\bar{U})$ for $\tau > 0$ (or $\tau < 0$). It then follows that we may construct the forward i-wave curve by $\Phi_i(\tau;\bar{U}) = W_i(\tau;\bar{U})$, for $\tau < 0$, and $\Phi_i(\tau;\bar{U}) = V_i(\tau;\bar{U})$, for $\tau > 0$. Similarly, the backward i-wave curve is defined by $\Psi_i(\tau;\bar{U}) = V_i(\tau;\bar{U})$, for $\tau < 0$, and $\Psi_i(\tau;\bar{U}) = W_i(\tau;\bar{U})$, for $\tau > 0$. These curves are $C^{2,1}$, on account of Theorem 8.2.2, and satisfy (9.3.2), (9.3.3) and (9.3.7), by Theorem 8.2.1.

In view of the above discussion, we have now established the existence of solution to the Riemann problem for systems with characteristic families that are either genuinely nonlinear or linearly degenerate:

9.4.1 Theorem. *Assume the system* (9.1.1) *is strictly hyperbolic and each characteristic family is either genuinely nonlinear or linearly degenerate. For* $|U_R - U_L|$ *sufficiently small, there exists a unique self-similar solution* (9.1.2) *of the Riemann problem* (9.1.1), (9.1.18), *with small total variation. This solution comprises* $n+1$ *constant states* $U_L = U_0\,, U_1\,, \cdots, U_{n-1}\,, U_n = U_R$. *When the i-characteristic family is linearly degenerate,* U_i *is joined to* U_{i-1} *by an i-contact discontinuity, while when the i-characteristic family is genuinely nonlinear,* U_i *is joined to* U_{i-1} *by either a centered i-rarefaction wave or a compressive i-shock.*

In particular, Theorem 9.4.1 applies to the Riemann problem for the system (7.1.12) of isentropic gas dynamics, when $2p'(\rho) + \rho p''(\rho) > 0$, so that both characteristic families are genuinely nonlinear; also for the system (7.1.8) of adiabatic thermoelasticity, under the assumption $p_{uu}(u,s) \neq 0$, in which case the 1- and the 3-characteristic families are genuinely nonlinear while the 2-characteristic family is linearly degenerate.

As a simple illustration, let us consider the system (7.1.11) of isentropic thermoelasticity, in Lagrangian coordinates, assuming that $\sigma(u)$ is defined on $(-\infty,\infty)$ and $0 < a \le \sigma'(u) \le b < \infty$, $\sigma''(u) < 0$. Under these conditions, the wave curves are globally defined and the Riemann problem may be solved even when the end-states (u_L, v_L) and (u_R, v_R) lie far apart. It is convenient to reparametrize the wave curves, employing u as the new parameter. In that case, the forward 1-wave curve Φ_1 and the backward 2-wave curve Ψ_2 through the typical point $(\bar{u}, \bar{v})$ of the state space may be represented as $v = \varphi(u; \bar{u}, \bar{v})$ and $v = \psi(u; \bar{u}, \bar{v})$, respectively. Recalling the form of the Hugoniot locus (8.2.11) and rarefaction wave curves (7.6.15) for this system, we deduce that

$$\varphi(u;\bar{u},\bar{v}) = \begin{cases} \bar{v} - \sqrt{[\sigma(u)-\sigma(\bar{u})](u-\bar{u})}\ , & u \leq \bar{u} \\ \bar{v} + \displaystyle\int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}d\omega, & u > \bar{u} \end{cases} \tag{9.4.1}$$

$$\psi(u;\bar{u},\bar{v}) = \begin{cases} \bar{v} + \sqrt{[\sigma(u)-\sigma(\bar{u})](u-\bar{u})}\ , & u \leq \bar{u} \\ \bar{v} - \displaystyle\int_{\bar{u}}^{u} \sqrt{\sigma'(\omega)}d\omega, & u > \bar{u}. \end{cases} \tag{9.4.2}$$

Figure 9.4.1 depicts a solution of the Riemann problem that comprises a compressive 1-shock and a centered 2-rarefaction wave. The intermediate constant state (u_M, v_M) is determined on the u-v plane as the intersection of the forward 1-wave curve Φ_1 through (u_L, v_L) with the backward 2-wave curve Ψ_2 through (u_R, v_R), namely by solving the equation

$$v_M = \varphi(u_M; u_L, v_L) = \psi(u_M; u_R, v_R). \tag{9.4.3}$$

For systems of two conservation laws it is often expedient to perform the construction of the intermediate constant state on the plane of Riemann invariants rather than in the original state space. The reason is that, as noted in Section 7.6, in the plane of Riemann invariants rarefaction wave curves become straight lines parallel to the coordinate axis. This facilitates considerably the task of locating the intersection of wave curves of different characteristic families. Figure 9.4.2 depicts the configuration of the wave curves of Figure 9.4.1 in the plane w-z of Riemann invariants.

In particular, Riemann problems with end-states (u_L, v_L) whose solutions consist, as above, of a compressive 1-shock and a 2-rarefaction wave, are generated by the collision of a compressive 1-shock, joining states (u_L, v_L) and (u_0, v_0), with a slower compressive 1-shock that joins (u_0, v_0) and (u_R, v_R). Similarly, the collision of two compressive 2-shocks leads to Riemann problems whose solutions consist of a 1-rarefaction wave and a compressive 2-shock.

The situation is more complicated when the medium in (7.1.11) is a gas, in which case $\sigma'(u) \to 0$ as $u \to \infty$. This will be discussed in Section 9.6.

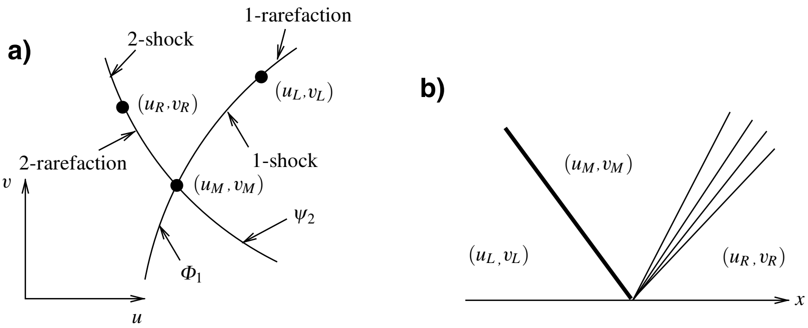

Fig. 9.4.1(a,b)

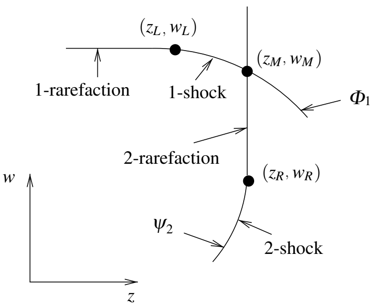

Fig. 9.4.2

The most celebrated case of a Riemann problem pertains to the system of conservation laws of mass, momentum and energy governing the adiabatic rectilinear flow of an inviscid gas in a duct. There is an exhaustive treatment of this problem in the texts cited in Section 9.11, so a brief presentation will suffice for present purposes. The problem is usually considered for the Eulerian form of the system – namely the one-dimensional version of (3.3.29) – with density ρ, velocity v and pressure p as state variables, and shock curves parametrized by p. Here, however, we deal with the Lagrangian form (7.1.8), with state variables the specific volume u, velocity v and entropy s. Furthermore, the shock curves will be parametrized by u. This approach is convenient for extending the shock curves into the realm of strong shocks by taking advantage of the representation of $s(u)$ as solution of the differential equation (8.2.13).

We make the assumptions $p_u < 0$, for hyperbolicity, $p_{uu} > 0$, so that the 1- and the 3-characteristic families are genuinely nonlinear, and $p_s = -\theta_u > 0$, which renders the differential equation (8.2.13), for the 1- and 3-shock curves, well-posed in the large. The 2-characteristic family is linearly degenerate.

Under the above assumptions, the solution to the Riemann problem, with end-states (u_L, v_L, s_L) and (u_R, v_R, s_R), is expected to comprise a compressive 1-shock or 1-rarefaction wave, joining (u_L, v_L, s_L) with a state (u_M^-, v_M^-, s_M^-), followed by a stationary contact discontinuity that joins (u_M^-, v_M^-, s_M^-) with a state (u_M^+, v_M^+, s_M^+), and a compressive 3-shock or 3-rarefaction wave, joining (u_M^+, v_M^+, s_M^+) with (u_R, v_R, s_R). The pressure associated with the above states will be denoted by p_L, p_M^-, p_M^+ and p_R. The Rankine-Hugoniot jump conditions for the contact discontinuity require that $v_M^- = v_M^+ = v_M$ and $p_M^- = p_M^+ = p_M$. Thus, for solving the Riemann problem it is expedient to track the projection of the wave curves on the v-p plane.

The 1- and 3-shock curves are determined by (8.2.13), (8.2.14), while the 1- and 3-rarefaction wave curves are given by (7.6.16). Hence the projections Φ and Ψ of the forward 1-wave curve and the backward 3-wave curve through the state $(\bar{u}, \bar{v}, \bar{s})$, parametrized by u, are

(9.4.4)
$$\Phi(u;\bar{u},\bar{v},\bar{s}) = \begin{cases} \left(\bar{v} - \sqrt{[-p(u,s(u)) + p(\bar{u},\bar{s})](u-\bar{u})},\ p(u,s(u))\right), & u \leq \bar{u} \\[2ex] \left(\bar{v} + \displaystyle\int_{\bar{u}}^{u} \sqrt{-p_\omega(\omega,\bar{s})}\,d\omega,\ p(u,\bar{s})\right), & u > \bar{u} \end{cases}$$

(9.4.5)
$$\Psi(u;\bar{u},\bar{v},\bar{s}) = \begin{cases} \left(\bar{v} + \sqrt{[-p(u,s(u)) + p(\bar{u},\bar{s})](u-\bar{u})},\ p(u,s(u))\right), & u \leq \bar{u} \\[2ex] \left(\bar{v} - \displaystyle\int_{\bar{u}}^{u} \sqrt{-p_\omega(\omega,\bar{s})}\,d\omega,\ p(u,\bar{s})\right), & u > \bar{u} \end{cases}$$

where $s(u)$ denotes the solution of (8.2.13) with initial condition $s(\bar{u}) = \bar{s}$. Solutions to the Riemann problem are generated by intersections of the curves $\Phi(\cdot, u_L, v_L, s_L)$ and $\Psi(\cdot, u_R, v_R, s_R)$:

(9.4.6)
$$(v_M, p_M) = \Phi(u_M^-; u_L, v_L, s_L) = \Psi(u_M^+; u_R, v_R, s_R).$$

As asserted by Theorem 9.4.1, when (u_L, v_L, s_L) and (u_R, v_R, s_R) are close to each other, there exists a unique local intersection, and thereby a unique solution to the Riemann problem. Whether the 1- and 3- waves will be compressive shocks or rarefaction waves will depend on the relative location of the end states in state space. We shall not attempt here to provide a complete classification of possible wave configurations, since this can be found in the references cited in Section 9.11. As an illustrative example, we consider the Riemann problem induced by the collision of two compressive shocks of the third family for an ideal gas (2.5.20), with $\gamma \leq 5/3$, whose solution comprises a compressive shock of the third family, a rarefaction wave of the first family and a contact discontinuity, as depicted in Figure 9.4.3. By contrast, when $\gamma > 5/3$ the collision of two weak compressive shocks of the same family yields compressive shocks of both families together with a contact discontinuity.

The wave curves Φ and Ψ are defined in the large and hence, as in the isentropic case, one may use them to solve the Riemann problem even when the end-states lie far apart. The study of this problem reveals that, depending on the equations of state and on the relative location of (u_L, v_L, s_L) and (u_R, v_R, s_R), the two wave curves may intersect at a single point, at multiple points, or not at all, resulting in a unique solution, multiple solutions, or no solution to the Riemann problem.

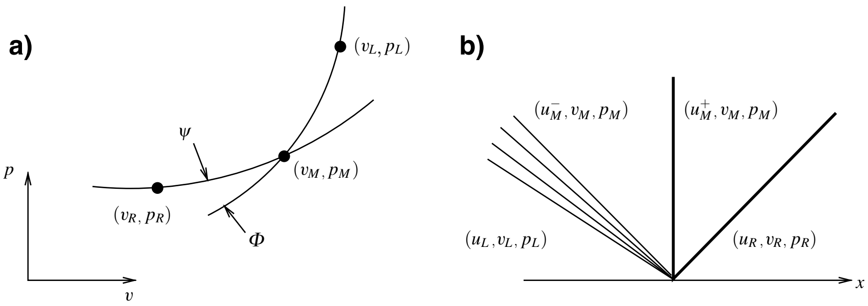

Fig. 9.4.3(a,b)

9.5 General Strictly Hyperbolic Systems

Our next task is to describe admissible wave fans, and construct the corresponding wave curves, for systems with characteristic families that are neither genuinely nonlinear nor linearly degenerate. In that case, the Lax E-condition is no longer sufficiently selective to single out a unique solution to the Riemann problem so the more stringent Liu E-condition will be imposed on shocks.

We begin with the scalar conservation law (7.1.2), where $f(u)$ may have inflection points. The Liu E-condition is now expressed by the Oleinik E-condition (8.4.3). By Theorem 9.3.1, the solution of the Riemann problem comprises two constant states u_L and u_R joined by a wave that is a sequence of shocks and/or centered rarefaction waves. There exists precisely one such wave with shocks satisfying the Oleinik E-condition, and it is constructed by the following procedure: When $u_L < u_R$ (or $u_L > u_R$), we let g denote the *convex* (or *concave*) *envelope* of f over the interval $[u_L, u_R]$ (or $[u_R, u_L]$); namely, $g(u)$ is the infimum (or supremum) of all convex combinations $\theta_1 f(u_1) + \theta_2 f(u_2)$, with $\theta_1 \geq 0$, $\theta_2 \geq 0$, $\theta_1 + \theta_2 = 1$, $u_1, u_2 \in [u_L, u_R]$ (or $[u_R, u_L]$) and $\theta_1 u_1 + \theta_2 u_2 = u$. Thus the graph of g may be visualized as the configuration of a flexible string anchored at the points $(u_L, f(u_L)), (u_R, f(u_R))$ and stretched under (or over) the "obstacle" $\{(u,v) : u_L \leq u \leq u_R, v \geq f(u)\}$ (or $\{(u,v) : u_R \leq u \leq u_L, v \leq f(u)\}$). The slope $\xi = g'(u)$ is a continuous nondecreasing (or nonincreasing) function whose inverse $u = \omega(\xi)$ generates the (generally composite) wave $u = \omega(x/t)$. In particular, the flat parts of $g'(u)$ give rise to the shocks while the intervals over which $g'(u)$ is strictly monotone generate the rarefaction waves. Figure 9.5.1 depicts an example in which the resulting wave consists of a centered rarefaction wave followed by a contact discontinuity, which is in turn followed by another centered rarefaction wave bordered on the right by a left contact discontinuity.

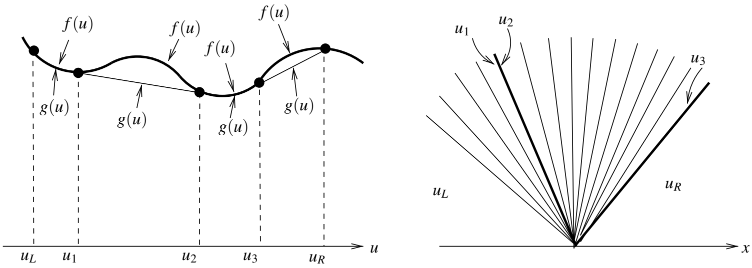

Fig. 9.5.1

To prepare the ground for the investigation of systems, we construct waves, and corresponding wave curves, for the simple system (7.1.11), where $\sigma(u)$ may have inflection points. The Liu E-condition here reduces to the Wendroff E-condition (8.4.4). As in the genuinely nonlinear case, we shall employ u as parameter and determine the forward 1-wave curve Φ_1 and the backward 2-wave curve Ψ_2, through the state $(\bar{u},\bar{v})$, in the form $v=\varphi(u;\bar{u},\bar{v})$ and $v=\psi(u;\bar{u},\bar{v})$, respectively. Recalling the equations (8.2.11) for the Hugoniot locus, the equations (7.6.15) for the rarefaction wave curves, and (8.4.4), one easily verifies that

(9.5.1) $$\varphi(u;\bar{u},\bar{v})=\bar{v}+\int_{\bar{u}}^{u}\sqrt{g'(\omega;u,\bar{u})}\,d\omega,$$

(9.5.2) $$\psi(u;\bar{u},\bar{v})=\bar{v}-\int_{\bar{u}}^{u}\sqrt{g'(\omega;u,\bar{u})}\,d\omega,$$

where $g'(\omega;u,\bar{u})$ is the derivative, with respect to ω, of the monotone increasing, continuously differentiable function $g(\omega;u,\bar{u})$ which is constructed by the following procedure: For fixed $u\le\bar{u}$ (or $u\ge\bar{u}$), $g(\cdot,u,\bar{u})$ is the convex (or concave) envelope of $\sigma(\cdot)$ over the interval $[u,\bar{u}]$ (or $[\bar{u},u]$). Indeed, as in the case of the scalar conservation law discussed above, the states $(\bar{u},\bar{v})$ and (u,v), $v=\phi(u;\bar{u},\bar{v})$, are joined by a (generally composite) 1-wave $(\omega(x/t),\upsilon(x/t))$, where $\omega(\xi)$ is the inverse of the function $\xi=\sqrt{g'(\omega;u,\bar{u})}$ and

(9.5.3) $$v(\xi)=\bar{v}+\int_{\bar{u}}^{\omega(\xi)}\sqrt{g'(\omega;u,\bar{u})}\,d\omega.$$

Again, the flat parts of g' give rise to shocks while the intervals over which g' is strictly monotone generate the rarefaction waves. In the genuinely nonlinear case, $\sigma''(u)<0$, (9.5.1) and (9.5.2) reduce to (9.4.1) and (9.4.2). Once ϕ and ψ have been determined, the Riemann problem is readily solved, as in the genuinely nonlinear case, by locating the intermediate constant state (u_M,v_M) through the equation (9.4.3).

After this preparation, we continue with a somewhat sketchy and informal description of the construction of wave curves for general systems. To avoid aggravating complications induced by various degeneracies, we limit the investigation to

i-characteristic families that are *piecewise genuinely nonlinear* in the sense that if U is a state of linear degeneracy, $\mathrm{D}\lambda_i(U)R_i(U) = 0$, then $\mathrm{D}(\mathrm{D}\lambda_i(U)R_i(U))R_i(U) \neq 0$. This implies, in particular, that the set of states of linear degeneracy of the i-characteristic family is locally a smooth manifold of codimension one, which is transversal to the vector field R_i. The scalar conservation law (7.1.2) and the system (7.1.11) of isentropic elasticity will satisfy this assumption when the functions $f(u)$ and $\sigma(u)$ have isolated, nondegenerate inflection points, i.e., $f'''(u)$ and $\sigma'''(u)$ are nonzero at any point u where $f''(u)$ and $\sigma''(u)$ vanish. Even after these simplifications, the construction is complicated. The ideas may become more transparent if the reader refers back to the model system (7.1.11) to illustrate each step. Familiarity with Lemma 8.2.4 and the remarks following its statement will also prove helpful.

Assuming the i-characteristic family is piecewise genuinely nonlinear, we consider the forward i-wave curve $\Phi_i(\tau;\bar{U})$ through a point $\bar{U}$ of genuine nonlinearity, say $\mathrm{D}\lambda_i(\bar{U})R_i(\bar{U}) = 1$. Then Φ_i starts out as in the genuinely nonlinear case, namely, for τ positive small it coincides with the i-rarefaction wave curve $V_i(\tau;\bar{U})$ through $\bar{U}$, while for τ negative, near zero, it coincides with the i-shock curve $W_i(\tau;\bar{U})$ through $\bar{U}$. In particular, (9.3.7) holds. We shall follow Φ_i along the positive τ-direction; the description for $\tau < 0$ is quite analogous.

For $\tau > 0, \Phi_i(\tau;\bar{U})$ will stay with the i-rarefaction wave curve $V_i(\tau;\bar{U})$ for as long as the latter dwells in the region of genuine nonlinearity: $\mathrm{D}\lambda_i(V_i)R_i(V_i) > 0$. Suppose now $V_i(\tau;\bar{U})$ first encounters the set of states of linear degeneracy of the i-characteristic family at the state $\tilde{U} = V_i(\tilde{\tau};\bar{U}) : \mathrm{D}\lambda_i(\tilde{U})R_i(\tilde{U}) = 0$. The set of states of linear degeneracy in the vicinity of $\tilde{U}$ forms a manifold $\mathscr{M}$ of codimension 1, transversal to the vector field R_i; see Fig. 9.5.2 (a,b).

The extension of Φ_i beyond $\tilde{U}$ is constructed as follows: For $\tau^* < \tilde{\tau}$, with $\tilde{\tau} - \tau^*$ small, we draw the i-shock curve $W_i(\zeta;U^*)$ through the state $U^* = V_i(\tau^*,\bar{U})$. On account of (8.2.1), $s_i(0;U^*) = \lambda_i(U^*)$ and since $\mathrm{D}\lambda_i(U^*)R_i(U^*) > 0$, (8.2.2) implies that for ζ negative, near 0, $\dot{s}_i(\zeta;U^*) > 0$ and $s_i(\zeta;U^*) < \lambda_i(W_i(\zeta;U^*))$. However, after crossing $\mathscr{M}, W_i(\zeta;U^*)$ enters the region where $\mathrm{D}\lambda_i(U)R_i(U) < 0$ and thus $\lambda_i(W_i(\zeta;U^*))$ will become decreasing. Eventually, ζ^* will be reached where $s_i(\zeta^*;U^*) = \lambda_i(W_i(\zeta^*;U^*))$. For $\zeta < \zeta^*$, by virtue of Lemma 8.2.4, we have $s_i(\zeta,U^*) > \lambda_i(W_i(\zeta;U^*))$ and $\dot{s}_i(\zeta,U^*) < 0$. Finally, a value $\zeta^\sharp$ will be attained such that $s_i(\zeta^\sharp;U^*) = \lambda_i(U^*)$. Then the state $U^\sharp = W_i(\zeta^\sharp;U^*)$, on the right, is joined to U^*, on the left, by a left i-contact discontinuity with speed $\lambda_i(U^*)$. This shock satisfies the Liu E-condition, since $s_i(\zeta;U^*) > \lambda_i(U^*)$ for $\zeta < \zeta^\sharp$. In particular, $\lambda_i(U^*) = s_i(\zeta^\sharp;U^*) > \lambda_i(U^\sharp)$. Consequently, $\bar{U}$, on the left, is joined to $U^\sharp$, on the right, by an admissible i-wave, comprising the i-rarefaction wave that joins U^* to $\bar{U}$ and the admissible left i-contact discontinuity that joins $U^\sharp$ to U^*. It can be shown that as U^* moves along the curve $V_i(\tau;\bar{U})$ from $\tilde{U}$ towards $\bar{U}$, the corresponding $U^\sharp$ traces a curve, say Γ. If $U^* = \tilde{U}$, then $U^\sharp = \tilde{U}$ so Γ starts out from $\tilde{U}$. Also Γ at $\tilde{U}$ is tangential to $R_i(\tilde{U})$. We adjoin Γ to $V_i(\tau;\bar{U})$ and consider it as the continuation of $\Phi_i(\tau;\bar{U})$ beyond $\tilde{U}$, with the proper parametrization.

$\Phi_i(\tau;\bar{U})$ will stay with Γ up until a state $\hat{U}$ is reached at which one of the following two alternatives first occurs:

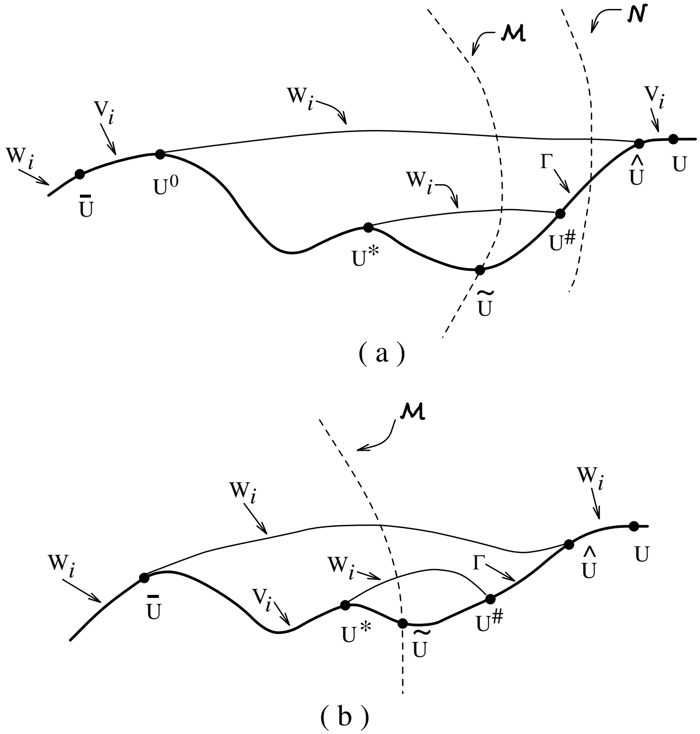

Fig. 9.5.2(a,b)

One possibility is depicted in Fig. 9.5.2(a): Γ crosses another manifold $\mathscr{N}$ of states of linear degeneracy of the i-characteristic family, entering the region $\mathrm{D}\lambda_i(U)R_i(U) > 0$, and eventually U^* backs up to a position U^0 so that the corresponding $U^\sharp$, denoted by $\hat{U}$, satisfies $\lambda_i(\hat{U}) = \lambda_i(U^0)$. In that case, $\Phi_i(\tau;\bar{U})$ is extended beyond $\hat{U}$ as the i-rarefaction curve $V_i(\zeta;\hat{U})$ through $\hat{U}$, properly reparametrized. Any state U on that curve is joined, on the right, to $\bar{U}$ by a composite i-wave comprising an i-rarefaction wave that joins U^0 to $\bar{U}$, an i-contact discontinuity that joins $\hat{U}$ to U^0, and a second i-rarefaction wave that joins U to $\hat{U}$.

The alternative is depicted in Fig. 9.5.2(b): U^* backs up all the way to $\bar{U}$ and the corresponding $U^\sharp$, denoted by $\hat{U}$, satisfies $\lambda_i(\hat{U}) < \lambda_i(\bar{U})$. In that case $\hat{U}$ lies on the i-shock curve through $\bar{U}$, say $\hat{U} = W_i(\hat{\tau};\bar{U})$. As $s_i(\hat{\tau};\bar{U}) = \lambda_i(\bar{U}) > \lambda_i(\hat{U})$, Lemma 8.2.4 implies $\dot{s}_i(\hat{\tau};\bar{U}) < 0$. Then $\Phi_i(\tau;\bar{U})$ is extended beyond $\hat{U}$ along the i-shock curve $W_i(\tau;\bar{U})$. Any state U on this arc of the curve is joined, on the right, to $\bar{U}$ by a single i-shock that satisfies the Liu E-condition.

By continuing this process we complete the construction of $\Phi_i(\tau;\bar{U})$ within the range of waves of moderate strength, and for certain systems even for strong waves. Furthermore, careful review of the construction verifies that the graph of Φ_i contains all states in a small neighborhood of $\bar{U}$ that may be joined to $\bar{U}$ by an admissible i-wave.

As we saw earlier, before crossing any manifold of states of linear degeneracy, Φ_i is $C^{2,1}$. Its regularity may be reduced to $C^{1,1}$ after the first crossing with such a manifold, and it may become merely Lipschitz beyond a second crossing (references in Section 9.12). Nevertheless, (9.3.2) will still hold, within the realm of waves of moderate strength, so that the range of τ for which the Lipschitz constant of $P_i(\tau;U)$ is small transcends the manifolds of states of linear degeneracy, and does not depend on their number. Consequently, one may trace wave curves for any strictly hyperbolic system whose flux may be realized as the C^1 limit of a sequence of fluxes with characteristic families that are piecewise genuinely nonlinear.

Later on, in Section 9.8, we will encounter an alternative construction of wave curves, for general strictly hyperbolic systems, without any requirement of piecewise genuine nonlinearity, which resembles the construction for the scalar conservation law described earlier in this section.

Once wave curves satisfying (9.3.2) are in place, one may employ the construction of the solution to the Riemann problem, described above, thus arriving at the following generalization of Theorem 9.4.1:

9.5.1 Theorem. *Assume the system* (9.1.1) *is strictly hyperbolic. For* $|U_R - U_L|$ *sufficiently small, there exists a unique self-similar solution* (9.1.2) *of the Riemann problem* (9.1.1), (9.1.18)*, with small total variation. This solution comprises* $n+1$ *constant states* $U_L = U_0\,, U_1\,, \cdots, U_{n-1}\,, U_n = U_R$*, and* U_i *is joined to* U_{i-1} *by an admissible i-wave, composed of i-rarefaction waves and (at most countable) i-shocks which satisfy the Liu E-condition.*

9.6 Failure of Existence or Uniqueness; Delta Shocks and Transitional Waves

The orderly picture painted by Theorem 9.5.1 breaks down when one leaves the realm of strictly hyperbolic systems and waves of small amplitude.

The following exemplifies the difficulties that may be encountered in the construction of solutions. We consider the isentropic flow of an infinitely long column of a polytropic gas, with equation of state $p = \frac{1}{\gamma}\rho^{\gamma}$, $\gamma > 1$, under the following initial conditions. The density is constant $\bar{\rho} > 0$, throughout the length of the column. The right half of the column is subjected to a uniform impulse $\bar{\rho}\bar{v} > 0$, while the left half is subjected to an equal and opposite impulse $-\bar{\rho}\bar{v}$. Thus, in Lagrangian coordinates, we have to solve the Riemann problem for the system (7.1.11), with $\sigma(u) = -\frac{1}{\gamma}u^{-\gamma}$ for initial data $(u_L, v_L) = (\bar{u}, -\bar{v})$ and $(u_R, v_R) = (\bar{u}, \bar{v})$, where $\bar{u} = 1/\bar{\rho}$.

With reference to (9.4.1) and (9.4.2), it is clear that any intersection of the forward 1-wave curve through $(\bar{u}, -\bar{v})$ with the backward 2-wave curve through $(\bar{u}, \bar{v})$ will take place at $u_M > \bar{u}$, so that the jump discontinuity at the origin will be resolved into two rarefaction waves. In that range,

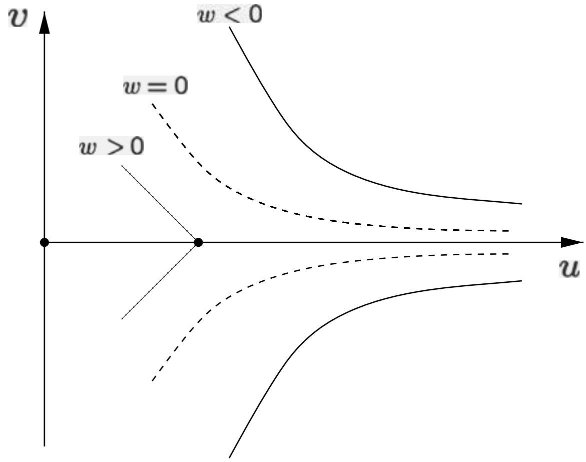

Fig. 9.6.1

(9.6.1)
$$\begin{cases} \varphi(u;\bar u,\bar v) = \frac{2}{1-\gamma}\left(u^{\frac{1-\gamma}{2}} - w\right), & u \geq \bar u, \\ \psi(u;\bar u,\bar v) = -\frac{2}{1-\gamma}\left(u^{\frac{1-\gamma}{2}} - w\right), & u \geq \bar u, \end{cases}$$

where we have set

(9.6.2)
$$w = \bar u^{\frac{1-\gamma}{2}} + \tfrac{1-\gamma}{2}\bar v.$$

The form of the solution will depend on the sign of w. Figure 9.6.1 depicts the wave curves when $w>0, w=0$ or $w<0$.

When $w>0$, the wave curves intersect at $u_M = w^{\frac{2}{1-\gamma}}$, $v_M = 0$, and the Riemann problem admits the solution

(9.6.3)
$$(u(\xi),v(\xi)) = \begin{cases} (\bar u, -\bar v), & -\infty < \xi \leq -\xi_F \\ \left(|\xi|^{-\frac{2}{\gamma+1}},\ -\frac{2}{\gamma-1}|\xi|^{\frac{\gamma-1}{\gamma+1}} + \frac{2}{\gamma-1}|\xi_0|^{\frac{\gamma-1}{\gamma+1}}\right), & -\xi_F < \xi \leq -\xi_S \\ (u_M, 0), & -\xi_S < \xi < \xi_S \\ \left(|\xi|^{-\frac{2}{\gamma+1}},\ \frac{2}{\gamma-1}|\xi|^{\frac{\gamma-1}{\gamma+1}} - \frac{2}{\gamma-1}|\xi_0|^{\frac{\gamma-1}{\gamma+1}}\right), & \xi_S \leq \xi < \xi_F \\ (\bar u, \bar v), & \xi_F \leq \xi < \infty, \end{cases}$$

where $\xi_F = \bar u^{-\frac{\gamma+1}{2}}$ and $\xi_S = w^{\frac{\gamma+1}{\gamma-1}}$.

When $w=0$, the two wave curves intersect at infinity. As $w \downarrow 0$, $(u(\xi),v(\xi))$ of (9.6.3) reduces to

(9.6.4)
$$(u(\xi),v(\xi)) = \begin{cases} (\bar{u},-\bar{v}), & \infty < \xi \le -\xi_F \\ \left(|\xi|^{-\frac{2}{\gamma+1}},\ \frac{2}{\gamma-1}\operatorname{sgn}\xi|\xi|^{\frac{\gamma-1}{\gamma+1}}\right), & -\xi_F < \xi < \xi_F \\ (\bar{u},\bar{v}), & \xi_F \le \xi < \infty. \end{cases}$$

Notice that the singularity of u at $\xi = 0$ is integrable while v is continuous. It is then easy to check that $(u(\xi),v(\xi))$, defined by (9.6.4), satisfies, in the sense of distributions, (9.1.3) for the system (7.1.11), namely

(9.6.5)
$$\begin{cases} (-v-\xi u)^{\cdot} + u = 0 \\ \left(\frac{1}{\gamma}u^{-\gamma} - \xi v\right)^{\cdot} + v = 0, \end{cases}$$

and thus solves the Riemann problem for $w = 0$. That $u(0) = \infty$ simply means that the density ρ vanishes along the line $x = 0$.

When $w < 0$, the two wave curves fail to intersect, even at infinity, and no standard solution to the Riemann problem may thus be constructed. The physical problem is of course still solvable. Indeed, we may reformulate and solve it, in Eulerian coordinates, as a Riemann problem for the system (7.1.13), where $\kappa = 1/\gamma$, with data: $\rho(x,0) = \bar{\rho}$, $x \in (-\infty,\infty)$; $v(x,0) = -\bar{v}$, $x \in (-\infty,0)$; and $v(x,0) = \bar{v}$, $x \in (0,\infty)$. The solution comprises two rarefaction waves whose tail ends recede from each other with respective speeds $\pm\frac{2}{\gamma-1}w$, leaving in the wake between them a *vacuum state* where ρ vanishes. The discussion of Section 2.2 does not cover the mapping of this flow to Lagrangian coordinates, because the determinant ρ^{-1} of the deformation gradient is unbounded on the vacuum state. In fact, under this change of coordinates the full sector in physical space-time occupied by the vacuum state is mapped to the single line $x = 0$ in the reference space-time. Consequently, $u = \rho^{-1}$ becomes very singular at $\xi = 0$:

(9.6.6)
$$(u(\xi),v(\xi)) = \begin{cases} (\bar{u},-\bar{v}), & -\infty < \xi \le -\xi_F \\ \left(|\xi|^{-\frac{2}{\gamma+1}} - \frac{4}{\gamma-1}w\delta_0,\ \frac{2}{\gamma-1}\operatorname{sgn}\xi\left[|\xi|^{\frac{\gamma-1}{\gamma+1}} - w\right]\right), & -\xi_F < \xi < \xi_F \\ (\bar{u},\bar{v}), & \xi_F \le \xi < \infty, \end{cases}$$

where δ_0 denotes the Dirac delta function at the origin. In fact, $(u(\xi),v(\xi))$ of (9.6.6) is a distributional solution of (9.6.5), providing one regards $u^{-\gamma}(\xi)$ as a continuous function that vanishes at the origin $\xi = 0$. In the (x,t) coordinates, (9.6.6) induces the stationary singularity $-\frac{4}{\gamma-1}wt\delta_0$ on u, along the t-axis. This new type of singularity that supports point masses is called a *delta shock*.

One might argue that the delta shock appeared here because we employed Lagrangian coordinates, which are ill-suited to this problem. It turns out, however, that delta shocks are often present in solutions of the Riemann problem, especially for systems that are not strictly hyperbolic. Relevant references are cited in Section 9.12. As we shall see in Section 9.8, the method of vanishing viscosity contributes insight into the formation of delta shocks.

Failure of strict hyperbolicity is also a source of difficulties in regard to uniqueness of solutions to the Riemann problem. The Liu E-condition is no longer sufficiently selective to single out a unique solution. To illustrate this, let us consider the Riemann problem for the model system (7.2.11), with data $(u_L, v_L) = (1,0)$ and $(u_R, v_R) = (a,0)$, where $a \in (-\frac{1}{2}, 0)$. One solution comprises the two constant states $(1,0)$ and $(a,0)$ joined by an overcompressive shock, of speed $s = 1 + a + a^2$, which satisfies the Liu E-condition. There is, however, another solution comprising three constant states, $(1,0), (-1,0)$ and $(a,0)$, where $(-1,0)$ is joined to $(1,0)$ by a 1-contact discontinuity of speed 1 and $(a,0)$ is joined to $(-1,0)$ by a 2-shock of speed $s = 1 - a + a^2$. Both shocks satisfy the Liu E-condition. Following the discussion on this system in Section 8.6, one may be inclined to disqualify overcompressive shocks, in which case the second solution of the Riemann problem emerges as the admissible one. This of course hinges on the premise that (8.6.4) is the proper dissipative form of (7.2.11).

The issue of nonuniqueness also arises in the context of (usually not strictly hyperbolic) systems that admit undercompressive shocks. Consider, for definiteness, such a system of two conservation laws. Any undercompressive shock is crossed by both 1-characteristics, from right to left, and 2-characteristics, from left to right. Consequently, such a shock may be incorporated into a wave fan that contains a compressive 1-shock, or 1-rarefaction wave, on its left, and a compressive 2-shock, or 2-rarefaction wave, on its right. In that capacity, the undercompressive shock serves as a "bridge" joining the two characteristic families and so is dubbed a *transitional wave*. It is also possible to have rarefaction transitional waves that are composites of a 1-rarefaction and a 2-rarefaction. These may occur when the 1-rarefaction wave curve and the 2-rarefaction wave curve meet tangentially on the line along which strict hyperbolicity fails, $\lambda_1(U) = \lambda_2(U)$. The possibility of including transitional waves renders the family of solutions to the Riemann problem richer and thereby the issue of uniqueness thornier. As pointed out in Chapter VIII, viscosity or viscosity-capillarity conditions, as well as kinetic relations, are being used as admissibility criteria for these undercompressive shocks. The importance of working with genuine physical systems cannot be overemphasized at this point.

9.7 The Entropy Rate Admissibility Criterion

According to the entropy shock admissibility criterion, the entropy production across shocks, defined by the left-hand side of (8.5.1), must be negative, so in particular the total entropy shall be decreasing. We have seen, however, that this requirement is generally insufficiently discriminating to rule out all spurious solutions. A wave fan admissibility criterion will be introduced here, which is a strengthened version of the entropy admissibility condition, as it stipulates that the combined entropy production of all shocks in the fan is not just negative but as small as possible, or equivalently, as it turns out, that the total entropy is not just decreasing but actually decreasing at the highest allowable rate.

We assume that our system (9.1.1) is endowed with a designated entropy-entropy flux pair $(\eta(U), q(U))$ and consider the admissibility of wave fans $U(x,t) = V(x/t)$, with prescribed end-states $V(-\infty) = U_L$ and $V(\infty) = U_R$. The combined entropy production of the shocks in V is given by

$$\mathscr{P}_V = \sum_{\xi} \{q(V(\xi+)) - q(V(\xi-)) - \xi\,[\eta(V(\xi+)) - \eta(V(\xi-))]\}, \tag{9.7.1}$$

where the summation runs over the at most countable set of points ξ of jump discontinuity of V.

Because of the Rankine-Hugoniot jump condition (9.1.8), for any $\Theta \in \mathbb{M}^{1\times n}$ and $a \in \mathbb{R}$, the entropy-entropy flux pair $(\eta(U) + \Theta U + a\,, q(U) + \Theta F(U))$ yields the same value for the combined entropy production as $(\eta(U), q(U))$. One may thus assume, without loss of generality, that

$$\eta\left(U_L\right) = \eta\left(U_R\right) = 0. \tag{9.7.2}$$

After this normalization, the rate of change of the total entropy in the wave fan is given by

$$\dot{\mathscr{H}}_V = \frac{d}{dt}\int_{-\infty}^{\infty} \eta(U(x,t))dx = \frac{d}{dt}\int_{-\infty}^{\infty} \eta\left(V\left(\frac{x}{t}\right)\right)dx = \int_{-\infty}^{\infty} \eta(V(\xi))d\xi. \tag{9.7.3}$$

Actually, $\mathscr{P}_V$ and $\dot{\mathscr{H}}_V$ are related through

$$\dot{\mathscr{H}}_V = \mathscr{P}_V + q\left(U_L\right) - q\left(U_R\right). \tag{9.7.4}$$

To verify this, begin with the identity

$$\eta(V(\xi)) = [\xi\,\eta(V(\xi)) - q(V(\xi))]^{\cdot} + \dot{q}(V(\xi)) - \xi\,\dot{\eta}(V(\xi)), \tag{9.7.5}$$

which holds in the sense of measures, and note that the generalized chain rule, Theorem 1.7.5, yields

$$\dot{q}(V) - \xi\,\dot{\eta}(V) = \left[\widetilde{\mathrm{D}q}(V) - \xi\,\widetilde{\mathrm{D}\eta}(V)\right]\dot{V}. \tag{9.7.6}$$

From (9.7.6), (7.4.1) and (9.1.6) it follows that the measure $\dot{q}(V) - \xi\,\dot{\eta}(V)$ is concentrated in the set of points of jump discontinuity of V. Therefore, combining (9.7.1), (9.7.2), (9.7.3) and (9.7.5), one arrives at (9.7.4).

9.7.1 Definition. A wave fan $U(x,t) = V(x/t)$, with $V(-\infty) = U_L\,, V(\infty) = U_R$, meets the *entropy rate admissibility criterion* if $\mathscr{P}_V \leq \mathscr{P}_{\bar{V}}$, or equivalently $\dot{\mathscr{H}}_V \leq \dot{\mathscr{H}}_{\bar{V}}$, holds for any other wave fan $\bar{U}(x,t) = \bar{V}(x/t)$ with the same end-states, namely $\bar{V}(-\infty) = U_L\,, \bar{V}(\infty) = U_R$.

In its connection to continuum physics, the entropy rate admissibility criterion is a more stringent version of the Second Law of thermodynamics: not only should the physical entropy increase, but it should be increasing at the maximum rate allowed by the balance laws of mass, momentum and energy. The kinetic theory seems to lend some credence to that thesis, at least for waves of small amplitude (references in Section 9.12). However, the status of the entropy rate principle shall ultimately be judged on the basis of its implications in the context of familiar systems, and its comparison to other, firmly established, admissibility conditions. This will be our next task.

We begin our investigation by testing the entropy rate criterion on the scalar conservation law. In that case a wave fan consists of a single, generally composite, wave.

9.7.2 Theorem. *For the scalar conservation law* (7.1.2), *a wave fan satisfies the entropy rate criterion for an arbitrary designated strictly convex entropy η if and only if every shock satisfies the Oleinik E-condition* (8.4.3).

Proof. For simplicity, we confine the proof to the class of unidirectional wave fans, even though the assertion is true even without that restriction. Let $u(x,t) = v(x/t)$ be any unidirectional wave fan for (7.1.2), with prescribed end-states u_L and u_R. Assuming for definiteness that $u_L < u_R$ (the opposite case, $u_L > u_R$, is similar), v is a nondecreasing function on $(-\infty,\infty)$ solving the differential equation

$$[f(v(\xi)) - \xi v(\xi)]^{\cdot} + v(\xi) = 0, \qquad -\infty < \xi < \infty, \tag{9.7.7}$$

with boundary conditions $v(-\infty) = u_L$ and $v(\infty) = u_R$.

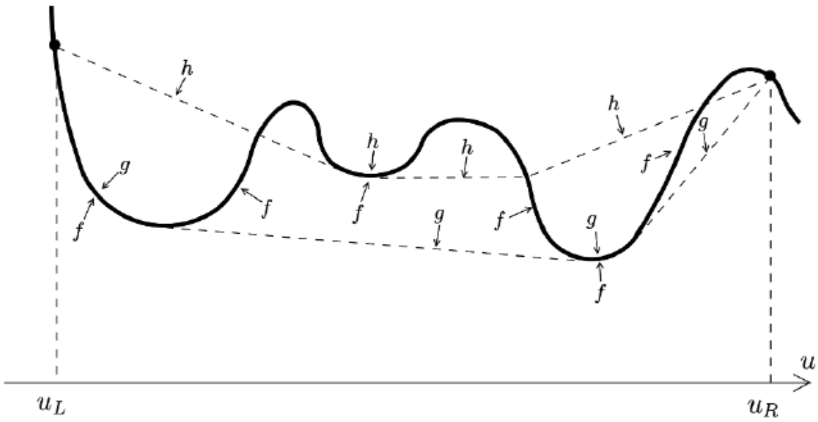

Fig. 9.7.1

We extend v into a maximal monotone multivalued function $\tilde{v}$, by assigning the interval $[v(\xi-), v(\xi+)]$ as values of $\tilde{v}$ at any point ξ of jump discontinuity of v. We then consider the (maximal monotone) inverse function ξ of $\tilde{v}$ and set

$$h(u) = f(u_L) + \int_{u_L}^{u} \xi(v)dv, \qquad u_L \le u \le u_R\,. \tag{9.7.8}$$

We notice that h is a convex function on $[u_L, u_R]$ with $h(u_L) = f(u_L)$, $h(u_R) = f(u_R)$. Furthermore, for any $u \in (u_L, u_R)$, either $h(u) = f(u)$ or $h(u) \neq f(u)$ and

$$h(u) = \frac{(u_+ - u)f(u_-) + (u - u_-)f(u_+)}{u_+ - u_-}\,, \tag{9.7.9}$$

with $u_L \le u_- < u_+ \le u_R$. Thus the graph of h, depicted in Fig. 9.7.1, is a concatenation of arcs of the graph of f, corresponding to the rarefaction waves of v, and chords of the graph of f, corresponding to the shocks of v. Furthermore, the constant states of v correspond to the points of jump discontinuity of h'. In particular, the h induced by the unique wave fan, constructed in Section 9.5, whose shocks satisfy the Oleinik E-condition, is the lower convex envelope of the graph of f over $[u_L, u_R]$, i.e., the function g depicted in Figs 9.5.1 and 9.7.1. Conversely, any convex function h on $[u_L, u_R]$ with the above properties generates a wave fan v of (7.1.2) with end-points u_L and u_R.

We now designate some convex entropy function η, with associated flux q, $q'(u) = \eta'(u)f'(u)$, and calculate the entropy production of the shocks in v through (9.7.1):

$$\begin{aligned} P_v &= \sum_{\text{shocks}} \{q(v(\xi+)) - q(v(\xi-)) - \xi[\eta(v(\xi+)) - \eta(v(\xi-))]\} \\ &= \int_{u_L}^{u_R} [q'(v) - h'(v)\eta'(v)]dv \\ &= \int_{u_L}^{u_R} [f'(v) - h'(v)]\eta'(v)dv \\ &= \int_{u_L}^{u_R} [h(v) - f(v)]\eta''(v)dv. \end{aligned} \tag{9.7.10}$$

Since $h(v) \ge g(v)$, for $v \in [u_L, u_R]$, P_v is minimized when $h = g$, i.e., v is the unique wave fan with shocks satisfying the Oleinik E-condition, constructed in Section 9.5. This completes the proof.

We now turn to general strictly hyperbolic systems but limit our investigation to shocks with small amplitude:

9.7.3 Theorem. *For any strictly hyperbolic system* (9.1.1) *of conservation laws, with designated entropy-entropy flux pair* (η, q), *where* η *is (locally) uniformly convex, a wave fan with waves of moderate strength may satisfy the entropy rate admissibility criterion only if every shock satisfies the Liu E-condition.*

Proof. We will establish the assertion of the theorem by contradiction: given any wave fan containing some shock that violates the Liu E-condition, we will construct a wave fan with lower entropy rate. In order to convey the essence of the argument with minimal technicalities, we will discuss, in full detail, the special case where the given wave fan consists of a single shock. Then we will describe briefly how to handle the general case.

Accordingly, assume that states U_L and U_R, with $|U_R - U_L| = \delta$, positive and small, are joined by an i-shock of speed s,

$$F(U_R) - F(U_L) = s[U_R - U_L], \tag{9.7.11}$$

that violates the Liu E-condition. Under the normalization assumption (9.7.2), the entropy rate for this shock is zero. The aim is to construct a wave fan V, with the same end-states U_L, U_R and $\dot{\mathscr{H}}_V < 0$.

Let $U_R = W_i(\tau_R; U_L)$. For definiteness, assume $\tau_R > 0$. Since the Liu E-condition is violated, the set of $\tau \in (0, \tau_R)$ with $s_i(\tau; U_L) < s$ is nonempty. Let τ_L be the infimum of that set, and assume $\tau_L > 0$, as the case $\tau_L = 0$ is simpler. Thus, $s_i(\tau_L; U_L) = s$, and $s_i(\tau; U_L)$ is decreasing at τ_L. We consider the generic case $\dot{s}_i(\tau_L; U_L) < 0$.

Upon setting $U_M = W_i(\tau_L; U_L)$,

$$F(U_M) - F(U_L) = s[U_M - U_L]. \tag{9.7.12}$$

Combining (9.7.11) and (9.7.12),

$$F(U_R) - F(U_M) = s[U_R - U_M], \tag{9.7.13}$$

which shows that U_M also lies on the i-shock curve emanating from U_R, namely that $U_M = W_i(\tau_R; U_R)$ and $s_i(\tau_R; U_R) = s$.

One may thus regard the i-shock joining U_L with U_R as the superposition of two i-shocks, one that joins U_L with U_M and one that joins U_M with U_R, both propagating with the same speed s. The aim is to perform a perturbation that splits the original shock into two shocks, one with speed slightly lower than s and the other with speed slightly higher than s, and then show that the resulting wave fan has negative entropy rate.

We begin by fixing a small positive number τ. The wave fan V will consist of $(n+2)$ constant states $U_L = U_0, \cdots, U_{i-1}, \tilde{U}, U_i, \cdots, U_n = U_R$, joined by shocks. For $j = 1, \cdots, i-1$, U_j is joined to U_{j-1} by a j-shock, $U_j = W_j(\varepsilon_j; U_{j-1})$, of speed s_j:

$$F(U_j) - F(U_{j-1}) = s_j[U_j - U_{j-1}], \qquad j = 1, \cdots, i-1. \tag{9.7.14}$$

$\tilde{U}$ is joined to U_{i-1} by an i-shock, $\tilde{U} = W_i(\tau_L + \tau; U_{i-1})$, of speed s_-:

$$F(\tilde{U}) - F(U_{i-1}) = s_-[\tilde{U} - U_{i-1}]. \tag{9.7.15}$$

$\tilde{U}$ is also joined to U_i by an i-shock, $\tilde{U} = W_i(\tau_R + \varepsilon_i; U_i)$, of speed s_+:

$$F(\tilde{U}) - F(U_i) = s_+[\tilde{U} - U_i]. \tag{9.7.16}$$

Finally, for $j = i+1, \cdots, n$, U_{j-1} is joined to U_j by a j-shock, $U_{j-1} = W_j(\varepsilon_j; U_j)$, of speed s_j:

$$F(U_{j-1}) - F(U_j) = s_j[U_{j-1} - U_j], \qquad j = i+1, \cdots, n. \tag{9.7.17}$$

The amplitudes $(\varepsilon_1, \cdots, \varepsilon_n)$ of the waves are computed from the equation

$$\Omega(\varepsilon_1, \cdots, \varepsilon_n; \tau) = 0, \tag{9.7.18}$$

where

$$\begin{aligned} \Omega(\varepsilon_1, \cdots, \varepsilon_n; \tau) = {} & W_i(\tau_R + \varepsilon_i; W_{i+1}(\varepsilon_{i+1}; \cdots W_n(\varepsilon_n; U_R) \cdots)) \\ & - W_i(\tau_L + \tau; W_{i-1}(\varepsilon_{i-1}; \cdots W_1(\varepsilon_1; U_L) \cdots)). \end{aligned} \tag{9.7.19}$$

Observing that

$$\begin{aligned} \Omega(\varepsilon_1, \cdots, \varepsilon_n; \tau) = {} & -\sum_{j=1}^{i-1} \varepsilon_j R_j(U_L) + \sum_{j=i+1}^{n} \varepsilon_j R_j(U_R) \\ & - \tau \dot{W}_i(\tau_L; U_L) + \varepsilon_i \dot{W}_i(\tau_R; U_R) + G(\varepsilon_1, \cdots, \varepsilon_n; \tau), \end{aligned} \tag{9.7.20}$$

where G and its first derivatives vanish at $(0 \cdots, 0; 0)$, we conclude that, for τ sufficiently small, (9.7.15) has a unique solution. One still needs to verify that $s_- = s_i(\tau_L + \tau; U_{i-1})$ is smaller than $s_+ = s_i(\tau_R + \varepsilon_i; U_i)$ and this will be done below.

By combining (9.7.11), (9.7.14), (9.7.15), (9.7.16) and (9.7.17), we deduce

$$\begin{aligned} & (s - s_-)[\tilde{U} - U_L] + (s - s_+)[U_R - \tilde{U}] \\ & \qquad = \sum_{j=1}^{i-1} (s_j - s_-)[U_j - U_{j-1}] + \sum_{j=i+1}^{n} (s_j - s_+)[U_j - U_{j-1}]. \end{aligned} \tag{9.7.21}$$

For $j = 1, \cdots, i-1, i+1, \cdots, n$,

$$U_j - U_{j-1} = a_j R_j(U_L) + S_j, \qquad |S_j| = O(\delta)|a_j|. \tag{9.7.22}$$

Thus, upon setting

$$\ell = |(s - s_-) L_i(U_L)[\tilde{U} - U_L]|, \tag{9.7.23}$$

we deduce, from (9.7.21):

$$|a_j| = O(\delta)\ell, \qquad j = 1, \cdots, i-1, i+1, \cdots, n, \tag{9.7.24}$$

$$(s - s_-)[\tilde{U} - U_L] + (s - s_+)[U_R - \tilde{U}] = O(\delta^2)\ell, \tag{9.7.25}$$

which implies, in particular, that $(s - s_-)$ and $(s - s_+)$ have opposite signs. Recalling that $\dot{s}_i(\tau_L; U_L) < 0$, we conclude that $s_- < s < s_+$.

We now compute $\dot{\mathscr{H}}_V$ from (9.7.3):

$$\dot{\mathscr{H}}_V = \sum_{j=1}^{i-2}(s_{j+1}-s_j)\eta(U_j)+(s_- - s_{i-1})\eta(U_{i-1})+(s_+ - s_-)\eta(\tilde{U}) + (s_{i+1}-s_+)\eta(U_i)+\sum_{j=i+1}^{n-1}(s_{j+1}-s_j)\eta(U_j). \tag{9.7.26}$$

Upon rearranging the terms in the above summations:

$$\dot{\mathscr{H}}_V = \sum_{j=1}^{i-1}(s_- - s_j)[\eta(U_j)-\eta(U_{j-1})]+(s_+ - s_-)\eta(\tilde{U}) + \sum_{j=i+1}^{n}(s_+ - s_j)[\eta(U_j)-\eta(U_{j-1})]. \tag{9.7.27}$$

By adding and subtracting terms, the above may be rewritten as

$$\begin{aligned}\dot{\mathscr{H}}_V = {} & (s-s_-)\{\eta(\tilde{U})-\mathrm{D}\eta(\tilde{U})[\tilde{U}-U_L]-\eta(U_L)\} \\ & +(s_+ - s)\{\eta(\tilde{U})-\mathrm{D}\eta(\tilde{U})[\tilde{U}-U_R]-\eta(U_R)\} \\ & +\sum_{j=1}^{i-1}(s_- - s_j)\{\eta(U_j)-\eta(U_{j-1})-\mathrm{D}\eta(\tilde{U})[U_j-U_{j-1}]\} \\ & +\sum_{j=i+1}^{n}(s_+ - s_j)\{\eta(U_j)-\eta(U_{j-1})-\mathrm{D}\eta(\tilde{U})[U_j-U_{j-1}]\}.\end{aligned} \tag{9.7.28}$$

Since η is strictly convex, and recalling (9.7.23) and (9.7.25),

$$\begin{cases} -(s-s_-)\{\eta(\tilde{U})-\mathrm{D}\eta(\tilde{U})[\tilde{U}-U_L]-\eta(U_L)\} \geq \alpha\ell|\tilde{U}-U_L| \\ -(s_+ - s)\{\eta(\tilde{U})-\mathrm{D}\eta(\tilde{U})[\tilde{U}-U_R]-\eta(U_R)\} \geq \alpha\ell|\tilde{U}-U_R|, \end{cases} \tag{9.7.29}$$

with $\alpha > 0$. On the other hand, by virtue of (9.7.24),

$$\begin{cases} |\eta(U_j)-\eta(U_{j-1})-\mathrm{D}\eta(\tilde{U})[U_j-U_{j-1}]| \leq O(\delta)\ell|\tilde{U}-U_L|, \quad j=1,\cdots,i-1 \\ |\eta(U_j)-\eta(U_{j-1})-\mathrm{D}\eta(\tilde{U})[U_j-U_{j-1}]| \leq O(\delta)\ell|\tilde{U}-U_R|, \quad j=i+1,\cdots,n. \end{cases} \tag{9.7.30}$$

Hence, $\dot{\mathscr{H}}_V < 0$, which establishes the assertion of the theorem in the special case where the wave fan consists of a single shock violating the Liu E-condition.

The general case where the shock that violates the Liu E-condition is part of a wave fan $\bar{V}$ is treated in a similar fashion. One perturbs $\bar{V}$ in such a way that the shock violating the Liu E-condition splits into two shocks, decreasing the entropy rate, while the change in the entropy rate resulting from the perturbation of the other waves is of higher order. The process of perturbing $\bar{V}$ is tedious but conceptually straightforward. The reader may consult the references cited in Section 9.12. This completes the proof.

Theorem 9.7.4 motivates an alternative construction of the admissible solution to the Riemann problem (9.1.1), (9.1.18) by minimizing the entropy rate functional $\dot{\mathscr{H}}_V$ over all wave fans with end-states U_L and U_R.

9.7.4 Theorem. *Consider any strictly hyperbolic system* (9.1.1) *that is endowed with a uniformly convex entropy $\eta(U)$. When $|U_R - U_L|$ is sufficiently small, there exists a solution $U(x,t) = V(x/t)$ of the Riemann problem* (9.1.1), (9.1.18), *where $V(\xi)$ minimizes the entropy rate $\dot{\mathscr{H}}_V$, or equivalently the total entropy production $\mathscr{P}_V$, over all wave fans with unidirectional i-waves of moderate strength and end-states U_L and U_R. Furthermore, this solution is identical to the unique solution with shocks satisfying the Liu E-condition, established by Theorem* 9.5.1.

Proof. Assume U_L and U_R lie in a ball $\mathscr{B}_{\delta^2}(\bar{U})$ with center at some state $\bar{U}$ and radius δ^2, where δ is a small positive number. We consider the family of outgoing wave fans with values in the ball $\mathscr{B}_{\delta}(\bar{U})$, end-states U_L, U_R and unidirectional i-waves. This family is nonempty, as it contains the wave fan with end-states U_L and U_R whose i-waves are (not necessarily admissible) i-shocks. Indeed, to construct this wave fan we define, in the place of (9.3.4),

$$\Omega(\varepsilon;U) = W_n(\varepsilon;W_{n-1}(\varepsilon;\dots W_1(\varepsilon_1,U)\dots)), \tag{9.7.31}$$

where W_i denotes the i-shock curve, and observe that (9.3.5) still holds with G and its first derivatives vanishing at $\varepsilon = 0$. Therefore, for δ sufficiently small there exists a unique n-tuple $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)$ with $\Omega(\varepsilon;U_L) = U_R$, and the resulting wave fan takes values in $\mathscr{B}_{\delta}(\bar{U})$.

We now fix some wave fan V in the above family, which as shown in Section 9.1, comprises n waves $[\zeta_1, \xi_1],\dots,[\zeta_n, \xi_n]$ separated by constant states $U_0, U_1, \dots, U_n$. In what follows, c stands for a generic positive constant that does not depend on δ and V.

We write

$$U_i - U_{i-1} = \sum_{k=1}^{n} a_{ik} R_k(\bar{U}), \qquad i = 1,\dots,n, \tag{9.7.32}$$

noting that $|a_{ik}| \le c\delta$, for $i,k = 1,\dots,n$. Furthermore, by (9.1.6), $|a_{ik}| \le c\delta^2$, for $i \neq k$. On the other hand, since

$$\sum_{i=1}^{n} (U_i - U_{i-1}) = U_R - U_L, \tag{9.7.33}$$

$$\sum_{i=1}^{n} a_{ik} = L_k(\bar{U})[U_R - U_L], \qquad k = 1, \dots, n. \tag{9.7.34}$$

Hence $|U_i - U_{i-1}| \leq c\delta^2$, for $i = 1, \dots, n$, which implies

$$TV_{(-\infty,\infty)}V(\cdot) \leq c\delta^2, \tag{9.7.35}$$

by virtue of (9.1.17).

It now follows from Helly's theorem that, within the family of wave fans under consideration, any minimizing sequence $\{V_\ell\}$ for the functional (9.7.3) contains a subsequence converging to a minimizer V. On account of Theorem 9.7.3, any shock of V must satisfy the Liu E-condition and thus V coincides with the unique admissible solution to the Riemann problem established by Theorem 9.5.1. This completes the proof.

The effectiveness of the entropy rate criterion established above in general strictly hyperbolic systems, for waves of moderate strength, does not necessarily extend to the realm of strong waves, as the following situation may arise. For a wave fan containing a rarefaction wave of moderate strength consider a modified wave fan in which the rarefaction wave has been replaced by a rarefaction shock. The resulting net change in the entropy rate will be the sum of the positive term contributed by the rarefaction shock and the possibly negative term accounting for the change in speed and amplitude of the remaining shocks in the wave fan. As noted in the closing paragraph of the proof of Theorem 9.7.3, the second term is of lower order than the first when all the waves are of moderate strength. However, this is no longer necessarily true in the presence of very strong waves, so that the modified wave fan, with the rarefaction shock, may have lower entropy rate than the original one. Whether the entropy rate criterion selects the same solution as other shock admissibility conditions, for strong shocks, will depend on the detailed structure of the Hugoniot locus for the particular system, and thus should be tested on a case by case basis. The conclusions of such tests are recorded below. The proofs, which usually require lengthy calculations, are found in the literature cited in Section 9.11.

Since the entropy rate criterion was motivated by the Second Law of thermodynamics, it is natural to test it on the systems of conservation laws of gas dynamics.

9.7.5 Theorem. *For the system of isentropic or isothermal rectilinear flow of an ideal gas, namely* (7.1.11) *with* $\sigma = -p$, $p = \kappa u^{-\gamma}$, $\gamma \geq 1$, *and designated entropy the mechanical energy, i.e.,* $\eta = \frac{1}{2}v^2 + \frac{\kappa}{\gamma-1}u^{1-\gamma}$, *when* $\gamma > 1$, *or* $\eta = \frac{1}{2}v^2 + \kappa \log u$, *when* $\gamma = 1$, *the standard solution to the Riemann problem, with shocks satisfying the Lax E-condition, minimizes the entropy rate.*

9.7.6 Theorem. *For the general system of rectilinear flow of an ideal gas, namely* (7.1.8) *with equations of state* (2.5.20), *for* $\gamma \geq 5/3$, *and designated entropy minus the physical entropy, i.e.,* $\eta = -s$, *the standard solution to the Riemann problem, with shocks satisfying the Lax E-condition, minimizes the entropy rate.*

The reader should note that $5/3$ is the value for the adiabatic exponent γ predicted by the kinetic theory in the case of a monatomic ideal gas. When $\gamma < 5/3$ (polyatomic gases), the situation is different. Consider a wave fan comprising three constant states (u_L, v_L, s_L), (u_M, v_M, s_M) and (u_R, v_R, s_R), where the first two are joined by a stationary 2-contact discontinuity, while the second and the third are joined by a 3-rarefaction wave. In particular, we have $v_M = v_L$, $p(u_M, s_M) = p(u_L, s_L)$, $s_M = s_R$, and $z(u_M, v_M, s_M) = z(u_R, v_R, s_R)$, where $z(u, v, s)$ denotes the second 3-Riemann invariant listed in (7.3.4). The total entropy production of this wave fan is of course zero. For u_R/u_L in a certain range, there is a second wave fan with the same end-states, which comprises four constant states (u_L, v_L, s_L), (u_1, v_1, s_1), (u_2, v_2, s_2) and (u_R, v_R, s_R), where the first two are joined by a 1-shock that satisfies the Lax E-condition, the second is joined to the third by a 2-contact discontinuity, while the last two are joined by a 3-shock that violates the Lax E-condition. It turns out that when u_M/u_L is not too large, i.e., the contact discontinuity is not too strong, the total entropy production of the second wave fan is positive, and hence the first wave fan has lower entropy rate. By contrast, when u_M/u_L is sufficiently large, the total entropy production of the second wave fan is negative and so the first wave fan no longer satisfies the entropy rate criterion.

Similar issues arise for systems that are not strictly hyperbolic. Let us consider our model system (7.2.11). Recall the two wave fans with the same end-states $(1,0)$ and $(a,0)$, $a \in (-\frac{1}{2}, 0)$, described in Section 9.6: The first one comprises the states $(1,0)$ and $(a,0)$, joined by an overcompressive shock of speed $1+a+a^2$. The second comprises three states, $(1,0), (-1,0)$ and $(a,0)$, where the first two are joined by a 1-contact discontinuity of speed 1, while the second is joined to the third by a 2-shock of speed $1-a+a^2$. If we designate the entropy-entropy flux pair

$$\eta = \tfrac{1}{2}(u^2+v^2), \qquad q = \tfrac{3}{4}(u^2+v^2)^2, \tag{9.7.36}$$

the entropy production of the overcompressive shock is $\frac{1}{4}(a^2-1)(1-a)^2$ while the entropy production of the second wave fan is $\frac{1}{4}(a^2-1)(1+a)^2$. Thus the entropy rate criterion favors the overcompressive shock, even though, as we saw in Section 8.6, this is incompatible with the stable shock profile condition. The reader should bear in mind, however, that these conclusions are tied to our selections for artificial viscosity and entropy. Whether (8.6.4) is the proper dissipative form and (9.7.36) is the natural entropy-entropy flux pair for (7.2.11) may be decided only when this system is considered in the context of some physical model.

9.8 Viscous Wave Fans

The viscous shock admissibility criterion, introduced in Section 8.6, characterizes admissible shocks for the hyperbolic system of conservation laws (9.1.1) as $\mu \downarrow 0$ limits of traveling wave solutions of the associated dissipative system (8.6.1). The aim here is to extend this principle from single shocks to general wave fans. The difficulty is that, in contrast to (9.1.1), the system (8.6.1) is not invariant under uniform stretching of the space-time coordinates and thus it does not possess traveling wave fans as solutions. As a remedy, it has been proposed that in the place of (8.6.1) one should employ a system with time-varying viscosity,

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu t \partial_x^2 U(x,t), \tag{9.8.1}$$

which is invariant under the transformation $(x,t) \mapsto (\alpha x, \alpha t)$. It is easily seen that $U = V_\mu(x/t)$ is a self-similar solution of (9.8.1) if and only if $V_\mu(\xi)$ satisfies the ordinary differential equation

$$\mu \ddot{V}_\mu(\xi) = \dot{F}(V_\mu(\xi)) - \xi \dot{V}_\mu(\xi). \tag{9.8.2}$$

A self-similar solution $U = V(x/t)$ of (9.1.1) is said to satisfy the *viscous wave fan admissibility criterion* if V is the almost everywhere limit, as $\mu \downarrow 0$, of a uniformly bounded family of solutions V_μ of (9.8.2).

In addition to serving as a test of admissibility, the viscous wave fan criterion suggests an alternative approach for constructing solutions to the Riemann problem (9.1.1), (9.1.12). Towards that end, one has to show that for any fixed $\mu > 0$ there exists some solution $V_\mu(\xi)$ of (9.8.2) on $(-\infty, \infty)$, with boundary conditions

$$V_\mu(-\infty) = U_L\,, \qquad V_\mu(+\infty) = U_R\,, \tag{9.8.3}$$

and then prove that the family $\{V_\mu(\xi) : 0 < \mu < 1\}$ has uniformly bounded variation on $(-\infty, \infty)$. In that case, by Helly's theorem (cf. Section 1.7), a convergent sequence $\{V_{\mu_m}\}$ may be extracted, with $\mu_m \downarrow 0$ as $m \to \infty$, whose limit V induces the solution $U(x,t) = V(x/t)$ to the Riemann problem.

The above program has been implemented successfully under a variety of conditions. One may solve the Riemann problem under quite general data U_L and U_R albeit for special systems, most notably for pairs of conservation laws. Alternatively, one may treat general systems but only in the context of waves of moderate strength, requiring that $|U_R - U_L|$ be sufficiently small. Let us consider this last situation first. The analysis is lengthy and technical so only the main ideas shall be outlined. For the details, the reader may consult the references cited in Section 9.11.

The crucial step is to establish a priori bounds on the total variation of $V_\mu(\xi)$ over $(-\infty, \infty)$, independent of μ. To prepare the ground for systems, let us begin with the scalar conservation law (7.1.2). Setting $\lambda(u) = f'(u)$ and $\dot{V}_\mu(\xi) = a(\xi)$, we write (9.8.2) in the form

$$\mu \dot{a} + [\xi - \lambda(V_\mu(\xi))]a = 0. \tag{9.8.4}$$

The solution of (9.8.4) is $a(\xi) = \tau\phi(\xi)$, where

$$\phi(\xi) = \frac{\exp[-\frac{1}{\mu}g(\xi)]}{\int_{-\infty}^{\infty} \exp[-\frac{1}{\mu}g(\zeta)]d\zeta}\,, \tag{9.8.5}$$

$$g(\xi) = \int_s^\xi [\zeta - \lambda(V_\mu(\zeta))]d\zeta. \tag{9.8.6}$$

The lower limit of integration s is selected so that $g(\xi) \geq 0$ for all ξ in $(-\infty,\infty)$. The amplitude τ is determined with the help of the assigned boundary conditions, that is $V_\mu(-\infty) = u_L$, $V_\mu(\infty) = u_R$, $\tau = u_R - u_L$. From (9.8.5) it follows that the L^1 norm of $a(\xi)$ is bounded, uniformly in μ, and so the family $\{V_\mu : 0 < \mu < 1\}$ has uniformly bounded variation on $(-\infty,\infty)$.

Turning now to general strictly hyperbolic systems (9.1.1), we realize $V_\mu(\xi)$ as the assemblage of (composite) waves associated with distinct characteristic families, by writing

$$\dot{V}_\mu(\xi) = \sum_{j=1}^{n} a_j(\xi) R_j(V_\mu(\xi)). \tag{9.8.7}$$

We substitute $\dot{V}_\mu$ from (9.8.7) into (9.8.2). Upon multiplying the resulting equation, from the left, by $L_i(V_\mu(\xi))$, we deduce

$$\mu \dot{a}_i + [\xi - \lambda_i(V_\mu(\xi))] a_i = \mu \sum_{j,k=1}^{n} \beta_{ijk}(V_\mu(\xi)) a_j a_k\,, \tag{9.8.8}$$

where

$$\beta_{ijk}(U) = -L_i(U) \mathrm{D}R_j(U) R_k(U). \tag{9.8.9}$$

In (9.8.8), the left-hand side coincides with the left-hand side of (9.8.4), for the scalar conservation law, while the right-hand side accounts for the interactions of distinct characteristic families. The reader should notice the analogy between (9.8.8) and (7.8.6). It should also be noted that when our system is endowed with a coordinate system $(w_1,\cdots,w_n)$ of Riemann invariants, $a_i(\xi) = \dot{w}_i(V_\mu(\xi))$. In that case, as shown in Section 7.3, for $j \neq k$, $\mathrm{D}R_j R_k$ lies in the span of $\{R_j, R_k\}$ and so (9.8.9) implies $\beta_{ijk} = 0$ when $i \neq j \neq k \neq i$. For special systems, such as (7.3.18), with coinciding shock and rarefaction wave curves, $\mathrm{D}R_j R_j$ is collinear to R_j and so $\beta_{ijk} = 0$ even when $i \neq j = k$ so that the equations in (9.8.8) decouple. In general, the thrust of the analysis is to demonstrate that in the context of solutions with small oscillation, i.e., a_i small, the effect of interactions, of quadratic order, will be even smaller.

The solution of (9.8.8) may be partitioned into

$$a_i(\xi) = \tau_i \phi_i(\xi) + \theta_i(\xi), \tag{9.8.10}$$

where

$$\phi_i(\xi) = \frac{\exp[-\frac{1}{\mu} g_i(\xi)]}{\int_{-\infty}^{\infty} \exp[-\frac{1}{\mu} g_i(\zeta)] d\zeta}\,, \tag{9.8.11}$$

$$g_i(\xi) = \int_{s_i}^{\xi} [\zeta - \lambda_i(V_\mu(\zeta))] d\zeta, \tag{9.8.12}$$

and $\theta_i(\xi)$ satisfies the equation

(9.8.13) $$\mu\dot{\theta}_i+[\xi-\lambda_i(V_\mu(\xi))]\theta_i=\mu\sum_{j,k=1}^{n}\beta_{ijk}(V_\mu(\xi))[\tau_j\phi_j(\xi)+\theta_j][\tau_k\phi_k(\xi)+\theta_k].$$

The differential equations (9.8.13) may be transformed into an equivalent system of integral equations by means of the variation of parameters formula:

(9.8.14)
$$\theta_i(\xi)=\phi_i(\xi)\int_{c_i}^{\xi}\phi_i^{-1}(\zeta)\beta_{ijk}(V_\mu(\zeta))[\tau_j\phi_j(\zeta)+\theta_j(\zeta)][\tau_k\phi_k(\zeta)+\theta_k(\zeta)]d\zeta.$$

Careful estimation shows that

(9.8.15) $$|\theta_i(\xi)|\leq c(\tau_1^2+\cdots+\tau_n^2)\sum_{j=1}^{n}\phi_j(\xi),$$

which verifies that, in (9.8.10), θ_i is subordinate to $\tau_i\phi_i$, i.e., the characteristic families decouple to leading order.

It can be shown, by means of a contraction argument, that for any fixed $(\tau_1,\cdots,\tau_n)$ in a small neighborhood of the origin, there exists some solution $V_\mu(\xi)$ of (9.8.2) on $(-\infty,\infty)$, which satisfies (9.8.7), (9.8.10) and (9.8.15). To solve the boundary value problem (9.8.2), (9.8.3), the $(\tau_1,\cdots,\tau_n)$ have to be selected so that

(9.8.16) $$\sum_{j=1}^{n}\int_{-\infty}^{\infty}[\tau_j\phi_j(\xi)+\theta_j(\xi)]R_j(V_\mu(\xi))d\xi=U_R-U_L\,.$$

It has been proved that (9.8.16) admits a unique solution $(\tau_1,\cdots,\tau_n)$, at least when $|U_R-U_L|$ is sufficiently small. The result is summarized in the following

9.8.1 Theorem. *Assume the system* (9.1.1) *is strictly hyperbolic on* $\mathscr{O}$ *and fix any state* $U_L\in\mathscr{O}$. *There is* $\delta>0$ *such that for any* $U_R\in\mathscr{O}$ *with* $|U_R-U_L|<\delta$ *and every* $\mu>0$, *the boundary value problem* (9.8.2), (9.8.3) *possesses a solution* $V_\mu(\xi)$, *which admits the representation* (9.8.7), (9.8.10) *with* $(\tau_1,\cdots,\tau_n)$ *close to the origin and* θ_i *obeying* (9.8.15). *Moreover, the family* $\{V_\mu(\xi):0<\mu<1\}$ *of solutions has uniformly bounded (and small) total variation on* $(-\infty,\infty)$. *In particular, one may extract a sequence* $\{V_{\mu_m}(\xi)\}$, *with* $\mu_m\downarrow 0$ *as* $m\to\infty$, *which converges, boundedly almost everywhere, to a function* $V(\xi)$ *such that the wave fan* $U=V(x/t)$ *solves the Riemann problem* (9.1.1), (9.1.18).

Careful analysis of the process that generates $V(\xi)$ as the limit of the sequence $\{V_{\mu_m}(\xi)\}$ reveals that $V(\xi)$ has the structure described in Theorem 9.3.1. Furthermore, for any point ξ of jump discontinuity of V, $V(\xi-)$, on the left, is connected to $V(\xi+)$, on the right, by a viscous shock profile, and so the viscous shock admissibility criterion is satisfied (with $B=I$), as discussed in Section 8.6. In particular, any shock of V satisfies the Liu E-condition and thus V coincides with the unique solution established by Theorem 9.5.1.

The construction of the solution $V_\mu(\xi)$ to the boundary value problem (9.8.2), (9.8.3) and the derivation of the bound on the total variation of the family $\{V_\mu\}$, asserted by Theorem 9.8.1, do not depend on the fact that the system (9.8.1) is conservative but apply equally well to any system

$$\mu \ddot{V}_\mu(\xi) = A(V_\mu(\xi))\dot{V}_\mu(\xi) - \xi \dot{V}_\mu(\xi), \tag{9.8.17}$$

so long as the matrix $A(U)$ has real distinct eigenvalues. If $V(\xi)$ is the $\mu \downarrow 0$ limit of $V_\mu(\xi)$, the function $U(x,t) = V(x/t)$ may be interpreted as a solution of the Riemann problem for the strictly hyperbolic, nonconservative system

$$\partial_t U + A(U)\partial_x U = 0, \tag{9.8.18}$$

even though it does not necessarily satisfy that system in the sense of distributions.

Viscous wave fans induce an alternative, implicit construction of wave curves for general strictly hyperbolic systems (9.1.1), without any requirement of piecewise genuine nonlinearity.

To trace the forward i-wave curve that emanates from some fixed state $\bar{U}$, assume that a state $\hat{U}$, on the right, is connected to $\bar{U}$, on the left, by an i-wave of moderate strength. Suppose this wave is the $\mu \downarrow 0$ limit of a family of viscous wave fans $V_\mu(\xi)$. Thus V_μ is defined for ξ in a small neighborhood of $\bar{\xi} = \lambda_i(\bar{U})$, it takes values near $\bar{U}$, and $\mu \dot{V}_\mu(\xi)$ is small. We stretch the domain by rescaling the variable, $\xi = \mu\zeta$. We also rescale the a_j in the expansion (9.8.7) by setting $w_j = \mu a_j$, and assemble the vector $W = (w_1, \cdots, w_n)$. Then we may recast (9.8.7), (9.8.8) into an autonomous first order system

$$\begin{cases} V' = \displaystyle\sum_{j=1}^{n} w_j R_j(V) \\ w_j' = [\lambda_j(V) - \xi]w_j + \displaystyle\sum_{k,\ell=1}^{n} \beta_{jk\ell}(V) w_k w_\ell\,, \qquad j = 1, \cdots, n \\ \xi' = \mu \\ \mu' = 0, \end{cases} \tag{9.8.19}$$

where the prime denotes differentiation with respect to ζ.

Linearization of (9.8.19) about the equilibrium point $V = \bar{U}$, $W = 0$, $\xi = \lambda_i(\bar{U})$, $\mu = 0$ yields the system

$$\begin{cases} V' = \displaystyle\sum_{j=1}^{n} w_j R_j(\bar{U}) \\ w_j' = [\lambda_j(\bar{U}) - \xi]w_j\,, \qquad j = 1, \cdots, n \\ \xi' = \mu \\ \mu' = 0. \end{cases} \tag{9.8.20}$$

The center subspace $\mathscr{N}$ of this system consists of all vectors $(V,W,\xi,\mu) \in \mathbb{R}^{2n+2}$ with $w_j = 0$ for $j \neq i$, and therefore has dimension $n+3$. By the center manifold theorem, any solution of (9.8.19) that dwells in the vicinity of the above equilibrium point must lie on a $(n+3)$-dimensional manifold $\mathscr{M}$, which is tangential to $\mathscr{N}$ at the equilibrium point, is invariant under the flow generated by (9.8.19), and admits the local representation

(9.8.21) $$w_j = \varphi_j(V,\omega,\xi;\mu), \qquad j \neq i,$$

where ω stands for w_i. By the theory of skew-product flows, the functions φ_j can be selected so that $\varphi_j(V,0,\xi;\mu) = 0$, for all V,ξ and μ close to $\bar{U}$, $\lambda_i(\bar{U})$ and 0. We may thus set

(9.8.22) $$w_j = \omega\psi_j(V,\omega,\xi;\mu), \qquad j \neq i,$$

where $\psi_j(\bar{U},0,\lambda_i(\bar{u});0) = 0$, since $\mathscr{M}$ is tangential to $\mathscr{N}$ at the equilibrium point. We also introduce a new variable τ such that

(9.8.23) $$\frac{d}{d\tau} = \frac{1}{\omega}\frac{d}{d\zeta}\,.$$

In order to see how the components $(V_\mu,\omega_\mu,\xi_\mu)$ of our solution evolve on $\mathscr{M}$ as functions of τ, we combine (9.8.19), (9.8.22) and (9.8.23) to deduce

(9.8.24) $$\frac{dV_\mu}{d\tau} = P_\mu(V_\mu,\omega_\mu,\xi_\mu),$$

(9.8.25) $$\frac{d\omega_\mu}{d\tau} = p_\mu(V_\mu,\omega_\mu,\xi_\mu) - \xi_\mu\,,$$

where we have set

(9.8.26) $$P_\mu(V,\omega,\xi) = R_i(V) + \sum_{j\neq i}\psi_j(V,\omega,\xi;\mu)R_j(V),$$

(9.8.27) $$p_\mu(V,\omega,\xi) = \lambda_i(V) + \sum_{k,\ell=1}^{n}\omega\beta_{ik\ell}(V)\psi_k(V,\omega,\xi;\mu)\psi_\ell(V,\omega,\xi;\mu).$$

In particular, $P_0(\bar{U},0,\lambda_i(\bar{U})) = R_i(\bar{U}),\ p_0(\bar{U},0,\lambda_i(\bar{U})) = \lambda_i(\bar{U})$.

To derive an equation for $\xi_\mu(\tau)$, we note that (9.8.23) together with $(9.8.19)_3$ yield $d\xi_\mu/d\tau = \xi_\mu'/\omega = \mu/\omega$. We differentiate this relation with respect to τ and use (9.8.25) to get

(9.8.28) $$\mu\frac{d^2\xi_\mu}{d\tau^2} = -\left(\frac{d\xi_\mu}{d\tau}\right)^2[p_\mu(V_\mu,\omega_\mu,\xi_\mu) - \xi_\mu].$$

As $\mu \downarrow 0$, $(V_\mu,\omega_\mu,\xi_\mu)$ converge uniformly to (V,ω,ξ). In particular, we have $V(0) = \bar{U},\ \omega(0) = 0$ and $V(s) = \hat{U}$, for some, say positive, small s. By virtue of

(9.8.28), $[0,s]$ is the union of an at most countable family of τ-intervals, associated with shocks, over which $d\xi/d\tau = 0$, and τ-intervals, associated with rarefaction waves, over which $\xi = p_0(V,\omega,\xi)$. Furthermore, at points of transition from shock to rarefaction (or rarefaction to shock), $d^2\xi_\mu/d\tau^2$ should be nonnegative (or nonpositive). It then follows that

$$\xi(\tau) = \frac{dg}{d\tau}(\tau), \qquad 0 \le \tau \le s, \tag{9.8.29}$$

where g is the convex envelope, over $[0,s]$, of the function

$$f(\tau) = \int_0^\tau p_0(V(\sigma),\omega(\sigma),\xi(\sigma))d\sigma, \qquad 0 \le \tau \le s, \tag{9.8.30}$$

i.e., $g(\tau) = \inf\{\theta_1 f(\tau_1) + \theta_2 f(\tau_2) : \theta_1 \ge 0, \theta_2 \ge 0, \theta_1 + \theta_2 = 1, 0 \le \tau_1 \le \tau_2 \le s, \theta_1\tau_1 + \theta_2\tau_2 = \tau\}$. Then (9.8.24) and (9.8.25) yield

$$V(t) = \bar{U} + \int_0^\tau P_0(V(\sigma),\omega(\sigma),\xi(\sigma))d\sigma, \qquad 0 \le \tau \le s, \tag{9.8.31}$$

$$\omega(\tau) = f(\tau) - g(\tau), \qquad 0 \le \tau \le s. \tag{9.8.32}$$

It can be shown that, once $P_0(V,\omega,\xi)$ and $p_0(V,\omega,\xi)$ are specified, the system of equations (9.8.29), (9.8.31) and (9.8.32) can be solved by Picard iteration to yield the functions $V(\tau)$, $\omega(\tau)$ and $\xi(\tau)$, over $[0,s]$, for any small positive s. The treatment of negative s is similar, except that now g is the concave envelope of f over $[s,0]$. Hence, these equations provide an implicit representation of the i-wave curve Φ_i emanating from $\bar{U}$, by setting $\Phi_i(s;\bar{U}) = V(s)$. By its definition through (9.8.23), τ is nearly equal to the projection of $V - \bar{U}$ on R_i. Thus, the above construction of the i-wave curve closely resembles the construction of the wave for the scalar conservation law described at the opening of Section 9.5.

Our next project is to construct, by the method of viscous wave fans, solutions to the Riemann problem for systems of just two conservation laws,

$$\begin{cases} \partial_t u + \partial_x f(u,v) = 0 \\ \partial_t v + \partial_x g(u,v) = 0, \end{cases} \tag{9.8.33}$$

albeit under unrestricted initial data

$$\big(u(x,0),v(x,0)\big) = \begin{cases} (u_L,v_L), & x<0 \\ (u_R,v_R), & x>0. \end{cases} \tag{9.8.34}$$

The crucial restriction will be that f_v and g_u have the same sign, say for definiteness

$$f_v(u,v) < 0, \qquad g_u(u,v) < 0, \qquad \text{for all } (u,v). \tag{9.8.35}$$

In particular, the system is strictly hyperbolic. Coupled symmetric systems and the system (7.1.11) of isentropic elastodynamics are typical representatives of this class. The analysis will demonstrate how delta shocks may emerge as "concentrations" in the limit of viscous profiles.

Equations (9.8.2), (9.8.3) here take the form

$$\begin{cases} \mu\ddot{u}_\mu(\xi) = \dot{f}\big(u_\mu(\xi), v_\mu(\xi)\big) - \xi\dot{u}_\mu(\xi) \\ \mu\ddot{v}_\mu(\xi) = \dot{g}\big(u_\mu(\xi), v_\mu(\xi)\big) - \xi\dot{v}_\mu(\xi), \end{cases} \tag{9.8.36}$$

$$\big(u_\mu(-\infty), v_\mu(-\infty)\big) = \big(u_L, v_L\big), \qquad \big(u_\mu(\infty), v_\mu(\infty)\big) = \big(u_R, v_R\big). \tag{9.8.37}$$

The importance of the assumption (9.8.35) stems from the following

9.8.2 Lemma. *Let* $\big(u_\mu(\xi), v_\mu(\xi)\big)$ *be a solution of* (9.8.36), *and* (9.8.37) *on* $(-\infty,\infty)$. *Then one of the following holds:*

(a) Both $u_\mu(\xi)$ *and* $v_\mu(\xi)$ *are constant on* $(-\infty,\infty)$.

(b) $u_\mu(\xi)$ *is strictly increasing (or decreasing), with no critical points on* $(-\infty,\infty)$; $v_\mu(\xi)$ *has at most one critical point on* $(-\infty,\infty)$, *which is necessarily a maximum (or minimum).*

(c) $v_\mu(\xi)$ *is strictly increasing (or decreasing), with no critical points on* $(-\infty,\infty)$; $u_\mu(\xi)$ *has at most one critical point on* $(-\infty,\infty)$, *which is necessarily a maximum (or minimum).*

Proof. Notice that $\dot{u}_\mu(\xi_0) = 0$ and $\ddot{u}_\mu(\xi_0) = 0$ imply $\dot{v}_\mu(\xi_0) = 0$; while $\dot{v}_\mu(\xi_0) = 0$ and $\ddot{v}_\mu(\xi_0) = 0$ imply $\dot{u}_\mu(\xi_0) = 0$. Therefore, by uniqueness of solutions to the initial value problem for ordinary differential equations, if either one of $u_\mu(\xi)$ and $v_\mu(\xi)$ has degenerate critical points, then both these functions must be constant on $(-\infty,\infty)$.

Turning to nondegenerate critical points, note that $\dot{u}_\mu(\xi_0) = 0$ and $\ddot{u}_\mu(\xi_0) < 0$ (or $\ddot{u}_\mu(\xi_0) > 0$) imply $\dot{v}_\mu(\xi_0) > 0$ (or $\dot{v}_\mu(\xi_0) < 0$); similarly, $\dot{v}_\mu(\xi_0) = 0$ and $\ddot{v}_\mu(\xi_0) < 0$ (or $\ddot{v}_\mu(\xi_0) > 0$) imply $\dot{u}_\mu(\xi_0) > 0$ (or $\dot{u}_\mu(\xi_0) < 0$).

Suppose now $v_\mu(\xi)$ has more than one nondegenerate critical point and pick two consecutive ones, a maximum at ξ_1 and a minimum at ξ_2. For definiteness, assume $\xi_1 < \xi_2$. Then $\dot{v}_\mu(\xi_1) = 0$, $\ddot{v}_\mu(\xi_1) < 0$, $\dot{v}_\mu(\xi_2) = 0$, $\ddot{v}_\mu(\xi_2) > 0$ and $\dot{v}_\mu(\xi) < 0$ for $\xi \in (\xi_1, \xi_2)$. Hence, $\dot{u}_\mu(\xi_1) > 0$ and $\dot{u}_\mu(\xi_2) < 0$. Therefore, there exists ξ_0 in (ξ_1, ξ_2) such that $\dot{u}_\mu(\xi_0) = 0$ and $\ddot{u}_\mu(\xi_0) < 0$. But this implies $\dot{v}_\mu(\xi_0) > 0$, which is a contradiction. The case $\xi_1 > \xi_2$ also leads to a contradiction. The same argument shows that $u_\mu(\xi)$ may have at most one nondegenerate critical point.

Finally, suppose both $u_\mu(\xi)$ and $v_\mu(\xi)$ have nondegenerate critical points, say at ξ_1 and ξ_2, respectively. For definiteness, assume $\xi_1 < \xi_2$ and ξ_2 is a maximum of $v_\mu(\xi)$. Then $\dot{v}_\mu(\xi) > 0$ for $\xi \in (-\infty, \xi_2)$ and $\dot{v}_\mu(\xi_2) = 0$, $\ddot{v}_\mu(\xi_2) < 0$. This implies $\dot{u}_\mu(\xi_2) > 0$, whence ξ_1 is necessarily a minimum of $u_\mu(\xi)$, with $\dot{u}_\mu(\xi_1) = 0$,

$\ddot{u}_\mu(\xi_1) > 0$. This in turn implies $\dot{v}_\mu(\xi_1) < 0$, which is a contradiction. All other possible combinations lead to similar contradictions. The proof is complete.

Because of the very special configuration of the graphs of $u_\mu(\xi)$ and $v_\mu(\xi)$, it is relatively easy to establish the existence of solutions to (9.8.36), (9.8.37). Indeed, it turns out that for that purpose it is sufficient to bound a priori the unique "peak" attained by $u_\mu(\xi)$ or $v_\mu(\xi)$, in terms of the given data (u_L, v_L), (u_R, v_R), and the parameter μ. The reader may find the derivation of such estimates, and resulting proof of existence, in the literature cited in Section 9.12, under the assumption that either the growth of $f(u,v)$ and $g(u,v)$ is restricted by

$$|f(u,v)| \leq h(v)(1+|u|)^p, \qquad |g(u,v)| \leq h(u)(1+|v|)^p, \tag{9.8.38}$$

where h is a continuous function and $p < 2$, or the system (9.8.33) is endowed with an entropy $\eta(u,v)$, with the property that the eigenvalues of the Hessian matrix $\mathrm{D}^2\eta(u,v)$ are bounded from below by $(1+|u|)^{-p}(1+|v|)^{-p}$, for some $p < 3$. The first class of systems contains in particular (7.1.11), and the second class includes all symmetric systems.

Assuming (u_μ, v_μ) exist, we pass to the limit, as $\mu \downarrow 0$, in order to obtain solutions of the Riemann problem. For that purpose, we shall need estimates independent of μ. Let us consider, for definiteness, the case where $v_\mu(\xi)$ is strictly increasing on $(-\infty, \infty)$, while $u_\mu(\xi)$ is strictly increasing on $(-\infty, \xi_\mu)$, attains its maximum at ξ_μ, and is strictly decreasing on (ξ_μ, ∞). All other possible configurations may be treated in a similar manner.

Let us set $\bar{u} = \max\{u_L, u_R\}$ and identify the points $\xi_\ell \in (-\infty, \xi_\mu) \cup \{-\infty\}$ and $\xi_r \in (\xi_\mu, \infty) \cup \{\infty\}$ with the property $u(\xi_\ell) = u(\xi_r) = \bar{u}$. For any open interval $(a,b) \subset (-\infty, \infty)$, using $(9.8.36)_1$ and (9.8.35), we deduce

$$\begin{aligned} \int_a^b [u_\mu(\xi) - \bar{u}]\, d\xi &\leq \int_{\xi_\ell}^{\xi_r} [u_\mu(\xi) - \bar{u}]\, d\xi = -\int_{\xi_\ell}^{\xi_r} \xi \dot{u}_\mu(\xi) d\xi \\ &= \mu \dot{u}_\mu(\xi_r) - \mu \dot{u}_\mu(\xi_\ell) - f(\bar{u}, v_\mu(\xi_r)) + f(\bar{u}, v_\mu(\xi_\ell)) \leq f(\bar{u}, v_L) - f(\bar{u}, v_R). \end{aligned} \tag{9.8.39}$$

By virtue of (9.8.39), there is a sequence $\{\mu_k\}$, $\mu_k \downarrow 0$ as $k \to 0$, such that $\{\xi_{\mu_k}\}$ converges to some point $\xi_0 \in (-\infty, \infty) \cup \{\pm\infty\}$, $\{v_{\mu_k}(\xi)\}$ converges, pointwise on $(-\infty, \infty)$, to a monotone increasing function $v(\xi)$, and $\{u_{\mu_k}(\xi)\}$ converges, pointwise on $(-\infty, \xi_0) \cup (\xi_0, \infty)$, to a locally integrable function $u(\xi)$, which is monotone increasing on $(-\infty, \xi_0)$ and monotone decreasing on (ξ_0, ∞). Furthermore, it is easily seen that $u(-\infty) = u_L$, $u(\infty) = u_R$, $v(-\infty) = v_L$ and $v(\infty) = v_R$.

When $\xi_0 = -\infty$ (or $\xi_0 = \infty$), $u(\xi)$ is a monotone increasing (or decreasing) function on $(-\infty, \infty)$, in which case $(u(\xi), v(\xi))$ is a standard solution to the Riemann problem. The situation becomes interesting when $\xi_0 \in (-\infty, \infty)$. In that case, as $k \to \infty$, $u_{\mu_k} \to u + \omega\delta_{\xi_0}$, in the sense of distributions, where δ_{ξ_0} denotes the Dirac delta function at ξ_0 and $\omega \geq 0$.

We multiply both equations in (9.8.36) by a test function $\varphi \in C_0^\infty(-\infty,\infty)$, integrate the resulting equations over $(-\infty,\infty)$, and integrate by parts to get

$$(9.8.40)\qquad \begin{cases} \displaystyle\int_{-\infty}^{\infty} \left\{\mu u_\mu \ddot{\varphi} + \left[f\left(u_\mu, v_\mu\right) - \xi u_\mu\right]\dot{\varphi} - u_\mu \varphi\right\} d\xi = 0, \\ \displaystyle\int_{-\infty}^{\infty} \left\{\mu v_\mu \ddot{\varphi} + \left[g\left(u_\mu, v_\mu\right) - \xi v_\mu\right]\dot{\varphi} - v_\mu \varphi\right\} d\xi = 0. \end{cases}$$

We apply (9.8.40) for test functions that are constant over some open interval containing ξ_0 and let $\mu \downarrow 0$ along the sequence $\{\mu_k\}$ thus obtaining

$$(9.8.41)\qquad \begin{cases} \displaystyle\int_{-\infty}^{\infty} [f(u(\xi),v(\xi)) - \xi u(\xi)]\dot{\varphi}(\xi) d\xi = \int_{-\infty}^{\infty} u(\xi)\varphi(\xi) d\xi + \omega \varphi(\xi_0), \\ \displaystyle\int_{-\infty}^{\infty} [g(u(\xi),v(\xi)) - \xi v(\xi)]\dot{\varphi}(\xi) d\xi = \int_{-\infty}^{\infty} v(\xi)\varphi(\xi) d\xi. \end{cases}$$

By shrinking the support of φ around ξ_0, one deduces that

$$(9.8.42)\qquad \begin{cases} \lim\limits_{\xi\uparrow\xi_0} [f(u(\xi),v(\xi)) - \xi u(\xi)] - \lim\limits_{\xi\downarrow\xi_0} [f(u(\xi),v(\xi)) - \xi u(\xi)] = \omega, \\ \lim\limits_{\xi\uparrow\xi_0} [g(u(\xi),v(\xi)) - \xi v(\xi)] - \lim\limits_{\xi\downarrow\xi_0} [g(u(\xi),v(\xi)) - \xi v(\xi)] = 0, \end{cases}$$

where all four limits exist (finite). In particular, this implies that the functions $f(u(\xi),v(\xi))$ and $g(u(\xi),v(\xi))$ are locally integrable on $(-\infty,\infty)$, and (9.8.41) holds for arbitrary $\varphi \in C_0^\infty(-\infty,\infty)$. Equivalently,

$$(9.8.43)\qquad \begin{cases} [f(u,v) - \xi u]^{\cdot} + u + \omega\delta_{\xi_0} = 0, \\ [g(u,v) - \xi v]^{\cdot} + v = 0, \end{cases}$$

in the sense of distributions. We thus conclude that when $\omega = 0$ then $(u(\xi),v(\xi))$ is a standard solution of the Riemann problem, possibly with $u(\xi_0) = \infty$, just like the solution (9.6.4) for the system (7.1.11). Whereas, when $\omega > 0$, $(u(\xi) + \omega\delta_{\xi_0}, v(\xi))$ may be interpreted as a nonstandard solution to the Riemann problem, containing a delta shock at ξ_0, like the solution (9.6.6) for the system (7.1.11).

We now assume that the system is endowed with an entropy-entropy flux pair (η,q), where $\eta(u,v)$ is convex, with superlinear growth,

$$(9.8.44)\qquad \frac{\eta(u,v)}{|u|+|v|} \to \infty, \quad \text{as } |u|+|v| \to \infty,$$

and show that $(u(\xi),v(\xi))$ is a standard solution to the Riemann problem, i.e., $\omega=0$. We multiply $(9.8.36)_1$ by $\eta_u(u_\mu,v_\mu)$, $(9.8.36)_2$ by $\eta_v(u_\mu,v_\mu)$, and add the resulting two equations to get

$$\mu\ddot{\eta}\left(u_\mu,v_\mu\right)-\mu\left[\eta_{uu}\dot{u}_\mu^2+2\eta_{uv}\dot{u}_\mu\dot{v}_\mu+\eta_{vv}\dot{v}_\mu^2\right]=\dot{q}\left(u_\mu,v_\mu\right)-\xi\dot{\eta}\left(u_\mu,v_\mu\right). \tag{9.8.45}$$

We let $\bar{\eta}=\max\{\eta\left(u_L,v_L\right),\eta\left(u_R,v_R\right)\}$ and then identify the greatest number ξ_L in $\left(-\infty,\xi_0\right)\cup\{-\infty\}$ and the smallest number ξ_R in $\left(\xi_0,\infty\right)\cup\{\infty\}$ with the property that $\eta\left(u_\mu(\xi_L),v_\mu(\xi_L)\right)=\eta\left(u_\mu\left(\xi_R\right),v_\mu\left(\xi_R\right)\right)=\bar{\eta}$. Using (9.8.45),

$$\begin{aligned}\int_{\xi_L}^{\xi_R}\left[\eta\left(u_\mu,v_\mu\right)-\bar{\eta}\right]d\xi &= -\int_{\xi_L}^{\xi_R}\xi\dot{\eta}\left(u_\mu,v_\mu\right)d\xi\\ &\le q\left(u_\mu\left(\xi_L\right),v_\mu\left(\xi_L\right)\right)-q\left(u_\mu\left(\xi_R\right),v_\mu\left(\xi_R\right)\right).\end{aligned} \tag{9.8.46}$$

The right-hand side of (9.8.46) is bounded, uniformly in $\mu>0$. Therefore, combining (9.8.46) with (9.8.44) yields

$$\int_{\{u_\mu\ge\bar{u}\}} u_\mu(\xi)d\xi\to 0,\quad \text{as } \bar{u}\to\infty, \tag{9.8.47}$$

uniformly in $\mu>0$, and hence $\omega=0$.

It is clear that the same argument applies to all possible configurations of $\left(u_\mu(\xi),v_\mu(\xi)\right)$. We have thus established

9.8.3 Theorem. *Assume that the system* (9.8.33), *where $f_v g_u>0$, is endowed with a convex entropy $\eta(u,v)$, exhibiting superlinear growth* (9.8.44). *Then sequences $\{\left(u_{\mu_k},v_{\mu_k}\right)\}$ of solutions to* (9.8.36), (9.8.37), *with $\mu_k\to 0$ as $k\to\infty$, converge pointwise, as well as in the sense of distributions, to standard solutions (u,v) of the Riemann problem* (9.8.33), (9.8.34). *At least one of the functions $u(\xi)$, $v(\xi)$ is monotone on $(-\infty,\infty)$, while the other may have at most one extremum, which may be bounded or unbounded.*

In particular, any symmetric system of two conservation laws, with $f_v=g_u\neq 0$, satisfies the assumptions of the above theorem. In the literature cited in Section 9.12, the reader will find assumptions on f and g under which the resulting solution to the Riemann problem is necessarily bounded. It has also been shown that any shock in these solutions satisfies the viscous shock admissibility criterion and thereby the Liu E-condition.

Following up on the discussion in Section 8.6, one may argue that wave fan solutions of the Riemann problem, with end-states U_L and U_R, should not be termed admissible unless they are captured through the $t\to\infty$ asymptotics of solutions of parabolic systems (8.6.1), under initial data $U_0(x)$ which decay sufficiently fast to U_L

and U_R, as $x \to \mp\infty$. In fact, the results reported in Section 8.6 on the asymptotic stability of viscous shock profiles address a special case of the above issue. The complementary special case, the asymptotic stability of rarefaction waves, has also been studied extensively (references in Section 9.12). The task of combining the above two ingredients so as to synthesize the full solution of the Riemann problem has not yet been accomplished in a definitive manner.

9.9 Interaction of Wave Fans

Up to this point, we have exploited the invariance of systems of conservation laws under uniform rescaling of the space-time coordinates in order to perform stretchings that reveal the local structure of solutions. However, one may also operate at the opposite end of the scale by performing contractions of the space-time coordinates that will provide a view of solutions from a large distance from the origin. It is plausible that initial data $U_0(x)$ which converge sufficiently fast to states U_L and U_R, as $x \to -\infty$ and $x \to \infty$, generate solutions that look from afar like centered wave fans joining the state U_L, on the left, with the state U_R, on the right. Actually, as we shall see in later chapters, this turns out to be true. Indeed, it seems that the quintessential property of hyperbolic systems of conservation laws in one space dimension is that the Riemann problem describes the asymptotics of solutions at both ends of the time scale: instantaneous and long-term.

The purpose here is to discuss a related question, which, as we shall see in Chapter XIII, is of central importance in the construction of solutions by the random choice method. We consider three wave fans: the first, joining a state U_L, on the left, with a state U_M, on the right; the second, joining the state U_M, on the left, with a state U_R, on the right; and the third, joining the state U_L, on the left, with the state U_R, on the right. These may be identified by their left states U_L, U_M and U_L, together with the respective n-tuples $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ of wave amplitudes. On the basis of the arguments presented above, it is natural to regard the wave fan ε as the result of the interaction of the wave fan α, on the left, with the wave fan β, on the right. Recalling (9.3.4), $U_M = \Omega(\alpha; U_L)$, $U_R = \Omega(\beta; U_M)$ and $U_R = \Omega(\varepsilon; U_L)$, whence we deduce

$$\Omega(\varepsilon; U_L) = \Omega(\beta; \Omega(\alpha; U_L)). \tag{9.9.1}$$

This determines implicitly the relation

$$\varepsilon = E(\alpha; \beta; U_L). \tag{9.9.2}$$

Our task is to study the properties of the function E in the vicinity of $(0; 0; U_L)$.

Let us first consider systems with characteristic families that are either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2), in which case the wave curves Φ_i, and thereby Ω and E, are all $C^{2,1}$ functions. Since $\Omega(0; \bar{U}) = \bar{U}$,

$$E(\alpha; 0; U_L) = \alpha, \qquad E(0; \beta; U_L) = \beta, \tag{9.9.3}$$

whence

$$\frac{\partial E_k}{\partial \alpha_i}(0;0;U_L) = \delta_{ik}\,, \qquad \frac{\partial E_k}{\partial \beta_j}(0;0;U_L) = \delta_{jk}\,, \tag{9.9.4}$$

namely, the Kronecker delta.

Starting from the identity

$$\begin{aligned} &E(\alpha;\beta;U_L) - E(\alpha;0;U_L) - E(0;\beta;U_L) + E(0;0;U_L) \\ &= \sum_{i,j=1}^{n} \{E(\alpha_1,\cdots,\alpha_i,0,\cdots,0;0,\cdots,0,\beta_j,\cdots,\beta_n;U_L) \\ &\quad -E(\alpha_1,\cdots,\alpha_{i-1},0,\cdots,0;0,\cdots,0,\beta_j,\cdots,\beta_n;U_L) \\ &\quad -E(\alpha_1,\cdots,\alpha_i,0,\cdots,0;0,\cdots,0,\beta_{j+1},\cdots,\beta_n;U_L) \\ &\quad +E(\alpha_1,\cdots,\alpha_{i-1},0,\cdots,0;0,\cdots,0,\beta_{j+1},\cdots,\beta_n;U_L)\}, \end{aligned} \tag{9.9.5}$$

one immediately deduces

$$E(\alpha;\beta;U_L) = \alpha + \beta + \sum_{i,j=1}^{n} \alpha_i\beta_j \int_0^1\int_0^1 \frac{\partial^2 E}{\partial\alpha_i\partial\beta_j} d\rho d\sigma, \tag{9.9.6}$$

with $\partial^2 E/\partial\alpha_i\partial\beta_j$ at $(\alpha_1,\cdots,\alpha_{i-1},\rho\alpha_i,0,\cdots,0;0,\cdots,0,\sigma\beta_j,\beta_{j+i},\cdots,\beta_n;U_L)$.

We say the i-wave of the wave fan α and the j-wave of the wave fan β are *approaching* when either (a) $i > j$ or (b) $i = j$, the i-characteristic family is genuinely nonlinear, and at least one of α_i, β_i is negative, i.e., corresponds to a shock. The *amount of wave interaction* of the fans α and β will be measured by the quantity

$$D(\alpha,\beta) = \sum_{\text{app}} |\alpha_i||\beta_j|, \tag{9.9.7}$$

where $\sum_{\text{app}}$ denotes summation over all pairs of approaching waves. The crucial observation is that when the wave fans α and β do not include any approaching waves, i.e., $D(\alpha,\beta) = 0$, then the wave fan ε is synthesized by "gluing together" the wave fan α, on the left, and the wave fan β, on the right; that is, $\varepsilon = \alpha + \beta$. In particular, whenever the i-wave of α and the j-wave of β are not approaching, either because $i < j$ or because $i = j$ and both α_i and β_i are positive (i.e., they correspond to rarefaction waves) then

$$\begin{aligned} &E(\alpha_1,\cdots,\alpha_i,0,\cdots,0;0,\cdots,0,\beta_j,\cdots,\beta_n;U_L) \\ &\qquad = (\alpha_1,\cdots,\alpha_i,0,\cdots,0) + (0,\cdots,0,\beta_j,\cdots,\beta_n), \end{aligned} \tag{9.9.8}$$

whence it follows that the corresponding (i,j)-term in the summation on the right-hand side of (9.9.6) vanishes. Thus (9.9.6) reduces to

$$\varepsilon = \alpha + \beta + \sum_{\text{app}} \alpha_i \beta_j \frac{\partial^2 E}{\partial \alpha_i \partial \beta_j}(0;0;U_L) + D(\alpha,\beta)O(|\alpha| + |\beta|). \tag{9.9.9}$$

The salient feature of (9.9.9), which will play a key role in Chapter XIII, is that the effect of wave interaction is induced solely by pairs of approaching waves and vanishes in the absence of such pairs. In order to determine the leading interaction term, of quadratic order, we first differentiate (9.9.1) with respect to β_j and set $\beta = 0$. Upon using (9.3.8), this yields

$$\sum_{k=1}^{n} \frac{\partial E_k}{\partial \beta_j}(\alpha;0;U_L) \frac{\partial \Omega}{\partial \varepsilon_k}(E(\alpha;0;U_L);U_L) = R_j(\Omega(\alpha;U_L)). \tag{9.9.10}$$

Next we differentiate (9.9.10) with respect to α_i and set $\alpha = 0$. Recall that we are interested only in the case where the i-wave of α and the j-wave of β are approaching, so in particular $i \geq j$. Therefore, upon using (9.9.3), (9.9.4), (9.3.8), (9.3.9), (9.3.10) and (7.2.15), we conclude

$$\sum_{k=1}^{n} \frac{\partial^2 E_k}{\partial \alpha_i \partial \beta_j}(0;0;U_L) R_k(U_L) = -[R_i(U_L), R_j(U_L)], \tag{9.9.11}$$

whence

$$\frac{\partial^2 E_k}{\partial \alpha_i \partial \beta_j}(0;0;U_L) = -L_k(U_L)[R_i(U_L), R_j(U_L)]. \tag{9.9.12}$$

In particular, when the system is endowed with a coordinate system of Riemann invariants, under the normalization (7.3.8) the Lie brackets $[R_i, R_j]$ vanish (cf. (7.3.10)), and hence the quadratic term in (9.9.9) drops out.

Upon combining (9.9.9) with (9.9.12), we arrive at

9.9.1 Theorem. *In a system with characteristic families that are either genuinely nonlinear or linearly degenerate, let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ be the wave fan generated by the interaction of the wave fan $\alpha = (\alpha_1, \cdots, \alpha_n)$, on the left, with the wave fan $\beta = (\beta_1, \cdots, \beta_n)$, on the right. Then*

$$\varepsilon = \alpha + \beta - \sum_{i>j} \alpha_i \beta_j L[R_i, R_j] + D(\alpha,\beta)O(|\alpha| + |\beta|), \tag{9.9.13}$$

where L denotes the $n \times n$ matrix with k-row vector the left eigenvector L_k, and $D(\alpha,\beta)$ is the amount of wave interaction of α and β. When the system is endowed with a coordinate system of Riemann invariants, the quadratic term vanishes.

We now consider wave interactions for systems with characteristic families that may be merely piecewise genuinely nonlinear, so that the incoming and outgoing wave fans will contain composite i-waves, each one comprising a finite sequence of elementary i-waves, namely i-shocks and i-rarefactions. There are two obstacles to overcome. The first is technical: As noted in Section 9.5, the wave curves Φ_i, and thereby the functions Ω and E, may now be merely Lipschitz continuous. Thus, the derivation, above, of (9.9.13) is no longer valid, as it relies on Taylor expansion. The most serious difficulty, however, is how to identify approaching waves. It is clear that an i-wave, on the left, and a j-wave, on the right, will be approaching if $i > j$ and not approaching if $i < j$. The situation is more delicate when both incoming waves belong to the same characteristic family. Recall that in the genuinely nonlinear case two incoming i-waves always approach when at least one of them is a shock and never approach when both are rarefactions. By contrast, here two incoming i-wave fans may include pairs of non-approaching i-shocks as well as pairs of approaching i-rarefaction waves. Consequently, the analog of Theorem 9.9.1 for such systems is quite involved:

9.9.2 Theorem. *In a system with characteristic families that are either piecewise genuinely nonlinear or linearly degenerate, let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be the wave fan generated by the interaction of the wave fan $\alpha = (\alpha_1, \dots, \alpha_n)$, on the left, with the wave fan $\beta = (\beta_1, \dots, \beta_n)$, on the right. Then*

$$\varepsilon = \alpha + \beta + O(1)D(\alpha, \beta), \tag{9.9.14}$$

where

$$D(\alpha, \beta) = \sum \theta |\gamma||\delta|, \tag{9.9.15}$$

with the summation running over all pairs of elementary waves, such that the first one, with amplitude γ, is part of an i-wave fan incoming from the left, while the second one, with amplitude δ, is part of a j-wave fan incoming from the right; and the weighting factor θ is selected according to the following rules.

(a) When $i < j$, then $\theta = 0$.

(b) When either $i > j$ or $i = j$ and $\gamma\delta < 0$, then $\theta = 1$.

(c) When $i = j$ and $\gamma\delta > 0$, then θ is determined as follows:

(c)$_1$ If both incoming elementary waves are i-shocks, with respective speeds σ_L and σ_R, then

$$\theta = (\sigma_L - \sigma_R)^+ . \tag{9.9.16$_1$}$$

(c)$_2$ If the elementary wave incoming from the left is an i-shock with speed σ_L, while the elementary wave incoming from the right is an i-rarefaction, joining states U_R and $V_i(\tau_R; U_R)$, then

$$\theta = \frac{1}{\tau_R} \int_0^{\tau_R} [\sigma_L - \lambda_i(V_i(\tau; U_R))]^+ d\tau. \tag{9.9.16$_2$}$$

(c)$_3$ If the elementary wave incoming from the left is an i-rarefaction, joining states U_L and $V_i(\tau_L;U_L)$, while the elementary wave incoming from the right is an i-shock with speed σ_R, then

$$\theta = \frac{1}{\tau_L}\int_0^{\tau_L}[\lambda_i(V_i(\tau';U_L)) - \sigma_R]^+ d\tau'. \tag{9.9.16$_3$}$$

(c)$_4$ If, finally, both incoming elementary waves are i-rarefactions, with the one on the right joining U_R and $V_i(\tau_R;U_R)$ and the one on the left joining U_L and $V_i(\tau_L;U_L)$, then

$$\theta = \frac{1}{\tau_L\tau_R}\int_0^{\tau_L}\int_0^{\tau_R}[\lambda_i(V_i(\tau';U_L)) - \lambda_i(V_i(\tau;U_R))]^+ d\tau d\tau'. \tag{9.9.16$_4$}$$

Sketch of Proof. The objective here is to explain why and how the weighting factor θ comes into play. For the case where an i-elementary wave, incoming from the left, is interacting with a j-elementary wave, incoming from the right, it is easy to understand, on the basis of our earlier discussions in this section, why it should be $\theta = 0$ when $i < j$ and $\theta = 1$ when $i > j$; the real difficulty arises when $i = j$.

It should be noted that if one accepts $(9.9.16)_1$ as the correct value for the weighting factor θ in the case of interacting shocks, then $(9.9.16)_2$, $(9.9.16)_3$ and $(9.9.16)_4$, which concern rarefaction waves, may be derived as follows. Any i-rarefaction wave is visualized as a fan of infinitely many (nonadmissible) i-rarefaction shocks, each with infinitesimal amplitude and characteristic speed, and then its contribution to the amount of wave interaction is evaluated by tallying the contributions of these infinitesimal shocks, using $(9.9.16)_1$.

In what follows, it will be shown that $(9.9.16)_1$ does indeed provide the correct value for the weighting factor when each incoming wave fan consists of a single i-shock. The proof for general incoming wave fans, which can be found in the references cited in Section 9.12, is long and technical.

Assume the i-shock incoming from the left joins U_L with U_M and has amplitude γ and speed σ_L, while the i-shock incoming from the right joins U_M with U_R and has amplitude δ and speed σ_R. By the Lax E-condition, $\sigma_L \geq \lambda_i(U_M) \geq \sigma_R$, so that the relative speed $\theta = \sigma_L - \sigma_R$ of the two incoming shocks is nonnegative. Notice that θ essentially measures the angle between these two shocks; accordingly, θ is dubbed the *incidence angle.*

The collision of the two incoming shocks will generate an outgoing wave fan $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)$, which is determined by solving the Riemann problem with end-states U_L and U_R. For simplicity, we assume that the i-wave fan of ε consists of a single i-shock, joining $\bar{U}_L$ with $\bar{U}_R$, having amplitude ε_i and speed σ.

There are two distinct possible wave configurations, as depicted in Fig. 9.9.1 (a) and (b), depending on whether γ and δ have the same or opposite signs. In either case we have $\varepsilon = \varepsilon(\gamma,\delta;U_L)$, where $\varepsilon(0,\delta;U_L) = (0,\dots,0,\delta,0,\dots,0)$ and where $\varepsilon(\gamma,0;U_L) = (0,\dots,0,\gamma,0,\dots,0)$. Therefore, $\varepsilon_i = \gamma+\delta+O(\gamma\delta)$ and $\varepsilon_j = O(\gamma\delta)$,

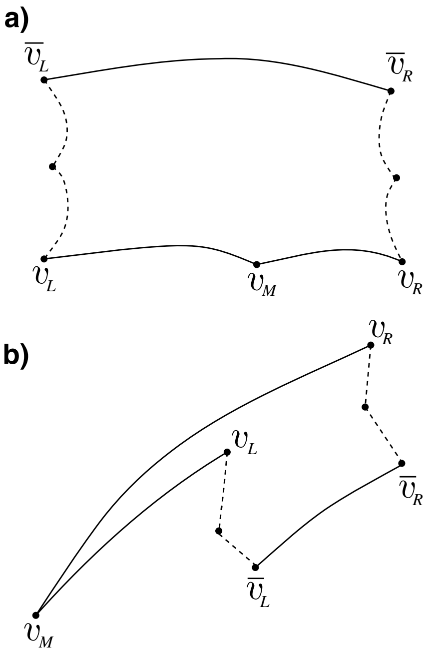

Fig. 9.9.1 (a, b)

for $j \neq i$. This relatively crude bound, $O(\gamma\delta)$, for the amount of wave interaction will suffice for the intended applications, in Chapter XIII, when $\gamma\delta < 0$, as in that case the cancellation in the linear term dominates. By contrast, when $\gamma\delta > 0$ a more refined estimate is needed. It is at this point that the incidence angle θ will come into play, as a measure of the rate the shock speed varies along the shock curve.

Recalling the discussion in Section 9.3,

$$\bar{U}_L = U_L + \sum_{j<i} \varepsilon_j R_j(U_M) + O(|\varepsilon_L|)\gamma + o(|\varepsilon_L|), \tag{9.9.17}$$

$$\bar{U}_R = U_R + \sum_{j>i} \varepsilon_j R_j(U_M) + O(|\varepsilon_R|)\delta + o(|\varepsilon_R|), \tag{9.9.18}$$

$$F(\bar{U}_L) = F(U_L) + \sum_{j<i} \varepsilon_j \lambda_j(U_M) R_j(U_M) + O(|\varepsilon_L|)\gamma + o(|\varepsilon_L|), \tag{9.9.19}$$

$$F(\bar{U}_R) = F(U_R) + \sum_{j>i} \varepsilon_j \lambda_j(U_M) R_j(U_M) + O(|\varepsilon_R|)\delta + o(|\varepsilon_R|), \tag{9.9.20}$$

where ε_L and ε_R stand for $(\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \dots, 0)$ and $(0, \dots, 0, \varepsilon_{i+1}, \dots, \varepsilon_n)$, respectively.

For convenience, we measure the amplitude of i-shocks by the projection of their jump on the left eigenvector $L_i(U_M)$. Thus

(9.9.21)

$$\gamma = L_i(U_M)[U_M - U_L], \qquad \delta = L_i(U_M)[U_R - U_M], \qquad \varepsilon_i = L_i(U_M)[\bar{U}_R - \bar{U}_L].$$

Starting out from the equation

$$[\bar{U}_R - \bar{U}_L] - [U_M - U_L] - [U_R - U_M] = [\bar{U}_R - U_R] - [\bar{U}_L - U_L], \tag{9.9.22}$$

multiplying it from the left by $L_i(U_M)$, and using (9.9.17), (9.9.18) and (9.9.21), we deduce

$$\varepsilon_i = \gamma + \delta + O(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta) + o(|\varepsilon_L| + |\varepsilon_R|). \tag{9.9.23}$$

Similarly, we consider the equation

(9.9.24)

$$\sigma[\bar{U}_R - \bar{U}_L] - \sigma_L[U_M - U_L] - \sigma_R[U_R - U_M] = [F(\bar{U}_R) - F(U_R)] - [F(\bar{U}_L) - F(U_L)],$$

which we get by combining the Rankine-Hugoniot jump conditions for the three shocks; we multiply it from the left by $L_i(U_M)$ and use (9.9.19) and (9.9.20) to get

$$\sigma\varepsilon_i = \sigma_L\gamma + \sigma_R\delta + O(|\varepsilon_L| + |\varepsilon_R|)(\gamma + \delta) + o(|\varepsilon_L| + |\varepsilon_R|). \tag{9.9.25}$$

Recalling that $\sigma_L - \sigma_R = \theta$, (9.9.25) together with (9.9.23) yield

$$\begin{cases} \sigma_L = \sigma + \dfrac{\delta\theta}{\gamma+\delta} + O(|\varepsilon_L| + |\varepsilon_R|) + o(|\varepsilon_L| + |\varepsilon_R|)(\gamma+\delta)^{-1} \\[2ex] \sigma_R = \sigma - \dfrac{\gamma\theta}{\gamma+\delta} + O(|\varepsilon_L| + |\varepsilon_R|) + o(|\varepsilon_L| + |\varepsilon_R|)(\gamma+\delta)^{-1}. \end{cases} \tag{9.9.26}$$

Substituting σ_L and σ_R from (9.9.26) into (9.9.24) and using (9.9.22), (9.9.17), (9.9.18), (9.9.19) and (9.9.20), we obtain

(9.9.27)
$$\sum_{j\neq i} \varepsilon_j |\lambda_j(U_M) - \sigma| R_j(U_M)$$
$$= \frac{-\delta\theta}{\gamma+\delta}[U_M - U_L] + \frac{\gamma\theta}{\gamma+\delta}[U_R - U_M] + O(|\varepsilon_L| + |\varepsilon_R|)(\gamma+\delta) + o(|\varepsilon_L| + |\varepsilon_R|).$$

Multiplying the above equation, from the left, by $L_j(U_M)$, $j \neq i$, and noting that

$$L_j(U_M)[U_M - U_L] = O(\gamma^2), \qquad L_j(U_M)[U_R - U_M] = O(\delta^2), \tag{9.9.28}$$

we deduce

$$\varepsilon_j = O(1)\theta\gamma\delta, \qquad j \neq i. \tag{9.9.29}$$

Equations (9.9.23) and (9.9.25) then give

$$\varepsilon_i = \gamma + \delta + O(1)\theta\gamma\delta, \tag{9.9.30}$$

$$\sigma\varepsilon_i = \sigma_L\gamma + \sigma_R\delta + O(1)\theta\gamma\delta. \tag{9.9.31}$$

We have thus established the assertion of the theorem, for the special case considered here.

One may regard the amplitude of a shock as its "mass" and the product of the amplitude with the speed of a shock as its "momentum". Thus, one may interpret (9.9.30) as balance of "mass" and (9.9.31) as balance of "momentum" under collision of two shocks. Equation (9.9.14) may then be interpreted as balance of "mass" under collision of wave fans. Similarly, one may define the "momentum" of a composite i-wave comprising, say, M i-shocks with amplitude γ_I and speed σ_I, $I = 1,\dots,M$, and N i-rarefaction waves, joining states U_J and $V_i(\tau_J;U_J)$, by tallying the "momenta" of its constituent elementary waves:

$$\Gamma_i = \sum_{I=1}^{M} \sigma_I\gamma_I + \sum_{J=1}^{N} \int_0^{\tau_J} \lambda_i(V_i(\tau;U_J))d\tau. \tag{9.9.32}$$

Then (9.9.31) admits the following extension. When two incoming wave fans α and β interact, the "momentum" Γ_i of the outgoing i-wave is related to the "momenta" Γ_i^- and Γ_i^+ of the incoming i-waves by

$$\Gamma_i = \Gamma_i^- + \Gamma_i^+ + O(1)D(\alpha,\beta), \qquad i = 1,\dots,n. \tag{9.9.33}$$

9.10 Breakdown of Weak Solutions

As we saw in the previous section, wave collisions may induce wave amplification. The following example shows that, as a result, there exist resonating wave patterns that drive the oscillation and/or total variation of weak solutions to infinity, in finite time.

Consider the system

$$\begin{cases} \partial_t u + \partial_x(uv + w) = 0 \\ \partial_t v + \partial_x(\frac{1}{16}v^2) = 0 \\ \partial_t w + \partial_x(u - uv^2 - vw) = 0. \end{cases} \tag{9.10.1}$$

The characteristic speeds are $\lambda_1 = -1$, $\lambda_2 = \frac{1}{8}v$, $\lambda_3 = 1$, so that strict hyperbolicity holds for $-8 < v < 8$. The first and third characteristic families are linearly degenerate, while the second characteristic family is genuinely nonlinear. Clearly, the system is partially decoupled: the second, Burgers-like, equation by itself determines v.

The Rankine-Hugoniot jump conditions for a shock of speed s, joining the state (u_-, v_-, w_-), on the left, with the state (u_+, v_+, w_+), on the right, here read

$$(9.10.2) \qquad \begin{cases} u_+v_+ - u_-v_- + w_+ - w_- = s(u_+ - u_-) \\ \frac{1}{16}v_+^2 - \frac{1}{16}v_-^2 = s(v_+ - v_-) \\ u_+ - u_- - u_+v_+^2 + u_-v_-^2 - v_+w_+ + v_-w_- = s(w_+ - w_-). \end{cases}$$

One easily sees that 1-shocks are 1-contact discontinuities, with $s = -1$, $v_- = v_+$ and

$$(9.10.3)_1 \qquad w_+ - w_- = -(v_\pm + 1)(u_+ - u_-).$$

Similarly, 3-shocks are 3-contact discontinuities, with $s = 1$, $v_- = v_+$ and

$$(9.10.3)_3 \qquad w_+ - w_- = -(v_\pm - 1)(u_+ - u_-).$$

Finally, for 2-shocks, $s = \frac{1}{16}(v_- + v_+)$, and $v_+ < v_-$, in order to satisfy the Lax E-condition.

Collisions between any two shocks, joining constant states, induce a jump discontinuity, which can be resolved by solving simple Riemann problems. In particular, when a 1-shock or a 3-shock collides with a 2-shock, the 2-shock remains undisturbed, as $(9.10.1)_2$ is decoupled from the other two equations of the system. This collision, however, produces both a 1- and a 2-outgoing shock, which may be interpreted as the "transmitted" and the "reflected" part of the incident 1- or 2-shock.

We now construct a piecewise constant, admissible solution of (9.10.1) with wave pattern depicted in Fig. 9.10.1: Two 2-shocks issue from the points $(-1,0)$ and $(1,0)$, with respective speeds $\frac{1}{4}$ and $-\frac{1}{4}$. On the left of the left 2-shock, $v = 4$; on the right of the right 2-shock, $v = -4$; and $v = 0$ between the two 2-shocks. A 1-shock issues from the origin $(0,0)$, and upon colliding with the left 2-shock it is partly transmitted as a 1-shock and partly reflected as a 3-shock. This 3-shock, upon impinging on the right 2-shock, is in turn partly transmitted as a 3-shock and partly reflected as a 1-shock, and the process is repeated ad infinitum.

By checking the Rankine-Hugoniot conditions (9.10.2), one readily verifies that, for instance, initial data

$$(9.10.4) \qquad (u(x,0), v(x,0), w(x,0)) = \begin{cases} (-65, +4, +225), & -\infty < x < -1 \\ (+15, 0, -15), & -1 < x < 0 \\ (-15, 0, +15), & 0 < x < 1 \\ (-63, -4, -225), & 1 < x < \infty \end{cases}$$

generate a solution with the above structure.

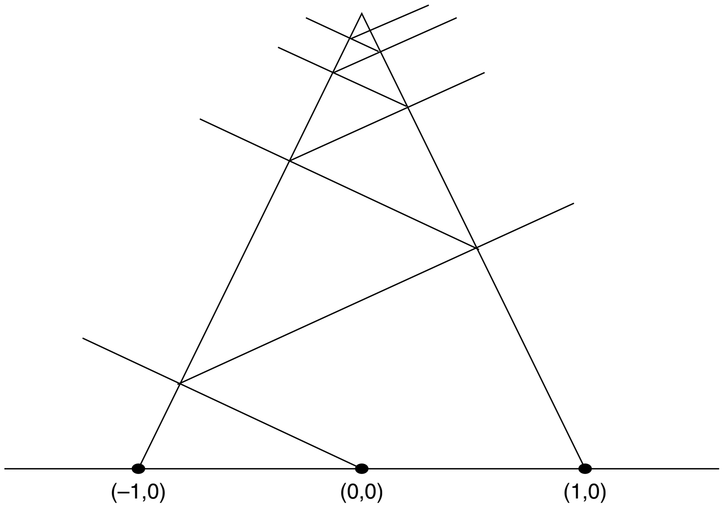

Fig. 9.10.1

The aim is to demonstrate that each reflection increases the strength of the shock by a constant factor. With collisions becoming progressively more frequent as the distance between the two 2-shocks is decreasing, until finally vanishing at $t = 4$, the conclusion will then be that the oscillation of the solution explodes as $t \uparrow 4$. It will be convenient to measure the strength of 1- and 3-shocks by the size of the jump of u across them.

Let us first examine the interaction depicted in Fig. 9.10.2, where a 1-shock hits the left 2-shock, from the right.

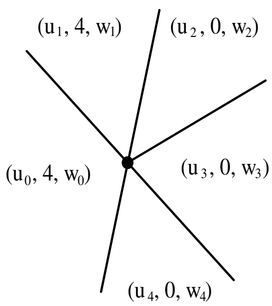

Fig. 9.10.2

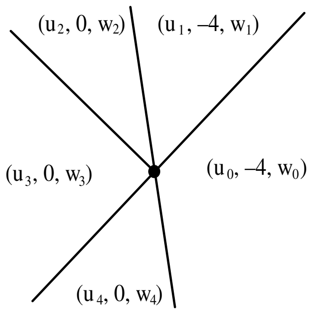

Fig. 9.10.3

We need to compare the strength $|u_3 - u_2|$ of the reflected 3-shock with the strength $|u_3 - u_4|$ of the incident 1-shock. We write the Rankine-Hugoniot conditions, (9.10.2) or (9.10.3), as applicable, for the five shocks involved in the interaction:

$$
(9.10.5)\qquad
\begin{cases}
w_3 - w_4 = -(u_3 - u_4) \\
w_1 - w_0 = -5(u_1 - u_0) \\
w_3 - w_2 = u_3 - u_2 \\
-4u_0 + w_4 - w_0 = \frac{1}{4}(u_4 - u_0) \\
u_4 - u_0 + 16u_0 + 4w_0 = \frac{1}{4}(w_4 - w_0) \\
-4u_1 + w_2 - w_1 = \frac{1}{4}(u_2 - u_1) \\
u_2 - u_1 + 16u_1 + 4w_1 = \frac{1}{4}(w_2 - w_1).
\end{cases}
$$

After elementary eliminations, one arrives at

$$
(9.10.6)\qquad u_3 - u_2 = -\tfrac{10}{9}(u_3 - u_4),
$$

which shows that as the 1-shock is reflected into a 3-shock, the strength increases by a factor 10/9.

Next we examine the interaction depicted in Fig. 9.10.3, where a 3-shock hits the right 2-shock from the left. By again writing the Rankine-Hugoniot conditions, completely analogous to (9.10.5), and after straightforward eliminations, one ends up once more with (9.10.6). Thus, the strength $|u_2 - u_3|$ of the reflected 1-shock exceeds the strength $|u_4 - u_3|$ of the incident 3-shock by a factor 10/9.

We have now confirmed that the oscillation of the solution blows up as $t \uparrow 4$. The above setting, which renders the calculation particularly simple, may appear at first as a singular, isolated example. However, after some reflection one realizes that the wave resonance persists under small perturbations of the equations and/or initial data, i.e., this kind of catastrophe may be generic.

Catastrophes of a different nature may occur as well: The total variation may blow up even though the oscillation remains bounded. This may be demonstrated in the context of the system

$$
(9.10.7)\qquad
\begin{cases}
\partial_t u + \partial_x(uv^2 + w) = 0 \\
\partial_t v + \partial_x(\frac{1}{16}v^2) = 0 \\
\partial_t w + \partial_x(u - uv^4 - v^2 w) = 0,
\end{cases}
$$

which has the same characteristic speeds as (9.10.1) and similarly admits piecewise constant solutions with the wave pattern depicted in Fig. 9.10.1. It is possible to adjust the speeds of the two 2-shocks in such a manner that after any two successive reflections, 1- and 3-shocks regain their original left and right states, i.e., the solution takes values in a finite set of states. On the other hand, as t approaches from below the time t^* of collision of the two 2-shocks, the number of shocks, of fixed strength, that cross the t-time line grows without bound, thus driving the total variation to infinity. Details may be found in the references cited in Section 9.12.

In view of the above, one should not expect global existence of weak solutions to the Cauchy problem for general systems of conservation laws and general initial data. Consequently, the aim of the theory should be to establish existence in the large, either for general systems under "small" initial data, or for special systems under general initial data. The hope is that this special class will include the systems arising in continuum physics, which are endowed with special features.

9.11 Notes

As pointed out in the historical introduction, the Riemann problem was originally formulated, and solved, by Riemann [1], in the context of the system (7.1.12) of isentropic gas dynamics.

For details on the structure of general self-similar solutions to hyperbolic conservation laws, outlined in Section 9.1, and proofs that, for genuinely nonlinear systems, such solutions are necessarily special functions of bounded variation, the reader is referred to Dafermos [28] and Elling and Roberts [2].

The method of shock and rarefaction wave curves, conceived by Riemann, was gradually developed in order to solve special Riemann problems, for the system of isentropic or adiabatic gas dynamics, describing wave interactions and shock tube experiments. This early research is surveyed in Courant and Friedrichs [1]. The distillation of that work led to the solution, by Lax [2], of the Riemann problem, with weak waves, for strictly hyperbolic systems of conservation laws with characteristic families that are genuinely nonlinear or linearly degenerate (Theorem 9.4.1). Detailed expositions of the solution to the Riemann problem for the system of adiabatic (nonisentropic) gas dynamics are found in the texts by Smoller [3], Serre [11], Godlewski and Raviart [2], Holden and Risebro [5], and especially in the monograph by Chang and Hsiao [3]. Early references addressing the issue of large data include Smoller [1,2], Smith [1] and Sever [1,2,3]. In particular, Sever [3] sounds the warning that, even for genuinely nonlinear systems, the standard shock admissibility criteria may be inadequate for securing uniqueness of solutions to the Riemann problem with large data.

Dealing with systems that are not genuinely nonlinear required additional effort. Following the prescription of the Oleinik E-condition, the form of the solution of the Riemann problem for the general scalar conservation law was described by Gelfand [1], through an example. Subsequently, Wendroff [1] solved the Riemann problem for the systems (7.1.11) and (7.1.8), when σ_{uu} may change sign. The construction of the solution for (7.1.2) and (7.1.11) described in this section, which employs the convex or concave envelope of f and σ, is found in Dafermos [2] and Leibovich [1]. The above results were apparently anticipated by research in China, in the 1960's, which did not circulate in the international scientific community until much later, e.g., Chang and Hsiao [1,2], and Hsiao and Zhang [1]. The monograph by Chang and Hsiao [3] provides a detailed exposition and many references. In particular, Chen and Young [2] investigate whether vacuum states may arise in the solution to the Riemann problem for the equations of nonisentropic flow of gases with general equations of state. The study of special systems motivated the solution by Tai-Ping Liu [1] of the Riemann problem for arbitrary piecewise genuinely nonlinear, strictly hyperbolic systems of conservation laws. A more precise analysis, by Iguchi and LeFloch [1], of the structure of wave fan curves associated with piecewise genuinely nonlinear characteristic families has led to the solution of the Riemann problem for more general strictly hyperbolic systems of conservation laws, not necessarily piecewise genuinely nonlinear. See also Ancona and Marson [4]. The observation that wave fan curves may be merely Lipschitz is due to Bianchini [6].

For systems with flux functions that are merely Lipschitz continuous, the Riemann problem is solved by Correia, LeFloch and Thanh [1]. For systems in which the flux function experiences jump discontinuities, see LeFloch and Thanh [2,4]. See also Holden and Risebro [4], Chalons, Raviart and Seguin [1], Colombo and Marcellini [1], and Andreianov [1]. The Riemann problem for balance laws is discussed in Hong and Temple [1], and in Goatin and LeFloch [4]

For early references on delta shocks, consult Keyfitz and Kranzer [2,3,4]. A detailed presentation is found in the memoir by Sever [12], and a "retrospective and prospective" in Keyfitz [4]. To a great extent, the theory of delta shocks was developed in conjunction with the study of the pressureless gas model. In that connection, see Li, Zhang and Yang [1], Feimin Huang [2], Sever [10,11], and the references cited in Section 11.14. Furthermore, Gui-Qiang Chen and Hailiang Liu [1,2] demonstrate that in the limit, as the response of the pressure to the density relaxes to zero, solutions of the Riemann problem for the system of isentropic or nonisentropic gas dynamics reduce to solutions of the equations of pressureless gas dynamics with delta shocks. Delta shocks are also discussed by Ercole [1], Joseph [1], Hayes and LeFloch [1], Tan [1], Tan, Zhang and Zheng [1], Sheng and Zhang [1], Hanchun Yang [1], Li and Yang [1], Brenier [3], Danilov and Shelkovich [1,2], Danilov and Mitrovich [2], Panov and Shelkovich [1], Shelkovich [1,2,3], Keyfitz and Tsikkou [1], Tsikkou [3], and Joseph and Sahoo [2].

The entropy rate admissibility criterion was proposed by Dafermos [3]. For motivation from the kinetic theory, see Ferziger and Kaper [1, §5.5] and Kohler [1]. Additional motivation is provided by the vanishing viscosity approach; see Bethuel, Despres and Smets [1]. A proof that the entropy rate criterion characterizes admissible L^∞ solutions for genuinely nonlinear scalar conservation laws is found in Blaser and Rivière [1]. Theorems 9.7.2 and 9.7.5 are from Dafermos [32], while Theorem 9.7.6 is due to Hsiao [1]. A detailed alternative proof of Theorem 9.7.3 appears in Dafermos [18]. Finally, Theorem 9.7.4 is taken from Dafermos [29].

The efficacy of the entropy rate criterion has also been tested on systems that change type, modeling phase transitions (Hattori [1,2,3,4,5,6,7], Pence [2]). See also Aavatsmark [1], Sever [7], and Krejčí and Straskraba [1].

The status of the entropy rate criterion for the Euler equations in two space dimensions is discussed in Chiodaroli and Kreml [1].

An alternative characterization of the solution of the Riemann problem by means of an entropy inequality is due to Heibig and Serre [1]. For other results on entropy minimization see Tadmor [1], Kröner, LeFloch and Thanh [1], Belletini, Bertini, Mariani and Novega [1], and Bonaschi, Carillo, DiFrancesco and Peletier [1].

The study of self-similar solutions of hyperbolic systems of conservation laws as limits of self-similar solutions of dissipative systems with time-dependent artificial viscosity was initiated, independently, by Kalasnikov [1], Tupciev [1,2], and Dafermos [4]. This approach was initially employed, in Dafermos [4,5] and Dafermos and DiPerna [1], for solving the Riemann problem in the case of strictly hyperbolic systems of two conservation laws. In particular, the treatment here of the system (9.8.33) has been adapted from Dafermos and DiPerna [1].

Theorem 9.8.1, due to Tzavaras [3], provided the earliest complete construction of solutions to the Riemann problem, for general strictly hyperbolic systems of conservation laws, without any assumption of piecewise genuine nonlinearity. A parallel construction, at the same level of generality, by means of the standard vanishing viscosity approach (see Chapter XV), was later provided by Bianchini and Bressan [5], and Bianchini [6]. In fact, the last reference motivated the construction of wave fan curves by the method of viscous wave fans, developed here, in Section 9.8. An alternative construction of the wave fan curves, also by the method of viscous wave fans, is found in Joseph and LeFloch [4].

For construction of solutions to the Riemann problem for particular hyperbolic systems through the viscous wave fan approach, the reader may consult the following references: Yong Jung Kim [1], Slemrod and Tzavaras [1,3], Tzavaras [2, 5], and Andreianov [2], for strictly hyperbolic systems; Boutin, Coquel and LeFloch, for coupled systems; Ercole [1], Keyfitz and Kranzer [3,4], Joseph [1], Tan [1], Tan, Zhang and Zheng [1], Sheng and Zhang [1], and Li and Yang [1], for nonstrictly hyperbolic systems with delta shocks; Slemrod [4], and Fan [1,2], for systems of mixed type; and Slemrod [7], for solutions with spherical symmetry to the system of isentropic gas dynamics.

Explicit constructions of solutions to Riemann problems for the systems of isentropic or nonisentropic gas flow, via the standard vanishing viscosity approach, are found in Lin and Yang [1], Huang, Jian and Wang [1], Huang, Wang and Yang [1,2], Huang, Li and Wang [1], Huang and Li [1], Huang, Wang, Wang and Yang [1], Zhang, Pan, Wang and Tan [1], and Zhang, Pan and Tan [1].

On the question of whether structurally stable solutions of a Riemann problem, even in the presence of strong and/or undercompressible shocks, may be approximated by viscous wave fans, see Schecter [4,5,6], Lin and Schecter [1], Schecter and Szmolyan [1], Weishi Liu [1], and Xiao-Biao Lin [1,2,3,4,5,6]. For related numerical computations, see Schecter, Marchesin and Plohr [3].

There are numerous extensions of the idea of modifying standard perturbations of systems of conservation laws so as to preserve self-similarity of solutions. In that connection, see Li, Zhang and Yang [1], for the system of pressureless gas dynamics in two space dimensios; Slemrod and Tzavaras [2], and Tzavaras [1], for the Broadwell system of gas dynamics; Joseph and LeFloch [1,2,3,4,5], and LeFloch and Rohde [1], for the Riemann problem on a half-plane or on a quarter-plane, for general viscosity matrices, or when viscosity is replaced by time-dependent viscosity-capillarity or relaxation; Jiequan Li and Peng Zhang [1], for the system (7.1.26) modeling combustion, with the constant reaction rate δ replaced by δ/t.

The current status of the theory of the Riemann problem for systems that are not strictly hyperbolic is far from definitive. Both existence and admissibility of solutions raise thorny issues, as wave fans may comprise a great variety of exotic waves such as overcompressive or undercompressive shocks, delta shocks, and oscillations. It is futile to aim for an all-encompassing theory; one should focus, instead, on specific systems arising in continuum physics, most notably in elasticity and multi-phase flows. Progress has been made on the classification of such systems and on the existence and uniqueness of admissible solutions; see Glimm [2], Azevedo and Marchesin [1], Azevedo, Marchesin, Plohr and Zumbrun [1,2], Freistühler [3], Isaacson, Marchesin and Plohr [1], Isaacson, Marchesin, Plohr and Temple [1,2], Isaacson and Temple [2], Schaeffer and Shearer [2], Schecter, Marchesin and Plohr [1,2], M. Shearer [4,5], Shearer, Schaeffer, Marchesin and Paes-Leme [1], Schecter and Shearer [1], Schulze and Shearer [1], Tang and Ting [1], Zhu and Ting [1], Čanić [1,2], Čanić and Peters [1], Peters and Čanić [1], Ercole [1], Keyfitz and Kranzer [1,2,3], Tan [1], Tan, Zhang and Zheng [1], Schecter [1,2,3], Chalons and Coquel [1], Hanche-Olsen, Holden and Risebro [1], Azevedo, de Souza, Furtado and Marchesin [1], Azevedo, Eschenazi, Marchesin and Palmeira [1], Lambert and Marchesin [2], Rodrigues-Bermúdez and Marchesin [1], Chapiro, Marchesin and Schecter [1], and Silva and Marchesin [1]. Ostrov [3] provides an instructive example, in which one obtains distinct solutions to the Riemann problem, via the vanishing viscosity method, by varying the relative size of viscosity coefficients.

The reader may get some idea of the wide variety of wave configurations encountered in solutions of the Riemann problem for systems arising in continuum physics from Tong Zhang and Yuxi Zheng [2], which treats the system of balance laws governing Chapman-Jouguet combustion. Interesting Riemann problems arise even in differential geometry; see Bascar and Prasad [1].

The solution of the Riemann problem for systems of mixed type, employed to model phase transitions, comprises phase boundaries, in addition to classical shocks and rarefaction waves. As already noted in Section 8.8, the admissibility of phase boundaries is dictated by kinetic relations. Solutions of Riemann problems of this type are found in Fan [1,2,5,6], Frid and Liu [1], Hattori [1,2,3,4,7,8], Holden [1], Hsiao [2], Hsiao and DeMottoni [1], Keyfitz [1], LeFloch and Thanh [1], Mercier and Piccoli [1,2], Pence [2], M. Shearer [1,3], Shearer and Yang [1], Slemrod [5], Corli and Fan [1], and Fan and Lin [1]. For an informative discussion and additional references, see the monograph by LeFloch [5].

The Riemann problem has also been posed for quasilinear hyperbolic systems (9.8.18) that are not in conservative form, and solved either by piecing together rarefaction waves and shocks defined by the approach outlined in Section 8.7 (see LeFloch and Liu [1], LeFloch and Tzavaras [1], and Colombo and Monti [1]) or via the vanishing viscosity approach (Bianchini and Bressan [5], Bianchini [6]). R. Young [10] exhibits systems in that form for which the Riemann problem admits multiple solutions containing rarefaction waves and no shocks.

For the solution of the Riemann problem on the quarter-plane by the vanishing viscosity method, see Bianchini and Spinolo [1]. Equally well, one may employ a construction via viscous wave fans, as explained in Section 9.8. See Christoforou and Spinolo [1,2,3]. For applications, see Andrianov and Warnecke [1,2], LeFloch and Thanh [2], LeFloch and Shearer [1], and Joseph and Sahoo [1].

So-called *generalized Riemann problems*, in which the initial data are smooth (rather than constant) on both sides of a jump discontinuity, are discussed in Ta-tsien Li and De-yin Kong [2], Ta-tsien Li and Libin Wang [2,4,5,6], Mentrelli and Ruggeri [1], Ben Artzi [1], and Ben-Artzi and Li [1]. For a comprehensive treatment, see the monograph by Ben-Artzi and Falcovitz [1].

The study of interactions of wave fans and the original proof of Theorem 9.9.1, for genuinely nonlinear systems, is due to Glimm [1]. The derivation presented here is taken from Yong [1]. For systems that are not genuinely nonlinear, wave interaction estimates were originally obtained by Tai-Ping Liu [15], who was the first to realize the key role of the incidence angle. For recent detailed and rigorous expositions see Iguchi and LeFloch [1], and Tai-Ping Liu and Tong Yang [7]. For a discussion on wave interactions in (nonlinear) elastic strings, see R. Young [5], and in the theory of relativity see Groah, Smoller and Temple [1]. For composite wave interactions in isentropic gas dynamics, producing solutions with surprising features, see R. Young [9]. A complete classification of possible wave interactions for the equations of non-isentropic ideal gas flow is found in Chang and Hsiao [3], and in Chen, Endres and Jenssen [1]. For a description of actual wave interactions, see Greenberg [1,2] for the system of isentropic elasticity, Liu and Zhang [1] for a scalar combustion model, and Luo and Yang [1] for the Euler equations of isentropic gas flow with frictional damping.

The example of breakdown of weak solutions presented in Section 9.10 is taken from Jenssen [1]. Additional examples were constructed by Baiti and Jenssen [3], R. Young [5,6], Young and Szeliga [1], and Jenssen and Young [1]. In particular, it is shown that even solutions starting out from initial data with arbitrarily small total variation may blow up in finite time when the characteristic speeds of distinct families are not uniformly separated on the range of the solution. For earlier work indicating rapid magnification, or even blowing up, in the supremum or the total variation of solutions, see Jeffrey [2], R. Young [2], and Joly, Métivier and Rauch [2]. No blowing up has been detected thus far in solutions of systems with physical interest, raising the hope that the special structure of these systems may offset the agents of instability. See, however, Tsikkou [1], and Bressan, Chen, Zhang and Zhu [1].

X

Generalized Characteristics

As already noted in Section 7.9, the function space of choice for weak solutions of hyperbolic systems of conservation laws in one space dimension is BV, since it is within its confines that one may discern shocks and study their propagation and interactions. The notion of characteristic, introduced in Section 7.2 for classical solutions, will here be extended to the framework of BV weak solutions. It will be established that generalized characteristics propagate either with classical characteristic speed or with shock speed. In particular, it will be shown that the extremal backward characteristics, emanating from any point in the domain of an admissible solution, always propagate with classical characteristic speed. The implications of these properties to the theory of weak solutions will be demonstrated in following chapters.

10.1 *BV* Solutions

We consider the strictly hyperbolic system

$$\partial_t U + \partial_x F(U) = 0 \tag{10.1.1}$$

of conservation laws. Throughout this chapter, U will denote a bounded measurable function on $(-\infty,\infty)\times(0,\infty)$, of class BV_{loc}, which is a weak solution of (10.1.1). By the theory of BV functions in Section 1.7, $(-\infty,\infty)\times(0,\infty) = \mathscr{C}\bigcup\mathscr{J}\bigcup\mathscr{I}$ where $\mathscr{C}$ is the set of points of approximate continuity of U, $\mathscr{J}$ denotes the set of points of approximate jump discontinuity (shock set) of U, and $\mathscr{I}$ stands for the set of irregular points of U. The one-dimensional Hausdorff measure of $\mathscr{I}$ is zero : $\mathscr{H}^1(\mathscr{I}) = 0$. The shock set $\mathscr{J}$ is essentially covered by the (at most) countable union of C^1 arcs. With any $(\bar{x},\bar{t}) \in \mathscr{J}$ are associated one-sided approximate limits $U_\pm$ and a "tangent" line of slope (shock speed) s which, as shown in Section 1.8, are related by the Rankine-Hugoniot jump condition (8.1.2).

We shall be assuming throughout that the Lax E-condition, introduced in Section 8.3, holds here in a strong sense: each shock is compressive but not overcompressive. That is, if $U_\pm$ are the one-sided limits and s is the corresponding shock speed associated with any point of the shock set, then there is $i \in \{1,\cdots,n\}$ such that

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(10.1.2) $$\lambda_{i-1}(U_\pm) < \lambda_i(U_+) \le s \le \lambda_i(U_-) < \lambda_{i+1}(U_\pm).$$

In (10.1.2), the first inequality is not needed when $i = 1$ and the last inequality is unnecessary when $i = n$. Moreover, since (10.1.1) is strictly hyperbolic, the first and the last inequalities will hold automatically whenever the oscillation of U is sufficiently small.

For convenience, we normalize U as explained in Section 1.7. In particular, at every point $(\bar{x},\bar{t}) \in \mathscr{C}$, $U(\bar{x},\bar{t})$ equals the corresponding approximate limit U_0. Recalling that $\mathscr{H}^1(\mathscr{I}) = 0$ and using Theorem 1.7.2, we easily conclude that there is a subset $\mathscr{N}$ of $(0,\infty)$, of measure zero, having the following properties: for any fixed $\bar{t} \notin \mathscr{N}$, the function $U(\cdot,\bar{t})$ has locally bounded variation on $(-\infty,\infty)$, and $(\bar{x},\bar{t}) \in \mathscr{C}$ if and only if $U(\bar{x}-,\bar{t}) = U(\bar{x}+,\bar{t})$, while $(\bar{x},\bar{t}) \in \mathscr{J}$ if and only if $U(\bar{x}-,\bar{t}) \neq U(\bar{x}+,\bar{t})$. In the latter case, $U_- = U(\bar{x}-,\bar{t})$ and $U_+ = U(\bar{x}+,\bar{t})$.

The above properties of U follow just from membership in BV. The fact that U is also a solution of (10.1.1) should induce additional structure. On the basis of experience with special systems, to be discussed in later chapters, it seems plausible to expect the following: U should be (classically) continuous on $\mathscr{C}$ and the one-sided limits $U_\pm$ at points of $\mathscr{J}$ should be attained in the classical sense. Moreover, $\mathscr{I}$ should be the (at most) countable set of endpoints of the arcs that comprise $\mathscr{J}$. Uniform stretching of the (x,t) coordinates about any point of $\mathscr{I}$ should yield, in the limit, a wave fan with the properties described in Section 9.1, i.e., $\mathscr{I}$ should consist of shock generation and shock interaction points. To what extent the picture painted above accurately describes the structure of solutions of general hyperbolic systems of conservation laws will be discussed in later chapters.

10.2 Generalized Characteristics

Characteristics associated with classical, Lipschitz continuous, solutions were introduced in Section 7.2, through Definition 7.2.1. They provide one of the principal tools of the classical theory for the study of analytical and geometric properties of solutions. It is thus natural to attempt to extend the notion to the framework of weak solutions.

Here we opt to define characteristics of the i-characteristic family, associated with the weak solution U, exactly as in the classical case, namely as integral curves of the ordinary differential equation (7.2.7), in the sense of Filippov:

10.2.1 Definition. A *generalized i-characteristic* for the system (10.1.1), associated with the (generally weak) solution U, on the time interval $[\sigma,\tau] \subset [0,\infty)$, is a Lipschitz function $\xi : [\sigma,\tau] \to (-\infty,\infty)$ which satisfies the differential inclusion

(10.2.1) $$\dot{\xi}(t) \in \Lambda_i(\xi(t),t), \qquad \text{a.e. on } [\sigma,\tau],$$

where

(10.2.2) $$\Lambda_i(\bar{x},\bar{t}) := \bigcap_{\varepsilon>0} \left[\, \operatorname*{ess\,inf}_{[\bar{x}-\varepsilon,\bar{x}+\varepsilon]} \lambda_i(U(x,\bar{t}))\ ,\ \operatorname*{ess\,sup}_{[\bar{x}-\varepsilon,\bar{x}+\varepsilon]} \lambda_i(U(x,\bar{t}))\,\right].$$

From the general theory of contingent equations like (10.2.1), one immediately infers the following

10.2.2 Theorem. *Through any fixed point* $(\bar{x},\bar{t}) \in (-\infty,\infty) \times [0,\infty)$ *pass two (not necessarily distinct) generalized i-characteristics, associated with U and defined on* $[0,\infty)$*, namely the minimal* $\xi_-(\cdot)$ *and the maximal* $\xi_+(\cdot)$*, with* $\xi_-(t) \le \xi_+(t)$ *for* $t \in [0,\infty)$*. The funnel-shaped region confined between the graphs of* $\xi_-(\cdot)$ *and* $\xi_+(\cdot)$ *comprises the set of points* (x,t) *that may be connected to* $(\bar{x},\bar{t})$ *by a generalized i-characteristic associated with* U*.*

Other standard properties of solutions of differential inclusions also have useful implications for the theory of generalized characteristics: If $\{\xi_m(\cdot)\}$ is a sequence of generalized i-characteristics, associated with U and defined on $[\sigma,\tau]$, which converges to some Lipschitz function $\xi(\cdot)$, uniformly on $[\sigma,\tau]$, then $\xi(\cdot)$ is necessarily a generalized i-characteristic associated with U. In particular, if $\xi_m(\cdot)$ is the minimal (or maximal) generalized i-characteristic through a point $(x_m,\bar{t})$ and $x_m \uparrow \bar{x}$ (or $x_m \downarrow \bar{x}$), as $m \to \infty$, then $\{\xi_m(\cdot)\}$ converges to the minimal (or maximal) generalized i-characteristic $\xi_-(\cdot)$ (or $\xi_+(\cdot)$) through the point $(\bar{x},\bar{t})$.

In addition to classical i-characteristics, i-shocks that satisfy the Lax E-condition are obvious examples of generalized i-characteristics. In fact, it turns out that these are the only possibilities. Indeed, even though Definition 10.2.1 would seemingly allow $\dot{\xi}$ to select any value in the interval Λ_i, the fact that U is a solution of (10.1.1) constrains generalized i-characteristics associated with U to propagate either with classical i-characteristic speed or with i-shock speed:

10.2.3 Theorem. *Let* $\xi(\cdot)$ *be a generalized i-characteristic, associated with U and defined on* $[\sigma,\tau]$*. The following holds for almost all* $t \in [\sigma,\tau]$ *: When* $(\xi(t),t) \in \mathscr{C}$*, then* $\dot{\xi}(t) = \lambda_i(U_0)$ *with* $U_0 = U(\xi(t)\pm,t)$*. When* $(\xi(t),t) \in \mathscr{J}$*, then* $\dot{\xi}(t) = s$*, where s is the speed of the i-shock that joins* U_-*, on the left, to* U_+*, on the right, with* $U_\pm = U(\xi(t)\pm,t)$*. In particular, s satisfies the Rankine-Hugoniot condition* (8.1.2) *as well as the Lax E-condition* (10.1.2).

Proof. Upon recalling the properties of BV solutions recorded in Section 10.1, it becomes clear that for almost all $t \in [\sigma,\tau]$ with $(\xi(t),t) \in \mathscr{C}$ the interval $\Lambda_i(\xi(t),t)$ reduces to the single point $\lambda_i(U(\xi(t)\pm,t))$ and so $\dot{\xi}(t) = \lambda_i(U(\xi(t)\pm,t))$, by virtue of (10.2.1).

Applying the measure equality (10.1.1) to arbitrary subarcs of the graph of ξ, and using Theorem 1.7.8 (in particular Equation (1.7.12)), yields

$$F(U(\xi(t)+,t)) - F(U(\xi(t)-,t)) = \dot{\xi}(t)[U(\xi(t)+,t) - U(\xi(t)-,t)], \tag{10.2.3}$$

almost everywhere on $[\sigma,\tau]$. Consequently, for almost all $t \in [\sigma,\tau]$ with $(\xi(t),t)$ in $\mathscr{J}$, we have $\dot{\xi}(t) = s$, where s is the speed of a shock that joins the two states $U_- = U(\xi(t)-,t)$ and $U_+ = U(\xi(t)+,t)$. Because of the structure of solutions, there is $j \in \{1,\cdots,n\}$ such that $\lambda_{j-1}(U_\pm) < \lambda_j(U_+) \le s \le \lambda_j(U_-) < \lambda_{j+1}(U_\pm)$. On the other hand, (10.2.1) implies that s lies in the interval with endpoints $\lambda_i(U_-)$ and $\lambda_i(U_+)$. Therefore, $j = i$ and (10.1.2) holds. This completes the proof.

The above theorem motivates the following terminology:

10.2.4 Definition. A generalized i-characteristic $\xi(\cdot)$, associated with U and defined on $[\sigma, \tau]$, is called *shock-free* if $U(\xi(t)-,t) = U(\xi(t)+,t)$, for almost all t in $[\sigma, \tau]$.

A consequence of the proof of Theorem 10.2.3 is that (10.2.1) is equivalent to

$$\dot{\xi}(t) \in [\lambda_i(U(\xi(t)+,t)) \, , \, \lambda_i(U(\xi(t)-,t))], \qquad \text{a.e. on } [\sigma, \tau]. \tag{10.2.4}$$

In what follows, an important role will be played by the special generalized characteristics that manage to propagate at the maximum or minimum allowable speed:

10.2.5 Definition. A generalized i-characteristic $\xi(\cdot)$, associated with U and defined on $[\sigma, \tau]$, is called a *left i-contact* if

$$\dot{\xi}(t) = \lambda_i(U(\xi(t)-,t)), \qquad \text{a.e. on } [\sigma, \tau], \tag{10.2.5}$$

and/or a *right i-contact* if

$$\dot{\xi}(t) = \lambda_i(U(\xi(t)+,t)), \qquad \text{a.e. on } [\sigma, \tau]. \tag{10.2.6}$$

Clearly, shock-free i-characteristics are left and right i-contacts. Note that, since they are generalized i-characteristics, left (or right) i-contacts should also satisfy the assertion of Theorem 10.2.3, namely $\dot{\xi}(t) = s$ for almost all $t \in [\sigma, \tau]$ with $(\xi(t),t)$ in $\mathscr{J}$. Of course this is impossible in systems that do not admit left (or right) contact discontinuities. In any such system, left (or right) contacts are necessarily shock-free. In particular, recalling Theorem 8.2.1, we conclude that when the i-characteristic family for the system (10.1.1) is genuinely nonlinear and the oscillation of U is sufficiently small, then any left or right i-contact is necessarily shock-free.

10.3 Extremal Backward Characteristics

With reference to some point $(\bar{x},\bar{t}) \in (-\infty, \infty) \times [0, \infty)$, a generalized characteristic through $(\bar{x},\bar{t})$ is dubbed *backward* when defined on $[0, \bar{t}]$, or *forward* when defined on $[\bar{t}, \infty)$. The extremal, minimal and maximal, backward and forward generalized characteristics through $(\bar{x},\bar{t})$ propagate at extremal speeds and are thus natural candidates for being contacts. This turns out to be true, at least for the backward extremal characteristics, in consequence of the Lax E-condition:

10.3.1 Theorem. *The minimal (or maximal) backward i-characteristic, associated with U, emanating from any point $(\bar{x},\bar{t})$ of the upper half-plane is a left (or right) i-contact.*

Proof. Let $\xi(\cdot)$ denote the minimal backward i-characteristic emanating from $(\bar{x},\bar{t})$ and defined on $[0,\bar{t}]$. We fix $\varepsilon > 0$ and select $\bar{t} = \tau_0 > \tau_1 > \cdots > \tau_k = 0$, for some $k \geq 1$, through the following algorithm: We start out with $\tau_0 = \bar{t}$. Assuming $\tau_m > 0$ has been determined, we let $\xi_m(\cdot)$ denote the minimal backward i-characteristic emanating from $(\xi(\tau_m) - \varepsilon, \tau_m)$. If $\xi_m(t) < \xi(t)$ for $0 < t \leq \tau_m$, we set $\tau_{m+1} = 0, m+1 = k$ and terminate. Otherwise, we locate $\tau_{m+1} \in (0, \tau_m)$ with the property $\xi_m(t) < \xi(t)$ for $\tau_{m+1} < t \leq \tau_m$ and $\xi_m(\tau_{m+1}) = \xi(\tau_{m+1})$. Clearly, this algorithm will terminate after a finite number of steps. Next we construct a left-continuous, piecewise Lipschitz function $\xi_\varepsilon(\cdot)$ on $[0,\bar{t}]$, with jump discontinuities (when $k \geq 2$) at $\tau_1, \cdots, \tau_{k-1}$, by setting $\xi_\varepsilon(t) = \xi_m(t)$ for $\tau_{m+1} < t \leq \tau_m$, with $m = 0, 1, \cdots, k-1$, and $\xi_\varepsilon(0) = \xi_{k-1}(0)$. Then

$$\text{(10.3.1)} \quad \xi_\varepsilon(\bar{t}) - \xi_\varepsilon(0) = (k-1)\varepsilon + \sum_{m=0}^{k-1} \int_{\tau_{m+1}}^{\tau_m} \dot{\xi}_m(t)dt \geq \int_0^{\bar{t}} \lambda_i(U(\xi_\varepsilon(t)+,t))dt.$$

As $\varepsilon \downarrow 0, \xi_\varepsilon(t) \to \xi(t)$, from the left, for any $t \in [0,\bar{t}]$. To see this, fix some $t \in [0,\bar{t}]$. Then $(\xi_\varepsilon(t),t)$ lies on the minimal backward characteristic emanating from some point $(\xi(\tau_\varepsilon) - \varepsilon, \tau_\varepsilon)$, with $\tau_\varepsilon \geq t$. Take any sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ and $\tau_{\varepsilon_k} \to \tau$, as $k \to \infty$. Thus $(\xi(\tau_{\varepsilon_k}) - \varepsilon_k, \tau_{\varepsilon_k}) \to (\xi(\tau), \tau)$. It then follows by the standard theory of contingent equations, like (10.2.1), that the sequence of minimal backward characteristics emanating from $(\xi(\tau_{\varepsilon_k}) - \varepsilon_k, \tau_{\varepsilon_k})$ must converge to $\xi(\cdot)$, uniformly on $[0,t]$. In particular, $\xi_{\varepsilon_k}(t) \to \xi(t)$, as $k \to \infty$. Therefore, letting $\varepsilon \downarrow 0$, (10.3.1) yields

$$\text{(10.3.2)} \qquad \xi(\bar{t}) - \xi(0) \geq \int_0^{\bar{t}} \lambda_i(U(\xi(t)-,t))dt.$$

On the other hand, $\dot{\xi}(t) \leq \lambda_i(U(\xi(t)-,t))$, almost everywhere on $[0,\bar{t}]$, and therefore $\dot{\xi}(t) = \lambda_i(U(\xi(t)-,t))$ for almost all $t \in [0,\bar{t}]$, i.e., $\xi(\cdot)$ is a left i-contact.

Similarly one shows that the maximal backward i-characteristic emanating from $(\bar{x},\bar{t})$ is a right i-contact. This completes the proof.

In view of the closing remarks in Section 10.2, Theorem 10.3.1 has the following corollary:

10.3.2 Theorem. *Assume the i-characteristic family for the system* (10.1.1) *is genuinely nonlinear and the oscillation of U is sufficiently small. Then the minimal and the maximal backward i-characteristics, emanating from any point* $(\bar{x},\bar{t})$ *of the upper half-plane, are shock-free.*

The implications of the above theorem will be seen in following chapters.

For future use, it will be expedient to introduce here a special class of backward characteristics emanating from infinity:

10.3.3 Definition. A *minimal* (or *maximal*) *i-separatrix*, associated with the solution U, is a Lipschitz function $\xi : [0,\bar{t}) \to (-\infty,\infty)$ such that $\xi(t) = \lim_{m\to\infty} \xi_m(t)$, uniformly on compact time intervals, where $\xi_m(\cdot)$ is the minimal (or maximal) backward

i-characteristic emanating from a point (x_m, t_m), with $t_m \to \bar{t}$, as $m \to \infty$. In particular, when $\bar{t} = \infty$, the i-separatrix $\xi(\cdot)$ is called a *minimal* (or *maximal*) *i-divide*.

Note that the graphs of any two minimal (or maximal) i-characteristics may run into each other but they cannot cross. Consequently, the graph of a minimal (or maximal) backward i-characteristic cannot cross the graph of any minimal (or maximal) i-separatrix. Similarly, the graphs of any two minimal (or maximal) i-separatrices cannot cross. In particular, any minimal (or maximal) i-divide divides the upper half-plane into two parts in such a way that no forward i-characteristic may cross from the left to the right (or from the right to the left).

Minimal or maximal i-separatrices are necessarily generalized i-characteristics, which, by virtue of Theorem 10.3.1, are left or right i-contacts. In particular, when the i-characteristic family is genuinely nonlinear and the oscillation of U is sufficiently small, Theorem 10.3.2 implies that minimal or maximal i-separatrices are shock-free.

The following propositions recount conditions under which divides exist:

10.3.4 Lemma. *Suppose that there are i-characteristics φ and ψ, associated with U, issuing, at $t = 0$, from the x-axis and satisfying the following properties: $\varphi(t) < \psi(t)$, for $t \in [0,\infty)$ and ψ is maximal (or ψ is minimal). Then there exists a minimal (or maximal) i-divide ξ such that $\varphi(t) \leq \xi(t) \leq \psi(t)$, for $t \in [0,\infty)$.*

Proof. Assuming φ is maximal, let ξ_m denote the minimal backward i-characteristic emanating from the point (x_m, t_m), where $\varphi(t_m) < x_m < \psi(t_m)$ and $t_m \to \infty$, $m \to \infty$. The graph of ξ_m cannot intersect the graph of φ, since φ is maximal, and even though it may meet the graph of ψ, it cannot cross it, because ξ_m is minimal. Therefore, $\varphi(t) < \xi_m(t) \leq \psi(t)$, for $t \in [0, t_m]$. It then follows from the Ascoli theorem that $\{\xi_m\}$ contains subsequences converging to a minimal i-divide ξ with the asserted properties. The case where ψ is minimal is treated in a similar fashion. This completes the proof.

10.3.5 Corollary. *When U is spatially periodic, $U(x+L,t) = U(x,t)$, at least one minimal and one maximal i-divide issue from any period-length interval of the x-axis.*

Indeed, to construct the minimal i-divide it suffices to apply Lemma 10.3.4, with φ the maximal i-characteristic issuing from $(0,0)$ and ψ defined by $\psi(t) = \varphi(t) + L$, for $t \in [0,\infty)$. Similarly, the existence of the maximal i-divide follows from Lemma 10.3.4 upon selecting ψ as the minimal i-characteristic issuing from $(L,0)$ and then defining φ by $\varphi(t) = \psi(t) - L$, for $t \in [0,\infty)$. Of course, the minimal and maximal i-divides may coincide.

This chapter will close with the following remark: Generalized characteristics were introduced here in connection to BV solutions of (10.1.1) defined on the entire upper half-plane. It is clear, however, that the notion and many of its properties are of purely local nature and thus apply to BV solutions defined on arbitrary open subsets of $\mathbb{R}^2$.

10.4 Notes

The presentation of the theory of generalized characteristics in this chapter follows Dafermos [19]. An exposition of the general theory of differential inclusions is found in the monograph by Filippov [1]. An early paper introducing generalized characteristics (for scalar conservation laws) as solutions of the classical characteristic equations, in the sense of Filippov, is Wu [1]. See also Hörmander [1]. Glimm and Lax [1] employ an alternative definition of generalized characteristics, namely Lipschitz curves propagating either with classical characteristic speed or with shock speed, constructed as limits of a family of "approximate characteristics." In view of Theorem 10.2.3, the two notions are closely related.

The notion of divide was introduced in Dafermos [21].

Even though generalized characteristics will be considered here in the setting of BV solutions, they may be defined and used within the broader context of L^∞ weak solutions, in which the Definition 10.2.1 makes sense. Classes of weak solutions, encountered in the applications, that may not be in BV and yet the approach via generalized characteristics is effective, include those with characteristic speeds in BV and those in which one-sided limits $U(x\pm,t)$ exist for all $x \in (-\infty,\infty)$ and almost all $t \in (0,\infty)$.

Generalized characteristics in several space dimensions are considered by Poupaud and Rascle [1], in the context of linear transport equations with discontinuous coefficients.

XI

Scalar Conservation Laws in One Space Dimension

Despite its apparent simplicity, the scalar conservation law in one space dimension possesses a surprisingly rich theory, which deserves attention not only for its intrinsic interest but also because it provides valuable insight in the behavior of systems. The discussion here will employ the theory of generalized characteristics developed in Chapter X.

The bulk of this chapter will be devoted to the genuinely nonlinear case, the special feature of which is that the extremal backward generalized characteristics are essentially classical characteristics, that is, straight lines along which the solution is constant. This property induces such a heavy constraint that one is able to derive very precise information on regularity and large time behavior of solutions.

Solutions are (classically) continuous at points of approximate continuity and locally Lipschitz continuous in the interior of the set of points of continuity. Points of approximate jump discontinuity lie on classical shocks. The remaining, irregular, points are at most countable and are formed by the collision of shocks and/or the focusing of compression waves. Generically, solutions with smooth initial data are piecewise smooth.

Genuine nonlinearity gives rise to a host of dissipative mechanisms that affect the large-time behavior of solutions. Entropy dissipation induces $O(t^{-\frac{p}{p+1}})$ decay of solutions with initial data in $L^p(-\infty,\infty)$. When the initial data have compact support, spreading of characteristics generates N-wave profiles. Confinement of characteristics under periodic initial data induces $O(t^{-1})$ decay in the total variation per period and the formation of sawtoothed profiles.

Another important feature of admissible weak solutions of the Cauchy problem for the genuinely nonlinear scalar conservation law is that they are related explicitly to their initial values, through the Lax function. This property, which will be established here by the method of generalized characteristics, may serve alternatively as the starting point for developing the general theory of solutions to the Cauchy problem.

Additional insight is gained from comparison theorems on solutions. It will be shown that the lap number of any admissible solution is nonincreasing with time.

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Moreover, the L^1 distance of any two solutions is generally nonincreasing, but typically conserved, whereas a properly weighted L^1 distance is strictly decreasing.

One of the advantages of the method of generalized characteristics is that it readily extends to inhomogeneous, genuinely nonlinear balance laws. This theory will be outlined here and two examples will be presented in order to demonstrate how inhomogeneity and source terms may affect the large-time behavior of solutions.

By contrast, one is faced with considerable complexity when genuine nonlinearity fails. This will be demonstrated in the context of the simplest case of a flux with a single inflection point. The presence of the inflection point has serious implications on the generic regularity and the long time behavior of solutions.

The tools provided by the theory of generalized characteristics will be employed for determining whether, in the absence of bounded variation, entropy may be produced at points of continuity of solutions.

11.1 Admissible *BV* Solutions and Generalized Characteristics

The next eight sections of this chapter deal with the scalar conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0 \tag{11.1.1}$$

that is genuinely nonlinear, $f''(u) > 0$, $-\infty < u < \infty$. We are assuming throughout that u is an admissible weak solution on $(-\infty,\infty) \times [0,\infty)$ with initial data u_0 that are bounded and have locally bounded variation on $(-\infty,\infty)$. By virtue of Theorem 6.2.6, u is in BV_{loc}, and for any $t \in [0,\infty)$ the function $u(\cdot,t)$ has locally bounded variation on $(-\infty,\infty)$.

As noted in Section 8.5, the entropy shock admissibility criterion will be satisfied almost everywhere (with respect to one-dimensional Hausdorff measure) on the shock set $\mathscr{J}$ of the solution u, for any entropy-entropy flux pair (η,q) with η convex. This in turn implies that the Lax E-condition will also hold almost everywhere on $\mathscr{J}$. Consequently, we have

$$u(x+,t) \leq u(x-,t), \tag{11.1.2}$$

for almost all $t \in (0,\infty)$ and all $x \in (-\infty,\infty)$.

On account of Theorem 10.2.3, a Lipschitz curve $\xi(\cdot)$, defined on the time interval $[\sigma,\tau] \subset [0,\infty)$, will be a generalized characteristic, associated with the solution u, if for almost all $t \in [\sigma,\tau]$

$$\dot{\xi}(t) = \begin{cases} f'(u(\xi(t)\pm,t)), & \text{when } \; u(\xi(t)+,t) = u(\xi(t)-,t) \\[2ex] \dfrac{f(u(\xi(t)+,t)) - f(u(\xi(t)-,t))}{u(\xi(t)+,t) - u(\xi(t)-,t)}, & \text{when } \; u(\xi(t)+,t) < u(\xi(t)-,t). \end{cases} \tag{11.1.3}$$

The special feature of genuinely nonlinear scalar conservation laws is that generalized characteristics that are shock-free are essentially classical characteristics:

11.1.1 Theorem. *Let $\xi(\cdot)$ be a generalized characteristic for* (11.1.1), *associated with the admissible solution u, shock-free on the time interval $[\sigma,\tau]$. Then there is a constant $\bar{u}$ such that*

$$u(\xi(\tau)+,\tau) \le \bar{u} \le u(\xi(\tau)-,\tau), \tag{11.1.4}$$

$$u(\xi(t)+,t) = \bar{u} = u(\xi(t)-,t), \quad \sigma < t < \tau, \tag{11.1.5}$$

$$u(\xi(\sigma)-,\sigma) \le \bar{u} \le u(\xi(\sigma)+,\sigma). \tag{11.1.6}$$

In particular, the graph of $\xi(\cdot)$ is a straight line segment with slope $f'(\bar{u})$.

Proof. Fix r and s, $\sigma \le r < s \le \tau$. For $\varepsilon > 0$, the conservation law for the domains $\{(x,t) : r < t < s,\ \xi(t) - \varepsilon < x < \xi(t)\}$ and $\{(x,t) : r < t < s,\ \xi(t) < x < \xi(t) + \varepsilon\}$ takes the form

$$\int_{\xi(s)-\varepsilon}^{\xi(s)} u(x,s)dx - \int_{\xi(r)-\varepsilon}^{\xi(r)} u(x,r)dx \tag{11.1.7}$$

$$= \int_r^s \{f(u(\xi(t)-\varepsilon+,t)) - f(u(\xi(t)-,t)) - \dot{\xi}(t)[u(\xi(t)-\varepsilon+,t) - u(\xi(t)-,t)]\}dt,$$

$$\int_{\xi(r)}^{\xi(r)+\varepsilon} u(x,r)dx - \int_{\xi(s)}^{\xi(s)+\varepsilon} u(x,s)dx \tag{11.1.8}$$

$$= \int_r^s \{f(u(\xi(t)+\varepsilon-,t)) - f(u(\xi(t)+,t)) - \dot{\xi}(t)[u(\xi(t)+\varepsilon-,t) - u(\xi(t)+,t)]\}dt.$$

By virtue of Definition 10.2.4, $\dot{\xi}(t) = f'(u(\xi(t)\pm,t))$, a.e. on $[r,s]$. Then, since f is convex, the right-hand sides of both (11.1.7) and (11.1.8) are nonnegative. Consequently, multiplying (11.1.7) and (11.1.8) by $1/\varepsilon$ and letting $\varepsilon \downarrow 0$ yields

$$u(\xi(s)-,s) \ge u(\xi(r)-,r), \;\; u(\xi(s)+,s) \le (u(\xi(r)+,r), \quad \sigma \le r < s \le \tau. \tag{11.1.9}$$

We now fix t_1 and t_2, $\sigma < t_1 < t_2 < r$, such that $u(\xi(t_1)-,t_1) = u(\xi(t_1)+,t_1)$, $u(\xi(t_2)-,t_2) = u(\xi(t_2)+,t_2)$. For any fixed $t \in (t_1,t_2)$, we apply (11.1.9) first with $r = t_1$, $s = t_2$, then with $r = t_1$, $s = t$, and finally with $r = t$, $s = t_2$. This yields (11.1.5). To complete the proof, we apply (11.1.9) for $s = \tau$, $r \in (\sigma,\tau)$, to obtain (11.1.4), and for $r = \sigma$, $s \in (\sigma,\tau)$, to deduce (11.1.6).

11.1.2 Corollary. *Assume $\xi(\cdot)$ and $\zeta(\cdot)$ are distinct generalized characteristics for* (11.1.1), *associated with the admissible weak solution u, which are shock-free on the time interval $[\sigma,\tau]$. Then $\xi(\cdot)$ and $\zeta(\cdot)$ cannot intersect for any $t \in (\sigma,\tau)$.*

The above two propositions have significant implications for extremal backward characteristics:

11.1.3 Theorem. *Let $\xi_-(\cdot)$ and $\xi_+(\cdot)$ denote the minimal and maximal backward characteristics, associated with some admissible solution u, emanating from any point $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. Then*

$$\begin{cases} u(\xi_-(t)-,t) = u(\bar{x}-,\bar{t}) = u(\xi_-(t)+,t) \\ u(\xi_+(t)-,t) = u(\bar{x}+,\bar{t}) = u(\xi_+(t)+,t) \end{cases} \qquad 0 < t < \bar{t}, \tag{11.1.10}$$

$$\begin{cases} u_0(\xi_-(0)-) \le u(\bar{x}-,\bar{t}) \le u_0(\xi_-(0)+) \\ u_0(\xi_+(0)-) \le u(\bar{x}+,\bar{t}) \le u_0(\xi_+(0)+). \end{cases} \tag{11.1.11}$$

In particular, $u(\bar{x}+,\bar{t}) \le u(\bar{x}-,\bar{t})$ holds for all $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ and $\xi_-(\cdot)$, $\xi_+(\cdot)$ coincide if and only if $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$.

Proof. By virtue of Theorem 10.3.2, both $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are shock-free on $[0,\bar{t}]$. We may then apply Theorem 11.1.1, with $\sigma = 0$ and $\tau = \bar{t}$. On account of (11.1.4), when $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$ we have $\bar{u} = u(\bar{x}\pm,\bar{t})$ and thus $\xi_-(\cdot)$, $\xi_+(\cdot)$ coincide. In the general case, consider an increasing (or decreasing) sequence $\{x_n\}$, converging to $\bar{x}$, such that $u(x_n+,\bar{t}) = u(x_n-,\bar{t})$, $n = 1,2,\cdots$. Let $\xi_n(\cdot)$ denote the unique backward characteristic emanating from $(x_n,\bar{t})$. Then $u(\xi_n(t)\pm,t) = u(x_n\pm,\bar{t})$ for all $t \in (0,\bar{t})$. As noted in Section 10.2, the sequence $\{\xi_n(\cdot)\}$ converges from below (or above) to $\xi_-(\cdot)$ (or $\xi_+(\cdot)$). Consequently, $u(\xi_-(t)-,t) = \lim u(x_n\pm,\bar{t}) = u(\bar{x}-,\bar{t})$ (or $u(\xi_+(t)+,t) = \lim u(x_n\pm,\bar{t}) = u(\bar{x}+,\bar{t})$). The proof is complete.

We now turn to the properties of forward characteristics:

11.1.4 Theorem. *A unique forward generalized characteristic, associated with an admissible solution u, issues from any point $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$.*

Proof. Suppose two distinct forward characteristics $\phi(\cdot)$ and $\psi(\cdot)$ issue from $(\bar{x},\bar{t})$, such that $\phi(s) < \psi(s)$ for some $s > \bar{t}$. Let $\xi(\cdot)$ denote the maximal backward characteristic emanating from $(\phi(s),s)$ and $\zeta(\cdot)$ denote the minimal backward characteristic emanating from $(\psi(s),s)$, both being shock-free on $[0,s]$. For $t \in [\bar{t},s]$, $\xi(t) \ge \phi(t)$ and $\zeta(t) \le \psi(t)$; hence $\xi(\cdot)$ and $\zeta(\cdot)$ must intersect at some $t \in [\bar{t},s)$, in contradiction to Corollary 11.1.2. This completes the proof.

Note that, by contrast, multiple forward characteristics may issue from points lying on the x-axis. In particular, the focus of any centered rarefaction wave must necessarily lie on the x-axis.

The next proposition demonstrates that, once they form, jump discontinuities propagate as shock waves for eternity:

11.1.5 Theorem. *Let $\chi(\cdot)$ denote the unique forward generalized characteristic, associated with the admissible solution u, issuing from a point $(\bar{x},\bar{t})$ such that $\bar{t} > 0$ and $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$. Then $u(\chi(s)+,s) < u(\chi(s)-,s)$ for all $s \in [\bar{t},\infty)$.*

Proof. Let $\xi_-(\cdot)$ and $\xi_+(\cdot)$ denote the minimal and maximal backward characteristics emanating from $(\bar{x},\bar{t})$. Since $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$, $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are distinct: $\xi_-(0) < \xi_+(0)$.

Fix any $s \in [\bar{t},\infty)$ and consider the minimal and maximal backward characteristics $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ emanating from $(\chi(s),s)$. For $t \in [0,\bar{t}]$, necessarily $\zeta_-(t) \leq \xi_-(t)$ and $\zeta_+(t) \geq \xi_+(t)$. Thus $\zeta_-(0) < \zeta_+(0)$ so that $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ are distinct. Consequently, $u(\chi(s)+,s) < u(\chi(s)-,s)$. This completes the proof.

In view of the above, it is possible to identify the points from which shocks originate:

11.1.6 Definition. We call $(\bar{x},\bar{t}) \in (-\infty,\infty) \times [0,\infty)$ a *shock generation point* if some forward generalized characteristic $\chi(\cdot)$ issuing from $(\bar{x},\bar{t})$ is a shock, i.e., $u(\chi(t)+,t) < u(\chi(t)-,t)$, for all $t > \bar{t}$, while every backward characteristic emanating from $(\bar{x},\bar{t})$ is shock-free.

When $(\bar{x},\bar{t})$ is a shock generation point with $\bar{t} > 0$, there are two possibilities: $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$ or $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$. In the former case, the shock starts out at $(\bar{x},\bar{t})$ with zero strength and develops as it evolves. In the latter case, distinct minimal and maximal backward characteristics $\xi_-(\cdot)$ and $\xi_+(\cdot)$ emanate from $(\bar{x},\bar{t})$. The sector confined between the graphs of $\xi_-(\cdot)$ and $\xi_+(\cdot)$ must be filled by characteristics, connecting $(\bar{x},\bar{t})$ with the x-axis, which, by definition, are shock-free and hence are straight lines. Thus in that case the shock is generated at the focus of a *centered compression wave*, so it starts out with positive strength.

11.2 The Spreading of Rarefaction Waves

We are already familiar with the destabilizing role of genuine nonlinearity: compression wave fronts get steeper and eventually break, generating shocks. It turns out, however, that at the same time, genuine nonlinearity also exerts a regularizing influence by inducing the spreading of rarefaction wave fronts. It is remarkable that this effect is purely geometric and is totally unrelated to the regularity of the initial data:

11.2.1 Theorem. *For any admissible solution u,*

$$\frac{f'(u(y\pm,t)) - f'(u(x\pm,t))}{y-x} \leq \frac{1}{t}, \quad -\infty < x < y < \infty\,,\ 0 < t < \infty\,. \tag{11.2.1}$$

Proof. Fix x,y and t with $x < y$ and $t > 0$. Let $\xi(\cdot)$ and $\zeta(\cdot)$ denote the maximal or minimal backward characteristics emanating from (x,t) and (y,t), respectively.

By virtue of Theorem 11.1.3, $\xi(0) = x - tf'(u(x\pm,t))$, $\zeta(0) = y - tf'(u(y\pm,t))$. Furthermore, $\xi(0) \leq \zeta(0)$, on account of Corollary 11.1.2. This immediately implies (11.2.1). The proof is complete.

Notice that (11.2.1) establishes a one-sided Lipschitz condition for $f'(u(\cdot,t))$, with Lipschitz constant independent of the initial data. By the general theory of scalar conservation laws, presented in Chapter VI, admissible solutions of (11.1.1) with initial data in $L^\infty(-\infty,\infty)$ may be realized as a.e. limits of sequences of solutions with initial data of locally bounded variation on $(-\infty,\infty)$. Consequently, (11.2.1) should hold even for admissible solutions with initial data that are merely in $L^\infty(-\infty,\infty)$. Clearly, (11.2.1) implies that, for fixed $t > 0$, $f'(u(\cdot,t))$, and thereby also $u(\cdot,t)$, have bounded variation over any bounded interval of $(-\infty,\infty)$. We have thus shown that, because of genuine nonlinearity, solutions are generally smoother than their initial data:

11.2.2 Theorem. *Admissible solutions of* (11.1.1), *with initial data in* $L^\infty(-\infty,\infty)$, *are in* BV_{loc} *on* $(-\infty,\infty)\times(0,\infty)$ *and satisfy the one-sided Lipschitz condition* (11.2.1).

11.3 Regularity of Solutions

The properties of generalized characteristics established in the previous section lead to a precise description of the structure and regularity of admissible weak solutions.

11.3.1 Theorem. *Let* $\chi(\cdot)$ *be the unique forward generalized characteristic and* $\xi_-(\cdot),\xi_+(\cdot)$ *the extremal backward characteristics, associated with an admissible solution* u, *emanating from any point* $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. *Then* $(\bar{x},\bar{t})$ *is a point of continuity of the function* $u(x-,t)$ *relative to the set* $\{(x,t) : 0 \leq t \leq \bar{t}, x \leq \xi_-(t)$ *or* $\bar{t} < t < \infty, x \leq \chi(t)\}$ *and also a point of continuity of the function* $u(x+,t)$ *relative to the set* $\{(x,t) : 0 \leq t \leq \bar{t}, x \geq \xi_+(t)$ or $\bar{t} < t < \infty, x \geq \chi(t)\}$. *Furthermore,* $\chi(\cdot)$ *is differentiable from the right at* $\bar{t}$ *and*

$$\text{(11.3.1)} \qquad \frac{d^+}{dt}\chi(\bar{t}) = \begin{cases} f'(u(\bar{x}\pm,\bar{t})), & \text{if } u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t}) \\[2ex] \dfrac{f(u(\bar{x}+,\bar{t})) - f(u(\bar{x}-,\bar{t}))}{u(\bar{x}+,\bar{t}) - u(\bar{x}-,\bar{t})}, & \text{if } u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t}). \end{cases}$$

Proof. Take any sequence $\{(x_n,t_n)\}$ in the set $\{(x,t) : 0 \leq t < \bar{t}, x \leq \xi_-(t)$ or $\bar{t} < t < \infty, x \leq \chi(t)\}$, which converges to $(\bar{x},\bar{t})$ as $n \to \infty$. Let $\xi_n(\cdot)$ denote the minimal backward characteristic emanating from (x_n,t_n). Clearly, $\xi_n(t) \leq \xi_-(t)$ for $t \leq \bar{t}$. Thus, as $n \to \infty$, $\{\xi_n(\cdot)\}$ converges from below to $\xi_-(\cdot)$. Hence, $\{u(x_n-,t_n)\}$ converges to $u(\bar{x}-,\bar{t})$.

Similarly, for any sequence $\{(x_n,t_n)\}$ in the set $\{(x,t) : 0 \leq t < \bar{t}, x \geq \xi_+(t)$ or $\bar{t} < t < \infty, x \geq \chi(t)\}$, converging to $(\bar{x},\bar{t})$, the sequence $\{u(x_n+,t_n)\}$ converges to $u(\bar{x}+,\bar{t})$.

For $\varepsilon > 0$,

(11.3.2)
$$\frac{1}{\varepsilon}[\chi(\bar{t}+\varepsilon)-\chi(\bar{t})] = \frac{1}{\varepsilon}\int_{\bar{t}}^{\bar{t}+\varepsilon}\dot{\chi}(t)dt,$$

where $\dot{\chi}(t)$ is determined through (11.1.3), with $\xi \equiv \chi$. As shown above, $\dot{\chi}(t)$ is continuous from the right at $\bar{t}$ and so, letting $\varepsilon \downarrow 0$ in (11.3.2), we arrive at (11.3.1). This completes the proof.

The above theorem has the following corollary:

11.3.2 Theorem. *Let u be an admissible solution and assume $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$, for some $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. Then $(\bar{x},\bar{t})$ is a point of continuity of u. A unique generalized characteristic $\chi(\cdot)$, associated with u, defined on $[0,\infty)$, passes through $(\bar{x},\bar{t})$. Furthermore, $\chi(\cdot)$ is differentiable at $\bar{t}$ and $\dot{\chi}(\bar{t}) = f'(u(\bar{x}\pm,\bar{t}))$.*

Next we focus attention on points of discontinuity.

11.3.3 Theorem. *Let u be an admissible solution and assume $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$, for some $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. When the extremal backward characteristics $\xi_-(\cdot)$, $\xi_+(\cdot)$ are the only backward generalized characteristics emanating from $(\bar{x},\bar{t})$ that are shock-free on $(0,\bar{t})$, then $(\bar{x},\bar{t})$ is a point of jump discontinuity of u in the following sense: There is a generalized characteristic $\chi(\cdot)$, associated with u, defined on $[0,\infty)$ and passing through $(\bar{x},\bar{t})$, such that $(\bar{x},\bar{t})$ is a point of continuity of the function $u(x-,t)$ relative to $\{(x,t) : 0 < t < \infty, x \leq \chi(t)\}$ and also a point of continuity of the function $u(x+,t)$ relative to $\{(x,t) : 0 < t < \infty, x \geq \chi(t)\}$. Furthermore, $\chi(\cdot)$ is differentiable at $\bar{t}$ and*

(11.3.3)
$$\dot{\chi}(\bar{t}) = \frac{f(u(\bar{x}+,\bar{t}))-f(u(\bar{x}-,\bar{t}))}{u(\bar{x}+,\bar{t})-u(\bar{x}-,\bar{t})}.$$

Proof. Fix any point on the x-axis, in the interval $(\xi_-(0),\xi_+(0))$, and connect it to $(\bar{x},\bar{t})$ by a characteristic $\chi(\cdot)$. Extend $\chi(\cdot)$ to $[\bar{t},\infty)$ as the unique forward characteristic issuing from $(\bar{x},\bar{t})$.

Take any sequence $\{(x_n,t_n)\}$ in the set $\{(x,t) : 0 < t < \infty\ , x \leq \chi(t)\}$, that converges to $(\bar{x},\bar{t})$, as $n \to \infty$. Let $\xi_n(\cdot)$ denote the minimal backward characteristic emanating from (x_n,t_n). As $n \to \infty$, $\{\xi_n(\cdot)\}$, or a subsequence thereof, will converge to some backward characteristic emanating from $(\bar{x},\bar{t})$, which is a straight line and shock-free. Since $\xi_n(t) \leq \chi(t)$, this implies that $\{\xi_n(\cdot)\}$ must necessarily converge to $\xi_-(\cdot)$. Consequently, $\{u(x_n-,t_n)\}$ converges to $u(\bar{x}-,\bar{t})$, as $n \to \infty$.

Similarly, for any sequence $\{(x_n,t_n)\}$ in $\{(x,t) : 0 < t < \infty, x \geq \chi(t)\}$, converging to $(\bar{x},\bar{t})$, the sequence $\{u(x_n+,t_n)\}$ converges to $u(\bar{x}+,\bar{t})$.

To verify (11.3.3), we start out again from (11.3.2), where now ε may be positive or negative. As shown above, $\bar{t}$ is a point of continuity of $\dot{\chi}(t)$ and so, letting $\varepsilon \to 0$, we arrive at (11.3.3). This completes the proof.

11.3.4 Theorem. *The set of irregular points of any admissible solution u is countable. $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ is an irregular point if and only if $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$ and, in addition to the extremal backward characteristics $\xi_-(\cdot)$, $\xi_+(\cdot)$, there is at least another, distinct, backward characteristic $\xi(\cdot)$, associated with u, emanating from $(\bar{x},\bar{t})$, which is shock-free on $(0,\bar{t})$. Irregular points are generated by the collision of shocks and/or by the focusing of centered compression waves.*

Proof. Necessity follows from Theorems 11.3.2 and 11.3.3. To show sufficiency, consider the subset $\mathscr{X}$ of the interval $[\xi_-(0),\xi_+(0)]$ such that, for $x \in \mathscr{X}$, the straight line segment connecting the points $(x,0)$ and $(\bar{x},\bar{t})$ is a characteristic associated with u, which is shock-free on $(0,\bar{t})$.

When $\mathscr{X} \equiv [\xi_-(0),\xi_+(0)]$, $(\bar{x},\bar{t})$ is the focus of a centered compression wave and the assertion of the theorem is clearly valid. In general, however, $\mathscr{X}$ will be a closed proper subset of $[\xi_-(0),\xi_+(0)]$, containing at least the three points $\xi_-(0)$, $\xi(0)$ and $\xi_+(0)$. The complement of $\mathscr{X}$ relative to $[\xi_-(0),\xi_+(0)]$ will then be the (at most) countable union of disjoint open intervals. Let (α_-,α_+) be one of these intervals, contained, say in $(\xi_-(0),\xi(0))$. The straight line segments connecting the points $(\alpha_-,0)$ and $(\alpha_+,0)$ with $(\bar{x},\bar{t})$ will be shock-free characteristics $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ along which u is constant, say u_- and u_+. Necessarily, $u(\bar{x}-,\bar{t}) \geq u_- > u_+ > u(\bar{x}+,\bar{t})$. Consider a characteristic $\chi(\cdot)$ connecting a point of (α_-,α_+) with $(\bar{x},\bar{t})$. Then $\zeta_-(t) < \chi(t) < \zeta_+(t)$, $0 \leq t < \bar{t}$. Take any sequence $\{(x_n,t_n)\}$ in the set $\{(x,t) : 0 \leq t < \bar{t},\, \zeta_-(t) \leq x \leq \chi(t)\}$, converging to $(\bar{x},\bar{t})$, as $n \to \infty$. If $\xi_n(\cdot)$ denotes the minimal backward characteristic emanating from (x_n,t_n), the sequence $\{\xi_n(\cdot)\}$ will necessarily converge to $\zeta_-(\cdot)$. In particular, this implies $u(x_n-,t_n) \longrightarrow u_-$, as $n \to \infty$. Similarly one shows that if $\{(x_n,t_n)\}$ is any sequence in the set $\{(x,t) : 0 \leq t < \bar{t},\, \chi(t) \leq x \leq \zeta_+(t)\}$ converging to $(\bar{x},\bar{t})$, then $u(x_n+,t_n) \longrightarrow u_+$, as $n \to \infty$. Thus, near $\bar{t}$ $\chi(\cdot)$ is a shock, which is differentiable from the left at $\bar{t}$ with

$$\frac{d^-}{dt}\chi(\bar{t}) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{11.3.4}$$

Since $f'(u_-) > \frac{d^-}{dt}\chi(\bar{t}) > f'(u_+)$, we conclude that $(\bar{x},\bar{t})$ is an irregular point of u.

We have thus shown that $(\bar{x},\bar{t})$ is a point of collision of shocks, one for each open interval of the complement of $\mathscr{X}$, and centered compression waves, when the measure of $\mathscr{X}$ is positive.

For fixed positive ε, we consider irregular points $(\bar{x},\bar{t})$, as above, with the additional property $\xi_+(0) - \xi(0) > \varepsilon$, $\xi(0) - \xi_-(0) > \varepsilon$. It is easy to see that one may fit an at most finite set of such points in any bounded subset of the upper half-plane. This in turn implies that the set of irregular points of any admissible solution is (at most) countable. The proof is complete.

The effect of genuine nonlinearity, reflected in the properties of characteristics, is either to smooth out solutions by rarefaction or to form jump discontinuities through compression. Aspects of this polarizing influence, which impedes the existence of solutions with "intermediate" regularity, are manifested in the following Theorems 11.3.5, 11.3.6 and 11.3.10.

To begin with, every admissible *BV* solution is necessarily a special function of bounded variation, in the sense of Definition 1.7.9:

11.3.5 Theorem. *There is an (at most) countable set* $\mathscr{T} \subset [0,\infty)$ *such that, for any* $t \in [0,\infty)\backslash\mathscr{T}$, $u(\cdot,t)$ *belongs to* $SBV_{\mathrm{loc}}(-\infty,\infty)$. *Furthermore, u belongs to* $SBV_{\mathrm{loc}}\big((-\infty,\infty)\times[0,\infty)\big)$.

The proof of the above proposition is found in the literature cited in Section 11.14. A rough explanation of this phenomenon runs as follows. Because of Theorem 11.2.2, for fixed $t>0$, the solution u may decrease, but not increase, rapidly with respect to x. Furthermore, by virtue of the properties of characteristics, a continuous, but not abrupt, rapid decrease may only occur shortly before a compression wave breaks (especially on the brink of focussing of a centered compression wave). Thus, the development of a Cantor part for $u(\cdot,t)$ may be induced by the almost simultaneous breaking of infinitely many compression waves, which is a rare, nongeneric, event

Next, we shall see that continuity is automatically upgraded to Lipschitz continuity:

11.3.6 Theorem. *Assume the set* $\mathscr{C}$ *of points of continuity of an admissible solution u has nonempty interior* $\mathscr{C}^0$. *Then u is locally Lipschitz on* $\mathscr{C}^0$.

Proof. Fix any point $(\bar{x},\bar{t}) \in \mathscr{C}^0$ and assume that the circle $\mathscr{B}_r$ of radius r, centered at $(\bar{x},\bar{t})$, is contained in $\mathscr{C}^0$. Consider any point (x,t) at a distance $\rho < r$ from $(\bar{x},\bar{t})$. The (unique) characteristics, associated with u, passing through $(\bar{x},\bar{t})$ and (x,t) are straight lines with slopes $f'(u(\bar{x},\bar{t}))$ and $f'(u(x,t))$, respectively, which cannot intersect inside the circle $\mathscr{B}_r$. Elementary trigonometric estimations then imply that $|f'(u(x,t)) - f'(u(\bar{x},\bar{t}))|$ cannot exceed $c\rho/r$, where c is any upper bound of $1+f'(u)^2$ over $\mathscr{B}_r$. Hence, if $a>0$ is a lower bound of $f''(u)$ over $\mathscr{B}_r$, $|u(x,t)-u(\bar{x},\bar{t})| \le \frac{c}{ar}\rho$. This completes the proof.

The reader should be aware that admissible solutions have been constructed whose set of points of continuity has empty interior.

We now investigate the regularity of admissible solutions with smooth initial data. In what follows, it shall be assumed that f is C^{k+1} and u is the admissible solution with C^k initial data u_0, for some $k \in \{1,2,\cdots,\infty\}$.

For $(x,t) \in (-\infty,\infty)\times(0,\infty)$, we let $y_-(x,t)$ and $y_+(x,t)$ denote the interceptors on the x-axis of the minimal and maximal backward characteristics, associated with u, emanating from the point (x,t). In particular,

$$x = y_-(x,t) + tf'(u_0(y_-(x,t)) = y_+(x,t) + tf'(u_0(y_+(x,t)), \tag{11.3.5}$$

$$u(x-,t) = u_0(y_-(x,t)), \qquad u(x+,t) = u_0(y_+(x,t)). \tag{11.3.6}$$

For fixed $t > 0$, both $y_-(\cdot,t)$ and $y_+(\cdot,t)$ are monotone nondecreasing and the first one is continuous from the left while the second is continuous from the right. Consequently,

$$1 + t\frac{d}{dy}f'(u_0(y)) \geq 0, \quad y = y_\pm(x,t), \tag{11.3.7}$$

holds for all $(x,t) \in (-\infty,\infty) \times (0,\infty)$.

Any point $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ of continuity of u is necessarily also a point of continuity of $y_\pm(x,t)$ and $y_-(\bar{x},\bar{t}) = y_+(\bar{x},\bar{t})$. Therefore, by virtue of (11.3.5), (11.3.6), and the implicit function theorem we deduce

11.3.7 Theorem. *If* $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ *is a point of continuity of u and*

$$1 + \bar{t}\frac{d}{dy}f'(u_0(y)) > 0, \quad y = y_\pm(\bar{x},\bar{t}), \tag{11.3.8}$$

then u is C^k *on a neighborhood of* $(\bar{x},\bar{t})$.

With reference to Theorem 11.3.3, if $(\bar{x},\bar{t})$ is a point of jump discontinuity of u, then $(\bar{x},\bar{t})$ is a point of continuity of $y_-(x,t)$ and $y_+(x,t)$ relative to the sets of points $\{(x,t) : 0 < t < \infty,\ x \leq \chi(t)\}$ and $\{(x,t) : 0 < t < \infty,\ x \geq \chi(t)\}$, respectively. Consequently, the implicit function theorem together with (11.3.5) and (11.3.6) yields

11.3.8 Theorem. *If* (11.3.8) *holds at a point* $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ *of jump discontinuity of u, then, in a neighborhood of* $(\bar{x},\bar{t})$, *the shock* $\chi(\cdot)$ *passing through* $(\bar{x},\bar{t})$ *is* C^{k+1} *and u is* C^k *on either side of the graph of* $\chi(\cdot)$.

Next we consider shock generation points, introduced by Definition 11.1.6.

11.3.9 Theorem. *If* $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ *is a shock generation point, then*

$$1 + \bar{t}\frac{d}{dy}f'(u_0(y)) = 0, \qquad y_-(\bar{x},\bar{t}) \leq y \leq y_+(\bar{x},\bar{t}). \tag{11.3.9}$$

Furthermore, when $k \geq 2$,

$$\frac{d^2}{dy^2}f'(u_0(y)) = 0, \qquad y_-(\bar{x},\bar{t}) \leq y \leq y_+(\bar{x},\bar{t}). \tag{11.3.10}$$

Proof. Recall that there are two types of shock generation points: points of continuity, in which case $y_-(\bar{x},\bar{t}) = y_+(\bar{x},\bar{t})$, and foci of centered compression waves, with $y_-(\bar{x},\bar{t}) < y_+(\bar{x},\bar{t})$. When $(\bar{x},\bar{t})$ is a point of continuity, (11.3.9) is a consequence of (11.3.7) and Theorem 11.3.7. When $(\bar{x},\bar{t})$ is the focus of a compression wave, $\bar{x} = y + \bar{t}f'(u_0(y))$ for any $y \in [y_-(\bar{x},\bar{t}), y_+(\bar{x},\bar{t})]$ and this implies (11.3.9).

When $y_-(\bar{x},\bar{t}) < y_+(\bar{x},\bar{t})$, differentiation of (11.3.9) with respect to y yields (11.3.10). To establish (11.3.10) for the case $(\bar{x},\bar{t})$ is a point of continuity, we take any sequence $\{x_n\}$ that converges from below (or above) to $\bar{x}$. Then $\{y_-(x_n,\bar{t})\}$ will approach from below (or above) $y_\pm(\bar{x},\bar{t})$. Because of (11.3.7), $1+\bar{t}\frac{d}{dy}f'(u_0(y)) \geq 0$ for $y = y_-(x_n,\bar{t})$; and this together with (11.3.9) imply that $y_\pm(\bar{x},\bar{t})$ is a critical point of $\frac{d}{dy}f'(u_0(y))$. The proof is complete.

For $k \geq 3$, the set of functions u_0 in C^k with the property that $\frac{d}{dy}f'(u_0(y))$ has infinitely many critical points in a bounded interval is of the first category. Therefore, generically, initial data $u_0 \in C^k$, with $k \geq 3$, induce solutions with a locally finite set of shock generation points and thereby with a locally finite set of shocks. In other words,

11.3.10 Theorem. *Generically, admissible solutions of* (11.1.1) *with initial data in* C^k, $k \geq 3$, *are piecewise* C^k *smooth functions and do not contain centered compression waves. In particular, solutions with analytic initial data are always piecewise analytic.*

11.4 Divides, Invariants and the Lax Formula

The theory of generalized characteristics will be used here to establish interesting and fundamental properties of admissible solutions of (11.1.1). The starting point will be a simple but, as we shall see, very useful identity.

Let us consider two admissible solutions u and u^*, with corresponding initial values u_0 and u_0^*, and trace one of the extremal backward characteristics $\xi(\cdot)$, associated with u, and one of the extremal backward characteristics $\xi^*(\cdot)$, associated with u^*, that emanate from any fixed point $(x,t) \in (-\infty,\infty) \times (0,\infty)$. Thus, $\xi(\cdot)$ and $\xi^*(\cdot)$ will be straight lines, and along $\xi(\cdot)$ u will be constant, equal to $u(x-,t)$ or $u(x+,t)$, while along $\xi^*(\cdot)$ u^* will be constant, equal to $u^*(x-,t)$ or $u^*(x+,t)$. In particular, $\dot{\xi}(\tau) = f'(u(x\pm,t))$ and $\dot{\xi}^*(\tau) = f'(u^*(x\pm,t))$, $0 < \tau < t$.

We write (11.1.1), first for u then for u^*, we combine the resulting two equations, integrate over the triangle with vertices $(x,t), (\xi(0),0), (\xi^*(0),0)$, and apply Green's theorem thus arriving at the identity

(11.4.1)

$$\int_0^t \{f(u(x\pm,t)) - f(u^*(\xi(\tau)-,\tau)) - f'(u(x\pm,t))[u(x\pm,t) - u^*(\xi(\tau)-,\tau)]\}d\tau$$

$$+\int_0^t \{f(u^*(x\pm,t)) - f(u(\xi^*(\tau)-,\tau)) - f'(u^*(x\pm,t))[u^*(x\pm,t) - u(\xi^*(\tau)-,\tau)]\}d\tau$$

$$= \int_{\xi^*(0)}^{\xi(0)} [u_0(y) - u_0^*(y)]dy.$$

As a consequence of the convexity of f, both integrals on the left-hand side of (11.4.1) are nonpositive. Thus (11.4.1) provides a comparison of the two solutions, which will find numerous applications in the sequel.

As a first application of (11.4.1), we use it to locate divides associated with an admissible solution u. The notion of divide was introduced by Definition 10.3.3. In the context of the genuinely nonlinear scalar conservation law, following the discussion in Section 10.3, divides are shock-free and hence, by virtue of Theorem 11.1.1, straight lines along which u is constant.

11.4.1 Theorem. *A divide, associated with the admissible solution u, with initial data u_0, along which u is constant $\bar{u}$, issues from the point $(\bar{x},0)$ of the x-axis if and only if*

$$\int_{\bar{x}}^{z}[u_0(y)-\bar{u}]dy \geq 0, \qquad -\infty < z < \infty\,. \tag{11.4.2}$$

Proof. Assume that (11.4.2) holds. We apply (11.4.1) with $u^* = \bar{u}$, $t \in (0,\infty)$, and $x = \bar{x}+tf'(\bar{u})$. In particular, $\xi^*(\tau) = \bar{x}+\tau f'(\bar{u})$ and $\xi^*(0) = \bar{x}$. Hence the right-hand side of (11.4.1) is nonnegative, on account of (11.4.2). But then both integrals on the left-hand side must vanish, so that $u(x\pm,t) = \bar{u}$. We have thus established that the straight line $x = \bar{x}+tf'(\bar{u})$ is a shock-free characteristic on $[0,\infty)$, which is a divide associated with u.

Conversely, assume the straight line $x = \bar{x}+tf'(\bar{u})$ is a divide associated with u. Take any $z \in (-\infty,\infty)$ and fix $\tilde{u}$ such that $\tilde{u} < \bar{u}$ if $z > \bar{x}$ and $\tilde{u} > \bar{u}$ if $z < \bar{x}$. The straight lines $z+tf'(\tilde{u})$ and $\bar{x}+tf'(\bar{u})$ will then intersect at a point (x,t) with $t > 0$. We apply (11.4.1) with $u^* \equiv \tilde{u}$, in which case $\xi(0) = \bar{x}$, $\xi^*(0) = z$. The left-hand side is nonpositive and so

$$\int_{z}^{\bar{x}}[u_0(y)-\tilde{u}]dy \leq 0. \tag{11.4.3}$$

Letting $\tilde{u} \to \bar{u}$ we arrive at (11.4.2). This completes the proof.

The above proposition has implications for the existence of important time invariants of solutions:

11.4.2 Theorem. *Assume u_0 is integrable over $(-\infty,\infty)$ and the maxima*

$$\max_x \int_x^{-\infty} u_0(y)dy = q_-\,, \qquad \max_x \int_x^{\infty} u_0(y)dy = q_+ \tag{11.4.4}$$

exist. If u is the admissible solution with initial data u_0, then, for any $t > 0$,

$$\max_x \int_x^{-\infty} u(y,t)dy = q_-\,, \qquad \max_x \int_x^{\infty} u(y,t)dy = q_+\,. \tag{11.4.5}$$

Proof. Notice that q_- exists if and only if q_+ exists and in fact, by virtue of Theorem 11.4.1, both maxima are attained on the set of $\bar{x}$ with the property that the straight line $x = \bar{x}+tf'(0)$ is a divide associated with u, along which u is constant, equal to zero. But then, again by Theorem 11.4.1, both maxima in (11.4.5) will be attained at $\hat{x} = \bar{x}+tf'(0)$.

We now normalize f by $f(0)=0$ and take the integral of (11.1.1), first over the domain $\{(y,\tau): 0<\tau<t, -\infty<y<\bar{x}+\tau f'(0)\}$ and then also over the domain $\{(y,\tau): 0<\tau<t, \bar{x}+\tau f'(0)<y<\infty\}$. Applying Green's theorem, and since u vanishes along the straight line $x=\bar{x}+\tau f'(0)$,

$$\int_{\hat{x}}^{-\infty} u(y,t)dy = \int_{\bar{x}}^{-\infty} u_0(y)dy, \qquad \int_{\hat{x}}^{\infty} u(y,t)dy = \int_{\bar{x}}^{\infty} u_0(y)dy, \tag{11.4.6}$$

which verifies (11.4.5). The proof is complete.

One of the most striking features of genuinely nonlinear scalar conservation laws is that admissible solutions may be determined explicitly from the initial data by the following procedure. We start out with the Legendre transform

$$g(v) = \max_u \,[uv - f(u)], \tag{11.4.7}$$

noting that the maximum is attained at $u=[f']^{-1}(v)$. With given initial data $u_0(\cdot)$ we associate the *Lax function*

$$G(y,x,t) = \int_0^y u_0(z)dz + tg\left(\frac{x-y}{t}\right), \tag{11.4.8}$$

defined for $(x,t)\in(-\infty,\infty)\times(0,\infty)$ and $y\in(-\infty,\infty)$.

11.4.3 Theorem. *For fixed $(x,t)\in(-\infty,\infty)\times(0,\infty)$, the Lax function $G(y,x,t)$ is minimized at a point $\bar{y}\in(-\infty,\infty)$ if and only if the straight line segment that connects the points (x,t) and $(\bar{y},0)$ is a generalized characteristic associated with the admissible solution u with initial data u_0, which is shock-free on $(0,t)$.*

Proof. We fix y and $\bar{y}$ in $(-\infty,\infty)$, integrate (11.1.1) over the triangle with vertices $(x,t),(y,0),(\bar{y},0)$, and apply Green's theorem to get

$$\begin{aligned} &\int_0^{\bar{y}} u_0(z)dz + \int_0^t \left[\frac{x-\bar{y}}{t}u(\bar{y}+\tau\frac{x-\bar{y}}{t}\pm,\tau) - f(u(\bar{y}+\tau\frac{x-\bar{y}}{t}\pm,\tau))\right]d\tau \\ &= \int_0^{y} u_0(z)dz + \int_0^t \left[\frac{x-y}{t}u(y+\tau\frac{x-y}{t}\pm,\tau) - f(u(y+\tau\frac{x-y}{t}\pm,\tau))\right]d\tau. \end{aligned} \tag{11.4.9}$$

By virtue of (11.4.7) and (11.4.8), the left-hand side of (11.4.9) is less than or equal to $G(\bar{y},x,t)$, with equality holding if and only if $f'(u(\bar{y}+\tau\frac{x-\bar{y}}{t}\pm,\tau))=\frac{x-\bar{y}}{t}$, almost everywhere on $(0,t)$, i.e., if and only if the straight line segment that connects the points (x,t) and $(\bar{y},0)$ is a shock-free characteristic. Similarly, the right-hand side of (11.4.9) is less than or equal to $G(y,x,t)$, with equality holding if and only if the straight line segment that connects the points (x,t) and $(y,0)$ is a shock-free characteristic. Assuming then that the straight line segment connecting (x,t) with $(\bar{y},0)$ is a shock-free characteristic, we deduce from (11.4.9) that $G(\bar{y},x,t)\le G(y,x,t)$ for any $y\in(-\infty,\infty)$.

Conversely, assume $G(\bar{y},x,t) \leq G(y,x,t)$, for all $y \in (-\infty,\infty)$. In particular, pick y so that $(y,0)$ is the intercept by the x-axis of the minimal backward characteristic emanating from (x,t). As shown above, y is a minimizer of $G(\cdot,x,t)$ and so $G(y,x,t) = G(\bar{y},x,t)$. Moreover, the right-hand side of (11.4.9) equals $G(y,x,t)$ and so the left-hand side equals $G(y,x,t)$. As explained above, this implies that the straight line segment connecting (x,t) with $(\bar{y},0)$ is a shock-free characteristic. The proof is complete.

The above proposition may be used to determine the admissible solution u from the initial data u_0: For fixed $(x,t) \in (-\infty,\infty) \times (0,\infty)$, we let y_- denote the smallest and y_+ denote the largest minimizer of $G(\cdot,x,t)$ over $(-\infty,\infty)$. We then have

$$u(x\pm,t) = [f']^{-1}\left(\frac{x-y_\pm}{t}\right). \tag{11.4.10}$$

On account of Theorems 11.3.2, 11.3.3 and 11.3.4, we conclude that (x,t) is a point of continuity of u if and only if $y_- = y_+$; a point of jump discontinuity of u if and only if $y_- < y_+$ and y_-, y_+ are the only minimizers of $G(\cdot,x,t)$; or an irregular of u if and only if $y_- < y_+$ and there exist additional minimizers of $G(\cdot,x,t)$ in the interval (y_-,y_+). One may develop the entire theory of the Cauchy problem for genuinely nonlinear scalar conservation laws on the basis of the above construction of admissible solutions, in lieu of the approach via generalized characteristics. It should be noted, however, that the method of generalized characteristics affords greater flexibility, as it applies to solutions defined on arbitrary open subsets of $\mathbb{R}^2$, not necessarily on the entire upper half-plane.

The change of variables $u = \partial_x v$, reduces the conservation law (11.1.1) to the *Hamilton-Jacobi equation*

$$\partial_t v(x,t) + f(\partial_x v(x,t)) = 0. \tag{11.4.11}$$

In that context, u is an admissible weak solution of (11.1.1) if and only if v is a *viscosity solution* of (11.4.11); see references in Section 11.14. In fact, Theorems 11.4.2 and 11.4.3 reflect properties of solutions of Hamilton-Jacobi equations rather than of hyperbolic conservation laws, in that they readily extend to the multi-space dimensional versions of the former though not of the latter.

11.5 Decay of Solutions Induced by Entropy Dissipation

Genuine nonlinearity gives rise to a multitude of dissipative mechanisms which, acting individually or collectively, affect the large time behavior of solutions. In this section we shall get acquainted with examples in which the principal agent of damping is entropy dissipation.

11.5.1 Theorem. *Let u be the admissible solution with initial data u_0 such that*

(11.5.1)
$$\int_x^{x+\ell} u_0(y)dy = O(\ell^r), \qquad \text{as } \ell \to \infty,$$

for some $r \in [0,1)$, *uniformly in* x *on* $(-\infty,\infty)$. *Then*

(11.5.2)
$$u(x\pm,t) = O\left(t^{-\frac{1-r}{2-r}}\right), \qquad \text{as } t \to \infty,$$

uniformly in x *on* $(-\infty,\infty)$.

Proof. We fix $(x,t) \in (-\infty,\infty) \times (0,\infty)$ and write (11.4.1) for $u^* \equiv 0$. Notice that $\xi(0) - \xi^*(0) = t[f'(u(x\pm,t)) - f'(0)]$. Also recall that both integrals on the left-hand side are nonpositive. Consequently, using (11.5.1), we deduce

(11.5.3)
$$\Phi(u(x\pm,t)) = O(t^{r-1}), \qquad \text{as } t \to \infty,$$

uniformly in x on $(-\infty,\infty)$, where we have set

(11.5.4)
$$\Phi(u) = \frac{f(0) - f(u) + uf'(u)}{|f'(u) - f'(0)|^r} = \frac{\int_0^u v f''(v)dv}{|\int_0^u f''(v)dv|^r}\,.$$

A simple estimation yields $\Phi(u) \geq K|u|^{2-r}$, with $K > 0$, and so (11.5.3) implies (11.5.2). This completes the proof.

In particular, when $u_0 \in L^p$ (11.5.1) holds with $r = 1 - \frac{1}{p}$, by virtue of Hölder's inequality. Therefore, Theorem 11.5.1 has the following corollary:

11.5.2 Theorem. *Let* u *be the admissible solution with initial data* u_0 *in* $L^p(-\infty,\infty)$, $1 \leq p < \infty$. *Then*

(11.5.5)
$$u(x\pm,t) = O\left(t^{-\frac{p}{p+1}}\right), \qquad \text{as } t \to \infty,$$

uniformly in x *on* $(-\infty,\infty)$.

In the above examples, the comparison function was the solution $u^* \equiv 0$. Next we consider the case where the comparison function is the solution of a Riemann problem comprising two constant states u_- and u_+, $u_- > u_+$, joined by a shock, namely,

(11.5.6)
$$u^*(x,t) = \begin{cases} u_-\,, & x < st \\ u_+\,, & x > st\,, \end{cases}$$

where

(11.5.7)
$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}\,.$$

11.5.3 Theorem. *Let u denote the admissible solution with initial data u_0 such that the improper integrals $\int_{-\infty}^{0}[u_0(y)-u_-]dy$ and $\int_0^\infty [u_0(y)-u_+]dy$ exist, for u_- and u_+ with $u_- > u_+$. Normalize the origin $x=0$ so that*

$$\int_{-\infty}^{0}[u_0(y)-u_-]dy+\int_0^\infty [u_0(y)-u_+]dy=0. \tag{11.5.8}$$

Consider any forward characteristic $\chi(\cdot)$ issuing from $(0,0)$. Then, as $t\to\infty$,

$$\chi(t)=st+o(1), \tag{11.5.9}$$

with s given by (11.5.7), and

$$u(x\pm,t)=\begin{cases} u_-+o(t^{-1/2}), & uniformly\ for\ x<\chi(t)\\ u_++o(t^{-1/2}), & uniformly\ for\ x>\chi(t). \end{cases} \tag{11.5.10}$$

Proof. Fix any $(x,t)\in(-\infty,\infty)\times(0,\infty)$ and write (11.4.1) for the solution u, with initial data u_0, and the comparison solution u^* given by (11.5.6). By virtue of $f'(u_-)>s>f'(u_+)$, as $t\to\infty$, $\xi^*(0)\to-\infty$, uniformly in x on $(-\infty,st)$, and $\xi^*(0)\to\infty$, uniformly in x on (st,∞). Similarly, as $t\to\infty$, $\xi(0)\to-\infty$, uniformly in x on $(-\infty,\chi(t))$, and $\xi(0)\to\infty$ uniformly in x on $(\chi(t),\infty)$. Indeed, in the opposite case one would be able to find a sequence $\{(x_n,t_n)\}$, with $t_n\to\infty$ as $n\to\infty$, such that the intercepts $\xi_n(0)$ of the minimal backward characteristics $\xi_n(\cdot)$ emanating from (x_n,t_n) are confined in a bounded set. But then some subsequence of $\{\xi_n(\cdot)\}$ would converge to a divide issuing from some point $(\bar{x},0)$. However, this is impossible, because, since $u_->u_+$, (11.5.8) is incompatible with (11.4.2), for any $\bar{x}\in(-\infty,\infty)$ and every $\bar{u}\in(-\infty,\infty)$.

In view of the above, (11.5.8) implies that the right-hand side of (11.4.1) is $o(1)$, as $t\to\infty$, uniformly in x on $(-\infty,\infty)$. The same will then be true for each integral on the left-hand side of (11.4.1), because they are of the same sign (nonpositive).

Consider first points $(x,t)\in(-\infty,\infty)\times(0,\infty)$ with $x<\min\{\chi(t),st\}$. Then $\xi(\tau)<s\tau$, $0<\tau<t$, and so the first integral on the left-hand side of (11.4.1) yields

$$t\{f(u(x\pm,t))-f(u_-)-f'(u(x\pm,t))[u(x\pm,t)-u_-]\}=o(1). \tag{11.5.11}$$

Since f is uniformly convex, (11.5.11) implies $u(x\pm,t)-u_-=o(t^{-1/2})$.

A similar argument demonstrates that for points $(x,t)\in(-\infty,\infty)\times(0,\infty)$ with $x>\max\{\chi(t),st\}$, we have $u(x\pm,t)-u_+=o(t^{-1/2})$.

Next, consider points $(x,t)\in(-\infty,\infty)\times(0,\infty)$ with $st\le x<\chi(t)$. Then $\xi(\cdot)$ will have to intersect the straight line $x=s\tau$, say at $\tau=r$, $r\in[0,t]$, in which case the first integral on the left-hand side of (11.4.1) gives

$$(t-r)\{f(u(x\pm,t))-f(u_+)-f'(u(x\pm,t))[u(x\pm,t)-u_+]\}=o(1), \tag{11.5.12}$$

$$r\{f(u(x\pm,t))-f(u_-)-f'(u(x\pm,t))[u(x\pm,t)-u_-]\}=o(1). \tag{11.5.13}$$

For $x < \chi(t)$, it was shown above that $\xi(0) \to -\infty$, as $t \to \infty$, and this in turn implies $r \to \infty$. Thus, by (11.5.13) and the convexity of f, $u(x\pm,t) = u_- + o(1)$. Then (11.5.12) implies that $t - r = o(1)$ so that $\chi(t) - st = o(1)$ and (11.5.13) yields (11.5.11). From (11.5.11) and the convexity of f we deduce, as before, $u(x\pm,t) - u_- = o(t^{-1/2})$.

A similar argument establishes that for points $(x,t) \in (-\infty,\infty) \times (0,\infty)$ with $\chi(t) < x \leq st$ we have $u(x\pm,t) - u_+ = o(t^{-1/2})$ and also $\chi(t) - st = o(1)$. This completes the proof.

11.6 Spreading of Characteristics and Development of N-Waves

Another feature of genuine nonlinearity, affecting the large-time behavior of solutions, is spreading of characteristics. In order to see the effects of this mechanism, we shall study the asymptotic behavior of solutions with initial data of compact support. We already know, on account of Theorem 11.5.2, that the amplitude decays to zero as $O(t^{-1/2})$. The closer examination here will reveal that asymptotically the solution attains the profile of an *N-wave*, namely, a centered rarefaction wave flanked on both sides by shocks whose amplitudes decay like $O(t^{-1/2})$.

11.6.1 Theorem. *Let u be the admissible solution with initial data u_0, such that $u_0(x) = 0$ for $|x| > \ell$. Consider the minimal forward characteristic $\chi_-(\cdot)$ issuing from $(-\ell,0)$ and the maximal forward characteristic $\chi_+(\cdot)$ issuing from $(\ell,0)$. Then*

$$u(x\pm,t) = 0, \qquad \text{for } t > 0 \text{ and } x < \chi_-(t) \text{ or } x > \chi_+(t). \tag{11.6.1}$$

As $t \to \infty$,

$$f'(u(x\pm,t)) = \frac{x}{t} + O(\frac{1}{t}), \qquad \text{for } \chi_-(t) < x < \chi_+(t), \tag{11.6.2}$$

$$u(x\pm,t) = \frac{1}{f''(0)}[\frac{x}{t} - f'(0)] + O(\frac{1}{t}), \;\; \text{for } \chi_-(t) < x < \chi_+(t), \tag{11.6.3}$$

$$\begin{cases} \chi_-(t) = tf'(0) - [2q_-tf''(0)]^{1/2} + O(1) \\ \chi_+(t) = tf'(0) + [2q_+tf''(0)]^{1/2} + O(1), \end{cases} \tag{11.6.4}$$

with q_- and q_+ given by (11.4.4). *Moreover, the decreasing variation of $u(\cdot,t)$ over the interval $[\chi_-(t),\chi_+(t)]$ is $O(t^{-1})$.*

Proof. Since $\chi_-(\cdot)$ is minimal and $\chi_+(\cdot)$ is maximal, the extremal backward characteristics emanating from any point (x,t) with $t > 0$ and $x < \chi_-(t)$ or $x > \chi_+(t)$ will be intercepted by the x-axis outside the support of u_0. This establishes (11.6.1).

On the other hand, the minimal or maximal backward characteristic $\xi(\cdot)$ emanating from a point (x,t) with $t>0$ and $\chi_-(t)<x<\chi_+(t)$ will be intercepted by the x-axis inside the interval $[-\ell,\ell]$, that is, $\xi(0)\in[-\ell,\ell]$. Consequently, as $t\to\infty$, $x-tf'(u(x\pm,t))=\xi(0)=O(1)$, which yields (11.6.2).[1]

On account of Theorem 11.5.2, u is $O(t^{-1/2})$, as $t\to\infty$, and thus, assuming f is C^3, $f'(u)=f'(0)+f''(0)u+O(t^{-1})$. Therefore, (11.6.3) follows from (11.6.2).

To derive the asymptotics of $\chi_\pm(t)$, as $t\to\infty$, we first observe that on account of $0\geq\dot{\chi}_-(t)-f'(0)\geq O(t^{-1/2})$, $0\leq\dot{\chi}_+(t)-f'(0)\leq O(t^{-1/2})$ and this in turn yields $0\geq\chi_-(t)-tf'(0)\geq O(t^{1/2})$, $0\leq\chi_+(t)-tf'(0)\leq O(t^{1/2})$. Next we appeal to Theorem 11.4.2: A divide $x=\bar{x}+tf'(0)$ originates from a point $(\bar{x},0)$, with $\bar{x}\in[-\ell,\ell]$, along which u is zero, and for any $t>0$,

$$\int_{\bar{x}+tf'(0)}^{\chi_-(t)}u(y,t)dy=q_-\,,\qquad\int_{\bar{x}+tf'(0)}^{\chi_+(t)}u(y,t)dy=q_+\,. \tag{11.6.5}$$

In (11.6.5) we insert u from its asymptotic form (11.6.3), and after performing the simple integration we deduce

$$\frac{1}{2q_\pm tf''(0)}[\chi_\pm(t)-tf'(0)]^2=1+O(t^{-1/2}) \tag{11.6.6}$$

whence (11.6.4) follows. The proof is complete.

11.7 Confinement of Characteristics and Formation of Saw-toothed Profiles

The confinement of the intercepts of extremal backward characteristics in a bounded interval of the x-axis induces bounds on the decreasing variation of characteristic speeds and thereby, by virtue of genuine nonlinearity, on the decreasing variation of the solution itself.

11.7.1 Theorem. *Let $\chi_-(\cdot)$ and $\chi_+(\cdot)$ be generalized characteristics on $[0,\infty)$, associated with an admissible solution u, and $\chi_-(t)<\chi_+(t)$ for $t\in[0,\infty)$. Then, for any $t>0$, the decreasing variation of the function $f'(u(\cdot,t))$ over the interval $(\chi_-(t),\chi_+(t))$ cannot exceed $[\chi_+(0)-\chi_-(0)]t^{-1}$. Thus the decreasing variation of $u(\cdot,t)$ over the interval $(\chi_-(t),\chi_+(t))$ is $O(t^{-1})$ as $t\to\infty$.*

Proof. Fix $t>0$ and consider any mesh $\chi_-(t)<x_1<x_2<\cdots<x_{2m}<\chi_+(t)$ such that (x_i,t) is a point of continuity of u and also $u(x_{2k-1},t)>u(x_{2k},t)$,

[1] As $t\to\infty$, the $\xi(0)$ accumulate at the set of points from which divides originate. In the generic case where (11.4.2) holds, with $\bar{u}=0$, at a single point $\bar{x}$, which we normalize so that $\bar{x}=0$, the $\xi(0)$ accumulate at the origin and hence in (11.6.2) $O(t^{-1})$ is upgraded to $o(t^{-1})$. When, in addition, u_0 is C^1 and $u_0'(0)>0$, then in (11.6.2) $O(t^{-1})$ is improved to $O(t^{-2})$ and, for t large, the profile $u(\cdot,t)$ is C^1 on the interval $(\chi_-(t),\chi_+(t))$.

$k = 1, \cdots, m$. Let $\xi_i(\cdot)$ denote the (unique) backward characteristic emanating from (x_i, t). Then $\chi_-(0) \le \xi_1(0) \le \cdots \le \xi_{2m}(0) \le \chi_+(0)$. Furthermore, we have that $\xi_i(0) = x_i - tf'(u(x_i,t))$ and so

$$\sum_{k=1}^{m} t[f'(u(x_{2k-1},t)) - f'(u(x_{2k},t))] \le \chi_+(0) - \chi_-(0) \tag{11.7.1}$$

whence the assertion of the theorem follows. This completes the proof.

In particular, referring to the setting of Theorem 11.6.1, we deduce that the decreasing variation of the N-wave profile $u(\cdot,t)$ over the interval $(\chi_-(t), \chi_+(t))$ is $O(t^{-1})$, as $t \to \infty$.

Another corollary of Theorem 11.7.1 is that when the initial data u_0, and thereby the solution u, are periodic in x, then the decreasing variation, and hence also the total variation, of $u(\cdot,t)$ over any period interval is $O(t^{-1})$ as $t \to \infty$. We may achieve finer resolution than $O(t^{-1})$ by paying closer attention to the initial data:

11.7.2 Theorem. *Let u be an admissible solution with initial data u_0. Assume $\chi_-(t) = x_- + tf'(\bar{u})$ and $\chi_+(t) = x_+ + tf'(\bar{u})$, $x_- < x_+$, are adjacent divides associated with u, that is* (11.4.2) *holds for $\bar{x} = x_-$ and $\bar{x} = x_+$ but for no $\bar{x}$ in the interval (x_-, x_+). Then*

$$\int_{\chi_-(t)}^{\chi_+(t)} u(x,t)dx = \int_{x_-}^{x_+} u_0(y)dy = (x_+ - x_-)\bar{u}, \qquad t \in [0,\infty). \tag{11.7.2}$$

Consider any forward characteristic $\psi(\cdot)$ issuing from the point $(\frac{x_- + x_+}{2}, 0)$. Then, as $t \to \infty$,

$$\psi(t) = \tfrac{1}{2}[\chi_-(t) + \chi_+(t)] + o(1), \tag{11.7.3}$$

$$u(x\pm,t) = \begin{cases} \bar{u} + \dfrac{1}{f''(\bar{u})} \dfrac{x - \chi_-(t)}{t} + o\left(\dfrac{1}{t}\right), & \textit{for } \chi_-(t) < x < \psi(t) \\[2ex] \bar{u} + \dfrac{1}{f''(\bar{u})} \dfrac{x - \chi_+(t)}{t} + o\left(\dfrac{1}{t}\right), & \textit{for } \psi(t) < x < \chi_+(t). \end{cases} \tag{11.7.4}$$

Moreover, the decreasing variation of $u(\cdot,t)$ over the intervals $(\chi_-(t), \psi(t))$ and $(\psi(t), \chi_+(t))$ is $o(t^{-1})$ as $t \to \infty$.

Proof. To verify the first equality in (11.7.2), it suffices to integrate (11.1.1) over the parallelogram $\{(x,\tau) : 0 < \tau < t, \chi_-(\tau) < x < \chi_+(\tau)\}$ and then apply Green's theorem. The second equality in (11.7.2) follows because (11.4.2) holds for both $\bar{x} = x_-$ and $\bar{x} = x_+$.

For $t > 0$, we let $\xi_-^t(\cdot)$ and $\xi_+^t(\cdot)$ denote the extremal backward characteristics emanating from the point $(\psi(t), t)$. As $t \uparrow \infty$, $\xi_-^t(0) \downarrow x_-$ and $\xi_+^t(0) \uparrow x_+$, because otherwise there would exist divides originating at points $(\bar{x},0)$ with $\bar{x} \in (x_-,x_+)$, contrary to our assumptions. It then follows from Theorem 11.7.1 that the decreasing variation of $f'(u(\cdot,t))$, and thereby also the decreasing variation of $u(\cdot,t)$ itself, over the intervals $(\chi_-(t),\psi(t))$ and $(\psi(t),\chi_+(t))$ is $o(t^{-1})$ as $t \to \infty$.

The extremal backward characteristics emanating from any point (x,t) with $\chi_-(t) < x < \psi(t)$ (or $\psi(t) < x < \chi_+(t)$) will be intercepted by the x-axis inside the interval $[x_-,\xi_-^t(0)]$ (or $[\xi_+^t(0),x_+]$) and thus

$$x - tf'(u(x\pm,t)) = \begin{cases} x_- + o(t^{-1}), & \text{for } \chi_-(t) < x < \psi(t) \\ x_+ + o(t^{-1}), & \text{for } \psi(t) < x < \chi_+(t). \end{cases} \tag{11.7.5}$$

Since $u(\chi_-(t),t) = u(\chi_+(t),t) = \bar{u}$, Theorem 11.7.1 implies $u - \bar{u} = O(t^{-1})$ and so, as $t \to \infty$, $f'(u) = f'(\bar{u}) + f''(\bar{u})(u-\bar{u}) + O(t^{-2})$. This together with (11.7.5) yield (11.7.4).

Finally, introducing u from (11.7.4) into (11.7.2) we arrive at (11.7.3). The proof is complete.

We shall employ the above proposition to describe the asymptotics of periodic solutions:

11.7.3 Theorem. *When the initial data u_0 are periodic, with mean $\bar{u}$, then, as the time $t \to \infty$, the admissible solution u tends, at the rate $o(t^{-1})$, to a periodic serrated profile consisting of wavelets of the form* (11.7.4). *The number of wavelets (or teeth) per period equals the number of divides per period or, equivalently, the number of points on any interval of the x-axis of period length at which the primitive of the function $u_0 - \bar{u}$ attains its minimum. In particular, in the generic case where the minimum of the primitive of $u_0 - \bar{u}$ is attained at a single point on each period interval, u tends to a sawtooth-shaped profile with a single tooth per period.*

Proof. It is an immediate corollary of Theorems 11.4.1 and 11.7.2. If u_0 is periodic, (11.4.2) may hold only when $\bar{u}$ is the mean of u_0 and is attained at points $\bar{x}$ where the primitive of $u_0 - \bar{u}$ is minimized. The set of such points is obviously invariant under period translations and contains at least one (generically precisely one) point in each interval of period length.

11.8 Comparison Theorems and L^1 Stability

The assertions of Theorem 6.2.3 will be reestablished here, in sharper form, for the special case of genuinely nonlinear scalar conservation laws (11.1.1), in one space dimension. The key factor will be the properties of the function

$$Q(u,v,w) = \begin{cases} f(v) - f(u) - \dfrac{f(u)-f(w)}{u-w}[v-u], & \text{if } u \neq w \\ f(v) - f(u) - f'(u)[v-u], & \text{if } u = w, \end{cases} \tag{11.8.1}$$

defined for u, v and w in $\mathbb{R}$. Clearly, $Q(u,v,w) = Q(w,v,u)$. Since f is uniformly convex, $Q(u,v,w)$ will be negative when v lies between u and w, and positive when v lies outside the interval with endpoints u and w. In particular, for the Burgers equation (4.2.1), $Q(u,v,w) = \frac{1}{2}(v-u)(v-w)$.

The first step is to refine the ordering property:

11.8.1 Theorem. *Let u and $\bar{u}$ be admissible solutions of* (11.1.1), *on the upper half-plane, with respective initial data u_0 and $\bar{u}_0$ such that*

$$u_0(x) \leq \bar{u}_0(x), \qquad \textit{for all } x \in (y, \bar{y}). \tag{11.8.2}$$

Let $\psi(\cdot)$ be any forward characteristic, associated with the solution u, issuing from the point $(y,0)$, and let $\bar{\psi}(\cdot)$ be any forward characteristic, associated with $\bar{u}$, issuing from $(\bar{y},0)$. Then, for any $t > 0$ with $\psi(t) < \bar{\psi}(t)$,

$$u(x,t) \leq \bar{u}(x,t), \qquad \textit{for all } x \in (\psi(t), \bar{\psi}(t)). \tag{11.8.3}$$

Proof. We fix any interval $(z, \bar{z})$ with $\psi(t) < z < \bar{z} < \bar{\psi}(t)$ and consider the maximal backward characteristic $\xi(\cdot)$, associated with the solution u, emanating from the point (z,t), and the minimal backward characteristic $\bar{\zeta}(\cdot)$, associated with $\bar{u}$, emanating from the point $(\bar{z},t)$. Thus, $\xi(0) \geq y$ and $\bar{\zeta}(0) \leq \bar{y}$.

Suppose first $\xi(0) < \bar{\zeta}(0)$. We integrate the equation

$$\partial_t[u - \bar{u}] + \partial_x[f(u) - f(\bar{u})] = 0 \tag{11.8.4}$$

over the trapezoid $\{(x,\tau) : 0 < \tau < t, \xi(\tau) < x < \bar{\zeta}(\tau)\}$ and apply Green's theorem to get

$$\begin{aligned} \int_z^{\bar{z}} [u(x,t) - \bar{u}(x,t)]dx - \int_{\xi(0)}^{\bar{\zeta}(0)} [u_0(x) - \bar{u}_0(x)]dx \\ = -\int_0^t Q(u(\xi(\tau),\tau), \bar{u}(\xi(\tau),\tau), u(\xi(\tau),\tau))d\tau \\ -\int_0^t Q(\bar{u}(\bar{\zeta}(\tau),\tau), u(\bar{\zeta}(\tau),\tau), \bar{u}(\bar{\zeta}(\tau),\tau))d\tau. \end{aligned} \tag{11.8.5}$$

Both integrals on the right-hand side of (11.8.5) are nonpositive. Hence, by virtue of (11.8.2), the integral of $u(\cdot,t) - \bar{u}(\cdot,t)$ over $(z,\bar{z})$ is nonpositive.

Suppose now $\xi(0) \geq \bar{\zeta}(0)$. Then the straight lines $\xi(\cdot)$ and $\bar{\zeta}(\cdot)$ must intersect at some time $s \in [0,t)$. In that case we integrate the Equation (11.8.4) over the triangle $\{(x,\tau) : s < \tau < t, \xi(\tau) < x < \bar{\zeta}(\tau)\}$ and employ the same argument as above to deduce that the integral of $u(\cdot,t) - \bar{u}(\cdot,t)$ over $(z,\bar{z})$ is again nonpositive.

Since $(z,\bar{z})$ is an arbitrary subinterval of $(\psi(t), \bar{\psi}(t))$, we conclude (11.8.3). The proof is complete.

As a corollary of the above theorem, we infer that the number of sign changes of the function $u(\cdot,t) - \bar{u}(\cdot,t)$ over $(-\infty,\infty)$ is nonincreasing with time. Indeed, assume there are points $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$ such that, on each interval $(y_i, y_{i+1}), u_0(\cdot) - \bar{u}_0(\cdot)$ is nonnegative when i is even and nonpositive when i is odd. Let $\psi_i(\cdot)$ be any forward characteristic, associated with the solution u, issuing from the point $(y_i, 0)$ with i odd, and $\bar{\psi}_i(\cdot)$ any forward characteristic, associated with $\bar{u}$, issuing from $(y_i, 0)$ with i even. These curves are generally assigned finite life spans, according to the following prescription. At the time t_1 of the earliest collision between some ψ_i and some $\bar{\psi}_j$, these two curves are terminated. Then, at the time t_2 of the next collision between any (surviving) ψ_k and $\bar{\psi}_\ell$, these two curves are likewise terminated; and so on. By virtue of Theorem 11.8.1, $u(\cdot,t) - \bar{u}(\cdot,t)$ undergoes n sign changes for any $t \in [0, t_1)$, $n-2$ sign changes for any $t \in [t_1, t_2)$, and so on. In particular, the so-called *lap number*, which counts the crossings of the graph of the solution $u(\cdot,t)$ with any fixed constant $\bar{u}$, is nonincreasing with time.

By Theorem 6.2.3, the spatial L^1 distance of any pair of admissible solutions of a scalar conservation law is nonincreasing with time. In the present setting, it will be shown that it is actually possible to determine under what conditions the L^1 distance is strictly decreasing and at what rate:

11.8.2 Theorem. *Let u and $\bar{u}$ be admissible solutions of* (11.1.1) *with initial data u_0 and $\bar{u}_0$ in $L^1(-\infty,\infty)$. Thus $\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(-\infty,\infty)}$ is a nonincreasing function of t which is locally Lipschitz on $(0,\infty)$. For any fixed $t \in (0,\infty)$, consider the (possibly empty and at most countable) sets*

$$
\left\{
\begin{aligned}
&\mathscr{J} = \{y \in (-\infty,\infty) : u_+ < \bar{u}_+ \le \bar{u}_- < u_-\},\\
&\bar{\mathscr{J}} = \{y \in (-\infty,\infty) : \bar{u}_+ < u_+ \le u_- < \bar{u}_-\},
\end{aligned}
\right.
\tag{11.8.6}
$$

where $u_\pm$ and $\bar{u}_\pm$ stand for $u(y\pm,t)$ and $\bar{u}(y\pm,t)$, respectively. Let

$$
u_* = \begin{cases}
u_\pm & \text{if } u_+ = u_- ,\\[2ex]
u_- & \text{if } u_+ < u_- \text{ and } \dfrac{f(u_+) - f(u_-)}{u_+ - u_-} \ge \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-},\\[2ex]
u_+ & \text{if } u_+ < u_- \text{ and } \dfrac{f(u_+) - f(u_-)}{u_+ - u_-} < \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-},
\end{cases}
\tag*{(11.8.7)$_1$}
$$

$$
\bar{u}_* = \begin{cases}
\bar{u}_\pm & \text{if } \bar{u}_+ = \bar{u}_- ,\\[2ex]
\bar{u}_- & \text{if } \bar{u}_+ < \bar{u}_- \text{ and } \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-} \ge \dfrac{f(u_+) - f(u_-)}{u_+ - u_-},\\[2ex]
\bar{u}_+ & \text{if } \bar{u}_+ < \bar{u}_- \text{ and } \dfrac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-} < \dfrac{f(u_+) - f(u_-)}{u_+ - u_-}.
\end{cases}
\tag*{(11.8.7)$_2$}
$$

Then

$$\text{(11.8.8)}\quad \frac{d^+}{dt}\|u(\cdot,t)-\bar{u}(\cdot,t)\|_{L^1(-\infty,\infty)} = 2\sum_{y\in\mathscr{J}} Q(u_-,\bar{u}_*,u_+) + 2\sum_{y\in\bar{\mathscr{J}}} Q(\bar{u}_-,u_*,\bar{u}_+).$$

Proof. First we establish (11.8.8) for the special case where $u(\cdot,t)-\bar{u}(\cdot,t)$ undergoes a finite number of sign changes on $(-\infty,\infty)$. We thus assume that there are points $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$ such that $u(\cdot,t)-\bar{u}(\cdot,t)$ is nonnegative on the intervals (y_i,y_{i+1}) with i even, and nonpositive on the intervals (y_i,y_{i+1}) with i odd. In particular, any $y\in\mathscr{J}$ must be one of the y_i, with i odd, and any $y\in\bar{\mathscr{J}}$ must be one of the y_i, with i even.

Let $\psi_i(\cdot)$ be the (unique) forward characteristic, associated with the solution u, issuing from the point (y_i,t) with i odd, and let $\bar{\psi}_i(\cdot)$ be the forward characteristic, associated with $\bar{u}$, issuing from (y_i,t) with i even. We fix $s>t$ with $s-t$ so small that no collisions of the above curves may occur on $[t,s]$, and integrate (11.8.4) over the domains $\{(x,\tau): t<\tau<s,\ \psi_i(\tau)<x<\bar{\psi}_{i+1}(\tau)\}$, for i odd, and over the domains $\{(x,\tau): t<\tau<s,\ \bar{\psi}_i(\tau)<x<\psi_{i+1}(\tau)\}$, for i even. We apply Green's theorem and employ Theorem 11.8.1, to deduce

$$\begin{aligned}
\text{(11.8.9)}\quad & \|u(\cdot,s)-\bar{u}(\cdot,s)\|_{L^1(-\infty,\infty)} - \|u(\cdot,t)-\bar{u}(\cdot,t)\|_{L^1(-\infty,\infty)} \\
&= \sum_{i\text{ even}}\int_{\bar{\psi}_i(s)}^{\psi_{i+1}(s)}[u(x,s)-\bar{u}(x,s)]dx + \sum_{i\text{ odd}}\int_{\psi_i(s)}^{\bar{\psi}_{i+1}(s)}[\bar{u}(x,s)-u(x,s)]dx \\
&\quad - \sum_{i\text{ even}}\int_{y_i}^{y_{i+1}}[u(x,t)-\bar{u}(x,t)]dx - \sum_{i\text{ odd}}\int_{y_i}^{y_{i+1}}[\bar{u}(x,t)-u(x,t)]dx \\
&= \sum_{i\text{ odd}}\int_t^s \{Q(u(\psi_i(\tau)-,\tau),\bar{u}(\psi_i(\tau)-,\tau),u(\psi_i(\tau)+,\tau)) \\
&\qquad\qquad + Q(u(\psi_i(\tau)+,\tau),\bar{u}(\psi_i(\tau)+,\tau),u(\psi_i(\tau)-,\tau))\}d\tau \\
&\quad + \sum_{i\text{ even}}\int_t^s \{Q(\bar{u}(\bar{\psi}_i(\tau)-,\tau),u(\bar{\psi}_i(\tau)-,\tau),\bar{u}(\bar{\psi}_i(\tau)+,\tau)) \\
&\qquad\qquad + Q(\bar{u}(\bar{\psi}_i(\tau)+,\tau),u(\bar{\psi}_i(\tau)+,\tau),\bar{u}(\bar{\psi}_i(\tau)-,\tau))\}d\tau.
\end{aligned}$$

By virtue of Theorem 11.3.1, as $s\downarrow t$ the integrand in the first integral on the right-hand side of (11.8.9) tends to zero, if $y_i\notin\mathscr{J}$, or to $2Q(u_-,\bar{u}_*,u_+)$, if $y_i\in\mathscr{J}$. Similarly, the integrand in the second integral on the right-hand side of (11.8.9) tends to zero, if $y_i\notin\bar{\mathscr{J}}$, or to $2Q(\bar{u}_-,u_*,\bar{u}_+)$, if $y_i\in\bar{\mathscr{J}}$. Therefore, upon dividing (11.8.9) by $s-t$ and letting $s\downarrow t$, we arrive at (11.8.8).

We now turn to the general situation, where $u(\cdot,t)-\bar{u}(\cdot,t)$ may undergo infinitely many sign changes over $(-\infty,\infty)$, observing that in that case, the open set

$\{x \in (-\infty,\infty) : u(x\pm,t) - \bar{u}(x\pm,t) < 0\}$ is the countable union of disjoint open intervals $(y_i, \bar{y}_i)$. For $m = 1,2,\cdots$, we let u_m denote the admissible solution of our conservation law (11.1.1) on $(-\infty,\infty) \times [t,\infty)$, with

$$u_m(x,t) = \begin{cases} \bar{u}(x,t), & x \in \bigcup_{i=m}^{\infty} (y_i, \bar{y}_i) \\ u(x,t), & \text{otherwise.} \end{cases} \tag{11.8.10}$$

Thus $u_m(\cdot,t) - \bar{u}(\cdot,t)$ undergoes a finite number of sign changes over $(-\infty,\infty)$ and so, for $\tau \geq t$, $\frac{d^+}{d\tau}\|u_m(\cdot,\tau) - \bar{u}(\cdot,\tau)\|_{L^1}$ is evaluated by the analog of (11.8.8). Moreover, the function $\tau \mapsto \frac{d^+}{d\tau}\|u_m(\cdot,\tau) - \bar{u}(\cdot,\tau)\|_{L^1}$ is right-continuous at t and the modulus of right continuity is independent of m. To verify this, note that the total contribution of small jumps to the rate of change of $\|u_m(\cdot,\tau) - \bar{u}(\cdot,\tau)\|_{L^1}$ is small, controlled by the total variation of $u(\cdot,t)$ and $\bar{u}(\cdot,t)$ over $(-\infty,\infty)$, while the contribution of the (finite number of) large jumps is right-continuous, on account of Theorem 11.3.1. Therefore, by passing to the limit, as $m \to \infty$, we establish (11.8.8) for general solutions u and $\bar{u}$. The proof is complete.

According to the above theorem, the L^1 distance of $u(\cdot,t)$ and $\bar{u}(\cdot,t)$ may decrease only when the graph of either of these functions happens to cross the graph of the other at a point of jump discontinuity. More robust contraction is realized in terms of a new metric which weighs the L^1 distance of two solutions by a weight specially tailored to them.

For v and $\bar{v}$ in $BV(-\infty,\infty)$, let

$$\begin{aligned} \rho(v,\bar{v}) = \int_{-\infty}^{\infty} \{ & (V(x) + \bar{V}(\infty) - \bar{V}(x))[v(x) - \bar{v}(x)]^+ \\ & + (\bar{V}(x) + V(\infty) - V(x))[\bar{v}(x) - v(x)]^+ \} dx, \end{aligned} \tag{11.8.11}$$

where the superscript $+$ denotes "positive part", $w^+ = \max\{w,0\}$, and V or $\bar{V}$ denotes the variation function of v or $\bar{v}$, $V(x) = TV_{(-\infty,x)}v(\cdot)$, $\bar{V}(x) = TV_{(-\infty,x)}\bar{v}(\cdot)$.

11.8.3 Theorem. *Let u and $\bar{u}$ be admissible solutions of* (11.1.1) *with initial data u_0 and $\bar{u}_0$ in $BV(-\infty,\infty)$. Then, for any fixed $t \in (0,\infty)$,*

$$\begin{aligned} \frac{d^+}{dt}\rho(u(\cdot,t),\bar{u}(\cdot,t)) \leq & -\int_{-\infty}^{\infty} Q(u(x,t),\bar{u}(x,t),u(x,t))dV_t^c(x) \\ & - \int_{-\infty}^{\infty} Q(\bar{u}(x,t),u(x,t),\bar{u}(x,t))d\bar{V}_t^c(x) \\ & - \sum_{y\in\mathscr{K}} (u_- - u_+)Q(u_-,\bar{u}_*,u_+) - \sum_{y\in\bar{\mathscr{K}}} (\bar{u}_- - \bar{u}_+)Q(\bar{u}_-,u_*,\bar{u}_+) \\ & + (V_t(\infty) + \bar{V}_t(\infty))\{\sum_{y\in\mathscr{J}} Q(u_-,\bar{u}_*,u_+) + \sum_{y\in\bar{\mathscr{J}}} Q(\bar{u}_-,u_*,\bar{u}_+)\}, \end{aligned} \tag{11.8.12}$$

where V_t or $\bar{V}_t$ is the variation function of $u(\cdot,t)$ or $\bar{u}(\cdot,t)$; V_t^c or $\bar{V}_t^c$ denotes the continuous part of V_t or $\bar{V}_t$; $u_\pm$ or $\bar{u}_\pm$ stand for $u(y\pm,t)$ or $\bar{u}(y\pm,t)$, u_* and $\bar{u}_*$ are again determined through (11.8.7)$_1$ *and* (11.8.7)$_2$; *the sets $\mathscr{J}$ and $\bar{\mathscr{J}}$ are defined by* (11.8.6) *and $\mathscr{K}$ or $\bar{\mathscr{K}}$ denotes the set of jump points of $u(\cdot,t)$ or $\bar{u}(\cdot,t)$:*

$$\begin{cases} \mathscr{K} = \{y \in (-\infty,\infty) : u_+ < u_-\} \\ \\ \bar{\mathscr{K}} = \{y \in (-\infty,\infty) : \bar{u}_+ < \bar{u}_-\}. \end{cases} \tag{11.8.13}$$

Proof. We begin as in the proof of Theorem 11.8.2: We assume there is a partition of the real line: $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$ such that, on each interval (y_i, y_{i+1}), $u(\cdot,t) - \bar{u}(\cdot,t)$ is nonnegative when i is even and nonpositive when i is odd. We consider the forward characteristic $\psi_i(\cdot)$, associated with u, issuing from each point (y_i,t), with i odd, and the forward characteristic $\bar{\psi}_i(\cdot)$, associated with $\bar{u}$, issuing from each (y_i,t), with i even.

We focus our attention on some (y_i, y_{i+1}) with i even. We shall discuss only the case $-\infty < y_i < y_{i+1} < \infty$, as the other cases are simpler. With the exception of $\bar{\psi}_i(\cdot)$, all characteristics to be considered below will be associated with the solution u. The argument varies somewhat, depending on whether the forward characteristic χ_0 issuing from (y_i,t) lies to the left or to the right of $\bar{\psi}_i(\cdot)$; for definiteness, we shall treat the latter case, which is slightly more complicated.

We fix ε positive small and identify all $z_1, \cdots, z_N$, $y_i < z_1 < \cdots < z_N < y_{i+1}$, such that $u(z_I-,t) - u(z_I+,t) \geq \varepsilon$, $I = 1,\cdots,N$. We consider the forward characteristic $\chi_I(\cdot)$ issuing from the point (z_I,t), $I = 1,\cdots,N$. Then we select $s > t$ with $s - t$ so small that the following hold:

(a) No intersection of any two of the characteristics $\chi_0, \chi_1, \cdots, \chi_N, \psi_{i+1}$ may occur on the time interval $[t,s]$.

(b) For $I = 1,\cdots,N$, if $\zeta_I(\cdot)$ and $\xi_I(\cdot)$ denote the minimal and the maximal backward characteristics emanating from the point $(\chi_I(s),s)$, then the total variation of $u(\cdot,t)$ over the intervals $(\zeta_I(t), z_I)$ and $(z_I, \xi_I(t))$ does not exceed ε/N.

(c) If $\zeta(\cdot)$ denotes the minimal backward characteristic emanating from $(\psi_{i+1}(s),s)$, then the total variation of $u(\cdot,t)$ over the interval $(\zeta(t), y_{i+1})$ does not exceed ε.

(d) If $\zeta_0(\cdot)$ is the minimal backward characteristic emanating from $(\psi_i(s),s)$ and $\xi_0(\cdot)$ is the maximal backward characteristic emanating from $(\chi_0(s),s)$, then the total variation of $u(\cdot,t)$ over the intervals $(\zeta_0(t), y_i)$ and $(y_i, \xi_0(t))$ does not exceed ε.

For $I = 0,\cdots,N-1$, and some k to be fixed later, we pick a mesh on the interval $[\chi_I(s), \chi_{I+1}(s)] : \chi_I(s) = x_I^0 < x_I^1 < \cdots < x_I^k < x_I^{k+1} = \chi_{I+1}(s)$; and likewise for $[\chi_N(s), \psi_{i+1}(s)] : \chi_N(s) = x_N^0 < x_N^1 < \cdots < x_N^k < x_N^{k+1} = \psi_{i+1}(s)$. For $I = 0,\cdots,N$ and $j = 1,\cdots,k$, we consider the maximal backward characteristic $\xi_I^j(\cdot)$ emanating from the point (x_I^j, s) and identify its intercept $z_I^j = \xi_I^j(t)$ by the t-time line. We also set $z_0^0 = y_i$, $z_N^{k+1} = y_{i+1}$ and $z_{I-1}^{k+1} = z_I^0 = z_I$, $I = 1, \cdots, N$.

We now note the identity

$$R - S = -D, \tag{11.8.14}$$

where

$$R = \int_{\bar{\psi}_i(s)}^{\chi_0(s)} V_t(y_i)[u(x,s) - \bar{u}(x,s)]dx \tag{11.8.15}$$
$$+ \sum_{I=0}^{N} \sum_{j=0}^{k} \int_{x_I^j}^{x_I^{j+1}} V_t(z_I^i+)[u(x,s) - \bar{u}(x,s)]dx,$$

$$S = \sum_{I=0}^{N} \sum_{j=0}^{k} \int_{z_I^j}^{z_I^{j+1}} V_t(z_I^j+)[u(x,t) - \bar{u}(x,t)]dx, \tag{11.8.16}$$

(11.8.17)
$$D = \sum_{I=0}^{N} \sum_{j=1}^{k} \int_t^s [V_t(z_I^j+) - V_t(z_I^{j-1}+)] Q(u(\xi_I^j(\tau),\tau), \bar{u}(\xi_I^j(\tau)-,\tau), u(\xi_I^j(\tau),\tau))d\tau$$
$$+ \sum_{I=1}^{N} \int_t^s [V_t(z_I+) - V_t(z_{I-1}^k+)] Q(u(\chi_I(\tau)-,\tau), \bar{u}(\chi_I(\tau)-,\tau), u(\chi_I(\tau)+,\tau))d\tau$$
$$+ \int_t^s [V_t(y_i+) - V_t(y_i)] Q(u(\chi_0(\tau)-,\tau), \bar{u}(\chi_0(\tau)-,\tau), u(\chi_0(\tau)+,\tau))d\tau$$
$$- \int_t^s V_t(y_i) Q(\bar{u}(\bar{\psi}_i(\tau)-,\tau), u(\bar{\psi}_i(\tau)+,\tau), \bar{u}(\bar{\psi}_i(\tau)+,\tau))d\tau$$
$$- \int_t^s V_t(z_N^k+) Q(u(\psi_{i+1}(\tau)-,\tau), \bar{u}(\psi_{i+1}(\tau)-,\tau), u(\psi_{i+1}(\tau)+,\tau))d\tau.$$

To verify (11.8.14), one first integrates (11.8.4) over the following domains: $\{(x,\tau) : t < \tau < s,\ \bar{\psi}_i(\tau) < x < \chi_0(\tau)\}$, $\{(x,\tau) : t < \tau < s,\ \xi_I^j(\tau) < x < \xi_I^{j+1}(\tau)\}$, $\{(x,\tau) : t < \tau < s,\ \chi_I(\tau) < x < \xi_I^1(\tau)\}$, $\{(x,\tau) : t < \tau < s,\ \xi_I^k(\tau) < x < \chi_{I+1}(\tau)\}$, $\{(x,\tau) : t < \tau < s,\ \xi_N^k(\tau) < x < \psi_{i+1}(\tau)\}$ and applies Green's theorem; then forms the weighted sum of the resulting equations, with respective weights $V_t(y_i), V_t(z_I^j+)$, $V_t(z_I+), V_t(z_I^k+), V_t(z_N^k+)$.

To estimate R, we note that $V_t(y_i) \geq V_s(\chi_0(s))$, and $V_t(z_I^j+) \geq V_s(x_I^j+)$, $I = 0, \cdots, N$, $j = 0, \cdots, k$. Hence, if we pick the $x_I^{j+1} - x_I^j$ sufficiently small, we can guarantee

(11.8.18) $$R \geq \int_{\bar{\psi}_i(s)}^{\psi_{i+1}(s)} V_s(x)[u(x,s) - \bar{u}(x,s)]dx - (s-t)\varepsilon.$$

To estimate S, it suffices to observe that $V_t(\cdot)$ is nondecreasing, and so

(11.8.19) $$S \leq \int_{\bar{\psi}_i(t)}^{\psi_{i+1}(t)} V_t(x)[u(x,t) - \bar{u}(x,t)]dx.$$

To estimate D, observe that by the properties of Q all five terms are nonnegative. For $I = 0, \cdots, N$ and $j = 1, \cdots, k$, $V_t(z_I^j+) - V_t(z_I^{j-1}+) \geq V_t^c(z_I^j) - V_t^c(z_I^{j-1})$. Furthermore,

(11.8.20)
$$Q(u(\xi_I^j(\tau),\tau), \bar{u}(\xi_I^j(\tau)-,\tau), u(\xi_I^j(\tau),\tau)) = Q(u(z_I^j,t), \bar{u}(p_\tau(z_I^j),t), u(z_I^j,t)),$$

where the monotone increasing function p_τ is determined through

(11.8.21) $$p_\tau(x) = x + (\tau - t)[f'(u(x,t)) - f'(\bar{u}(p_\tau(x),t))].$$

Upon choosing the $x_I^{j+1} - x_I^j$ so small that the oscillation of $V_t^c(\cdot)$ over each one of the intervals (z_I^j, z_I^{j+1}) does not exceed ε, the standard estimates on Stieltjes integrals imply

(11.8.22) $$\sum_{I=0}^{N}\sum_{j=1}^{k}[V_t(z_I^j+) - V_t(z_I^{j-1}+)]Q(u(\xi_I^j(\tau),\tau), \bar{u}(\xi_I^j(\tau)-,\tau), u(\xi_I^j(\tau),\tau))$$
$$\geq \int_{y_i}^{y_{i+1}} Q(u(x,t), \bar{u}(p_\tau(x),t), u(x,t))dV_t^c(x) - c\varepsilon.$$

We now combine (11.8.14) with (11.8.18), (11.8.19), (11.8.17) and (11.8.22), then we divide the resulting inequality by $s - t$, we let $s \downarrow t$, and finally we let $\varepsilon \downarrow 0$. This yields

(11.8.23)
$$\frac{d^+}{dt}\int_{\bar{\psi}_i(t)}^{\psi_{i+1}(t)} V_t(x)[u(x,t) - \bar{u}(x,t)]dx$$
$$\leq -\int_{y_i}^{y_{i+1}} Q(u(x,t), \bar{u}(x,t), u(x,t))dV_t^c(x)$$
$$-\sum(u_- - u_+)Q(u_-, \bar{u}_*, u_+) + V_t(y_i)Q(\bar{u}_-, u_*, \bar{u}_+)$$
$$+V_t(y_{i+1})Q(u_-, \bar{u}_*, u_+),$$

where the summation runs over all y in $\mathscr{K}\bigcap(y_i, y_{i+1})$ and also over y_i if $y_i \in \mathscr{K}$ and χ_0 lies to the right of $\bar{\psi}_i$. The $u_\pm$, $\bar{u}_\pm$, u_* and $\bar{u}_*$ are of course evaluated at the corresponding y

Next we focus attention on intervals $(y_i\,, y_{i+1})$ with i odd. A completely symmetrical argument yields, in the place of (11.8.23),

(11.8.24)
$$
\begin{aligned}
\frac{d^+}{dt}\int_{\psi_i(t)}^{\bar{\psi}_{i+1}(t)} &(V_t(\infty)-V_t(x))[\bar{u}(x,t)-u(x,t)]dx \\
&\le -\int_{y_i}^{y_{i+1}} Q(u(x,t),\bar{u}(x,t),u(x,t))dV_t^c(x) \\
&\quad -\sum(u_- - u_+)Q(u_-,\bar{u}_*,u_+)+(V_t(\infty)-V_t(y_i+))Q(u_-,\bar{u}_*,u_+) \\
&\quad +(V_t(\infty)-V_t(y_{i+1}))Q(\bar{u}_-,u_*,\bar{u}_+),
\end{aligned}
$$

where the summation runs over all y in $\mathscr{K}\bigcap(y_i, y_{i+1})$, and also over y_{i+1} if $y_{i+1} \in \mathscr{K}$ and the forward characteristic, associated with u, issuing from the point (y_{i+1}, t) lies to the left of $\bar{\psi}_{i+1}$.

We thus write (11.8.23), for all i even, then (11.8.24), for all i odd, and sum over $i = 0, \cdots, n$. This yields

(11.8.25)
$$
\begin{aligned}
&\frac{d^+}{dt}\int_{-\infty}^{\infty}\{V_t(x)[u(x,t)-\bar{u}(x,t)]^+ + (V_t(\infty)-V_t(x))[\bar{u}(x,t)-u(x,t)]^+\}dx \\
&\le -\int_{-\infty}^{\infty} Q(u(x,t),\bar{u}(x,t),u(x,t))dV_t^c(x) \\
&\quad -\sum_{y\in\mathscr{K}}(u_- - u_+)Q(u_-,\bar{u}_*,u_+)+V_t(\infty)\{\sum_{y\in\mathscr{J}}Q(u_-,\bar{u}_*,u_+)+\sum_{y\in\bar{\mathscr{J}}}Q(\bar{u}_-,u_*,\bar{u}_+)\}.
\end{aligned}
$$

By employing a technical argument, as in the proof of Theorem 11.8.2, one shows that (11.8.25) remains valid even when $u(\cdot,t)-\bar{u}(\cdot,t)$ is allowed to undergo infinitely many sign changes on $(-\infty,\infty)$.

We write the inequality resulting from (11.8.25) by interchanging the roles of u and $\bar{u}$, and then combine it with (11.8.25). This yields (11.8.12). The proof is complete.

The estimate (11.8.12) is sharp, in that it holds as equality, at least for piecewise smooth solutions. All terms on the right-hand side of (11.8.12) are negative, with the exception of the terms $-(u_- - u_+)Q(u_-,\bar{u}_*,u_+)$, for $y \in \mathscr{J}$, and $-(\bar{u}_- - \bar{u}_+)Q(\bar{u}_-,u_*,\bar{u}_+)$, for $y \in \bar{\mathscr{J}}$. However, even these positive terms are offset by the negative terms $V_t(\infty)Q(u_-,\bar{u}_*,u_+)$ and $\bar{V}_t(\infty)Q(\bar{u}_-,u_*,\bar{u}_+)$. Thus, the distance function $\rho(u(\cdot,t),\bar{u}(\cdot,t))$ is generally strictly decreasing.

An analog of the functional ρ will be employed in Chapter XIV for establishing L^1 stability of solutions for systems of conservation laws.

11.9 Genuinely Nonlinear Scalar Balance Laws

The notion of generalized characteristic may be extended in a natural way to general systems of balance laws, and may be used, in particular, for deriving a precise description of the structure of solutions of genuinely nonlinear, scalar balance laws

$$\partial_t u(x,t) + \partial_x f(u(x,t),x,t) + g(u(x,t),x,t) = 0. \tag{11.9.1}$$

Extending the analysis from (11.1.1) to (11.9.1) is rather straightforward, so it will suffice to outline here the main steps, with few proofs.

We assume that f and g are, respectively, C^2 and C^1 given functions, defined on $(-\infty,\infty)\times(-\infty,\infty)\times[0,\infty)$, and the genuine nonlinearity condition, namely $f_{uu}(u,x,t) > 0$, holds for all (u,x,t). We will be dealing with solutions $u(x,t)$ of (11.9.1), of class BV_{loc} on the upper half-plane $(-\infty,\infty)\times[0,\infty)$, such that $u(\cdot,t)$ has locally bounded variation in x on $(-\infty,\infty)$, for any fixed $t\in[0,\infty)$, and the Lax E-condition (11.1.2) holds for almost all $t\in[0,\infty)$ and all $x\in(-\infty,\infty)$. Solutions in this class may be constructed by solving the Cauchy problem with initial data that are bounded and have locally bounded variation on $(-\infty,\infty)$, for instance by the vanishing viscosity method expounded in Chapter VI. Restrictions have to be imposed on f and g in order to prevent the blowing up of the solution in finite time. For that purpose, it is sufficient to assume $|f_u|\le A$, for u in bounded intervals, and $f_x - g_u \le B$, for all u, uniformly on the upper half-plane. The reader may find details in the references cited in Section 11.14.

Along the lines of Definition 10.2.1, a generalized characteristic of (11.9.1), associated with the solution u, is a Lipschitz curve $\xi(\cdot)$, defined on some closed time interval $[\sigma,\tau]\subset[0,\infty)$, and satisfying the differential inclusion

$$\dot\xi(t) \in \left[f_u(u(\xi(t)+,t),\xi(t),t),\ f_u(u(\xi(t)-,t),\xi(t),t)\right], \tag{11.9.2}$$

for almost all $t\in[\sigma,\tau]$. As in Section 11.1, it can be shown that (11.9.2) is actually equivalent to

(11.9.3)

$$\dot\xi(t) = \begin{cases} f_u(u(\xi(t)\pm,t),\xi(t),t), & \text{if } u(\xi(t)+,t) = u(\xi(t)-,t) \\[2ex] \dfrac{f(u(\xi(t)+,t),\xi(t),t) - f(u(\xi(t)-,t),\xi(t),t)}{u(\xi(t)+,t) - u(\xi(t)-,t)}, & \text{if } u(\xi(t)+,t) < u(\xi(t)-,t) \end{cases}$$

for almost all $t\in[\sigma,\tau]$; compare with (11.1.3).

As with Definition 10.2.4, the characteristic $\xi(\cdot)$ is called *shock-free* on $[\sigma,\tau]$ if $u(\xi(t)-,t) = u(\xi(t)+,t)$, almost everywhere on $[\sigma,\tau]$. The key result is the following generalization of Theorem 11.1.1.

11.9.1 Theorem. *Let $\xi(\cdot)$ be a generalized characteristic for* (11.9.1)*, associated with the admissible solution u, which is shock-free on $[\sigma,\tau]$. Then there is a C^1 function v on $[\sigma,\tau]$ such that*

$$u(\xi(\tau)+,\tau) \le v(\tau) \le u(\xi(\tau)-,\tau), \tag{11.9.4}$$

$$u(\xi(t)+,t) = v(t) = u(\xi(t)-,t), \qquad \sigma < t < \tau, \tag{11.9.5}$$

$$u(\xi(\sigma)-,\sigma) \le v(\sigma) \le u(\xi(\sigma)+,\sigma). \tag{11.9.6}$$

Furthermore, $(\xi(\cdot), v(\cdot))$ *satisfy the classical characteristic equations*

$$\begin{cases} \dot{\xi} = f_u(v,\xi,t) \\ \dot{v} = -f_x(v,\xi,t) - g(v,\xi,t) \end{cases} \tag{11.9.7}$$

on (σ,τ). *In particular,* $\xi(\cdot)$ *is* C^1 *on* $[\sigma,\tau]$.

Proof. Let $I = \{t \in (\sigma,\tau) : u(\xi(t)-,t) = u(\xi(t)+,t)\}$. For $t \in I$, set $v(t) = u(\xi(t)\pm,t)$. In particular, (11.9.3) implies

$$\dot{\xi}(t) = f_u(v(t),\xi(t),t), \quad \text{a.e. on } (\sigma,\tau). \tag{11.9.8}$$

Fix r and s, $\sigma \le r < s \le \tau$. For $\varepsilon > 0$, we integrate the measure equality (11.9.1) over the set $\{(x,t) : r < t < s, \xi(t) - \varepsilon < x < \xi(t)\}$, apply Green's theorem, and use (11.9.8) and $f_{uu} > 0$ to get

$$\begin{aligned} &\int_{\xi(s)-\varepsilon}^{\xi(s)} u(x,s)dx - \int_{\xi(r)-\varepsilon}^{\xi(r)} u(x,r)dx + \int_r^s \int_{\xi(t)-\varepsilon}^{\xi(t)} g(u(x,t),x,t)dx\,dt \\ &= \int_r^s \{f(u(\xi(t)-\varepsilon+,t),\xi(t)-\varepsilon,t) - f(v(t),\xi(t),t) \\ &\qquad\qquad - f_u(v(t),\xi(t),t)[u(\xi(t)-\varepsilon+,t) - v(t)]\}\,dt \\ &\ge \int_r^s \{f(u(\xi(t)-\varepsilon+,t),\xi(t)-\varepsilon,t) - f(u(\xi(t)-\varepsilon+,t),\xi(t),t)\}\,dt. \end{aligned} \tag{11.9.9}$$

Multiplying (11.9.9) by $1/\varepsilon$ and letting $\varepsilon \downarrow 0$ yields

$$u(\xi(s)-,s) \ge u(\xi(r)-,r) - \int_r^s \{f_x(v(t),\xi(t),t) + g(v(t),\xi(t),t)\}\,dt. \tag{11.9.10}$$

Next we integrate (11.9.1) over the set $\{(x,t) : r < t < s, \xi(t) < x < \xi(t) + \varepsilon\}$ and follow the same procedure, as above, to deduce

$$u(\xi(s)+,s) \le u(\xi(r)+,r) - \int_r^s \{f_x(v(t),\xi(t),t) + g(v(t),\xi(t),t)\}\,dt. \tag{11.9.11}$$

For any $t \in (\sigma,\tau)$, we apply (11.9.10) and (11.9.11), first for $r = t$, $s \in I \cap (t,\tau)$, then for $s = t$, $r \in I \cap (\sigma,t)$. This yields $u(\xi(t)-,t) = u(\xi(t)+,t)$. Therefore, I coincides with (σ,τ) and (11.9.5) holds. For any r and s in (σ,τ), (11.9.10) and (11.9.11) combine into

$$v(s) = v(r) - \int_r^s \{f_x(v(t),\xi(t),t) + g(v(t),\xi(t),t)\}\,dt. \tag{11.9.12}$$

In conjunction with (11.9.8), (11.9.12) implies that $(\xi(\cdot),v(\cdot))$ are C^1 functions on $[\sigma,\tau]$ which satisfy the system (11.9.7).

To verify (11.9.4) and (11.9.6), it suffices to write (11.9.10), (11.9.11), first for $s = \tau$, $r \in (\sigma,\tau)$ and then for $r = \sigma$, $s \in (\sigma,\tau)$. This completes the proof.

11.9.2 Remark. When the balance law is a conservation law, $g \equiv 0$, and f does not depend explicitly on t, (11.9.7) implies $\dot{f}(v,\xi) = 0$, that is, f stays constant along shock-free characteristics.

The family of backward generalized characteristics emanating from any point $(\bar{x},\bar{t})$ of the upper half-plane spans a funnel bordered by the minimal backward characteristic $\xi_-(\cdot)$ and the maximal backward characteristic $\xi_+(\cdot)$. Theorem 10.3.2 is readily extended to systems of balance laws, and in the present context yields that both $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are shock-free on $(0,\bar{t})$. Thus, upon substituting Theorem 11.9.1 for Theorem 11.1.1, one easily derives the following generalization of Theorem 11.1.3:

11.9.3 Theorem. *Let u be an admissible solution of* (11.9.1) *with initial data u_0. Given any point $(\bar{x},\bar{t})$ on the upper half-plane, consider the solutions $(\xi_-(\cdot),v_-(\cdot))$ and $(\xi_+(\cdot),v_+(\cdot))$ of the system* (11.9.7)*, satisfying initial conditions $\xi_-(\bar{t}) = \bar{x}$, $v_-(\bar{t}) = u(\bar{x}-,\bar{t})$ and $\xi_+(\bar{t}) = \bar{x}$, $v_+(\bar{t}) = u(\bar{x}+,\bar{t})$. Then $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are respectively the minimal and the maximal backward characteristics emanating from $(\bar{x},\bar{t})$. Furthermore,*

$$\begin{cases} u(\xi_-(t)-,t) = v_-(t) = u(\xi_-(t)+,t) \\ u(\xi_+(t)-,t) = v_+(t) = u(\xi_+(t)+,t) \end{cases} \qquad 0 < t < \bar{t}, \tag{11.9.13}$$

$$\begin{cases} u_0(\xi_-(0)-) \geq v_-(0) \geq u_0(\xi_-(0)+) \\ u_0(\xi_+(0)-) \geq v_+(0) \geq u_0(\xi_+(0)+). \end{cases} \tag{11.9.14}$$

In particular, $u(\bar{x}+,\bar{t}) \leq u(\bar{x}-,\bar{t})$ holds for all $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ and $\xi_-(\cdot)$, $\xi_+(\cdot)$ coincide if and only if $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$.

Theorems 11.1.4 and 11.1.5, which describe properties of forward characteristics for homogeneous conservation laws, can also be readily extended to nonhomogeneous balance laws:

11.9.4 Theorem. *A unique forward generalized characteristic $\chi(\cdot)$, associated with an admissible solution u, issues from any point $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. Furthermore, if $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$, then $u(\chi(s)+,s) < u(\chi(s)-,s)$ for all $s \in [\bar{t},\infty)$.*

Solutions of the inhomogeneous balance law (11.9.1) have similar structure, and enjoy similar regularity properties with the solutions of the homogeneous conservation law (11.1.1), described in Section 11.3. A number of relevant propositions are stated below. The reader may find the proofs in the literature cited in Section 11.14.

11.9.5 Theorem. *Let u be an admissible solution and assume $u(\bar{x}+,\bar{t}) = u(\bar{x}-,\bar{t})$, for some $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. Then $(\bar{x},\bar{t})$ is a point of continuity of u. A unique generalized characteristic $\chi(\cdot)$, associated with u, defined on $[0,\infty)$, passes through $(\bar{x},\bar{t})$. Furthermore, $\chi(\cdot)$ is differentiable at $\bar{t}$ and $\dot{\chi}(\bar{t}) = f_u(u(\bar{x}\pm,\bar{t}),\bar{x},\bar{t})$.*

11.9.6 Theorem. *Let u be an admissible solution and assume $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$, for some $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$. When the extremal backward characteristics $\xi_-(\cdot)$, $\xi_+(\cdot)$ are the only backward generalized characteristics emanating from $(\bar{x},\bar{t})$ that are shock-free, then $(\bar{x},\bar{t})$ is a point of jump discontinuity of u in the following sense: There is a generalized characteristic $\chi(\cdot)$, associated with u, defined on $[0,\infty)$ and passing through $(\bar{x},\bar{t})$, such that $(\bar{x},\bar{t})$ is a point of continuity of the function $u(x-,t)$ relative to $\{(x,t) : 0 < t < \infty,\, x \le \chi(t)\}$ and also a point of continuity of the function $u(x+,t)$ relative to $\{(x,t) : 0 < t < \infty,\, x \ge \chi(t)\}$. Furthermore, $\chi(\cdot)$ is differentiable at $\bar{t}$ and*

$$\dot{\chi}(\bar{t}) = \frac{f(u(\bar{x}+,\bar{t}),\bar{x},\bar{t}) - f(u(\bar{x}-,\bar{t}),\bar{x},\bar{t})}{u(\bar{x}+,\bar{t}) - u(\bar{x}-,\bar{t})}\,. \tag{11.9.15}$$

11.9.7 Theorem. *The set of irregular points of any admissible solution u is countable. $(\bar{x},\bar{t}) \in (-\infty,\infty)\times(0,\infty)$ is an irregular point if and only if $u(\bar{x}+,\bar{t}) < u(\bar{x}-,\bar{t})$ and, in addition to the extremal backward characteristics $\xi_-(\cdot)$, $\xi_+(\cdot)$, there is at least another, distinct, backward characteristic $\xi(\cdot)$, associated with u, emanating from $(\bar{x},\bar{t})$, which is shock-free. Irregular points are generated by the collision of shocks and/or by the focusing of centered compression waves.*

11.9.8 Theorem. *Assume that f and g are, respectively, C^{k+1} and C^k functions on $(-\infty,\infty)\times(-\infty,\infty)\times[0,\infty)$, for some $3 \le k \le \infty$. Let u be an admissible solution with initial data u_0 in C^k. Then $u(x,t)$ is C^k on the complement of the closure of the shock set. Furthermore, generically, u is piecewise smooth and does not contain centered compression waves.*

The large-time behavior of solutions of inhomogeneous, genuinely nonlinear balance laws can be widely varied, and the method of generalized characteristics provides an efficient tool for determining the asymptotic profile. Two typical, very simple, examples will be presented in the following two sections to demonstrate the effect of source terms or inhomogeneity on the asymptotics of solutions with periodic initial data.

11.10 Balance Laws with Linear Excitation

We consider the balance law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) - u(x,t) = 0, \tag{11.10.1}$$

with $f''(u) > 0$, $-\infty < u < \infty$. For convenience, we normalize f and the space-time frame so that $f(0) = 0$ and $f'(0) = 0$.

The aim is to demonstrate that, as a result of the competition between the destabilizing action of the source term and the damping effect of genuine nonlinearity, periodic solutions with zero mean become asymptotically standing waves of finite amplitude.

In what follows, $u(x,t)$ will denote an admissible solution of (11.10.1), of locally bounded variation on the upper half-plane, with initial values $u(x,0) = u_0(x)$ of locally bounded variation on $(-\infty,\infty)$.

The system (11.9.7) for shock-free characteristics here takes the form

$$\begin{cases} \dot{\xi} = f'(v) \\ \\ \dot{v} = v. \end{cases} \tag{11.10.2}$$

In particular, divides are characteristics that are shock-free on $[0,\infty)$. Clearly, u grows exponentially along divides, with the exception of stationary ones, $x = \bar{x}$, along which u vanishes. The following proposition, which identifies the points of origin of stationary divides, should be compared with Theorem 11.4.1.

11.10.1 Lemma. *The line $x = \bar{x}$ is a stationary divide, associated with the solution u, if and only if*

$$\int_{\bar{x}}^{z} u_0(x)dx \geq 0, \qquad -\infty < z < \infty, \tag{11.10.3}$$

i.e., $\bar{x}$ is a minimizer of the primitive of $u_0(\cdot)$.

Proof. The reason it is possible here to locate the point of origin of divides with such precision is that the homogeneous balance law (11.10.1) may be regarded equally well as an inhomogeneous conservation law:

$$\partial_t \left[e^{-t} u(x,t)\right] + \partial_x \left[e^{-t} f(u(x,t))\right] = 0. \tag{11.10.4}$$

Assume first (11.10.3) holds. Fix any $\bar{t} > 0$ and consider the minimal backward characteristic $\xi(\cdot)$ emanating from the point $(\bar{x},\bar{t})$. We integrate (11.10.4) over the set bordered by the graph of $\xi(\cdot)$, the line $x = \bar{x}$ and the x-axis. Applying Green's theorem and using Theorem 11.9.3 and (11.10.2) yields

$$(11.10.5)\qquad \int_0^{\bar{t}} e^{-t}\left\{f'(u(\xi(t),t)u(\xi(t),t)-f(u(\xi(t),t))\right\}dt$$
$$+\int_0^{\bar{t}} e^{-t}f(u(\bar{x}\pm,t))dt+\int_{\bar{x}}^{z}u_0(x)dx=0.$$

All three terms on the left-hand side of (11.10.5) are nonnegative and hence they should all vanish. Thus $u(\bar{x}\pm,t)=0$, $t\in(0,\infty)$, and $x=\bar{x}$ is a stationary divide.

Conversely, assume $x=\bar{x}$ is a stationary divide, along which u vanishes. Fix any $z<\bar{x}$. For $\varepsilon>0$, let $\chi(\cdot)$ be the curve issuing from the point $(z,0)$ and having slope $\dot{\chi}(t)=f'\left(\varepsilon e^t\right)$. Suppose $\chi(\cdot)$ intersects the line $x=\bar{x}$ at time $\bar{t}$. We integrate (11.10.4) over the set $\{(x,t):0\le t\le\bar{t},\chi(t)\le x\le\bar{x}\}$ and apply Green's theorem. Upon adding and subtracting terms that depend solely on t, we end up with

$$(11.10.6)\qquad \int_0^{\bar{t}} e^{-t}\left\{f\left(\varepsilon e^t\right)-f(u(\chi(t)+,t))-f'\left(\varepsilon e^t\right)\left[\varepsilon e^t-u(\chi(t)+,t)\right]\right\}dt$$
$$-\int_0^{\bar{t}} e^{-t}f\left(\varepsilon e^t\right)dt=\int_z^{\bar{x}}\left[u_0(x)-\varepsilon\right]dx.$$

Both terms on the left-hand side of the above equation are nonpositive, and hence so also is the right-hand side. Letting $\varepsilon\downarrow 0$, we arrive at (11.10.3), for any $z<\bar{x}$. The case $z>\bar{x}$ is handled by the same method. This completes the proof.

Next we show that between adjacent stationary divides the solution attains asymptotically a standing wave profile of finite amplitude. The following proposition should be compared with Theorem 11.7.2.

11.10.2 Lemma. *Assume $x=x_-$ and $x=x_+$, $x_-<x_+$, are adjacent divides, associated with the solution u, i.e.,* (11.10.3) *holds for $\bar{x}=x_-$ and $\bar{x}=x_+$, but not for any $\bar{x}$ in the interval $\left(x_-,x_+\right)$. Consider any forward characteristic $\psi(\cdot)$ issuing from the point $\left(\frac{x_-+x_+}{2},0\right)$. Then, as $t\to\infty$,*

$$(11.10.7)\qquad u(x\pm,t)=\begin{cases} v_-(x)+o(1), & uniformly\ for\ \ x_-<x<\psi(t)\\ v_+(x)+o(1), & uniformly\ for\ \ \psi(t)<x<x_+,\end{cases}$$

where $v_-(x)$ and $v_+(x)$ are solutions of the differential equation $\partial_x f(v)=v$, with $v_-(x_-)=0$ and $v_+(x_+)=0$. Furthermore,

$$(11.10.8)\qquad \psi(t)=x_0+o(1),$$

where x_0 is determined by the condition

$$\int_{x_-}^{x_0} v_-(y)dy + \int_{x_0}^{x_+} v_+(y)dy = 0. \tag{11.10.9}$$

In particular, if u_0 is differentiable at $x_\pm$ and $u_0'(x_\pm) > 0$, then the order $o(1)$ in (11.10.7) *and* (11.10.8) *is upgraded to exponential: $O(e^{-t})$.*

Proof. As $t \to \infty$, the minimal backward characteristic $\zeta(\cdot)$ emanating from the point $(\psi(t),t)$ converges to a divide which is trapped inside the interval $[x_-,x_+)$ and thus is stationary. Since x_- and x_+ are adjacent, $\zeta(\cdot)$ must converge to x_-. In particular, $\zeta(0) = x_- + o(1)$, as $t \to \infty$.

We fix $t > 0$ and pick any $x \in (x_-, \psi(t)]$. Let $\zeta(\cdot)$ denote the minimal backward characteristic emanating from (x,t); it is intercepted by the x-axis at $\xi(0) = \xi_0$, with $x_- \le \xi_0 \le \zeta(0)$. In particular, $\xi_0 = x_- + o(1)$, as $t \to \infty$. Recalling Theorem 11.9.3, we integrate the system (11.10.2) to get $v(\tau) = \bar{u}e^{\tau}$, $0 \le \tau \le t$, for some $\bar{u}$ such that $u_0(\xi_0-) \le \bar{u} \le u_0(\xi_0+)$, and

$$x - \xi_0 = \int_0^t f'(v(\tau))d\tau = \int_{\bar{u}}^{u(x-,t)} \frac{f'(v)}{v}dv. \tag{11.10.10}$$

Now, (11.10.10) implies $u(x-,t) = O(1)$ whence $\bar{u} = e^{-t}u(x-,t) = O(e^{-t})$. In turn, by virtue of $\xi_0 = x_- + o(1)$ and $\bar{u} = O(e^{-t})$, (11.10.10) yields the upper half of (11.10.7). When $u_0'(x_-) > 0$, then $\bar{u} = O(e^{-t})$ implies $\xi_0 = x_- + O(e^{-t})$ and so $o(1)$ is upgraded to $O(e^{-t})$. The lower half of (11.10.7) is treated by the same method.

Integrating (11.10.1) over $[x_-,x_+] \times [0,t]$, we deduce $\int_{x_-}^{x_+} u(x,t)dx = 0$, so that (11.10.7) yields (11.10.8), (11.10.9). The proof is complete.

When the initial data $u_0(\cdot)$ are periodic, with mean M, then the solution $u(\cdot,t)$, at time t, is also periodic, with mean Me^t, and thus blows up as $t \to \infty$, unless $M = 0$. If $M = 0$, (11.10.3) is satisfied for at least one $\bar{x}$ in each period interval. Therefore, Lemma 11.10.2 has the following corollary, akin to Theorem 11.7.3.

11.10.3 Theorem. *When the initial data u_0 are periodic, with mean zero, then, as $t \to \infty$, the solution u tends to a periodic serrated profile consisting of wavelets of the form* (11.10.7). *The number of wavelets per period equals the number of points $\bar{x}$ in any period interval for which* (11.10.3) *holds. In the generic case where* (11.10.3) *is satisfied at a single point $\bar{x}$ in each period interval, u tends to a sawtooth profile with a single tooth per period.*

11.11 An Inhomogeneous Conservation Law

Here we discuss the large-time behavior of periodic solutions of an inhomogeneous conservation law

(11.11.1) $$\partial_t u(x,t) + \partial_x f(u(x,t),x) = 0,$$

where f is a C^2 function with the following properties:

(a) Periodicity in x: $f(u,x+1) = f(u,x),\ -\infty < u < \infty,\ -\infty < x < \infty.$

(b) Genuine nonlinearity: $f_{uu}(u,x) \geq \mu > 0,\ -\infty < u < \infty,\ -\infty < x < \infty.$

(c) The set of critical points consists of minima and saddles. For some $\bar{u}$ in $(-\infty,\infty)$, b in $(0,1)$, $x_k = b+k$ and $k = 0,\pm1,\pm2,\dots$ the following conditions hold: $f_u(\bar{u},x_k) = f_x(\bar{u},x_k) = 0$, $f_{uu}(\bar{u},x_k)f_{xx}(\bar{u},x_k) - f_{ux}^2(\bar{u},x_k) > 0$, and also $f_u(0,k) = f_x(0,k) = 0$, $f_{uu}(0,k)f_{xx}(0,k) - f_{ux}^2(0,k) < 0$.

(d) Normalization: $f(0,k) = 0$, hence $f(\bar{u},x_k) < 0$.

A typical example of such a function is $f(u,x) = u^2 - \sin^2(\pi x)$.

The system (11.9.7), for shock-free characteristics, here takes the form

(11.11.2) $$\begin{cases} \dot{\xi} = f_u(v,\xi) \\ \dot{v} = -f_x(v,\xi). \end{cases}$$

As noted in Remark 11.9.2, orbits of (11.11.2) are level curves of the function $f(u,x)$. By virtue of the properties of f, the phase portrait of (11.11.2) has the form depicted in Fig. 11.11.1. Orbits dwelling on level curves $f = p$, with $p > 0$, are unidirectional, from left to right or from right to left. By contrast, orbits dwelling on level curves $f = p$, with $p < 0$, are periodic. Finally, orbits dwelling on the level curves $f = 0$ are heteroclinic, joining neighboring saddle points; and in particular those dwelling on the nonnegative branch, $v = v_+(x)$, join $(k+1,0)$ to $(k,0)$, while those dwelling on the nonpositive branch, $v = v_-(x)$, join $(k,0)$ to $(k+1,0)$, $k = 0,\pm1,\pm2,\dots$.

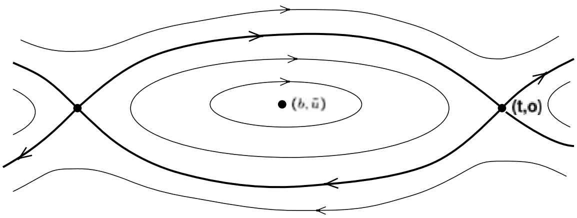

Fig. 11.11.1

For $p \in [f(\bar{u},b),\infty)\backslash\{0\}$, we define $T(p)$ as follows: If $p < 0$, $T(p)$ is the period around the level curve $f = p$. If $p > 0$, $T(p)$ is the time it takes to traverse a ξ-interval of length two, along the level curve $f = p$. The flow along any orbit moves at a swift pace, except near the equilibrium points $(k,0)$, where it slows down. In the linearized system about $(k,0)$, the sojourn in the vicinity of the equilibrium point, along the orbit on the level p, lasts for $-\lambda_0^{-1}\log|p|$ time units, where $\pm\lambda_0$ are the eigenvalues of the Jacobian matrix of the vector field $(f_u, -f_x)$, evaluated at the saddle point $(0,k)$, i.e., $\lambda_0 = \left[f_{ux}^2(k,0) - f_{uu}(k,0)f_{xx}(k,0)\right]^{1/2}$. Therefore,

$$\frac{T(p)}{\log|p|} = -\frac{2}{\lambda_0} + o(1), \qquad \text{as } p \to 0. \tag{11.11.3}$$

For any $M \in (-\infty,\infty)$, the equation (11.11.1) admits a unique admissible periodic stationary solution $u_M(x)$, with mean M. Let

$$M_\pm = \int_0^1 v_\pm(x)dx. \tag{11.11.4}$$

For $M \geq M_+$ or $M \leq M_-$, $u_M(x)$ is just the unique level curve $f = p \geq 0$ with mean M. By contrast, for $M_- < M < M_+$, u_M is a weak solution containing a single admissible stationary shock per period:

$$u_M(x) = \begin{cases} v_+(x), & k \leq x < k+a \\ v_-(x), & k+a < x < k+1, \end{cases} \qquad k = 0, \pm1, \pm2, \dots \tag{11.11.5}$$

where $a \in (0,1)$ is determined by

$$\int_0^a v_+(x)dx + \int_a^1 v_-(x)dx = M. \tag{11.11.6}$$

The aim is to show that, as $t \to \infty$, 1-periodic solutions of (11.11.1), with mean M, converge to u_M. We shall only discuss the interesting case $M_- < M < M_+$.

11.11.1 Theorem. *Let $u(x,t)$ be the admissible solution of* (11.11.1)*, on the upper half-plane, with initial data $u_0(x)$ which are* 1*-periodic functions with mean M in (M_-, M_+). Then, as $t \to \infty$, for any $\lambda < \lambda_0$,*

$$f(u(x\pm,t),x) = o\left(e^{-\lambda t}\right), \ \textit{uniformly on } (-\infty,\infty), \tag{11.11.7}$$

$$u(x\pm,t) = \begin{cases} v_+(x) + o\left(e^{-\frac{1}{2}\lambda t}\right), & k \leq x < \chi_k(t) \\ v_-(x) + o\left(e^{-\frac{1}{2}\lambda t}\right), & \chi_k(t) < x \leq k+1, \end{cases} \qquad k = 0, \pm1, \pm2, \dots \tag{11.11.8}$$

where

$$\chi_k(t) = k + a + o\left(e^{-\frac{1}{2}\lambda t}\right), \tag{11.11.9}$$

with a determined through (11.11.6).

Proof. We fix any $k = 0, \pm 1, \pm 2, \dots$, and note that

$$\int_k^{k+1} u(x,t)dx = M. \tag{11.11.10}$$

Since $M \in (M_-, M_+)$, (11.11.10) implies that there are $x \in (k, k+1)$ such that $f(u(x-,t),x) = p < 0$. For such an x, the minimal backward characteristic $\zeta(\cdot)$ emanating from (x,t) is the restriction to $[0,t]$ of the $T(p)$-periodic orbit that dwells on the level curve $f = p$. The minimal backward characteristic $\bar{\zeta}(\cdot)$ emanating from the point $(\bar{x},t)$, where $\bar{x} = x - \varepsilon$ with ε positive and small, will likewise be the restriction to $[0,t]$ of a periodic orbit dwelling on some level curve $f = \bar{p}$, with $|p - \bar{p}|$ small. It is now clear from the phase portrait, Fig. 11.11.1, that if t is larger than the period $T(p)$ the graphs of $\zeta(\cdot)$ and $\bar{\zeta}(\cdot)$ must intersect at some time $\tau \in (0,t)$, in contradiction to Theorem 11.9.4. Thus $t \le T(p)$ and hence $f(u(x-,t),x) \to 0$, as $t \to \infty$, by virtue of (11.11.3).

Suppose next there is $x \in [k, k+1]$ with $f(u(x-,t),x) = p > 0$. We fix $\bar{x}$ such that $\bar{x} < x < \bar{x} + 1$ and $f(u(\bar{x}-,t),\bar{x}) < 0$. If $\zeta(\cdot)$ and $\xi(\cdot)$ denote the minimal backward characteristics emanating from the points $(\bar{x},t)$ and (x,t), respectively, then we have $\zeta(\tau) < \xi(\tau) < \zeta(\tau) + 1$, for $0 < \tau < t$. Hence, $|x - \xi(0)| < 2$. But then $t \le T(p)$ and hence $f(u(x-,t),x) \to 0$, as $t \to \infty$, in this case as well.

By genuine nonlinearity and $f(u(x-,t),x) = o(1)$, for t large, $u(x-,t)$ must be close to either $v_-(x)$ or $v_+(x)$. Since admissible solutions are allowed to jump only downwards, there exists a characteristic $\chi_k(\cdot)$, with $\chi_k(t) \in (k, k+1)$ for $t \in [0, \infty)$, such that, for t large, $u(x-,t)$ is close to $v_-(x)$ if $k \le x < \chi_k(t)$, and close to $v_+(x)$ if $\chi_k(t) < x \le k+1$. Minimal backward characteristics emanating from points (x,t), with $\chi_{k-1}(t) < x < \chi_k(t)$ and $f(u(x-,t)) = p \gtrless 0$, are trapped between $\chi_{k-1}(\cdot)$ and $\chi_k(\cdot)$, so that our earlier estimate $t \le T(p)$ becomes sharper: $t \le \frac{1}{2}T(p) + O(1)$. This together with (11.11.3) implies (11.11.7), which in turn yields (11.11.8). Finally, by combining (11.11.8) with (11.11.10), we arrive at (11.11.9), where a is determined through (11.11.6). The proof is complete.

A more detailed picture of the asymptotic behavior of the above solution $u(x,t)$ emerges by locating its divides. On account of (11.11.7), any divide must be dwelling on the level curve $f = 0$. We shall see that the point of origin of any divide within the period interval $[\chi_{k-1}(0), \chi_k(0)]$ may be determined explicitly from the initial data. For that purpose we introduce the function

$$v_k(x) = \begin{cases} v_+(x), & -\infty < x \le k \\ v_-(x), & k < x < \infty, \end{cases} \tag{11.11.11}$$

which is a steady-state solution of (11.11.1):

$$\partial_t v_k(x) + \partial_x f(v_k(x), x) = 0. \tag{11.11.12}$$

11.11.2 Theorem. *Under the assumptions of Theorem* 11.11.1, *a divide associated with the solution $u(x,t)$ issues from the point $(\bar{x},0)$, with $\chi_{k-1}(0) \leq \bar{x} \leq \chi_k(0)$, if and only if $\bar{x}$ is a minimizer of the function*

$$\Phi_k(z) = \int_k^z [u_0(x) - v_k(x)]\,dx, \tag{11.11.13}$$

over $(-\infty,\infty)$.

Proof. Assume first $\bar{x} \in [\chi_{k-1}(0), \chi_k(0)]$ minimizes Φ_k over $(-\infty,\infty)$. We construct the characteristic $\xi(\cdot)$, associated with the solution v_k, issuing from the point $(\bar{x},0)$. Thus, $\xi(\cdot)$ will be determined by solving the system (11.11.2) with initial conditions $\xi(0) = \bar{x}$ and $v(0) = v_-(\bar{x})$ if $\bar{x} \geq k$, or $v(0) = v_+(\bar{x})$ if $\bar{x} < k$. In either case, $\dot{\xi}(t) = f_u(v_k(\xi(t)), \xi(t))$. We fix any $\bar{t} > 0$ and consider the minimal backward characteristic $\zeta(\cdot)$, associated with the solution $u(x,t)$, emanating from the point $(\xi(\bar{t}), \bar{t})$ and intercepted by the x-axis at $\zeta(0) = z \in [\chi_{k-1}(0), \chi_k(0)]$. Thus, $\dot{\zeta}(t) = f_u(u(\zeta(t)-,t), \zeta(t))$. We subtract (11.11.12) from (11.11.1) and integrate the resulting equation over the set bordered by the x-axis and the graphs of $\xi(\cdot)$ and $\zeta(\cdot)$ over $[0,\bar{t}]$. Applying Green's theorem yields

$$\begin{aligned}
&\int_0^{\bar{t}} \{ f(u(\xi(t)-,t), \xi(t)) - f(v_k(\xi(t)), \xi(t)) \\
&\qquad\qquad - f_u(v_k(\xi(t)), \xi(t))\,[u(\xi(t)-,t) - v_k(\xi(t))] \}\,dt \\
&- \int_0^{\bar{t}} \{ f(u(\zeta(t)-,t), \zeta(t)) - f(v_k(\zeta(t)), \zeta(t)) \\
&\qquad\qquad - f_u(u(\zeta(t)-,t), \zeta(t))\,[u(\zeta(t)-,t) - v_k(\zeta(t))] \}\,dt \\
&= \int_z^{\bar{x}} [u_0(x) - v_k(x)]\,dx = \Phi_k(\bar{x}) - \Phi_k(z).
\end{aligned} \tag{11.11.14}$$

Both terms on the left-hand side of the above equation are nonnegative, while the right-hand side is nonpositive. Thus, all three terms must vanish and $\xi(\cdot)$ is indeed a divide associated with $u(x,t)$.

Conversely, assume $(\bar{x},0)$ is the point of origin of a divide $\xi(\cdot)$ associated with the solution $u(x,t)$. Thus $\xi(\cdot)$ will solve the system (11.11.2) with initial conditions $\xi(0) = \bar{x}$ and $v(0) = v_-(\bar{x})$ if $\bar{x} \geq k$, or $v(0) = v_+(\bar{x})$ if $\bar{x} < k$. In either case, $u(\xi(t)\pm,t) = v_k(\xi(t))$, $t \in (0,\infty)$. We fix any $z \in (k-1, k+1)$ and construct the characteristic $\zeta(\cdot)$, associated with the solution v_k, that issues from the point $(z,0)$. Thus, $v_k(\zeta(t)) = v_+(\zeta(t))$ if $z \leq k$, or $v_k(\zeta(t)) = v_-(\zeta(t))$ if $z \geq k$. In either case, $\dot{\zeta}(t) = f_u(v_k(\zeta(t)), \zeta(t))$ and $\zeta(t) - \xi(t) \to 0$, as $t \to \infty$. We subtract (11.11.12) from (11.11.1) and integrate the resulting equation over the set bordered by the x-axis and the graphs of $\xi(\cdot)$ and $\zeta(\cdot)$ on $[0,\infty)$. Applying Green's theorem yields

$$(11.11.15)\quad \int_0^\infty \{f(u(\zeta(t)-,t),\zeta(t)) - f(v_k(\zeta(t)),\zeta(t))$$
$$- f_u(v_k(\zeta(t)),\zeta(t))\,[u(\zeta(t)-,t) - v_k(\zeta(t))]\}\,dt$$
$$= \int_{\bar{x}}^{z} [u_0(x) - v_k(x)]\,dx = \Phi_k(z) - \Phi_k(\bar{x}).$$

The left-hand side, and thereby also the right-hand side, of (11.11.15) is nonnegative. Therefore, $\bar{x}$ minimizes Φ_k over $(k-1,k+1)$, and hence even over $(-\infty,\infty)$, as $M \in (M_-,M_+)$. The proof is complete.

As $t \to \infty$, the family of minimal backward characteristics emanating from points $(\chi_k(t),t)$ converges monotonically to the divide that issues from $(x_+,0)$, where x_+ is the largest of the minimizers of Φ_k. Similarly, the family of maximal backward characteristics emanating from the points $(\chi_{k-1}(t),t)$ converges monotonically to the divide that issues from $(x_-,0)$, where x_- is the smallest of the minimizers of Φ_k. Generically, Φ_k should attain its minimum at a single point, in which case $x_- = x_+$.

11.12 When Genuine Nonlinearity Fails

As shown in the previous sections of this chapter, jump discontinuities in admissible BV solutions of scalar conservation laws with convex flux are necessarily compressive shocks. It is this property that induces the special features of extremal backward characteristics, recorded in Theorem 11.1.3, which are instrumental for rendering the structure of BV solutions simple and elegant. By contrast, when the flux possesses inflection points, the emergence of contact discontinuities causes considerable complications. The tool of generalized characteristics is still effective, but the analysis becomes quite cumbersome.

In this section we discus the simplest case of a scalar conservation law

$$(11.12.1)\qquad \partial_t u(x,t) + \partial_x f(u(x,t)) = 0,$$

with smooth flux f possessing a single inflection point at the origin,

$$(11.12.2)\qquad uf''(u) < 0, \qquad u \neq 0,$$

which is nondegenerate with order some even natural number m. For convenience, we scale u so that

$$(11.12.3)\qquad f^{(j)}(0) = 0, \quad j = 0,1,\dots,m, \qquad f^{(m+1)}(0) = -m!\,.$$

The reader should bear in mind that this example exhibits only part of the complexity that may be encountered in the absence of genuine nonlinearity. To begin with, under the assumption (11.12.2), the Oleinik E-condition only allows for left contact discontinuities; right contact discontinuities are inadmissible. Moreover, centered rarefaction waves cannot be generated by wave interactions. Still, rarefaction simple waves that are not centered may radiate out of left contact discontinuities.

We let u be the solution of (11.12.1) on $(-\infty,\infty)\times[0,\infty)$ with initial data u_0 taking values in an interval $[-\bar{u},\bar{u}]$ and having locally bounded variation on $(-\infty,\infty)$. We normalize $u_0(\cdot)$ and $u(\cdot,t)$ by making them continuous from the left.

A Lipschitz curve $\xi(\cdot)$, defined on the time interval $[\sigma,\tau]\subset[0,\infty)$, will be a generalized characteristic associated with the solution u if (11.1.3) holds for almost all $t\in[\sigma,\tau]$.

The solution u satisfies the admissibility condition

$$\partial_t\eta(u(x,t))+\partial_x q(u(x,t))\le 0, \tag{11.12.4}$$

for any convex function η, in the role of entropy, with entropy flux q determined by $q'(u)=\eta'(u)f'(u)$. If $\xi(\cdot)$ is a characteristic defined on $[\sigma,\tau]$, one easily derives, from (11.1.3) and (11.12.4) that

(11.12.5)
$$q(u(\xi(t)+,t))-q(u(\xi(t),t))-\dot{\xi}(t)[\eta(u(\xi(t)+,t))-\eta(u(\xi(t),t))]\le 0,$$

for almost all $t\in[\sigma,\tau]$. Recalling Section 8.4, (11.12.5) implies that the Oleinik E-condition (8.4.3) holds for almost all $t\in[\sigma,\tau]$ with $u(\xi(t)+,t)\neq u(\xi(t),t)$.

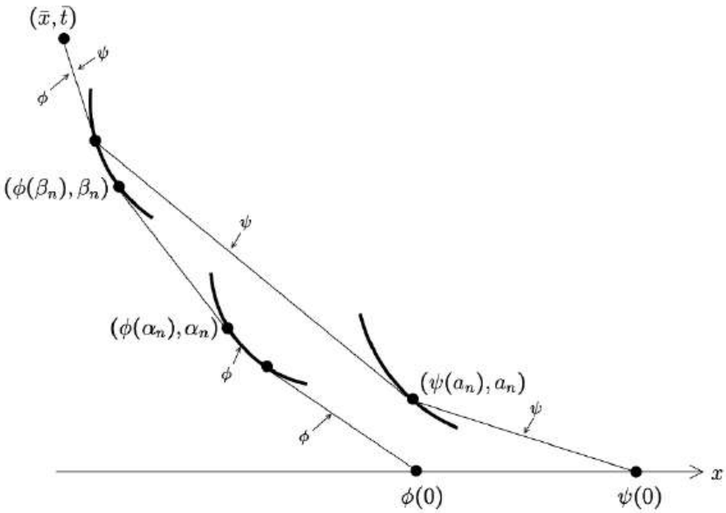

Fig. 11.12.1

We now consider the minimal and the maximal backward characteristics $\phi(\cdot)$ and $\psi(\cdot)$ emanating from some fixed point $(\bar{x},\bar{t})$ of the upper half-plane. By Theorem 10.3.1, $\phi(\cdot)$ is a left contact and $\psi(\cdot)$ is a right contact. Actually, since right contact discontinuities are ruled out by the Oleinik E-condition, $\psi(\cdot)$ must be shock-free and thereby also a left contact. The above properties induce structure to $\phi(\cdot)$ and $\psi(\cdot)$ depicted in Fig. 11.12.1 and roughly described as follows. Both $\phi(\cdot)$ and $\psi(\cdot)$ are convex functions on $[0,\bar{t}]$. The graph of $\phi(\cdot)$ is a C^1 curve composed of linear segments and/or arcs of left contact discontinuities, while the graph of $\psi(\cdot)$ is a polygonal chain with vertices at the points of intersection with left contact discontinuities. The function $u(\phi(t),t)$ is continuous and monotone nonincreasing when $u(\bar{x},\bar{t}) > 0$, or monotone nondecreasing when $u(\bar{x},\bar{t}) < 0$. By contrast, $u(\psi(t),t)$ is constant along each line segment of the polygonal chain, with alternating signs on adjacent segments. The full description of the structure of extremal backward characteristics is expounded in the following two propositions:

11.12.1 Theorem. *Let $\phi(\cdot)$ be the minimal backward characteristic emanating from the point $(\bar{x},\bar{t})$ of $(-\infty,\infty)\times(0,\infty)$. Then $u(\phi(t),t)$ is a continuous function on $(0,\bar{t})$, which is nondecreasing when $u(\bar{x},\bar{t}) < 0$, nonincreasing when $u(\bar{x},\bar{t}) > 0$ and constant, equal to zero, when $u(\bar{x},\bar{t}) = 0$. For $t \in (0,\bar{t})$,*

$$\dot{\phi}(t) = f'(u(\phi(t),t)), \tag{11.12.6}$$

so that $\phi(\cdot)$ is a convex C^1 function. Furthermore, the interval $(0,\bar{t})$ is the union of disjoint subsets $\mathscr{O}$ and $\mathscr{C}$ with the following properties. $\mathscr{O}$ is the countable union of pairwise disjoint open intervals, $\mathscr{O} = \bigcup(\alpha_n,\beta_n)$, with the property that, for all t in *(α_n,β_n), $u(\phi(t),t) = u(\phi(t)+,t) = u(\phi(\alpha_n),\alpha_n) = u(\phi(\beta_n),\beta_n)$, so the restriction of the graph of $\phi(\cdot)$ to (α_n,β_n) is a straight line segment with slope $f'(u(\phi(\alpha_n),\alpha_n))$. On the other hand, for any $t \in \mathscr{C}$, $u(\phi(t)+,t) \neq u(\phi(t),t)$ and*

$$f'(u(\phi(t),t)) = \frac{f(u(\phi(t)+,t)) - f(u(\phi(t),t))}{u(\phi(t)+,t) - u(\phi(t),t)}. \tag{11.12.7}$$

11.12.2 Theorem. *Let $\psi(\cdot)$ be the maximal backward characteristic emanating from the point $(\bar{x},\bar{t})$ of $(-\infty,\infty)\times(0,\infty)$. Then $\psi(\cdot)$ is convex and shock-free on $(0,\bar{t})$. Furthermore,*

(i) *When $u(\bar{x}+,\bar{t}) \neq 0$, there is a finite mesh $0 = a_0 < a_1 < \cdots < a_{k+1} = \bar{t}$ such that the graph of $\psi(\cdot)$ is a polygonal chain with vertices at the points $(\psi(a_n),a_n)$, for $n = 0,\dots,k+1$. Moreover,*

$$u(\psi(t),t) = u(\psi(t)+,t) = u(\psi(a_{n+1})+,a_{n+1}),\ \ a_n < t < a_{n+1},\ \ n = 0,\dots,k, \tag{11.12.8}$$

$$u(\psi(a_n),a_n) = u(\psi(a_{n+1})+,a_{n+1}), \quad n = 1,\dots,k, \tag{11.12.9}$$

$$\begin{cases} u_0(\psi(0)) \geq u(\psi(a_1)+,a_1), & \text{if } u(\psi(a_1)+,a_1) > 0, \\ u_0(\psi(0)) \leq u(\psi(a_1)+,a_1), & \text{if } u(\psi(a_1)+,a_1) < 0, \end{cases} \tag{11.12.10}$$

$$\text{(11.12.11)} \qquad \dot{\psi}(t) = f'(u(\psi(a_{n+1})+, a_{n+1})), \quad a_n < t < a_{n+1}, \quad n = 0, \ldots, k,$$

(11.12.12)
$$f'(u(\psi(a_n), a_n)) = \frac{f(u(\psi(a_n)+, a_n)) - f(u(\psi(a_n), a_n))}{u(\psi(a_n)+, a_n) - u(\psi(a_n), a_n)}, \quad n = 1, \ldots, k.$$

(ii) *When* $u(\bar{x}+, \bar{t}) = u(\bar{x}, \bar{t}) = 0$, *there is* $a \in [0, \bar{t}]$ *such that* $\psi(t) = \bar{x}$ *for* t *in* $[a, \bar{t}]$ *and* $u(\psi(t)+, t) = u(\psi(t), t) = 0$ *for* t *in* $(a, \bar{t}]$. *Moreover, if* $a > 0$, *there is an increasing sequence* $0 = a_0 < a_1 < \ldots$, *with* $a_n \to a$ *and* $\psi(a_n) \to \bar{x}$, *as* $n \to \infty$, *such that* (11.12.8), (11.12.9), (11.12.10), (11.12.11) *and* (11.12.12) *all hold, for* $n = 1, 2, \ldots$. *In particular,*

$$\text{(11.12.13)} \qquad u(\psi(t), t) \to 0, \quad f'(u(\psi(t), t)) \to 0, \quad \text{as } t \to a.$$

The complete proof of the above theorems, which is quite lengthy, is found in the bibliography cited in Section 11.14. Here we provide a sketch, bypassing the more technical steps.

Consider the entropy function

$$\text{(11.12.14)} \qquad \eta(u) = \exp[-\mu f'(u)],$$

which is convex on $[-\bar{u}, \bar{u}]$ when μ is fixed sufficiently large. It induces an entropy flux q satisfying

$$\text{(11.12.15)} \qquad q(w) - q(v) - f'(v)[\eta(w) - \eta(v)] = \int_v^w \eta'(\omega)[f'(\omega) - f'(v)]d\omega$$

$$= -\mu \exp[-\mu f'(v)] \int_0^z \theta \exp(-\mu\theta) d\theta \leq 0,$$

for any v, w, and $z = f'(w) - f'(v)$.

Suppose now $\xi(\cdot)$ is any left contact characteristic defined on some time interval $[\sigma, \tau] \subset [0, \infty)$. In particular, $\dot{\xi}(t) = f'(u(\xi(t), t))$, for almost all $t \in [\sigma, \tau]$. We fix r, s and ε, with $\sigma \leq r < s \leq \tau$ and $\varepsilon > 0$, and integrate (11.12.4), for the entropy (11.12.14), over the set $\{(x,t) \,:\, r < t < s,\ \xi(t) - \varepsilon < x < \xi(t)\}$. Upon using (11.12.15),

$$\text{(11.12.16)} \qquad \int_{\xi(s)-\varepsilon}^{\xi(s)} \eta(u(x,s))dx - \int_{\xi(r)-\varepsilon}^{\xi(r)} \eta(u(x,r))dx$$

$$\leq \int_r^s \{q(u(\xi(t)-\varepsilon, t)) - q(u(\xi(t), t)) - \dot{\xi}(t)[\eta(u(\xi(t)-\varepsilon, t)) - \eta(u(\xi(t), t))]\} \leq 0.$$

Thus, multiplying (11.12.16) by $1/\varepsilon$ and passing to the limit as $\varepsilon \to 0$, we deduce that $f'(u(\xi(r), r)) \leq f'(u(\xi(s), s))$.

Since both $\phi(\cdot)$ and $\psi(\cdot)$ are left contacts, we conclude that $f'(u(\phi(t), t))$ and $f'(u(\psi(t), t))$ are nondecreasing functions on $(0, \bar{t})$, whence $\phi(\cdot)$ and $\psi(\cdot)$ are convex.

A key step in the proof is to verify that $u(\phi(t),t)$ cannot switch signs on $(0,\bar{t}]$. Notice that when $u(\bar{x}-,\bar{t})$ does not vanish, and since $f'(u(\phi(t),t)$ is nondecreasing on $(0,\bar{t}]$, sign changes of $u(\phi(t),t)$ cannot occur by passing through zero. Thus any sign change, say at time $\tau \in (0,\bar{t})$, would incur a jump on $u(\phi(t),t)$ thereby imparting a jump on the variation of $u(\cdot,t)$ across the time line $t=\tau$. On the basis of this observation, a long and tedious argument, found in the bibliography listed in Section 11.14, and based on the interplay between the properties of minimal and maximal backward characteristics, demonstrates that $u(\phi(t),t)$ is continuous at every t. Thus $u(\phi(t),t)$ is a monotone continuous function on $(0,\bar{t}]$, which preserves the sign of $u(\bar{x}-,\bar{t})$.

By contract, $u(\psi(t),t)$ may undergo sign changes, but when $u(\bar{x}+,\bar{t}) \neq 0$ their number is (at most) finite. To see this, suppose there are $0 < b_1 < \cdots < b_k < \bar{t}$ such that $u_i = u(\psi(b_i),b_i)$ is positive, for i even, and negative, for i odd. If $\phi_i(\cdot)$ denotes the minimal backward characteristic emanating from $(\psi(b_i),b_i)$, then $u_0(\phi_i(0)) \geq u_i$, for i even, and $u_0(\phi_i(0)) \leq u_i$, for i odd. Hence, the sum of $|u_{i+1}-u_i|$, for $i=1,\dots,k$, is bounded by the total variation of $u_0(\cdot)$ over the interval $[\phi(0),\psi(0)]$. On the other hand, if $u(\bar{x}+,\bar{t}) \neq 0$ and since $f'(u(\psi(t),t))$ is nondecreasing, $|u_{i+1}-u_i| > \delta$, where δ is some positive number, independent of i. Thus k cannot be too large.

We conclude that there exist $0 = a_0 < a_1 < \cdots < a_{k+1} = \bar{t}$ such that $u(\psi(t),t)$ has a constant sign on each interval (a_n,a_{n+1}), alternating on adjacent intervals. Since f is convex on $(-\infty,0)$ and concave on $(0,\infty)$, Theorem 11.1.1 applies and yields (11.12.8). In particular, the graph of $\psi(\cdot)$ is a polygonal chain with vertices at the points $(\psi(a_n),a_n)$.

Notice that $\psi(\cdot)$ is the minimal forward characteristics issuing from the vertex $(\psi(a_n),a_n)$, while the maximal forward characteristic is some shock $\chi(\cdot)$. Recalling the properties of ψ and since χ must satisfy the Lax E-condition, we deduce

$$
\begin{aligned}
(11.12.17) \qquad f'(u(\psi(a_n),a_n)) &\leq f'(u(\psi(a_{n+1})+,a_{n+1})) \leq \dot{\psi}(a_n) \leq \dot{\chi}(a_n) \\
&= \frac{f(u(\psi(a_n)+,a_n)) - f(u(\psi(a_n),a_n))}{u(\psi(a_n)+,a_n) - u(\psi(a_n),a_n)} \leq f'(u(\psi(a_n),a_n)),
\end{aligned}
$$

which verifies (11.12.12). Thus the vertices of the chain are the points of intersection of $\psi(\cdot)$ with left contact discontinuities.

We return to the minimal backward characteristic $\phi(\cdot)$, assuming for definiteness that $u(\bar{x},\bar{t})$, and thereby also $u(\phi(t),t)$, are positive. Suppose that the set $\mathscr{O}$ of t in $(0,\bar{t})$ with $u(\phi(t)+,t) > 0$ is nonempty. The stability estimates established in Chapter VI, namely

$$
(11.12.18) \qquad TV_{[y,z]}u(\cdot,t) \leq TV_{[y-\lambda(t-\tau),\, z+\lambda(t-\tau)]}u(\cdot,\tau),
$$

$$
(11.12.19) \qquad \int_y^z |u(x,t)-u(x,\tau)|dx \leq \lambda(t-\tau)TV_{[y-\lambda(t-\tau),\, z+\lambda(t-\tau)]}u(\cdot,\tau),
$$

which hold for any $-\infty < y < z < \infty$ and $0 \leq \tau < t < \infty$, with λ an upper bound of $|f'|$ on $[-\bar{u},\bar{u}]$, imply that if $\tau \in \mathscr{O}$ then $t \in \mathscr{O}$ for all t in some interval $[\tau,\tau+\delta)$. Thus

$\mathscr{O}$ has nonempty interior, which will be the (at most) countable union of pairwise disjoint open intervals (α_n, β_n). We may then invoke Theorem 11.1.1 to conclude that $u(\phi(t),t)$ is constant on each interval (α_n, β_n) and the restriction of the graph of ϕ to (α_n, β_n) is a straight line segment. The complement $\mathscr{C}$ of $\mathscr{O}$ is the set of $t \in (0, \bar{t})$ with $u(\phi(t)+,t) < 0$. It is clear that $\beta_n \in \mathscr{C}$. It can also be shown that $\alpha_n \in \mathscr{C}$. Since $\phi(\cdot)$ is a left contact, (11.12.7) holds for almost all $t \in \mathscr{C}$. It can be shown that in fact (11.12.7) holds for all $t \in \mathscr{C}$. This completes the sketch of the proof of Theorems 11.12.1 and 11.12.2.

In the literature cited in Section 11.14, the reader will find a systematic discussion of the regularity of BV solutions to the Cauchy problem for the conservation law (11.12.1), under the assumptions (11.12.2), (11.12.3). The tools are the properties of the extremal backward characteristics, expounded above, but the analysis is quite technical and the emerging structure of solutions is considerably more complex than what was encountered in Section 11.3 for the genuinely nonlinear case. It is still true that, generically, C^∞ initial data generate solutions that are piecewise C^∞ smooth. However, in addition to a finite collection of compressive shocks and left contact discontinuities such solutions may now contain a finite number of weak waves. A *weak wave* of order m is a characteristic across which $\partial_x^k u$ is continuous, for $k = 0, \dots, m-1$, but $\partial_x^m u$ experiences jump discontinuities. Weak waves are straight line segments emerging tangentially out of left contact discontinuities and terminating upon colliding with a compressive shock or left contact discontinuity. A weak wave of order 1 is triggered by a "grazing ray", i.e., by the glancing collision of a characteristic, from the left, on a left contact discontinuity, while a weak wave of order $m > 1$ is emitted when a weak wave of order $m-1$ collides, from the right, with a left contact discontinuity. See Fig. 11.12.2.

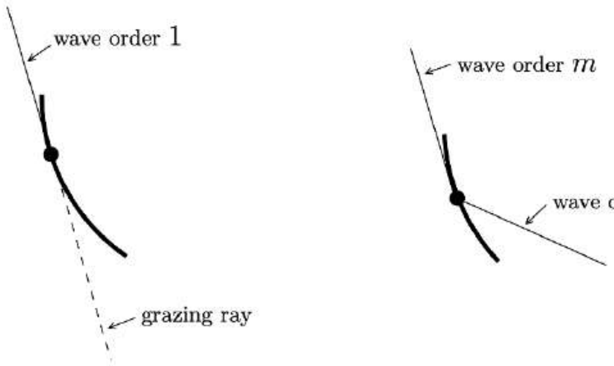

Fig. 11.12.2

We now turn to the question of the long time behavior of solutions. The following proposition provides the first indication on the regularizing and dissipative effect of the nonlinearity of the flux f.

11.12.3 Theorem. *Let u be the admissible BV solution to the Cauchy problem for* (11.12.1), *with initial data u_0 in $L^1(-\infty,\infty)$. Then*

$$TV_{(-\infty,\infty)}F(u(\cdot,t)) \leq \frac{2}{t}\int_{-\infty}^{\infty}|u_0(x)|dx, \qquad 0<t<\infty, \tag{11.12.20}$$

where F is the monotone increasing function defined by

$$F(u)=f(u)-uf'(u), \qquad -\infty<u<\infty. \tag{11.12.21}$$

Proof. Fix $t>0$ and take any mesh $-\infty<x_{-\ell}<\cdots<x_0<\cdots<x_s<\infty$ such that $u(x_i+,t)=u(x_i,t)$, for $i=-\ell,\dots,s$, and $u(x_i,t)<u(x_{i+1},t)$, when i is even, and $u(x_i,t)>u(x_{i+1},t)$, when i is odd.

With each $i=-\ell,\dots,s$, we associate a backward characteristic $\chi_i(\cdot)$ emanating from (x_i,t), which is a left contact on $(0,t)$ and is determined as follows.

(a) When i is even and $u(x_i,t)<0$ or when i is odd and $u(x_i,t)>0$, then $\chi_i(\cdot)$ is the minimal backward characteristic $\phi_i(\cdot)$ emanating from (x_i,t).

(b) When i is even and $u(x_i,t)>0$ or when i is odd and $u(x_i,t)<0$, then $\chi_i(\cdot)$ is identified through the following process. We consider the maximal backward characteristic $\psi_i(\cdot)$ emanating from (x_i,t), whose graph, by Theorem 11.12.2, is a polygonal chain with vertices $(\psi(a_n),a_n)$, where $0=a_0<a_1<\cdots<a_{k+1}=t$. If $k=0$ or $k=1$, we identify $\chi_i(\cdot)$ with $\psi_i(\cdot)$ on $[0,t]$. On the other hand, if $k\geq 2$, we construct the minimal backward characteristic $\phi_i(\cdot)$ emanating from $(\psi_i(a_{k-1}),a_{k-1})$ and identify $\chi_i(\cdot)$ with $\psi_i(\cdot)$ on the time interval $[a_{k-1},t]$ and with $\phi(\cdot)$ on the time interval $[0,a_{k-1})$.

It is easy to see that the graphs of $\chi_i(\cdot)$ and $\chi_{i+1}(\cdot)$ cannot intersect on $(0,t]$. We then integrate (11.12.1) over the set $\{(x,\tau):0<\tau<t,\ \chi_i(\tau)<x<\chi_{i+1}(\tau)\}$. Since both $\chi_i(\cdot)$ and $\chi_{i+1}(\cdot)$ are left contacts, recalling (11.12.21) we deduce

$$\begin{aligned}\int_0^t F(u(\chi_{i+1}(\tau),\tau))d\tau-\int_0^t F(u(\chi_i(\tau),\tau))d\tau \\ =\int_{\chi_i(0)}^{\chi_{i+1}(0)}u_0(x)dx-\int_{x_i}^{x_{i+1}}u(x,t)dx.\end{aligned}\tag{11.12.22}$$

We now show that

$$\begin{aligned}\left|\int_0^t F(u(\chi_{i+1}(\tau),\tau))d\tau-\int_0^t F(u(\chi_i(\tau),\tau))d\tau\right| \\ \geq t|F(u(x_{i+1},t))-F(u(x_i,t))|.\end{aligned}\tag{11.12.23}$$

To verify (11.12.23), one has to investigate the different situations that may arise, depending on whether i is even or odd, in connection with the signs of $u(x_i,t)$ and $u(x_{i+1},t)$. It will suffice to discuss here two cases that are representative of the rest.

Suppose first $u(x_i,t) < 0 < u(x_{i+1},t)$ (in particular, i is even). In that case, $\chi_i(\cdot)$ and $\chi_{i+1}(\cdot)$ are the minimal backward characteristics $\phi_i(\cdot)$ and $\phi_{i+1}(\cdot)$ emanating from (x_i,t) and (x_{i+1},t). By virtue of Theorem 11.12.2, for $\tau \in (0,t)$,we have $u(\chi_i(\tau),\tau) \leq u(x_i,t)$ and $u(\chi_{i+1}(\tau),\tau) \geq u(x_{i+1},t)$. Since F is increasing, we arrive at (11.12.23).

Suppose next $0 < u(x_i,t) < u(x_{i+1},t)$ (in particular, i is even). In that case, $\chi_{i+1}(\cdot)$ is still the minimal backward characteristic $\phi_i(\cdot)$ emanating from (x_{i+1},t) and hence, as above, $u(\chi_{i+1}(\tau),\tau) \geq u(x_{i+1},t)$. On the other hand, $\chi_i(\cdot)$ coincides with the maximal backward characteristic $\psi_i(\cdot)$, emanating from (x_i,t), over the time interval $[a_{k-1},t]$, and with the minimal backward characteristic $\phi(\cdot)$, emanating from $(\psi_i(a_{k-1},a_{k-1}))$, over the time interval $[0,a_{k-1})$. We now observe that $u(\chi_i(\tau),\tau) = u(x_i,t)$, for $\tau \in [a_k,t]$, and $u(\chi_i(\tau),\tau) \leq 0$, for $\tau \in (0,a_k)$. Thus (11.12.23) holds in this case as well.

Once (11.12.23) has been established, we combine it with (11.12.22) and sum over $i = -\ell,\dots,s-1$. Recalling Theorem 6.2.7, we arrive at (11.12.20). This completes the proof.

The assumption that the initial data have locally bounded variation did not play any role in the derivation of the estimate (11.12.20). We may thus infer, by completion, that if u is the admissible solution to the Cauchy problem for (11.12.1), with initial data u_0 that are merely in $L^\infty(-\infty,\infty)$, then, for any fixed $t > 0$, the function $F(u(\cdot,t))$ has locally bounded variation on $(-\infty,\infty)$. This implies, in particular, that one-sided limits $u(x\pm,t)$ exist, for all (x,t) in $(-\infty,\infty)\times(0,\infty)$. It is then possible to extend the notion and theory of generalized characteristics to the realm of these solutions. Furthermore, by virtue of (11.12.3), we have $|v-w|^{m+1} \leq c|F(v)-F(w)|$, whence, for any fixed $t > 0$, $u(\cdot,t)$ belongs to the space of function with bounded variation of fractional order $\frac{1}{m+1}$. In fact, as shown in the references listed in Section 11.14, for any $t \geq 0$, the characteristic speed $f'(u(\cdot,t))$ is of locally bounded variation on $(-\infty,\infty)$. Thus $u(\cdot,t)$ belongs to the space of functions of locally bounded variation of fractional order $\frac{1}{m}$. This is the counterpart of Theorem 11.2.2, from the genuinely nonlinear to the present case.

Before embarking on the next project, which is the investigation of the long time behavior of solutions with initial data that are either periodic or have compact support, we need the following preparation.

With any $u \neq 0$, we associate the unique $u^* \neq u$ with the property

$$f'(u) = \frac{f(u^*)-f(u)}{u^*-u}. \tag{11.12.24}$$

Thus u^* is the unique state that may be joined to u, on the right, by a left contact discontinuity. We know that u and u^* have opposite signs and that $f'(u^*) < f'(u)$. It can also be shown that, by virtue of (11.12.3),

$$\lim_{u\to 0}\frac{f'(u)}{f'(u^*)} = \rho_m\,, \tag{11.12.25}$$

where ρ_m is the unique positive solution of the equation

$$m\rho^{\frac{m+1}{m}} + (m+1)\rho - 1 = 0. \tag{11.12.26}$$

For example, when $m = 2$, $\rho_2 = \frac{1}{4}$. It is important to note that $\rho_m < \frac{1}{m+1}$.

We begin with the case of initial data with compact support:

11.12.4 Theorem. *Let u be the admissible BV solution to the Cauchy problem for* (11.12.1), *with initial data u_0 that are supported in the interval $[-\ell, 0]$ and carry mass*

$$\int_{-\ell}^{0} u_0(x)dx = M. \tag{11.12.27}$$

Then, as $t \to \infty$,

$$f'(u(x,t)) = \begin{cases} 0, & -\infty < x \le \chi(t) \\ \dfrac{x}{t} + O(t^{r-1}), & \chi(t) < x \le 0 \\ 0, & 0 < x < \infty \end{cases} \tag{11.12.28}$$

where r is any constant with $\rho_m < r < \frac{1}{m+1}$ and

$$\chi(t) = -\left[\frac{m+1}{m}|M|\right]^{\frac{m}{m+1}} t^{\frac{1}{m+1}} + O(t^r). \tag{11.12.29}$$

Proof. Clearly, $x = 0$ is the maximal forward characteristic issuing from $(0,0)$ and hence $u(x,t) = 0$ for any (x,t) with $x > 0$. Similarly, letting $\chi(\cdot)$ denote the minimal forward characteristic issuing from $(-\ell, 0)$, we have $u(x,t) = 0$, for all (x,t) with $x < \chi(t)$.

Fix any r in the interval $\left(\rho_m, \frac{1}{m+1}\right)$. By virtue of (11.12.25), there is $\delta > 0$ such that

$$f'(u) > rf'(u^*), \qquad -\delta \le u \le \delta. \tag{11.12.30}$$

On account of Theorem 11.12.3 and (11.12.3), $u(x,t) = O(t^{-\frac{1}{m+1}})$, as $t \to \infty$, uniformly in x. Thus there exists $T > 0$ such that $|u(x,t)| \le \delta$, for $-\infty < x < \infty$ and $t \ge T$.

We now fix any point (x,t), with $t > T$, $\chi(t) < x < 0$ and $u(x+,t) = u(x,t) \neq 0$, and proceed to verify (11.12.28). We consider the minimal backward characteristic $\phi(\cdot)$ emanating from (x,t), with the properties recounted in Theorem 11.12.1. Since $\phi(\cdot)$ is convex, $x - tf'(u(x,t)) \le \phi(0) \le 0$, which provides the upper bound to $f'(u(x,t))$ asserted by (11.12.28). We shall derive the lower bound through the following procedure.

We identify the point $\phi(T)$ of interception of $\phi(\cdot)$ by the time line $t = T$. One immediately gets the desired lower bound to $f'(u(x,t))$ when $\phi(\cdot)$ is shock free on (T,t), since in that case

$$x - tf'(u(x,t)) = \phi(T) - Tf'(u(x,t)) \geq \phi(T) \geq -\ell - \lambda T, \tag{11.12.31}$$

where λ is any upper bound to $-f'$ on $[-\bar{u}, \bar{u}]$.

The alternative, more difficult, situation arises when $\phi(\cdot)$ is shock-free on a time interval (α,t), with $\alpha \in (T,t)$, but $u(\phi(\alpha)+,\alpha) \neq u(\phi(\alpha),\alpha)$. In that case,

$$x - tf'(u(x,t)) = \phi(\alpha) - \alpha f'(u(x,t)) \geq \phi(\alpha), \tag{11.12.32}$$

so we need a lower bound for $\phi(\alpha)$. To that end, we show that $\phi(\cdot)$ satisfies the differential inequality

$$\dot{\phi}(\tau) \geq \frac{r}{\tau}\phi(\tau), \qquad T \leq \tau \leq \alpha. \tag{11.12.33}$$

We first verify (11.12.33) for any τ with $u(\phi(\tau)+,\tau) \neq u(\phi(\tau),\tau)$. Let $\psi(\cdot)$ denote the maximal backward characteristic emanating from $(\phi(\tau),\tau)$. By virtue of Theorem 11.12.2 and (11.12.30), recalling that $|u(\phi(\tau),\tau)| \leq \delta$,

$$\dot{\phi}(\tau) = f'(u(\phi(\tau),\tau)) \geq rf'(u(\phi(\tau)+,\tau)) = r\dot{\psi}(\tau) \geq r\dot{\psi}(s), \tag{11.12.34}$$

for $s \in [0,\tau]$. Integrating (11.12.34), with respect to s, over $[0,\tau]$, and recalling that $\psi(0) \leq 0$, $\psi(\tau) = \phi(\tau)$, we deduce

$$\tau\dot{\phi}(\tau) \geq r\phi(\tau), \tag{11.12.35}$$

which establishes (11.12.33) for this class of τ.

Next we consider any $\tau \in [T,\alpha)$ with $u(\phi(\tau)+,\tau) = u(\phi(\tau),\tau)$. In that case, by Theorem 11.12.1, there exists $\beta \in (\tau,\alpha]$ such that $\phi(\cdot)$ is shock-free on $[\tau,\beta)$ but $u(\phi(\beta)+,\beta) \neq u(\phi(\beta),\beta)$. In particular,

$$\phi(\beta) - \beta\dot{\phi}(\beta) = \phi(\tau) - \tau\dot{\phi}(\beta) \tag{11.12.36}$$

and $\dot{\phi}(\tau) = \dot{\phi}(\beta)$. Furthermore, as shown above, $\beta\dot{\phi}(\beta) \geq r\phi(\beta)$. It then follows that

$$r\phi(\tau) - \tau\dot{\phi}(\tau) = r\phi(\beta) - \beta\dot{\phi}(\beta) + (1-r)(\beta-\tau)\dot{\phi}(\beta) \leq 0, \tag{11.12.37}$$

which establishes (11.12.33) for this class of τ as well.

Upon integrating the differential inequality (11.12.33), we conclude that $\phi(\alpha)$ is $O(\alpha^r)$, and hence a fortiori $O(t^r)$. This together with (11.12.32) completes the proof of (11.12.28).

Finally, we determine $\chi(t)$ by appealing to mass conservation:

$$\int_{\chi(t)}^{0} u(x,t)dx = \int_{-\ell}^{0} u_0(x)dx = M. \tag{11.12.38}$$

Notice that (11.12.28) allows for two values:

$$u(x,t) = \pm \left[-\frac{x}{t} \right]^{\frac{1}{m}} + O\left(t^{r-\frac{2}{m+1}} \right). \tag{11.12.39}$$

However, shock admissibility rules out jumps from $(-x/t)^{\frac{1}{m}}$ to $-(-x/t)^{\frac{1}{m}}$ or vice versa, whence u retains a constant sign on $(\chi(t),0)$, namely positive when $M>0$ or negative when $M<0$. In either case, combining (11.12.39) with (11.12.38) we arrive at (11.12.29). This completes the proof.

We conclude this section with a discussion of the periodic case.

11.12.5 Theorem. *Let u be the admissible BV solution to the Cauchy for* (11.2.1), *with initial data u_0 that are periodic*,

$$u_0(x+\ell) = u_0(x), \qquad -\infty < x < \infty, \tag{11.12.40}$$

and have zero mean

$$\int_0^\ell u_0(x)dx = 0. \tag{11.12.41}$$

Then

$$-f'(u(x,t)) \le \frac{k\ell}{t}, \qquad -\infty < x < \infty,\ \ 0 < t < \infty. \tag{11.12.42}$$

Proof. Recalling (11.12.25) for u^* constructed via (11.12.24), we infer

$$f'(u) > \rho f'(u^*), \qquad -\bar{u} \le u \le \bar{u}, \tag{11.12.43}$$

for some $\rho < 1$. We proceed to show that

$$-tf'(u(y+,t)) \le \hat{k}\ell, \tag{11.12.44}$$

for every (y,t) such that $u(y,t)$ and $u(y+,t)$ have opposite signs. The constant $\hat{k}$ in (11.12.42) depends solely on ρ.

We consider the maximal backward characteristic $\psi(\cdot)$ emanating from (y,t). By Theorem 11.12.2, the graph of $\psi(\cdot)$ is a polygonal chain with vertices at points $(\psi(a_n),a_n)$, where $0 = a_0 < a_1 < \cdots < a_{k+1} = t$. For fixed $n = 1,\dots,k+1$, we let $\phi_n(\cdot)$ denote the minimal backward characteristic emanating from the vertex $(\psi(a_n),a_n)$ of the chain. The key point is to show

$$\dot{\phi}_n(\tau) \ge \rho \frac{a_n - a_{n-1}}{\tau - a_{n-1}} f'(u(\psi(a_n)+,a_n)), \qquad \tau \in (a_{n-1},a_n]. \tag{11.12.45}$$

We first verify (11.12.45) for any $\tau \in (a_{n-1},a_n]$ with $u(\phi_n(\tau)+,\tau) \ne u(\phi_n(\tau),\tau)$. Let $\chi(\cdot)$ denote the maximal backward characteristic emanating from $(\phi_n(\tau),\tau)$. Then, on account of Theorems 11.12.1, 11.12.2, recalling (11.12.43), (11.12.46) $$\dot{\phi}_n(\tau) = f'(u(\phi_n(\tau),\tau)) \geq \rho f'(u(\phi_n(\tau)+,\tau)) = \rho\dot{\chi}(\tau) \geq \rho\dot{\chi}(s),$$

for any $s \in [a_{n-1},\tau]$, whence

(11.12.47) $$(\tau - a_{n-1})\dot{\psi}(\tau) \geq \rho\chi(\tau) - \rho\chi(a_{n-1}).$$

Combining (11.12.47) with

(11.12.48) $$\chi(a_{n-1}) \leq \psi(a_{n-1}) = \psi(a_n) - (a_n - a_{n-1})f'(u(\psi(a_n)+,a_n))$$

and since $\chi(\tau) - \phi_n(\tau) \geq \phi_n(a_n) = \psi(a_n)$, we arrive at (11.12.45).

Next we consider any $\tau \in (a_{n-1},a_n)$ with $u(\phi_n(\tau)+,\tau) = u(\phi_n(\tau),\tau)$. By virtue of Theorem 11.12.2, there is $\beta \in (\tau,a_n]$ such that $\phi_n(\cdot)$ is shock free on $[\tau,\beta)$, $\dot{\phi}_n(\tau) = \dot{\phi}_n(\beta)$, but $u(\phi_n(\beta)+,\beta) \neq u(\phi_n(\beta),\beta)$. As shown above, (11.12.45) holds for $\tau = \beta$ and so

(11.12.49) $$\begin{aligned}\dot{\phi}_n(\tau) = \dot{\phi}_n(\beta) &\geq \rho\frac{a_n - a_{n-1}}{\beta - a_{n-1}} f'(u(\psi(a_n)+,a_n)) \\ &\geq \rho\frac{a_n - a_{n-1}}{\tau - a_{n-1}} f'(u(\psi(a_n)+,a_n)),\end{aligned}$$

which verifies (11.12.45) for τ in that class, as well.

Integrating (11.12.45), we find that, for any $\tau \in (a_{n-1},a_n)$,

(11.12.50) $$\phi_n(a_n) - \phi_n(\tau) \geq \rho(a_n - a_{n-1})\log\,\frac{a_n - a_{n-1}}{\tau - a_{n-1}} f'(u(\psi(a_n)+,a_n)).$$

At the same time we have

(11.12.51) $$\psi(\tau) - \psi(a_n) = -(a_n - \tau)f'(u(\psi(a_n)+,a_n)).$$

We add (11.12.50) and (11.12.51), recalling that $\phi_n(a_n) = \psi(a_n)$. We also note that $\psi(\tau) - \phi_n(\tau) < \ell$, since otherwise the maximal backward characteristic emanating from $(\phi(a_n) - \ell, a_n)$ would intersect $\phi_n(\cdot)$ on the interval (a_{n-1},a_n). Thus, upon setting $\tau = a_{n-1} + \rho(a_n - a_{n-1})$, we conclude

(11.12.52) $$-(a_n - a_{n-1})f'(u(\psi(a_n)+,a_n)) \leq (1 - \rho - \rho\log\rho)^{-1}\ell.$$

By virtue of (11.12.12), (11.12.25) and (11.12.43),

(11.12.53) $$\rho^{k+1-n} f'(u(\psi(a_n)+,a_n)) \leq f'(u(y+,t)),$$

so (11.12.52) yields

(11.12.54) $$-(a_n - a_{n-1})f'(u(y+,t)) \leq \rho^{k+1-n}(1 - \rho + \rho\log\rho)^{-1}\ell.$$

By summing (11.12.54) over $n = 1\ldots,k+1$, we arrive at (11.12.44), with constant $\hat{k} = [(1-\rho)(1-\rho+\rho\log\rho)]^{-1}$.

We now proceed to the proof of (11.12.42). We fix (x,t) with $-\infty < x < \infty$ and $t > 0$. Since $u(\cdot,t)$ has zero mean, there is $y \in [x-\ell, x)$ with the property $u(y,t) \leq 0$, $u(y+,t) \geq 0$. Let $\chi(\cdot)$ and $\psi(\cdot)$ denote the maximal backward characteristics emanating from (x,t) and (y,t). For $\tau \in [t/2,t)$, $\dot{\chi}(\tau) \leq f'(u(x,t))$. Furthermore, using (11.12.44), $-\tau\dot{\psi}(\tau) \leq 2\hat{k}\ell$, for $\tau \in [t/2,t)$. Hence

$$\chi\left(\frac{t}{2}\right) \geq x - \frac{t}{2} f'(u(x,t)), \tag{11.12.55}$$

$$\psi\left(\frac{t}{2}\right) \leq y + \hat{k}\ell. \tag{11.12.56}$$

Subtracting (11.12.56) from (11.12.55) and using that $x > y$ and $\chi(t/2) - \psi(t/2) \leq \ell$, we arrive at (11.12.42), with $k = 2(\hat{k}+1)$. This completes the proof.

11.13 Entropy Production

In view of the central role of entropy dissipation in the theory of hyperbolic conservation laws, it is important to determine the location of entropy sinks. We have already seen that in the realm of BV solutions entropy is solely produced on the shock set: the measure (4.5.1) vanishes on the set of points of approximate continuity. Nevertheless, there are indications that entropy concentration is tied to the BV structure of the characteristic speed of the solution rather than of the solution itself. Indeed, it has been shown (references in Section 11.14) that in the case of scalar conservation laws in one spatial dimension entropy may only be produced at points of discontinuity of L^∞ solutions. To convey a taste in a simple setting, we show here that no entropy may be produced by continuous, not necessarily BV, solutions.

By Theorem 11.3.6, any continuous solution of a genuinely nonlinear scalar conservation law has bounded variation. Nevertheless, continuous solutions with unbounded variation exist when the flux possesses inflection points. This is also the case in the setting of certain balance laws with convex flux, where the source induces continuity, without bounded variation, on solutions (references in Section 11.14).

We thus consider the conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \tag{11.13.1}$$

with f an arbitrary C^2 function. We let $\mathscr{U}_-$, $\mathscr{U}_+$ and $\mathscr{U}_0$ denote the (possibly empty) sets of u with $f''(u) < 0$, $f''(u) > 0$ and $f''(u) = 0$. We will prove the following

11.13.1 Theorem. *Assume u is a continuous function on some open set $\mathscr{O}$ of $\mathbb{R}^2$ that satisfies* (11.13.1) *in the sense of distributions. Then*

$$\partial_t \eta(u(x,t)) + \partial_x q(u(x,t)) = 0 \tag{11.13.2}$$

holds, in the sense of distributions, for every entropy-entropy flux pair (η, q). In particular, any continuous solution and its time reversal satisfy the admissibility condition 6.2.1.

The assertion (11.13.2) is localized and thus, in order to simplify the presentation, we may assume, without loss of generality, that u is defined and continuous on some strip $(-\infty,\infty)\times[0,s]$. The proof of the above proposition will employ the theory of generalized characteristics. Characteristics associated with the continuous solution u of (11.13.1) are classical solutions of the ordinary differential equation $\frac{dx}{dt}=f'(u(x,t))$. In the absence of Lipschitz continuity, the standard theory of ordinary differential equations guarantees existence but not uniqueness of solutions. Nevertheless, we here have

11.13.2 Lemma. *Let $\xi(\cdot)$ be any characteristic associated with the continuous solution u of* (11.13.1), *defined on $[0,s]$. Then $u(\xi(t),t)$ is constant on $[0,s]$, and thus the graph of $\xi(\cdot)$ is a straight line segment. In particular, a unique characteristic passes through any point of the strip $(-\infty,\infty)\times[0,s]$.*

Proof. We set $u(\xi(s),s)=\bar{u}$. When $\bar{u}\in\mathscr{U}_+$, we fix $r\in[0,s)$ and $\varepsilon>0$, with $s-r$ and ε so small that $u(x,t)\in\mathscr{U}_+$ on $\{(x,t):t\in[r,s],\,\xi(t)-\varepsilon\le x\le\xi(t)+\varepsilon\}$. We then apply the conservation law to the domains $\{(x,t):r\le t\le s,\,\xi(t)-\varepsilon\le x\le\xi(t)\}$ and $\{(x,t):r\le t\le s,\,\xi(t)\le x\le\xi(t)+\varepsilon\}$ which yields (11.1.7) and (11.1.8). The right-hand sides of both (11.1.7) and (11.1.8) are nonnegative and thus, upon dividing by ε and letting $\varepsilon\to 0$, we deduce $u(\xi(r),r)=u(\xi(s),s)=\bar{u}$. A straightforward continuation argument then yields $u(\xi(t),t)=\bar{u}$, for all $t\in[0,s]$.

By the same argument, one reaches the same conclusion: $u(\xi(t),t)=\bar{u}$, $t\in[0,s]$, when $\bar{u}$ belongs to $\mathscr{U}_-$, in which case the right-hand sides of (11.1.7) and (11.1.8) are nonpositive, or when $\bar{u}$ belongs to the interior of $\mathscr{U}_0$, in which case the right-hand sides of (11.1.7) and (11.1.8) vanish. Finally, since the boundary points of $\mathscr{U}_0$ are not interconnected, the assertion $u(\xi(t),t)=\bar{u}$, for $t\in[0,s]$, still holds when $\bar{u}$ lies on the boundary of $\mathscr{U}_0$. This completes the proof.

Since the graphs of characteristics are nonintersecting straight line segments, we deduce

11.13.3 Corollary. *For any $-\infty<x<y<\infty$ and $0<t<s$,*

$$-\frac{1}{s-t}\le\frac{f'(u(y,t))-f'(u(x,t))}{y-x}\le\frac{1}{t}\,. \tag{11.13.3}$$

The next step is to establish entropy conservation for a particular family of domains. We fix any smooth entropy-entropy flux pair (η,q), with $q'(u)=\eta'(u)f'(u)$, and state the following

11.13.4 Lemma. *Let us fix $-\infty<a<b<\infty$ and $0<t<\bar{t}<s$, and set $u_a=u(a,t)$, $u_b=u(b,t)$, $\bar{a}=a+(\bar{t}-t)f'(u_a)$, $\bar{b}=b+(\bar{t}-t)f'(u_b)$. Then*

$$(11.13.4)\qquad \int_{\bar a}^{\bar b} \eta(u(x,\bar t))dx - \int_a^b \eta(u(x,t))dx \\ +(\bar t - t)[q(u_b) - f'(u_b)\eta(u_b) - q(u_a) + f'(u_a)\eta(u_a)] = 0.$$

Proof. Fix ε positive small and consider any finite partition of the interval $[a,b]$ by points $a = a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le a_n = b$ satisfying the following conditions. Whenever $a_i < b_i$, the open interval (a_i, b_i) is contained in either $\mathscr{U}_+$ or $\mathscr{U}_-$. Moreover,

$$(11.13.5)\qquad \mu\left(\mathscr{U}_0^c \bigcap \bigcup_{i=1}^{n-1} [b_i, a_{i+1}]\right) < \varepsilon.$$

For $i = 1, \dots, n$, we set $u_i = u(a_i, t)$, $v_i = u(b_i, t)$, and then $\bar a_i = a_i + (\bar t - t) f'(u_i)$, $\bar b_i = b_i + (\bar t - t) f'(v_i)$. We note the identities

$$(11.13.6)\qquad \int_{u_i}^{v_i} f''(w)\eta(w)dw = q(u_i) - f'(u_i)\eta(u_i) - q(v_i) + f'(v_i)\eta(v_i),$$

$$(11.13.7)\qquad \int_{v_i}^{u_{i+1}} f''(w)\eta(w)dw = q(v_i) - f'(v_i)\eta(v_i) - q(u_{i+1}) + f'(u_{i+1})\eta(u_{i+1}).$$

By the mean value theorem,

$$(11.13.8)\qquad \int_{u_i}^{v_i} f''(w)\eta(w)dw = \eta(w_i)[f'(v_i) - f'(u_i)] \\ = \frac{1}{\bar t - t}\eta(w_i)[(\bar b_i - \bar a_i) - (b_i - a_i)],$$

for some w_i lying between u_i and v_i. Furthermore, setting $w_i^* = \frac{1}{2}(v_i + u_{i+1})$,

$$(11.13.9)\qquad \eta(w_i^*)[f'(u_{i+1}) - f'(v_i)] = \frac{1}{\bar t - t}\eta(w_i^*)[(\bar a_{i+1} - \bar v_i) - (a_{i+1} - v_i)].$$

On account of (11.13.5) and (11.13.3),

$$(11.13.10)\qquad \sum_{i=1}^{n-1} TV_{[b_i, a_{i+1}]} f'(u(\cdot, t)) \le c\varepsilon.$$

In particular,

$$(11.13.11)\qquad \sum_{i=1}^{n-1} |f'(u_{i+1}) - f'(v_i)| \le c\varepsilon.$$

Furthermore, from the Stieltjes integral

$$\int_{v_i}^{u_{i+1}} f''(w)\eta(w)dw = \int_{b_i}^{a_{i+1}} \eta(u(\cdot,t))df'(u(\cdot,t)) \tag{11.13.12}$$

and (11.13.7), (11.13.10) implies

$$\sum_{i=1}^{n-1} |q(v_i) - f'(v_i)\eta(v_i) - q(u_{i+1}) + f'(u_{i+1})\eta(u_{i+1})| \leq c\varepsilon. \tag{11.13.13}$$

We now combine (11.13.8), (11.13.9), (11.13.11) and (11.13.13) to get

$$\begin{aligned} &|\sum_{i=1}^{n} \eta(w_i)(\bar{b}_i - \bar{a}_i) + \sum_{i=1}^{n-1} \eta(w_i^*)(\bar{a}_{i+1}) - \bar{b}_i) \\ &\quad - \sum_{i=1}^{n} \eta(w_i)(b_i - a_i) - \sum_{i=1}^{n-1} \eta(w_i^*)(a_{i+1} - b_i) \\ &\quad + (\bar{t} - t)[q(u_b) - f'(u_b)\eta(u_b) - q(u_a) + f'(u_a)\eta(u_a)| \leq c\varepsilon, \end{aligned} \tag{11.13.14}$$

where c does not depend on the chosen partition of the interval $[a,b]$ or on ε. As the partition of $[a,b]$ gets finer, the first two sums on the left-hand side of (11.13.14) converge to the integral of $\eta(u(\cdot,\bar{t}))$ over $[\bar{a},\bar{b}]$ while the next two sums converge to the integral of $\eta(u(\cdot,t))$ over $[a,b]$. Since ε is arbitrarily small, we arrive at (11.13.4). This completes the proof.

Next we fix some $\tau \in (0,s)$ and consider the coordinate change $(x,t) \mapsto (z,t)$ defined by

$$z = x - (t - \tau)f'(u(x,\tau)). \tag{11.13.15}$$

By virtue of Lemma 11.13.2 and Corollary 11.13.3, the above transformation is a bilipschitz homeomorphism with inverse

$$x = z + (t - \tau)f'(u(z,\tau)). \tag{11.13.16}$$

Notice that in the new coordinate system characteristics reduce to $z =$ constant and u becomes a function of z alone: $u(x(z,t),t) = \hat{u}(z)$. Thus it is natural to interpret (x,t) and (z,t), respectively, as "Eulerian" and "Lagrangian" representations of the flow generated by (11.13.1).

Proof of Theorem 11.13.1. We set

$$\theta(z,t) = \eta(\hat{u}(z))\partial_z x(z,t), \tag{11.13.17}$$

$$p(z) = q(\hat{u}(z)) - f'(\hat{u}(z))\eta(\hat{u}(z)), \tag{11.13.18}$$

and notice that (11.13.4) implies

$$\int_{\alpha}^{\beta}[\theta(z,\bar{t})-\theta(z,t)]dz+(\bar{t}-t)[p(\beta)-p(\alpha)]=0, \tag{11.13.19}$$

for all $-\infty<\alpha<\beta<\infty$ and $0<t<\bar{t}<s$. Thus p is Lipschitz on $(-\infty,\infty)$ and

$$\frac{\theta(z,\bar{t})-\theta(z,t)}{\bar{t}-t}+\frac{d}{dz}p(z)=0 \tag{11.13.20}$$

holds for almost all z, as well as in the sense of distributions, for any $0<t<\bar{t}<s$. It then follows that

$$\partial_t\theta(z,t)+\partial_z p(z)=0, \tag{11.13.21}$$

in the sense of distributions on $(-\infty,\infty)\times(0,s)$.

Fix any C^∞ test function $\phi(x,t)$ with compact support on $(-\infty,\infty)\times(0,s)$ and set $\psi(z,t)=\phi(x(z,t),t)$. Thus, ψ is a Lipschitz function with compact support on $(-\infty,\infty)\times(0,s)$ and derivatives $\partial_z\psi=\partial_x\phi\partial_z x$, $\partial_t\psi=\partial_t\phi+\partial_x\phi\partial_t x$. Considering that $\partial_t x=f'(u)$ and upon recalling (11.13.17) and (11.13.18),

$$\theta\partial_t\psi+p\partial_z\psi=\partial_z x[\eta(u)\partial_t\phi+q(u)\partial_x\phi]. \tag{11.13.22}$$

Therefore, by (11.13.21) and since $\partial_x z=[\partial_z x]^{-1}$,

$$\int_0^s\int_{-\infty}^{\infty}[\eta(u)\partial_t\phi+q(u)\partial_x\phi]dxdt=\int_0^s\int_{-\infty}^{\infty}[\theta\partial_t\psi+p\partial_z\psi]dzdt=0, \tag{11.13.23}$$

which establishes (11.13.2). This completes the proof.

11.14 Notes

There is voluminous literature on the scalar conservation law in one space dimension, especially the genuinely nonlinear case, beginning with the seminal paper of Hopf [1], on the Burgers equation, already cited in earlier chapters.

In the 1950's, the qualitative theory was developed by the Russian school, headed by Oleinik [1,2,4], based on the vanishing viscosity approach as well as on the Lax-Friedrichs finite difference scheme (Lax [1]). It is in that context that Theorem 11.2.2 was originally established. The reader may find an exposition in the text by Smoller [3]. The culmination of that approach was the development of the theory of scalar conservation laws in several space dimensions, discussed in Chapter VI.

In a different direction, Lax [2] discovered the explicit representation (11.4.10) for solutions to the Cauchy problem and employed it to establish the existence of invariants (Theorem 11.4.2), the development of N-waves under initial data of compact support (Theorem 11.6.1), and the formation of sawtooth profiles under periodic initial data (Theorem 11.7.3). One may thus base the entire theory of the homogeneous, genuinely nonlinear scalar conservation law on Lax's formula. Oleinik [1], derives a generalization of the above formula that applies to inhomogeneous, genuinely nonlinear scalar conservation laws. As noted in Section 11.4, these formulas actually express properties of the Hamilton-Jacobi equation (see Lions [1]) obtained by integration of the conservation law, and thus cannot be extended to (even) genuinely nonlinear balance laws, or to conservation laws that are not genuinely nonlinear. By contrast, the method of generalized characteristics, employed here, applies (though not with equal ease) to all scalar balance laws, regardless of whether they are homogeneous and genuinely nonlinear.

The treatment of the homogeneous, genuinely nonlinear scalar conservation law in this chapter follows Dafermos [7], which discusses the more general situation where $f_{uu} \geq 0$, and one-sided limits $u(x\pm,t)$ exist for all $x \in (-\infty,\infty)$ and almost all $t \in (0,\infty)$, even though $u(\cdot,t)$ may not be a function of bounded variation. However, many of the results reported here had been established, by different means, before the theory of generalized characteristics emerged.

Theorem 11.3.5 is due to Ambrosio and De Lellis [2]. For an alternative proof that also applies to the case of genuinely nonlinear systems of hyperbolic conservation laws, see Bianchini [10], Bianchini and Yu [1], and Bianchini and Caravenna [1]. On the other hand, using the method of generalized characteristics, Robyr [1] establishes the *SBV* property of solutions to, not necessarily genuinely nonlinear, scalar balance laws.

Generic piecewise regularity of solutions (a weaker version of Theorem 11.3.10) was originally established by Schaeffer [1].

For other manifestations of the regularizing effects of genuine nonlinearity, see Golse [1] and Golse and Perthame [1].

The optimal convergence rate to N-waves is established by Yong Jung Kim [2]. The interesting, metastable status of N-waves for the Burgers equation with viscosity is demonstrated in Kim and Tzavaras [1]. Asymptotics in terms of the Wasserstein metric is discussed by Carrillo, DiFrancesco and Lattanzio [1,2].

The property that the lap number of solutions of conservation laws (8.6.2) with viscosity is nonincreasing with time was discovered independently by Nickel [1] and Matano [1]. The L^1 contraction property for piecewise smooth solutions in one space dimension was noted by Quinn [1]. The functional (11.8.11), in alternative, albeit completely equivalent, form was designed by Tai-Ping Liu and Tong Yang [3], who employ it to establish Theorem 11.8.3, for piecewise smooth solutions. For an alternative derivation, see Goatin and LeFloch [1]. For more general functionals, see Hongxia Liu and Tong Yang [1], and Jiang and Yang [1].

The treatment of genuinely nonlinear, homogeneous or inhomogeneous, scalar balance laws, in Sections 11.10 and 11.11, follows the approach of Dafermos [8]. In particular, Section 11.10 improves on an earlier result of Lyberopoulos [1], while the example discussed in Section 11.11 was originally published in this book.The effects of inhomogeneity and source terms on the large-time behavior of solutions are also discussed, by the method of generalized characteristics, in Dafermos [17,33], Lyberopoulos [2], Fan and Hale [1,2], Härterich [1], Ehrt and Härterich [1], Mascia and Sinestrari [1], and Fan, Jin and Teng [1]. Problems of this type are also treated by different methods in Tai-Ping Liu [23], Dias and LeFloch [1], and Sinestrari [1].

The presentation, in Section 11.12, of properties of solutions to scalar conservation laws with non convex flux has been abridged from Dafermos [13].

For the case of scalar conservation laws in one spatial dimension, Bianchini and Marconi [1], and De Lellis and Rivière [1] sharpen the results on the structure of solutions outlined in Section 6.8, showing that L^∞ solutions are continuous, except possibly on a rectifiable set of Hausdorff dimension one, on which entropy production is concentrated. The discussion, in Section11.13, of continuous solutions follows Dafermos [26]. It should be noted that even though solutions that are merely continuous are rarely encountered in conservation laws, they naturally arise in the setting of a class of balance laws with regularizing source. Interesting representatives are the Hunter-Saxton equation and the Camassa-Holm equation. For treatment of these equations in the spirit of the techniques expounded in this chapter, see Dafermos [31], and Bressan, Chen and Zhang [2]. For recent, sophisticated treatment of continuous solutions to balance laws with nonconvex flux, see Alberti, Bianchini and Caravenna [1].

So much is known about the scalar conservation and balance law in one space dimension that it would be hopeless to attempt to provide comprehensive coverage. What follows is just a sample of relevant results.

For a probabilistic interpretation of generalized characteristics, see Rezakhanlou [1]. For an interesting application of the method of generalized characteristics in elastostatics, under incompressibility and inextensibility constraints, see Choksi [1]. Further applications are found in Dafermos [26], and Shearer and Dafermos [1].

An explicit representation of admissible solutions on the quarter-plane, analogous to Lax's formula for the upper half-plane, is presented in LeFloch [1], and LeFloch and Nédélec [1]. An analog of Lax's formula has also been derived for the special systems with coinciding shock and rarefaction wave curves; see Benzoni-Gavage [1].

The analog of (11.2.1) holds for scalar conservation laws (6.1.1), in several space variables, if $g_\alpha(u) = f(u)v_\alpha$, where v is a constant vector (Hoff [1]).

For a contraction property in a transport distance, see Esselborn, Gigli and Otto [1].

For a Chapman-Enskog type regularization of the scalar conservation law, see Schochet and Tadmor [1]. Another regularization that has attracted considerable attention is performed by coupling the scalar conservation law with an elliptic equation, which results in a model system for the so-called radiating gas equations. See Serre [20], and Lattanzio and Marcati [3].

A kinetic formulation, different from the one discussed in Section 6.7, is presented in Brenier and Corrias [1]. See also Vasseur [2].

Panov [1] and, independently, De Lellis, Otto and Westickenberg [2] show that the entropy inequality for just one uniformly convex entropy suffices for singling out the unique admissible weak solution in L^∞.

Regularity of solutions in Besov spaces is established in Lucier [2]. For the rate of convergence of numerical schemes see e.g. Nessyahu and Tadmor [1] and Osher and Tadmor [1].

The connection of the scalar conservation law with the system of "pressureless gas" (7.1.14) and the related model of "sticky particles" is investigated in E, Rykov and Sinai [1], Brenier and Grenier [1], Bouchut and James [1], and Tonon [1]. The interesting theory of the pressureless gas is developed in Wang and Ding [1], Wang, Huang and Ding [1], Huang and Wang [1], Li and Warnecke [1], Ding and Huang [1], Huang [2], and Sever [9,10,11,12]. The theory of generalized characteristics plays a useful role in a number of these papers.

Homogenization effects under random periodic forcing are demonstrated in E [2,3], E and Serre [1], E, Khanin, Mazel and Sinai [1], and Amadori [2].

For stochastic effects, see Holden and Risebro [1], Avelaneda and E [1], Bertoin [1], Jong Uhm Kim [1], Menon and Pego [1], and Chen, Ding and Karlsen [1].

For boundary value problems and boundary control problems associated with the scalar conservation law, see Andreianov and Sbihi [1,2], and Ancona and Marson [1,2].

The case where the flux is discontinuous is discussed in Lyons [1], Risebro [2], Klingenberg and Risebro [1], Klausen and Risebro [1], Garavello, Natalini, Piccoli and Terracina [1], Chen, Even and Klingenberg [1], Diehl [1], Carillo [1], Adimurthi and Gowda [1], Adimurthi, Dutta, Ghoshal and Gowda [1], Adimurthi, Mishra and Gowda [1,2,3], Ostrov [2], Coclite and Risebro [1], Andreianov, Karlsen and Risebro [2], Holden, Karlsen and Mitrovic [1], and Crasta, De Cicco and de Philippis [1].

As we saw in Section 11.12, when f has inflection points the structure of solutions is considerably more intricate, as a result of the formation of contact discontinuities, which become sources of signals propagating into the future. The general structure of solutions has been investigated by Bianchini and Yu [2,3]. See also Dafermos [13], Jenssen and Sinestrari [1], Marson [1], Ballou [1,2], Guckenheimer [1], Cheverry [4], Tang, Wang and Zhao [1], Jinghua Wang [1], Kim and Lee [1], Kim and Kim [1], and Ha and Kim [1]. The large-time behavior is investigated in Dafermos [1,11], Greenberg and Tong [1], Conlon [1], Kuo Shung Cheng [1,2,3], Weinberger [1], Sinestrari [2], Yong Jung Kim [3,4], and Baiti and Jenssen [1]. In particular, for a proof that $f'(u(\cdot,t))$ is of locally bounded variation on $(-\infty,\infty)$, even when the intial data are merely bounded, see Kuo Shung Cheng [3].

In the special case $f(u) = u^m$, the properties of solutions may be studied effectively with the help of the underlying self-similarity transformation; see Bénilan and Crandall [1], and Liu and Pierre [1]. This last paper also considers initial data that are merely measures. For developments in that direction, see Vasseur [6], and Chasseigne [1]. The limit behavior as $m \to \infty$ is discussed in Xiangsheng Xu [1].

For the case of the balance law, the regularizing effect of the source, the time-asymptotic convergence of solutions to a traveling wave, and the relaxation limit of solutions have been established by Mascia [1,2,3], and Mascia and Terracina [1]. See also Isaacson and Temple [5].

Nonhomogeneous scalar conservation laws or balance laws may be reformulated and treated as resonant systems of two conservation laws or balance laws; see Isaacson and Temple [5], and Amadori, Gosse and Guerra [2].

For other, relevant developments, see Glass [2], Hayes and Shearer [2], Tadmor and Tang [1], Colombo and Goatin [1], Holden, Priuli and Risebro [1], Adimurthi, Dutta, Ghoshal and Gowda [2,3,4], Andreianov and Seguin [1], Benzoni-Gavage [7], Coclite and Coclite [1,2], Ancona, Glass and Nguyen [1], Bourdarias, Gisclon and Junca [4], Colombo, Mercier and Rosini [1], Colombo and Rossi [1], Corli and Rohde [1], Bianchini and Modena [1], and Bank and Ben-Artzi [1].

XII

Genuinely Nonlinear Systems of Two Conservation Laws

The theory of solutions of genuinely nonlinear, strictly hyperbolic systems of two conservation laws will be developed in this chapter at a level of precision comparable to that for genuinely nonlinear scalar conservation laws, expounded in Chapter XI. This will be achieved by exploiting the presence of coordinate systems of Riemann invariants and the induced rich family of entropy-entropy flux pairs. The principal tools in the investigation will be generalized characteristics and entropy estimates.

The analysis will reveal a close similarity in the structure of solutions of scalar conservation laws and pairs of conservation laws. Thus, as in the scalar case, jump discontinuities are generally generated by the collision of shocks and/or the focusing of compression waves, and are then resolved into wave fans approximated locally by the solution of associated Riemann problems.

The total variation of the trace of solutions along space-like curves is controlled by the total variation of the initial data, and spreading of rarefaction waves affects total variation, as in the scalar case.

The dissipative mechanisms encountered in the scalar case are at work here as well, and have similar effects on the large-time behavior of solutions. Entropy dissipation induces $O(t^{-1/2})$ decay of solutions with initial data in $L^1(-\infty,\infty)$. When the initial data have compact support, the two characteristic families asymptotically decouple; the characteristics spread and form a single N-wave profile for each family. Finally, as in the scalar case, confinement of characteristics under periodic initial data induces $O(t^{-1})$ decay in the total variation per period and formation of sawtoothed profiles, one for each characteristic family.

12.1 Notation and Assumptions

We consider a genuinely nonlinear, strictly hyperbolic system of two conservation laws,

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \tag{12.1.1}$$

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on some disk $\mathscr{O}$ centered at the origin. The eigenvalues of DF (characteristic speeds) will here be denoted by λ and μ, with $\lambda(U) < 0 < \mu(U)$ for $U \in \mathscr{O}$, and the associated eigenvectors will be denoted by R and S.

The system is endowed with a coordinate system (z,w) of Riemann invariants, vanishing at the origin $U = 0$, and normalized according to (7.3.8):

$$\mathrm{D}zR = 1, \quad \mathrm{D}zS = 0, \quad \mathrm{D}wR = 0, \quad \mathrm{D}wS = 1. \tag{12.1.2}$$

The condition of genuine nonlinearity is now expressed by (7.5.4), which here reads

$$\lambda_z < 0, \qquad \mu_w > 0. \tag{12.1.3}$$

The direction in the inequalities (12.1.3) has been selected so that z increases across admissible weak 1-shocks while w decreases across admissible weak 2-shocks.

For definiteness, we will consider systems with the property that the interaction of any two shocks of the same characteristic family produces a shock of the same family and a rarefaction wave of the opposite family. Note that this condition is here expressed by

$$S^\top \mathrm{D}^2 z S > 0, \qquad R^\top \mathrm{D}^2 w R > 0. \tag{12.1.4}$$

Indeed, in conjunction with (8.2.24), (12.1.3) and Theorem 8.3.1, the inequalities (12.1.4) imply that z increases across admissible weak 2-shocks while w decreases across admissible weak 1-shocks. Therefore, the admissible shock and rarefaction wave curves emanating from the state $(\bar{z},\bar{w})$ have the shape depicted in Fig. 12.1.1. Consequently, as seen in Fig. 12.1.2(a), a 2-shock that joins the state (z_ℓ,w_ℓ), on the left, with the state (z_m,w_m), on the right, interacts with a 2-shock that joins (z_m,w_m), on the left, with the state (z_r,w_r), on the right, to produce a 1-rarefaction wave, joining (z_ℓ,w_ℓ), on the left, with a state (z_0,w_ℓ), on the right, and a 2-shock joining (z_0,w_ℓ), on the left, with (z_r,w_r), on the right, as depicted in Fig. 12.1.2(b). Similarly, the interaction of two 1-shocks produces a 1-shock and a 2-rarefaction wave.

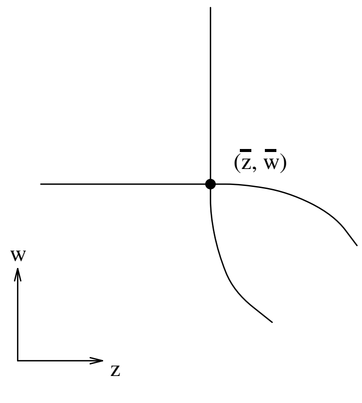

Fig. 12.1.1

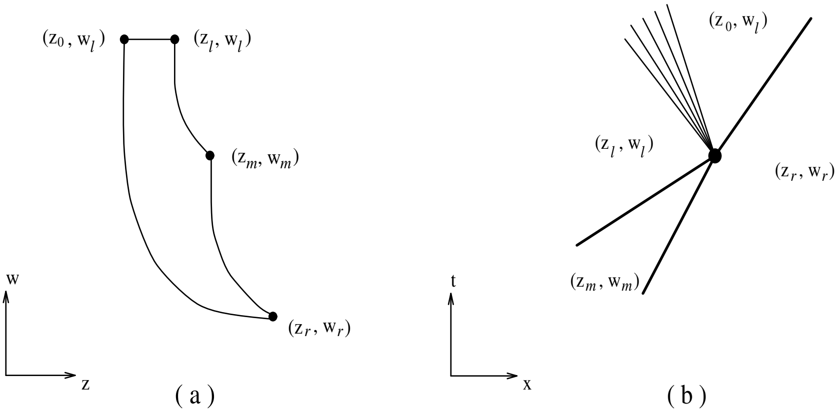

Fig. 12.1.2 (a,b)

Also for definiteness, we assume

$$\lambda_w < 0, \qquad \mu_z > 0, \tag{12.1.5}$$

or equivalently, by virtue of (7.3.14) and (7.4.15),

$$R^\top \mathrm{D}^2 z S > 0, \qquad S^\top \mathrm{D}^2 w R > 0. \tag{12.1.6}$$

The prototypical example is the system (7.1.11) of isentropic thermoelasticity, which satisfies all three assumptions (12.1.3), (12.1.4) and (12.1.6), with Riemann invariants (7.3.2), provided $\sigma''(u) < 0$, i.e., the elastic medium is a soft spring or a gas. When the medium is a hard spring, i.e., $\sigma''(u) > 0$, the sign of the Riemann invariants in (7.3.2) has to be reversed.

12.2 Entropy-Entropy Flux Pairs and the Hodograph Transformation

As explained in Section 7.4, our system is endowed with a rich family of entropy-entropy flux pairs (η, q), which may be determined as functions of the Riemann invariants (z, w) by solving the system (7.4.12), namely

$$q_z = \lambda \eta_z \,, \qquad q_w = \mu \eta_w \,. \tag{12.2.1}$$

The integrability condition (7.4.13) now takes the form

$$\eta_{zw} + \frac{\lambda_w}{\lambda - \mu}\eta_z + \frac{\mu_z}{\mu - \lambda}\eta_w = 0. \tag{12.2.2}$$

The entropy $\eta(z, w)$ will be a convex function of the original state variable U when the inequalities (7.4.16) hold, that is,

$$\begin{cases} \eta_{zz} + (R^\top \mathrm{D}^2 z R)\eta_z + (R^\top \mathrm{D}^2 w R)\eta_w \geq 0 \\ \eta_{ww} + (S^\top \mathrm{D}^2 z S)\eta_z + (S^\top \mathrm{D}^2 w S)\eta_w \geq 0 \,. \end{cases} \tag{12.2.3}$$

In the course of our investigation, we shall face the need to construct entropy-entropy flux pairs with prescribed specifications, by solving (12.2.1) or (12.2.2) under assigned side conditions. To verify that the constructed entropy satisfies the condition (12.2.3), for convexity, it usually becomes necessary to estimate the second derivatives η_{zz} and η_{ww} in terms of the first derivatives η_z and η_w. For that purpose, one may employ the equations obtained by differentiating (12.2.2) with respect to z and w:

(12.2.4)
$$\begin{cases} \eta_{zzw} + \dfrac{\lambda_w}{\lambda - \mu}\eta_{zz} = \dfrac{(\mu-\lambda)\lambda_{zw} + \lambda_z\lambda_w - 2\lambda_w\mu_z}{(\lambda-\mu)^2}\eta_z + \dfrac{(\lambda-\mu)\mu_{zz} - \lambda_z\mu_z + 2\mu_z^2}{(\lambda-\mu)^2}\eta_w \\ \eta_{wwz} + \dfrac{\mu_z}{\mu-\lambda}\eta_{ww} = \dfrac{(\mu-\lambda)\lambda_{ww} - \lambda_w\mu_w + 2\lambda_w^2}{(\mu-\lambda)^2}\eta_z + \dfrac{(\lambda-\mu)\mu_{zw} + \mu_z\mu_w - 2\lambda_w\mu_z}{(\mu-\lambda)^2}\eta_w \,. \end{cases}$$

As an illustration, we consider the important family of *Lax entropy-entropy flux pairs*

$$\begin{cases} \eta(z,w) = e^{kz}\left[\phi(z,w) + \dfrac{1}{k}\chi(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \\ q(z,w) = e^{kz}\lambda(z,w)\left[\psi(z,w) + \dfrac{1}{k}\theta(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \end{cases} \tag{12.2.5}$$

$$\begin{cases} \eta(z,w) = e^{kw}\left[\alpha(z,w) + \dfrac{1}{k}\beta(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \\ q(z,w) = e^{kw}\mu(z,w)\left[\gamma(z,w) + \dfrac{1}{k}\delta(z,w) + O\left(\dfrac{1}{k^2}\right)\right], \end{cases} \tag{12.2.6}$$

where k is a parameter. These are designed to vary stiffly with one of the two Riemann invariants so as to be employed for decoupling the two characteristic families. To construct them, one substitutes η and q from (12.2.5) or (12.2.6) into the system (12.2.1), thus deriving recurrence relations for the coefficients, and then shows that the remainder is $O(k^{-2})$. The recurrence relations for the coefficients of the family (12.2.5), read as follows:

$$\psi = \phi, \tag{12.2.7}$$

$$\lambda\theta + (\lambda\psi)_z = \lambda\chi + \lambda\phi_z\,, \tag{12.2.8}$$

(12.2.9)
$$(\lambda \psi)_w = \mu \phi_w .$$

Combining (12.2.7) with (12.2.9) yields

(12.2.10)
$$(\mu - \lambda)\phi_w = \lambda_w \phi ,$$

which may be satisfied by selecting

(12.2.11)
$$\phi(z,w) = \exp \int_0^w \frac{\lambda_w(z,\omega)}{\mu(z,\omega) - \lambda(z,\omega)} d\omega .$$

In particular, this ϕ is positive, uniformly bounded away from zero on compact sets. Hence, for k sufficiently large, the inequalities (12.2.3) will hold, the second one by virtue of (12.1.4). Consequently, for k large the Lax entropy is a convex function of U.

Important implications of (12.2.7) and (12.2.8) are the estimates

(12.2.12)
$$q - \lambda\eta = \frac{1}{k} e^{kz} \left[-\lambda_z \phi + O\left(\frac{1}{k}\right) \right] ,$$

(12.2.13)
$$q - (\lambda + \varepsilon)\eta = -e^{kz} \left[\varepsilon\phi + O\left(\frac{1}{k}\right) \right] ,$$

whose usefulness will become clear later.

There is a curious formal analogy between maps $(z,w) \mapsto (\eta,q)$, which carry pairs of Riemann invariants into entropy-entropy flux pairs, and *hodograph transformations* $(z,w) \mapsto (x,t)$, constructed by the following procedure: Suppose that $z(x,t)$ and $w(x,t)$ are the Riemann invariants of a C^1 solution of (12.1.1), on some domain $\mathscr{D}$ of the x-t plane. In the vicinity of any point of $\mathscr{D}$ where the Jacobian determinant $J = z_x w_t - w_x z_t$ does not vanish, the map $(x,t) \mapsto (z,w)$ admits a C^1 inverse $(z,w) \mapsto (x,t)$; with partial derivatives $x_z = J^{-1} w_t$, $t_z = -J^{-1} w_x$, $x_w = -J^{-1} z_t$, and $t_w = J^{-1} z_x$. Since $z_t + \lambda z_x = 0$ and $w_t + \mu w_x = 0$ on $\mathscr{D}$, we deduce $J = (\lambda - \mu) z_x w_x$ and

(12.2.14)
$$x_z = \mu t_z , \quad x_w = \lambda t_w ,$$

which should be compared and contrasted to (12.2.1). Elimination of x between the two equations in (12.2.14) yields

(12.2.15)
$$t_{zw} + \frac{\mu_w}{\mu - \lambda} t_z + \frac{\lambda_z}{\lambda - \mu} t_w = 0,$$

namely the analog of (12.2.2)[1]. One may thus construct (classical) solutions of the nonlinear system (12.1.1) of two conservation laws by solving the linear system (12.2.14), or equivalently the linear second order hyperbolic equation (12.2.15). Numerous important special solutions of the system of isentropic gas dynamics, and other systems of two conservation laws arising in mathematical physics, have been derived through that process.

[1] In the system (7.1.11), (12.2.2) and (12.2.15) coincide, as $\lambda = -\mu$.

12.3 Local Structure of Solutions

Throughout this chapter, U will denote a function of locally bounded variation, defined on $(-\infty,\infty)\times[0,\infty)$ and taking values in a disk of small radius, centered at the origin, which is a weak solution of (12.1.1) satisfying the Lax E-condition, in the sense described in Section 10.1. In particular,

$$\partial_t\eta(U(x,t))+\partial_x q(U(x,t))\le 0 \tag{12.3.1}$$

will hold, in the sense of measures, for any entropy-entropy flux pair (η,q), with η convex.

The notion of generalized characteristic, developed in Chapter X, will play a pivotal role in the discussion.

12.3.1 Definition. A Lipschitz curve, with graph $\mathscr{A}$ embedded in the upper half-plane, is called *space-like* relative to U when every point $(\bar{x},\bar{t})\in\mathscr{A}$ has the following property: The set $\{(x,t): 0\le t<\bar{t},\ \zeta(t)<x<\xi(t)\}$ of points confined between the graphs of the maximal backward 2-characteristic $\zeta(\cdot)$ and the minimal backward 1-characteristic $\xi(\cdot)$, emanating from $(\bar{x},\bar{t})$, has empty intersection with $\mathscr{A}$.

Clearly, any generalized characteristic, of either family, associated with U, is space-like relative to U. Similarly, all time lines, $t=$ constant, are space-like.

The solution U will be conveniently monitored through its induced Riemann invariant coordinates (z,w). In Section 12.5, it is shown that the total variation of the trace of z and w along space-like curves is controlled by the total variation of their initial data. In anticipation of that result, we shall be assuming henceforth that, for any space-like curve $t=t^*(x)$, $z(x\pm,t^*(x))$ and $w(x\pm,t^*(x))$ are functions of bounded variation, with total variation bounded by a positive constant θ. Since the oscillation of the solution is small and all arguments will be local, we may assume without further loss of generality that θ is small.

In order to describe the local structure of the solution, we associate with the generic point $(\bar{x},\bar{t})$ of the upper half-plane eight, not necessarily distinct, curves (see Fig. 12.3.1) determined as follows:

For $t<\bar{t}$: $\xi_-(\cdot)$ and $\xi_+(\cdot)$ are the minimal and the maximal backward 1-characteristics emanating from $(\bar{x},\bar{t})$; similarly, $\zeta_-(\cdot)$ and $\zeta_+(\cdot)$ are the minimal and the maximal backward 2-characteristics emanating from $(\bar{x},\bar{t})$.

For $t>\bar{t}$: $\phi_+(\cdot)$ is the maximal forward 1-characteristic and $\psi_-(\cdot)$ is the minimal forward 2-characteristic issuing from $(\bar{x},\bar{t})$. To determine the remaining two curves $\phi_-(\cdot)$ and $\psi_+(\cdot)$, we consider the minimal backward 1-characteristic $\xi(\cdot)$ and the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from the generic point (x,t) and locate the points $\xi(\bar{t})$ and $\zeta(\bar{t})$ where these characteristics are intercepted by the $\bar{t}$-time line. Then $\phi_-(t)$ is determined by the property that $\xi(\bar{t})<\bar{x}$ when $x<\phi_-(t)$ and $\xi(\bar{t})\ge\bar{x}$ when $x>\phi_-(t)$. Similarly, $\psi_+(t)$ is characterized by the property that $\zeta(\bar{t})\le\bar{x}$ when $x<\psi_+(t)$ and $\zeta(\bar{t})>\bar{x}$ when $x>\psi_+(t)$. In particular, $\phi_-(t)\le\phi_+(t)$ and if $\phi_-(t)<x<\phi_+(t)$ then $\xi(\bar{t})=\bar{x}$. Similarly, we infer that $\psi_-(t)\le\psi_+(t)$ and $\psi_-(t)<x<\psi_+(t)$ implies $\zeta(\bar{t})=\bar{x}$.

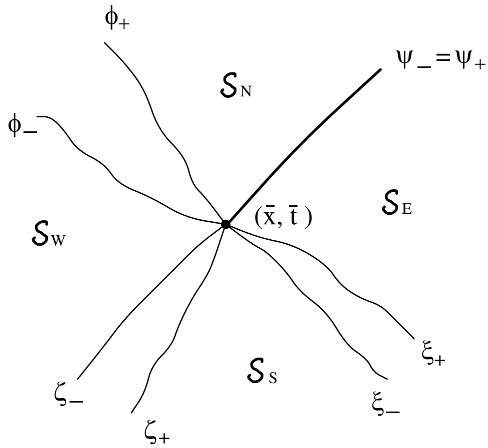

Fig. 12.3.1

We fix $\tau > \bar{t}$ and let $\xi_\tau(\cdot)$ denote the minimal backward 1-characteristic emanating from the point $(\phi_-(\tau),\tau)$. We also consider any sequence $\{x_m\}$ converging from above to $\phi_-(\tau)$ and let $\xi_m(\cdot)$ denote the minimal backward 1-characteristic emanating from (x_m,τ). Then the sequence $\{\xi_m(\cdot)\}$, or some subsequence thereof, will converge to some backward 1-characteristic $\hat{\xi}_\tau(\cdot)$ emanating from $(\phi_-(\tau),\tau)$. Moreover, for any $\bar{t} \le t \le \tau$, it is $\xi_\tau(t) \le \phi_-(t) \le \hat{\xi}_\tau(t)$. In particular, this implies that $\phi_-(\cdot)$ is a Lipschitz continuous space-like curve, with slope in the range of λ. Similarly, $\psi_+(\cdot)$ is a Lipschitz continuous space-like curve, with slope in the range of μ.

Referring again to Fig. 12.3.1, we see that the aforementioned curves border regions:

(12.3.2) $$\mathscr{S}_W = \{(x,t) : x < \bar{x},\ \zeta_-^{-1}(x) < t < \phi_-^{-1}(x)\},$$

(12.3.3) $$\mathscr{S}_E = \{(x,t) : x > \bar{x},\ \xi_+^{-1}(x) < t < \psi_+^{-1}(x)\},$$

(12.3.4) $$\mathscr{S}_N = \{(x,t) : t > \bar{t},\ \phi_+(t) < x < \psi_-(t)\},$$

(12.3.5) $$\mathscr{S}_S = \{(x,t) : t < \bar{t},\ \zeta_+(t) < x < \xi_-(t)\}.$$

12.3.2 Definition. The solution is called *locally regular* at the point $(\bar{x},\bar{t})$ of the upper half-plane when the following hold:

(a) As (x,t) tends to $(\bar{x},\bar{t})$ through any one of the regions $\mathscr{S}_W, \mathscr{S}_E, \mathscr{S}_N$, or $\mathscr{S}_S$, $(z(x\pm,t), w(x\pm,t))$ tend to respective limits $(z_W, w_W), (z_E, w_E), (z_N, w_N)$, or (z_S, w_S), where, in particular, it is $z_W = z(\bar{x}-,\bar{t})$, $w_W = w(\bar{x}-,\bar{t})$, $z_E = z(\bar{x}+,\bar{t})$, $w_E = w(\bar{x}+,\bar{t})$.

(b)$_1$ If $p_\ell(\cdot)$ and $p_r(\cdot)$ are any two backward 1-characteristics emanating from $(\bar{x},\bar{t})$, with $\xi_-(t) \le p_\ell(t) < p_r(t) \le \xi_+(t)$, for $t < \bar{t}$, then

$$z_S = \lim_{t\uparrow\bar{t}} z(\xi_-(t)\pm,t) \le \lim_{t\uparrow\bar{t}} z(p_\ell(t)-,t) \le \lim_{t\uparrow\bar{t}} z(p_\ell(t)+,t) \le \lim_{t\uparrow\bar{t}} z(p_r(t)-,t) \le \lim_{t\uparrow\bar{t}} z(p_r(t)+,t) \le \lim_{t\uparrow\bar{t}} z(\xi_+(t)\pm,t) = z_E\,, \tag{12.3.6$_1$}$$

$$w_S = \lim_{t\uparrow\bar{t}} w(\xi_-(t)\pm,t) \ge \lim_{t\uparrow\bar{t}} w(p_\ell(t)-,t) \ge \lim_{t\uparrow\bar{t}} w(p_\ell(t)+,t) \ge \lim_{t\uparrow\bar{t}} w(p_r(t)-,t) \ge \lim_{t\uparrow\bar{t}} w(p_r(t)+,t) \ge \lim_{t\uparrow\bar{t}} w(\xi_+(t)\pm,t) = w_E\,. \tag{12.3.7$_1$}$$

(b)$_2$ If $q_\ell(\cdot)$ and $q_r(\cdot)$ are any two backward 2-characteristics emanating from $(\bar{x},\bar{t})$, with $\zeta_-(t) \le q_\ell(t) < q_r(t) \le \zeta_+(t)$, for $t < \bar{t}$, then

$$w_W = \lim_{t\uparrow\bar{t}} w(\zeta_-(t)\pm,t) \ge \lim_{t\uparrow\bar{t}} w(q_\ell(t)-,t) \ge \lim_{t\uparrow\bar{t}} w(q_\ell(t)+,t) \ge \lim_{t\uparrow\bar{t}} w(q_r(t)-,t) \ge \lim_{t\uparrow\bar{t}} w(q_r(t)+,t) \ge \lim_{t\uparrow\bar{t}} w(\zeta_+(t)\pm,t) = w_S\,, \tag{12.3.6$_2$}$$

$$z_W = \lim_{t\uparrow\bar{t}} z(\zeta_-(t)\pm,t) \le \lim_{t\uparrow\bar{t}} z(q_\ell(t)-,t) \le \lim_{t\uparrow\bar{t}} z(q_\ell(t)+,t) \le \lim_{t\uparrow\bar{t}} z(q_r(t)-,t) \le \lim_{t\uparrow\bar{t}} z(q_r(t)+,t) \le \lim_{t\uparrow\bar{t}} z(\zeta_+(t)\pm,t) = z_S\,. \tag{12.3.7$_2$}$$

(c)$_1$ If $\phi_-(t) = \phi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then $z_W \le z_N$, $w_W \ge w_N$. On the other hand, if $\phi_-(t) < \phi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then $w_W = w_N$ and as (x,t) tends to $(\bar{x},\bar{t})$ through the region $\{(x,t) : t > \bar{t},\ \phi_-(t) < x < \phi_+(t)\}, w(x\pm,t)$ tends to w_W. Furthermore, if $p_\ell(\cdot)$ and $p_r(\cdot)$ are any two forward 1-characteristics issuing from $(\bar{x},\bar{t})$, with $\phi_-(t) \le p_\ell(t) \le p_r(t) \le \phi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then

$$z_W = \lim_{t\downarrow\bar{t}} z(\phi_-(t)\pm,t) \ge \lim_{t\downarrow\bar{t}} z(p_\ell(t)-,t) = \lim_{t\downarrow\bar{t}} z(p_\ell(t)+,t) \ge \lim_{t\downarrow\bar{t}} z(p_r(t)-,t) = \lim_{t\downarrow\bar{t}} z(p_r(t)+,t) \ge \lim_{t\downarrow\bar{t}} z(\phi_+(t)\pm,t) = z_N\,. \tag{12.3.8$_1$}$$

(c)$_2$ If $\psi_-(t) = \psi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then $w_N \ge w_E$, $z_N \le z_E$. On the other hand, if $\psi_-(t) < \psi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then $z_N = z_E$ and as (x,t) tends to $(\bar{x},\bar{t})$ through the region $\{(x,t) : t > \bar{t},\ \psi_-(t) < x < \psi_+(t)\}, z(x\pm,t)$ tends to z_E. Furthermore, if $q_\ell(\cdot)$ and $q_r(\cdot)$ are any two forward 2-characteristics issuing from $(\bar{x},\bar{t})$, with $\psi_-(t) \le q_\ell(t) \le q_r(t) \le \psi_+(t)$, for $\bar{t} < t < \bar{t}+s$, then

$$w_N = \lim_{t\downarrow\bar{t}} w(\psi_-(t)\pm,t) \le \lim_{t\downarrow\bar{t}} w(q_\ell(t)-,t) = \lim_{t\downarrow\bar{t}} w(q_\ell(t)+,t) \le \lim_{t\downarrow\bar{t}} w(q_r(t)-,t) = \lim_{t\downarrow\bar{t}} w(q_r(t)+,t) \le \lim_{t\downarrow\bar{t}} w(\psi_+(t)\pm,t) = w_E\,. \tag{12.3.8$_2$}$$

The motivation for the above definition lies in

12.3.3 Theorem. *For θ sufficiently small, the solution is locally regular at any point of the upper half-plane.*

The proof will be provided in the next section. However, the following remarks are in order here. Definition 12.3.2 is motivated by experience with piecewise smooth solutions. Indeed, at points of local regularity, incoming waves of the two characteristic families collide to generate a jump discontinuity, which is then resolved into an outgoing wave fan. Statements $(b)_1$ and $(b)_2$ regulate the incoming waves, allowing for any combination of admissible shocks and focusing compression waves. Statements $(c)_1$ and $(c)_2$ characterize the outgoing wave fan. In particular, $(c)_1$ implies that the state (z_W, w_W), on the left, may be joined with the state (z_N, w_N), on the right, by a 1-rarefaction wave or admissible 1-shock; while $(c)_2$ implies that the state (z_N, w_N), on the left, may be joined with the state (z_E, w_E), on the right, by a 2-rarefaction wave or admissible 2-shock. Thus, the outgoing wave fan is locally approximated by the solution of the Riemann problem with end-states $(z(\bar{x}-,\bar{t}), w(\bar{x}-,\bar{t}))$ and $(z(\bar{x}+,\bar{t}), w(\bar{x}+,\bar{t}))$.

A simple corollary of Theorem 12.3.3 is that $\phi_-(\cdot)$ is a 1-characteristic while $\psi_+(\cdot)$ is a 2-characteristic.

Definition 12.3.2 and Theorem 12.3.3 apply even to points on the initial line, $\bar{t} = 0$, after the irrelevant parts of the statements, pertaining to $t < \bar{t}$, are discarded. It should be noted, however, that there is an important difference between $\bar{t} = 0$ and $\bar{t} > 0$. In the former case, $(z(\bar{x}\pm,0), w(\bar{x}\pm,0))$ are unrestricted, being induced arbitrarily by the initial data, and hence the outgoing wave fan may comprise any combination of shocks and rarefaction waves. By contrast, when $\bar{t} > 0$, statements $(b)_1$ and $(b)_2$ in Definition 12.3.2 induce the restrictions $z_W \leq z_E$ and $w_W \geq w_E$. This, combined with statements $(c)_1$ and $(c)_2$, rules out the possibility that both outgoing waves may be rarefactions.

12.4 Propagation of Riemann Invariants Along Extremal Backward Characteristics

The theory of the genuinely nonlinear scalar conservation law, expounded in Chapter XI, owes its simplicity to the observation that extremal backward generalized characteristics are essentially classical characteristics, namely straight lines along which the solution stays constant. It is thus natural to investigate whether solutions U of systems (12.1.1) exhibit similar behavior. When U is Lipschitz continuous, the Riemann invariants z and w stay constant along 1-characteristics and 2-characteristics, respectively, by virtue of Theorem 7.3.4. One should not expect, however, that this will hold for weak solutions, because Riemann invariants generally jump across shocks of both characteristic families. In the context of piecewise smooth solutions, Theorem 8.2.3 implies that, under the current normalization conditions, the trace of z (or w) along shock-free 1-characteristics (or 2-characteristics) is a nonincreasing step function.

The jumps of z (or w) occur at the points where the characteristic crosses a shock of the opposite family, and are of cubic order in the strength of the crossed shock. It is remarkable that this property essentially carries over to general weak solutions:

12.4.1 Theorem. *Let $\xi(\cdot)$ be the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from any fixed point $(\bar{x},\bar{t})$ of the upper half-plane. Set*

$$\bar{z}(t) = z(\xi(t)-,t), \quad \bar{w}(t) = w(\xi(t)+,t), \quad 0 \le t \le \bar{t}. \tag{12.4.1}$$

Then $\bar{z}(\cdot)$ (or $\bar{w}(\cdot)$) is a nonincreasing saltus function whose variation is concentrated in the set of points of jump discontinuity of $\bar{w}(\cdot)$ (or $\bar{z}(\cdot)$). Furthermore, if $\tau \in (0,\bar{t})$ is any point of jump discontinuity of $\bar{z}(\cdot)$ (or $\bar{w}(\cdot)$), then

$$\bar{z}(\tau-) - \bar{z}(\tau+) \le a[\bar{w}(\tau+) - \bar{w}(\tau)]^3, \tag{12.4.2$_1$}$$

or

$$\bar{w}(\tau-) - \bar{w}(\tau+) \le a[\bar{z}(\tau+) - \bar{z}(\tau)]^3, \tag{12.4.2$_2$}$$

where a is a positive constant depending solely on F.

The proof of the above proposition will be intermingled with the proof of Theorem 12.3.3, on local regularity of the solution, and will be partitioned into several steps. The assumption that the trace of (z,w) along space-like curves has bounded variation will be employed only for special space-like curves, namely, generalized characteristics and time lines, t =constant.

12.4.2 Lemma. *When $\xi(\cdot)$ is the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from $(\bar{x},\bar{t})$, $\bar{z}(\cdot)$ (or $\bar{w}(\cdot)$) is nonincreasing on $[0,\bar{t}]$.*

Proof. The two cases are quite similar, so it will suffice to discuss the first one, namely where $\xi(\cdot)$ is a 1-characteristic. Then, by virtue of Theorem 10.3.2, $\xi(\cdot)$ is shock-free and hence

$$\dot{\xi}(t) = \lambda(U(\xi(t)\pm,t)), \quad \text{a.e. on } [0,\bar{t}]. \tag{12.4.3}$$

We fix numbers τ and s, with $0 \le \tau < s \le \bar{t}$. For ε positive and small, we let $\xi_\varepsilon(\cdot)$ denote the minimal Filippov solution of the ordinary differential equation

$$\frac{dx}{dt} = \lambda(U(x,t)) + \varepsilon, \tag{12.4.4}$$

on $[\tau,s]$, with initial condition $\xi_\varepsilon(s) = \xi(s) - \varepsilon$. Applying (12.1.1), as equality of measures, to arcs of the graph of $\xi_\varepsilon(\cdot)$ and using Theorem 1.7.8, we deduce

(12.4.5)

$$F(U(\xi_\varepsilon(t)+,t)) - F(U(\xi_\varepsilon(t)-,t)) - \dot{\xi}_\varepsilon(t)[U(\xi_\varepsilon(t)+,t) - U(\xi_\varepsilon(t)-,t)] = 0,$$

a.e. on $[\tau,s]$. Therefore, $\xi_\varepsilon(\cdot)$ propagates with speed $\lambda(U(\xi_\varepsilon(t)\pm,t))+\varepsilon$, at points of approximate continuity, or with 1-shock speed, at points of approximate jump discontinuity. In particular, $\lambda(U(\xi_\varepsilon(t)+,t)) \leq \lambda(U(\xi_\varepsilon(t)-,t))$, almost everywhere on $[\tau,s]$, and so, by the definition of Filippov solutions of (12.4.4),

$$\dot{\xi}_\varepsilon(t) \geq \lambda(U(\xi_\varepsilon(t)+,t)) + \varepsilon, \quad \text{a.e. on } [\tau,s]. \tag{12.4.6}$$

For any entropy-entropy flux pair (η,q), with η convex, integrating (12.3.1) over the region $\{(x,t) : \tau < t < s,\ \xi_\varepsilon(t) < x < \xi(t)\}$ and applying Green's theorem yields

$$\begin{aligned}\int_{\xi_\varepsilon(s)}^{\xi(s)} \eta(U(x,s))dx - \int_{\xi_\varepsilon(\tau)}^{\xi(\tau)} \eta(U(x,\tau))dx& \\ \leq -\int_\tau^s \{q(U(\xi(t)-,t)) - \dot{\xi}(t)\eta(U(\xi(t)-,t))\}dt& \\ + \int_\tau^s \{q(U(\xi_\varepsilon(t)+,t)) - \dot{\xi}_\varepsilon(t)\eta(U(\xi_\varepsilon(t)+,t))\}dt&.\end{aligned} \tag{12.4.7}$$

In particular, we write (12.4.7) for the Lax entropy-entropy flux pair (12.2.5). For k large, the right-hand side of (12.4.7) is nonpositive, by virtue of (12.4.3), (12.4.6), (12.2.12), (12.1.3) and (12.2.13). Hence

$$\int_{\xi_\varepsilon(s)}^{\xi(s)} \eta(z(x,s),w(x,s))dx \leq \int_{\xi_\varepsilon(\tau)}^{\xi(\tau)} \eta(z(x,\tau),w(x,\tau))dx. \tag{12.4.8}$$

We raise (12.4.8) to the power $1/k$ and then let $k \to \infty$. This yields

$$\text{ess sup}_{(\xi_\varepsilon(s),\xi(s))}\, z(\cdot,s) \leq \text{ess sup}_{(\xi_\varepsilon(\tau),\xi(\tau))}\, z(\cdot,\tau). \tag{12.4.9}$$

Finally, we let $\varepsilon \downarrow 0$. By standard theory of Filippov solutions, the family $\{\xi_\varepsilon(\cdot)\}$ contains a sequence that converges, uniformly on $[\tau,s]$, to some Filippov solution $\xi_0(\cdot)$ of the equation $dx/dt = \lambda(U(x,t))$, with initial condition $\xi_0(s) = \xi(s)$. But then $\xi_0(\cdot)$ is a backward 1-characteristic emanating from the point $(\xi(s),s)$. Moreover, $\xi_0(t) \leq \xi(t)$, for $\tau \leq t \leq s$. Since $\xi(\cdot)$ is minimal, $\xi_0(\cdot)$ must coincide with $\xi(\cdot)$ on $[\tau,s]$. Thus (12.4.9) implies $\bar{z}(s) \leq \bar{z}(\tau)$ and so $\bar{z}(\cdot)$ is nonincreasing on $[\tau,s]$. The proof is complete.

12.4.3 Lemma. *Let $\xi(\cdot)$ be the minimal (or maximal) backward 1-characteristic (or 2-characteristic) emanating from $(\bar{x},\bar{t})$. Then, for any $\tau \in (0,\bar{t}]$,*

$$z(\xi(\tau)-,\tau) \leq \bar{z}(\tau-) \leq z(\xi(\tau)+,\tau), \tag*{(12.4.10)$_1$}$$

or

$$w(\xi(\tau)-,\tau) \geq \bar{w}(\tau-) \geq w(\xi(\tau)+,\tau). \tag*{(12.4.10)$_2$}$$

In particular,

$$z(x-,t) \le z(x+,t), \; w(x-,t) \ge w(x+,t), \quad -\infty < x < \infty, \; 0 < t < \infty. \tag{12.4.11}$$

This will be established in conjunction with

12.4.4 Lemma. *Let $\xi(\cdot)$ be the minimal (or maximal) backward* 1*-characteristic (or* 2*-characteristic) emanating from $(\bar{x},\bar{t})$. For any $0<\tau<s\le\bar{t}$,*

$$z(\xi(\tau)+,\tau) - z(\xi(s)+,s) \le b\,\mathrm{osc}_{[\tau,s]}\bar{w}(\cdot)\, TV_{[\tau,s]}\bar{w}(\cdot), \tag{12.4.12$_1$}$$

or

$$w(\xi(\tau)-,\tau) - w(\xi(s)-,s) \le b\,\mathrm{osc}_{[\tau,s]}\bar{z}(\cdot)\, TV_{[\tau,s]}\bar{z}(\cdot), \tag{12.4.12$_2$}$$

where b is a positive constant depending on F. Furthermore, if $\bar{w}(\tau+)>\bar{w}(\tau)$ (or $\bar{z}(\tau+)>\bar{z}(\tau)$), then (12.4.2)$_1$ *(or* (12.4.2)$_2$*) holds.*

Proof. It suffices to discuss the case where $\xi(\cdot)$ is a 1-characteristic. Consider any convex entropy η with associated entropy flux q. We fix ε positive and small and integrate (12.3.1) over the region $\{(x,t): \tau<t<s,\; \xi(t)<x<\xi(t)+\varepsilon\}$. Notice that both curves $x=\xi(t)$ and $x=\xi(t)+\varepsilon$ have slope $\lambda(\bar{z}(t),\bar{w}(t))$, almost everywhere on (τ,s). Therefore, Green's theorem yields

$$\begin{aligned}\int_{\xi(s)}^{\xi(s)+\varepsilon} \eta(z(x,s),w(x,s))dx - \int_{\xi(\tau)}^{\xi(\tau)+\varepsilon} \eta(z(x,\tau),w(x,\tau))dx \\ \le -\int_{\tau}^{s} H(z(\xi(t)+\varepsilon+,t), w(\xi(t)+\varepsilon+,t), \bar{z}(t), \bar{w}(t))dt,\end{aligned} \tag{12.4.13}$$

under the notation

$$H(z,w,\bar{z},\bar{w}) = q(z,w) - q(\bar{z},\bar{w}) - \lambda(\bar{z},\bar{w})[\eta(z,w)-\eta(\bar{z},\bar{w})]. \tag{12.4.14}$$

One easily verifies, with the help of (12.2.1), that

$$H_z(z,w,\bar{z},\bar{w}) = [\lambda(z,w)-\lambda(\bar{z},\bar{w})]\eta_z(z,w), \tag{12.4.15}$$

$$H_w(z,w,\bar{z},\bar{w}) = [\mu(z,w)-\lambda(\bar{z},\bar{w})]\eta_w(z,w), \tag{12.4.16}$$

$$H_{zz}(z,w,\bar{z},\bar{w}) = \lambda_z(z,w)\eta_z(z,w) + [\lambda(z,w)-\lambda(\bar{z},\bar{w})]\eta_{zz}(z,w), \tag{12.4.17}$$

$$H_{zw}(z,w,\bar{z},\bar{w}) = \lambda_w(z,w)\eta_z(z,w) + [\lambda(z,w)-\lambda(\bar{z},\bar{w})]\eta_{zw}(z,w), \tag{12.4.18}$$

$$H_{ww}(z,w,\bar{z},\bar{w}) = \mu_w(z,w)\eta_w(z,w) + [\mu(z,w)-\lambda(\bar{z},\bar{w})]\eta_{ww}(z,w). \tag{12.4.19}$$

Let us introduce the notation $z_0 = z(\xi(\tau)+,\tau), w_0 = w(\xi(\tau)+,\tau) = \bar{w}(\tau)$, $z_1 = z(\xi(s)+,s), w_1 = w(\xi(s)+,s) = \bar{w}(s)$ and set $\delta = \operatorname{osc}_{[\tau,s]}\bar{w}(\cdot)$. We then apply (12.4.13) for the entropy η constructed by solving the Goursat problem for (12.2.2), with data

$$\text{(12.4.20)} \qquad \begin{cases} \eta(z,w_0) = -(z-z_0)+\beta(z-z_0)^2\,, \\[2ex] \eta(z_0,w) = -3\beta\delta(w-w_0)+\beta(w-w_0)^2\,, \end{cases}$$

where β is a positive constant, sufficiently large for the following to hold on a small neighborhood of the point (z_0,w_0):

$$\text{(12.4.21)} \qquad \eta \text{ is a convex function of } U,$$

$$\text{(12.4.22)} \qquad \eta(z,w) \text{ is a convex function of } (z,w),$$

$$\text{(12.4.23)} \qquad H(z,w,\bar{z},\bar{w}) \text{ is a convex function of } (z,w).$$

It is possible to satisfy the above requirements when $|z-z_0|$, $|w-w_0|$ and δ are sufficiently small. In particular, (12.4.21) will hold by virtue of (12.2.3), (12.1.4), (12.4.20), (12.2.2) and (12.2.4). Similarly, (12.4.22) follows from (12.4.20), (12.2.2) and (12.2.4). Finally, (12.4.23) is verified by combining (12.4.17), (12.4.18), (12.4.19), (12.4.20), (12.2.2) and (12.2.4).

By virtue of (12.4.23), (12.4.15) and (12.4.16),

$$\text{(12.4.24)} \qquad H(z,w,\bar{z},\bar{w}) \geq [\mu(\bar{z},\bar{w})-\lambda(\bar{z},\bar{w})]\eta_w(\bar{z},\bar{w})[w-\bar{w}].$$

One may estimate $\eta_w(\bar{z}(t),\bar{w}(t))$ by integrating (12.2.2), as an ordinary differential equation for η_w, along the line $w=\bar{w}(t)$, starting out from the initial value $\eta_w(z_0,\bar{w}(t))$ at $z=z_0$. Because $|\bar{w}(t)-w_0|\leq\delta$, one easily deduces from (12.4.20) that $-5\beta\delta \leq \eta_w(z_0,\bar{w}(t)) \leq -\beta\delta < 0$. Since $\lambda_w < 0$ and $\eta_z < 0$, (12.2.2) then implies $\eta_w(z,\bar{w}(t)) < 0$, for $z \leq z_0$. In anticipation of $(12.4.10)_1$, we now assume $z_0 \geq \bar{z}(\tau)$, which we already know will apply for almost all choices of τ in $(0,s)$, namely when $z(\xi(\tau)-,\tau) = z(\xi(\tau)+,\tau)$. By Lemma 12.4.2, $\bar{z}(t) \leq \bar{z}(\tau)$ and so $\eta_w(\bar{z}(t),\bar{w}(t)) < 0$, for $\tau \leq t \leq s$.

For $t \in [\tau,s]$, let $\zeta_t(\cdot)$ denote the maximal backward 2-characteristic emanating from the point $(\xi(t)+\varepsilon,t)$ (Fig. 12.4.1). We also draw the maximal forward 2-characteristic $\psi(\cdot)$, issuing from the point $(\xi(\tau),\tau)$, which collides with the curve $x=\xi(t)+\varepsilon$ at time r, where $0<r-\tau<c_0\varepsilon$.

For $t \in (r,s)$, the graph of $\zeta_t(\cdot)$ intersects the graph of $\xi(\cdot)$ at time σ_t. By Lemma 12.4.2,

$$\text{(12.4.25)} \qquad w(\xi(t)+\varepsilon+,t) = w(\zeta_t(t)+,t) \leq w(\zeta_t(\sigma_t)+,\sigma_t) = w(\xi(\sigma_t)+,\sigma_t) = \bar{w}(\sigma_t).$$

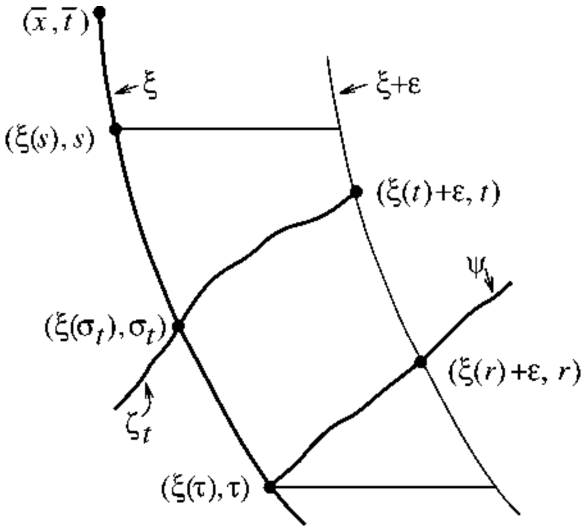

Fig. 12.4.1

Since $\eta_w(\bar{z}(t),\bar{w}(t))<0$, (12.4.24) and (12.4.25) together imply

$$(12.4.26)\qquad H(z(\xi(t)+\varepsilon+,t),w(\xi(t)+\varepsilon+,t),\bar{z}(t),\bar{w}(t))$$
$$\geq [\mu(\bar{z}(t),\bar{w}(t))-\lambda(\bar{z}(t),\bar{w}(t))]\eta_w(\bar{z}(t),\bar{w}(t))[\bar{w}(\sigma_t)-\bar{w}(t)].$$

Because the two characteristic speeds λ and μ are strictly separated, $0<t-\sigma_t<c_1\varepsilon$ and so (12.4.26) yields

$$(12.4.27)\qquad -\int_r^s H(z(\xi(t)+\varepsilon+,t),w(\xi(t)+\varepsilon+,t),\bar{z}(t),\bar{w}(t))dt$$
$$\leq c_2\varepsilon\sup_{(\tau,s)}|\eta_w(\bar{z}(\cdot),\bar{w}(\cdot))|NV_{(\tau,s)}\bar{w}(\cdot),$$

with NV denoting negative (i.e., decreasing) variation.

Next, we restrict t to the interval (τ,r). Then, $\zeta_t(\cdot)$ is intercepted by the τ-time line at $\zeta_t(\tau)\in[\xi(\tau),\xi(\tau)+\varepsilon)$. By virtue of Lemma 12.4.2,

$$(12.4.28)\quad w(\xi(t)+\varepsilon+,t)=w(\zeta_t(t)+,t)\leq w(\zeta_t(\tau)+,\tau)=w_0+o(1)\ ,\ \text{as } \varepsilon\downarrow 0.$$

On the other hand, upon setting $z_+=\bar{z}(\tau+)$, $w_+=\bar{w}(\tau+)$, we readily observe that $\bar{z}(t)=z_++o(1)$, $\bar{w}(t)=w_++o(1)$, as $\varepsilon\downarrow 0$. Therefore, combining (12.4.24) with (12.4.28) yields

$$(12.4.29)\qquad -\int_\tau^r H(z(\xi(t)+\varepsilon+,t),w(\xi(t)+\varepsilon+,t),\bar{z}(t),\bar{w}(t))dt$$
$$\leq -[\mu(z_+,w_+)-\lambda(z_+,w_+)]\eta_w(z_+,w_+)[w_0-w_+](r-\tau)+o(\varepsilon).$$

We now multiply (12.4.13) by $1/\varepsilon$ and then let $\varepsilon \downarrow 0$. Using (12.4.27) and (12.4.28), and recalling that $0 < r - \tau < c_0\varepsilon$, we deduce

$$\eta(z_1, w_1) - \eta(z_0, w_0) \leq c_3 \sup_{(\tau,s)} |\eta_w(\bar{z}(\cdot), \bar{w}(\cdot))| NV_{[\tau,s)} \bar{w}(\cdot). \tag{12.4.30}$$

In particular, s is the limit of an increasing sequence of τ with the property $z(\xi(\tau)-, \tau) = z(\xi(\tau)+, \tau)$, for which (12.4.30) is valid. This in turn implies that $\eta(z_1, w_1) \leq \eta(\bar{z}(s-), \bar{w}(s-))$. Now applying (12.4.25) for $t = s$, and letting $\varepsilon \downarrow 0$, yields $w_1 \leq \bar{w}(s-)$. Also, $\eta_w < 0$, $\eta_z < 0$. Hence, $\bar{z}(s-) \leq z_1$. By Lemma 12.4.2, $\bar{z}(s) \leq \bar{z}(s-)$ and so $z(\xi(s)-, s) = \bar{z}(s) \leq \bar{z}(s-) \leq z_1 = z(\xi(s)+, s)$. Since s is arbitrary, we may write these inequalities for $s = \tau$ and this verifies $(12.4.10)_1$. Lemma 12.4.3 has now been proved. Furthermore, $z_0 \geq \bar{z}(\tau)$ has been established and hence (12.4.30) is valid for all τ and s with $0 < \tau < s \leq \bar{t}$.

From (12.4.22) and (12.4.20) it follows

$$\eta(z_1, w_1) - \eta(z_0, w_0) \geq z_0 - z_1 - 3\beta\delta(w_1 - w_0). \tag{12.4.31}$$

Combining (12.4.30) with (12.4.31),

$$z_0 - z_1 \leq 3\beta\delta(w_1 - w_0) + c_3 \sup_{(\tau,s)} |\eta_w(\bar{z}(\cdot), \bar{w}(\cdot))| NV_{[\tau,s)} \bar{w}(\cdot). \tag{12.4.32}$$

To establish $(12.4.12)_1$ for general τ and s, it would suffice to verify it just for τ and s with $s - \tau$ so small that $TV_{[\tau,s]} \bar{w}(\cdot) < 2\delta$. For such τ and s, (12.4.32) gives the preliminary estimate $z_0 - z_1 \leq c_4\delta$, and in fact $z_0 - \bar{z}(t) \leq c_4\delta$, for all $t \in (\tau, s)$. But then, since $|\eta_w(z_0, \bar{w}(t))| \leq 5\beta\delta$, (12.2.2) implies $\sup_{(\tau,s)} |\eta_w(\bar{z}(\cdot), \bar{w}(\cdot))| \leq c_5\delta$. Inserting this estimate into (12.4.32), we arrive at $(12.4.12)_1$, with $b = 3\beta + c_3c_5$.

Finally, we assume $\bar{w}(\tau+) > \bar{w}(\tau)$, say $w_+ - w_0 = \delta_0 > 0$, and proceed to verify $(12.4.2)_1$. Keeping τ fixed, we choose $s - \tau$ so small that $TV_{[\tau,s]} \bar{w}(\cdot) < 2\delta_0$ and hence $\delta < 2\delta_0$. We need to improve the estimate (12.4.29), and thus we restrict t to the interval $[\tau, r]$.

On account of (12.4.23), (12.4.15) and (12.4.16),

$$\begin{aligned} H(z(\xi(t)+\varepsilon+, t), w(\xi(t)+\varepsilon+, t), \bar{z}(t), \bar{w}(t)) &\geq H(z_+, w_0, \bar{z}(t), \bar{w}(t)) \\ &-[\lambda(z_+, w_0) - \lambda(\bar{z}(t), \bar{w}(t))][z(\xi(t)+\varepsilon+, t) - z_+] \\ &-3\beta\delta[\mu(z_+, w_0) - \lambda(\bar{z}(t), \bar{w}(t))][w(\xi(t)+\varepsilon+, t) - w_0]. \end{aligned} \tag{12.4.33}$$

We have already seen that, as $\varepsilon \downarrow 0$, $\bar{z}(t) = z_+ + o(1), \bar{w}(t) = w_+ + o(1)$. In particular, for ε small, $\lambda(z_+, w_0) - \lambda(\bar{z}(t), \bar{w}(t)) > 0$, by virtue of (12.1.5). Furthermore, if $\hat{\xi}(\cdot)$ denotes the minimal backward 1-characteristic emanating from any point (x, t) with $\xi(t) < x < \xi(t) + 2\varepsilon$, by Lemma 12.4.2, it is $z(x-, t) \leq z(\hat{\xi}(\tau)-, \tau) = z_0 + o(1)$, as $\varepsilon \downarrow 0$. On the other hand, $(12.4.12)_1$, with $s \downarrow \tau$, implies $z_0 - z_+ \leq b\delta_0^2$. Therefore, as $\varepsilon \downarrow 0$, $z(\xi(t)+\varepsilon+, t) \leq z_+ + b\delta_0^2 + o(1)$. Finally, we recall (12.4.28). Collecting the above, we deduce from (12.4.33):

(12.4.34)
$$H(z(\xi(t)+\varepsilon+,t),w(\xi(t)+\varepsilon+,t),\bar{z}(t),\bar{w}(t))$$
$$\geq H(z_+,w_0,z_+,w_+)-c_6\delta_0^3+o(1), \quad \text{as } \varepsilon\downarrow 0.$$

To estimate the right-hand side of (12.4.34), let us visualize q as a function of (z,η). By the chain rule and (12.2.1), we deduce $q_\eta = \mu$, $q_{\eta\eta}=\mu_w/\eta_w$. For $w\in[w_0,w_+]$, $q_{\eta\eta}<0$. Hence

(12.4.35)
$$H(z_+,w_0,z_+,w_+)\geq[\mu(z_+,w_0)-\lambda(z_+,w_+)][\eta(z_+,w_0)-\eta(z_+,w_+)].$$

The next step is to show

(12.4.36)
$$\frac{r-\tau}{\varepsilon}\geq\frac{1}{\mu(z_0,w_+)-\lambda(z_+,w_+)}+o(1), \qquad \text{as } \varepsilon\downarrow 0.$$

To see this, let us begin with

(12.4.37)
$$\varepsilon=\psi(r)-\xi(r)=\int_\tau^r[\dot{\psi}(t)-\dot{\xi}(t)]dt$$
$$\leq\int_\tau^r[\mu(z(\psi(t)-,t),w(\psi(t)-,t))-\lambda(\bar{z}(t),\bar{w}(t))]dt.$$

As shown above, $z(\psi(t)-,t)\leq z_0+o(1)$, as $\varepsilon\downarrow 0$. On the other hand, the maximal backward 2-characteristic $\zeta(\cdot)$, emanating from a point (x,t) with $\xi(t)<x<\psi(t)$, will intersect the graph of $\xi(\cdot)$ at time $\sigma\in(\tau,r]$ and hence, by Lemma 12.4.2, $w(x+,t)\leq\bar{w}(\sigma)$. In particular, $w(\psi(t)-,t)\leq w_++o(1)$, as $\varepsilon\downarrow 0$. Since $\mu_z>0$ and $\mu_w>0$, (12.4.37) implies $\varepsilon\leq(r-\tau)[\mu(z_0,w_+)-\lambda(z_+,w_+)+o(1)]$ whence (12.4.36) immediately follows.

Once again we multiply (12.4.13) by $1/\varepsilon$, let $\varepsilon\downarrow 0$ and then also let $s\downarrow\tau$. Combining (12.4.27), (12.4.34), (12.4.35) and (12.4.36), we conclude:

(12.4.38)
$$\eta(z_+,w_0)-\eta(z_0,w_0)\leq\frac{\mu(z_0,w_+)-\mu(z_+,w_0)}{\mu(z_0,w_+)-\lambda(z_+,w_+)}[\eta(z_+,w_0)-\eta(z_+,w_+)]+c_7\delta_0^3.$$

By virtue of (12.4.20), $\eta(z_+,w_0)-\eta(z_0,w_0)\geq z_0-z_+$. The right-hand side of (12.4.38) is bounded by $a\delta_0^3$, because $\eta_w=O(\delta_0)$. Therefore, $z_0-z_+\leq a\delta_0^3$. Now $\bar{z}(\tau-)\leq z_0$, on account of $(12.4.10)_1$. Hence $\bar{z}(\tau-)-\bar{z}(\tau+)\leq a\delta_0^3$, which establishes $(12.4.2)_1$.

Since total variation is additive, we deduce immediately

12.4.5 Corollary. *In* $(12.4.12)_1$ *(or* $(12.4.12)_2$*),* $\mathrm{osc}_{[\tau,s]}\bar{w}(\cdot)$ *(or* $\mathrm{osc}_{[\tau,s]}\bar{z}(\cdot)$*) may be replaced by the local oscillation of* $\bar{w}(\cdot)$ *(or* $\bar{z}(\cdot)$*) in the interval* $[\tau,s]$*, which is measured by the maximum jump of* $\bar{w}(\cdot)$ *(or* $\bar{z}(\cdot)$*) in* $[\tau,s]$*. In particular,* $\bar{z}(\cdot)$ *(or* $\bar{w}(\cdot)$*) is a* *saltus function whose variation is concentrated in the set of points of jump discontinuity of* $\bar{w}(\cdot)$ *(or* $\bar{z}(\cdot)$*).*

We have thus verified all the assertions of Theorem 12.4.1, except that (12.4.2) has been established under the extraneous assumption $\bar{w}(\tau) < \bar{w}(\tau+)$. By Lemma 12.4.3, $\bar{w}(\tau) = w(\xi(\tau)+,\tau) \leq w(\xi(\tau)-,\tau)$. On the other hand, when $(\xi(\tau),\tau)$ is a point of local regularity of the solution, Condition $(c)_1$ of Definition 12.3.2 implies $\bar{w}(\tau+) = w(\xi(\tau)-,\tau)$. Hence, by establishing Theorem 12.3.3, we will justify, in particular, the assumption $\bar{w}(\tau) < \bar{w}(\tau+)$.

We thus turn to the proof of Theorem 12.3.3. Our main tool will be the estimate (12.4.12). In what follows, δ will denote an upper bound of the oscillation of z and w on the upper half-plane. We fix any point $(\bar{x},\bar{t})$ of the upper half-plane and construct the curves $\xi_\pm(\cdot)$, $\zeta_\pm(\cdot)$, $\phi_\pm(\cdot)$ and $\psi_\pm(\cdot)$, as described in Section 12.3 and sketched in Fig. 12.3.1. The first step is to verify the part of Condition (a) of Definition 12.3.2 pertaining to the "western" sector $\mathscr{S}_W$.

12.4.6 Lemma. *For θ sufficiently small, as (x,t) tends to $(\bar{x},\bar{t})$ through the region $\mathscr{S}_W$, defined by* (12.3.2), *$(z(x\pm,t),w(x\pm,t))$ converge to (z_W,w_W), where we set $z_W = z(\bar{x}-,\bar{t})$, $w_W = w(\bar{x}-,\bar{t})$.*

Proof. We shall construct a sequence $x_0 < x_1 < x_2 < \cdots < \bar{x}$ such that, for every $m = 0,1,2,\cdots$,

$$\operatorname{osc}_{\mathscr{S}_W\cap\{x>x_m\}} z \leq (3b\theta)^m\delta, \qquad \operatorname{osc}_{\mathscr{S}_W\cap\{x>x_m\}} w \leq (3b\theta)^m\delta, \tag{12.4.39}$$

where b is the constant appearing in (12.4.12). Clearly, (12.4.39) will readily imply the assertion of the proposition, provided $3b\theta < 1$.

For $m = 0$, (12.4.39) is satisfied with $x_0 = -\infty$. Arguing by induction, let us assume $x_0 < x_1 < \cdots < x_{k-1} < \bar{x}$ have already been fixed so that (12.4.39) holds for $m = 0,\cdots,k-1$. We proceed to determine x_k. We fix $\hat{t} \in (0,\bar{t})$ with $\bar{t} - \hat{t}$ so small that $\zeta_-(\hat{t}) > x_{k-1}$ and the oscillation of $z(\zeta_-(\tau)\pm,\tau)$ over the interval $[\hat{t},\bar{t})$ does not exceed $\frac{1}{3}(3b\theta)^k\delta$. Next we locate $\hat{x} \in (x_{k-1},\zeta_-(\hat{t}))$ with $\zeta_-(\hat{t}) - \hat{x}$ so small that the oscillation of $w(y-,\hat{t})$ over the interval $(\hat{x},\zeta_-(\hat{t})]$ is similarly bounded by $\frac{1}{3}(3b\theta)^k\delta$.

By the construction of $\phi_-(\cdot)$, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from any point (x,t) in $\mathscr{S}_W \cap \{x > x_k\}$ stays to the left of the graph of $\phi_-(\cdot)$. At the same time, as (x,t) tends to $(\bar{x},\bar{t})$ through $\mathscr{S}_W$, the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from it will tend to some backward 2-characteristic emanating from $(\bar{x},\bar{t})$, which necessarily lies to the right of the minimal characteristic $\zeta_-(\cdot)$ or coincides with $\zeta_-(\cdot)$. It follows that when $\bar{x} - x_k$ is sufficiently small, $\xi(\cdot)$ will have to cross the graph of $\zeta_-(\cdot)$ at some time $t^* \in (\hat{t},\bar{t})$, while $\zeta(\cdot)$ must intersect either the graph of $\zeta_-(\cdot)$ at some time $\tilde{t} \in (\hat{t},\bar{t})$ or the $\hat{t}$-time line at some $\tilde{x} \in (\hat{x},\zeta_-(\hat{t})]$.

By virtue of Lemmas 12.4.2 and 12.4.3,

$$z(x-,t) \leq z(\xi(t^*)-,t^*) = z(\zeta_-(t^*)-,t^*) \leq z(\zeta_-(t^*)+,t^*). \tag{12.4.40}$$

On account of (12.4.39), for $m = k-1$, and the construction of $\hat{t}$, the oscillation of $w(\xi(\tau)+,\tau)$ over the interval $[t^*,t]$ does not exceed $(3b\theta)^{k-1}\delta + \frac{1}{3}(3b\theta)^k\delta$, which in turn is majorized by $2(3b\theta)^{k-1}\delta$. Then $(12.4.12)_1$ yields

$$z(x+,t) \geq z(\xi(t^*)+,t^*) - 2b\theta(3b\theta)^{k-1}\delta = z(\zeta_-(t^*)+,t^*) - \frac{2}{3}(3b\theta)^k\delta. \tag{12.4.41}$$

Recalling that the oscillation of $z(\zeta_-(\tau)+,\tau)$ over $[\hat{t},\bar{t})$ is bounded by $\frac{1}{3}(3b\theta)^k\delta$, (12.4.40) and (12.4.41) together imply the bound (12.4.39) on the oscillation of z, for $m = k$.

The argument for w is similar: Assume, for example, that $\xi(\cdot)$ intersects the $\hat{t}$-time line, rather than the graph of $\zeta_-(\cdot)$. By virtue of Lemmas 12.4.2 and 12.4.3,

$$w(x+,t) \leq w(\zeta(\hat{t})+,\hat{t}) = w(\tilde{x}+,\hat{t}) \leq w(\tilde{x}-,\hat{t}). \tag{12.4.42}$$

The oscillation of $z(\zeta(\tau)-,\tau)$ over the interval $[\hat{t},t]$ does not exceed $(3b\theta)^{k-1}\delta$, on account of (12.4.39), for $m = k-1$. Then $(12.4.12)_2$ implies

$$w(x-,t) \geq w(\zeta(\hat{t})-,\hat{t}) - b\theta(3b\theta)^{k-1}\delta = w(\tilde{x}-,\hat{t}) - \frac{1}{3}(3b\theta)^k\delta. \tag{12.4.43}$$

The bound (12.4.39) on the oscillation of w, for $m = k$, now easily follows from (12.4.42), (12.4.43) and the construction of $\hat{t}$ and $\hat{x}$. The proof is complete.

The part of Condition (a) of Definition 12.3.2 pertaining to the "eastern" sector $\mathscr{S}_E$ is validated by a completely symmetrical argument. The next step is to check the part of Condition (a) that pertains to the "southern" sector $\mathscr{S}_S$.

12.4.7 Lemma. *For θ sufficiently small, as (x,t) tends to $(\bar{x},\bar{t})$ through the region $\mathscr{S}_S$, defined by* (12.3.5), *$(z(x\pm,t), w(x\pm,t))$ tend to a constant state (z_S, w_S).*

Proof. As in the proof of Lemma 12.4.6, the aim is to find $t_0 < t_1 < \cdots < \bar{t}$ such that

$$\operatorname{osc}_{\mathscr{S}_S \cap \{t > t_m\}} z \leq (4b\theta)^m\delta, \qquad \operatorname{osc}_{\mathscr{S}_S \cap \{t > t_m\}} w \leq (4b\theta)^m\delta, \tag{12.4.44}$$

for $m = 0, 1, 2, \cdots$. For $m = 0$, (12.4.44) is satisfied with $t_0 = 0$. Arguing by induction, we assume $t_0 < t_1 < \cdots < t_{k-1} < \bar{t}$ have already been fixed so that (12.4.44) holds for $m = 0, \cdots, k-1$, and proceed to locate t_k. We fix $\hat{t} \in (t_{k-1}, \bar{t})$ with $\bar{t} - \hat{t}$ sufficiently small that the oscillation of $z(\zeta_+(\tau)-,\tau)$, $w(\zeta_+(\tau)+,\tau), z(\xi_-(\tau)-,\tau)$, $w(\xi_-(\tau)+,\tau)$ over the interval $[\hat{t},\bar{t})$ does not exceed $\frac{1}{4}(4b\theta)^k\delta$. Next we locate $\hat{x}$ and $\tilde{x}$ in the interval $(\zeta_+(\hat{t}), \xi_-(\hat{t}))$ with $\hat{x} - \zeta_+(\hat{t})$ and $\xi_-(\hat{t}) - \tilde{x}$ so small that the oscillation of $z(y-,\hat{t})$ over the interval $(\tilde{x}, \xi_-(\hat{t})]$ and the oscillation of $w(y+,\hat{t})$ over the interval $[\zeta_+(\hat{t}), \hat{x})$ do not exceed $\frac{1}{4}(4b\theta)^k\delta$.

Since $\xi_-(\cdot)$ is the minimal backward 1-characteristic and $\zeta_+(\cdot)$ is the maximal backward 2-characteristic emanating from $(\bar{x},\bar{t})$, we can find $t_k \in (\hat{t},\bar{t})$ with $\bar{t} - t_k$ so small that the following holds for any (x,t) in $\mathscr{S}_S \cap \{t > t_k\}$: (a) the minimal backward 1-characteristic $\xi(\cdot)$ emanating from (x,t) must intersect either the $\hat{t}$-time line at $x' \in (\tilde{x}, \xi_-(\hat{t})]$ or the graph of $\xi_-(\cdot)$ at time $t' \in (\hat{t},\bar{t})$; and (b) the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from (x,t) must intersect either the $\hat{t}$-time line at $x^* \in [\zeta_+(\hat{t}),\hat{x})$ or the graph of $\zeta_+(\cdot)$ at some time $t^* \in (\hat{t},\bar{t})$. One then repeats the argument employed in the proof of Lemma 12.4.6 to verify that (12.4.44) is indeed satisfied for $m = k$, with t_k determined as above. The proof is complete.

To conclude the validation of Condition (a) of Definition 12.3.2, it remains to check the part pertaining to the "northern" sector $\mathscr{S}_N$.

12.4.8 Lemma. *For θ sufficiently small, as (x,t) tends to $(\bar{x},\bar{t})$ through the region $\mathscr{S}_N$, defined by* (12.3.4), $(z(x\pm,t),w(x\pm,t))$ *tend to a constant state* (z_N,w_N).

Proof. For definiteness, we treat the typical configuration depicted in Fig. 12.3.1, where $\psi_- \equiv \psi_+$, so that $\psi_-(\cdot)$ is a 2-shock of generally positive strength at $t = \bar{t}$, while $\phi_-(t) < \phi_+(t)$, for $t > \bar{t}$, in which case, as we shall see in Lemma 12.4.10, it is $\lim_{t\downarrow\bar{t}} z(\phi_+(t)-,t) = \lim_{t\downarrow\bar{t}} z(\phi_+(t)+,t)$ and $\lim_{t\downarrow\bar{t}} w(\phi_+(t)-,t) = \lim_{t\downarrow\bar{t}} w(\phi_-(t)+,t)$. Only slight modifications in the argument are needed for the case of alternative feasible configurations.

The aim is to find $t_0 > t_1 > \cdots > \bar{t}$ such that

$$\text{(12.4.45)}\qquad \operatorname{osc}_{\mathscr{S}_N\cap\{t<t_m\}} z \leq a(ab\theta)^m\delta, \qquad \operatorname{osc}_{\mathscr{S}_N\cap\{t<t_m\}} w \leq 3(ab\theta)^m\delta,$$

for $m = 0,1,2,\cdots$, where $a \geq 1$ is a constant, independent of m and θ, to be specified below. Clearly, (12.4.45) is satisfied for $m = 0$, with $t_0 = \infty$. Arguing by induction, we assume $t_0 > t_1 > \cdots > t_{k-1} > \bar{t}$ have already been fixed so that (12.4.45) holds for $m = 0,\cdots,k-1$, and proceed to determine t_k.

We select $t_k \in (\bar{t},t_{k-1})$ with $t_k - \bar{t}$ so small that the oscillation of $z(\phi_+(\tau)-,\tau)$ over the interval $(\bar{t},t_k)$ does not exceed $a(ab\theta)^{k-1}\delta$, the oscillation of $w(\phi_+(\tau)-,\tau)$ over $(\bar{t},t_k)$ is bounded by $(ab\theta)^k\delta$, and the oscillation of $U(\psi_-(\tau)-,\tau)$ over $(\bar{t},t_k)$ is majorized by $(ab\theta)^{2k}\delta^2$.

The bound (12.4.45) on the oscillation of w, for $m = k$, will be established by the procedure employed in the proof of Lemmas 12.4.6 and 12.4.7. We thus fix any (x,t) in $\mathscr{S}_N \cap \{t < t_k\}$ and consider the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from it, which intersects the graph of $\phi_+(\cdot)$ at some time $\tilde{t} \in (\bar{t},t_k)$. By virtue of Lemmas 12.4.2 and 12.4.3:

$$\text{(12.4.46)}\qquad w(x+,t) \leq w(\zeta(\tilde{t})+,\tilde{t}) = w(\phi_+(\tilde{t})+,\tilde{t}) \leq w(\phi_+(\tilde{t})-,\tilde{t}).$$

On account of (12.4.45), for $m = k-1$, and the construction of t_k, the oscillation of $z(\zeta(\tau)-,\tau)$ over the interval $[\tilde{t},t]$ does not exceed $2a(ab\theta)^{k-1}\delta$. Then $(12.4.12)_2$ implies

$$\text{(12.4.47)}\qquad w(x-,t) \geq w(\zeta(\tilde{t})-,\tilde{t}) - 2(ab\theta)^k\delta = w(\phi_+(\tilde{t})-,\tilde{t}) - 2(ab\theta)^k\delta.$$

The inequalities (12.4.46), (12.4.47), coupled with the condition that the oscillation of $w(\phi_+(\tau)-,\tau)$ over $(\bar{t},t_k)$ is majorized by $(ab\theta)^k\delta$, readily yield the bound (12.4.45) on the oscillation of w, for $m = k$.

To derive the corresponding bound on the oscillation of z requires an entirely different argument. Let us define $\bar{U} = \lim_{t \downarrow \bar{t}} U(\psi_-(t)-,t)$, with induced values $(\bar{z},\bar{w})$ for the Riemann invariants, and then set $\Delta z = z - \bar{z}$, $\Delta w = w - \bar{w}$. On $\mathscr{S}_N \cap \{t < t_k\}$, as shown above,

$$|\Delta w| \leq 3(ab\theta)^k \delta. \tag{12.4.48}$$

We construct the minimal backward 1-characteristic $\xi(\cdot)$, emanating from any point (y,t) of approximate continuity in $\mathscr{S}_N \cap \{t < t_k\}$, which is intercepted by the graph of $\psi_-(\cdot)$ at time $t^* \in (\bar{t}, t_k)$. Then $z(y,t) \leq z(\xi(t^*)-,t^*) = z(\psi_-(t^*)-,t^*)$, by Lemma 12.4.2, and this in conjunction with the selection of t_k yields

$$\Delta z(y,t) \leq c_1(ab\theta)^{2k}\delta^2\,, \tag{12.4.49}$$

for some constant c_1 independent of k and θ.

We now fix any point of approximate continuity (x,t) in $\mathscr{S}_N \cap \{t < t_k\}$. We consider, as above, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from (x,t), which is intercepted by the graph of $\psi_-(\cdot)$ at time $t^* \in (\bar{t}, t_k)$, and integrate the conservation law (12.1.1) over the region $\{(y,\tau) : t^* < \tau < t,\ \xi(\tau) < y < \psi_-(\tau)\}$. By Green's theorem,

$$\begin{aligned} &\int_x^{\psi_-(t)} [U(y,t) - \bar{U}]dy \\ &+ \int_{t^*}^t \{F(U(\psi_-(\tau)-,\tau)) - F(\bar{U}) - \dot{\psi}_-(\tau)[U(\psi_-(\tau)-,\tau) - \bar{U}]\}d\tau \\ &- \int_{t^*}^t \{F(U(\xi(\tau)+,\tau)) - F(\bar{U}) - \lambda(U(\xi(\tau)+,\tau))[U(\xi(\tau)+,\tau) - \bar{U}]\}d\tau = 0. \end{aligned} \tag{12.4.50}$$

Applying repeatedly (7.3.12), we obtain, for $U = U(z,w)$,

$$U = \bar{U} + \Delta z R(\bar{U}) + \Delta w S(\bar{U}) + O(\Delta z^2 + \Delta w^2), \tag{12.4.51}$$

$$\begin{aligned} F(U) - F(\bar{U}) - \lambda(U)[U - \bar{U}] &= \Delta w[\mu(\bar{U}) - \lambda(\bar{U})]S(\bar{U}) \\ &- \frac{1}{2}\Delta z^2 \lambda_z(\bar{U})R(\bar{U}) - \Delta z \Delta w \lambda_z(\bar{U})S(\bar{U}) + O(\Delta w^2 + |\Delta z|^3). \end{aligned} \tag{12.4.52}$$

We also note that the oscillation of $w(\xi(\tau)+,\tau)$ over the interval $(t^*,t]$ is bounded by $3(ab\theta)^k\delta$ and so, on account of $(12.4.12)_1$ and Lemma 12.4.3, we have

$$0 \leq \Delta z(\xi(\tau)+,\tau) - \Delta z(x,t) \leq 3b\theta(ab\theta)^k\delta \leq 3(ab\theta)^k\delta, \tag{12.4.53}$$

for any $\tau \in (t^*,t)$.

We substitute from (12.4.51) and (12.4.52) into (12.4.50) and then multiply the resulting equation, from the left, by $Dz(\bar{U})$. By using (12.1.2), (12.4.49), (12.4.48), (12.1.3), (12.4.53), and the properties of t_k, we end up with

(12.4.54) $$\Delta z^2(x,t) \le c(ab\theta)^{2k}\delta^2,$$

where c is a constant independent of (x,t), k and θ. Consequently, upon selecting $a = \max\{1, 2\sqrt{c}\}$, we arrive at the desired bound (12.4.45) on the oscillation of z, for $m = k$. This completes the proof.

To establish Condition (b) of Definition 12.3.2, we demonstrate

12.4.9 Lemma. *Let $p_\ell(\cdot)$ and $p_r(\cdot)$ be any backward 1-characteristics emanating from $(\bar{x},\bar{t})$, with $p_\ell(t) < p_r(t)$, for $t < \bar{t}$. If θ is sufficiently small, then*

(12.4.55) $$\lim_{t\uparrow\bar{t}} z(p_\ell(t)+,t) \le \lim_{t\uparrow\bar{t}} z(p_r(t)-,t),$$

(12.4.56) $$\lim_{t\uparrow\bar{t}} w(p_\ell(t)+,t) \ge \lim_{t\uparrow\bar{t}} w(p_r(t)-,t).$$

Proof. Consider any sequence $\{(x_n,t_n)\}$ with $t_n \uparrow \bar{t}$, as $n \to \infty$, and x_n in $(p_\ell(t_n), p_r(t_n))$ so close to $p_r(t_n)$ that $\lim_{n\to\infty}[w(x_n+,t_n) - w(p_r(t_n)-,t_n)] = 0$. Let $\zeta_n(\cdot)$ denote the maximal backward 2-characteristic emanating from (x_n,t_n), which intersects the graph of $p_\ell(\cdot)$ at time t_n^*. Then $w(x_n+,t_n) \le w(\zeta_n(t_n^*)+,t_n^*) = w(p_\ell(t_n^*)+,t_n^*)$, by Lemma 12.4.2. Since $t_n^* \uparrow \bar{t}$, as $n \to \infty$, this establishes (12.4.56).

To verify (12.4.55), we begin with another sequence $\{(x_n,t_n)\}$, with $t_n \uparrow \bar{t}$, as $n \to \infty$, and $x_n \in (p_\ell(t_n), p_r(t_n))$ such that $\lim_{n\to\infty}[z(x_n-,t_n) - z(p_\ell(t_n)+,t_n)] = 0$. We construct the minimal backward 1-characteristics $\xi_n(\cdot)$ and $\xi_n^*(\cdot)$, emanating from the points (x_n,t_n) and $(p_r(t_n),t_n)$, respectively. Because of minimality, we now have $\xi_n(t) \le \xi_n^*(t) \le p_r(t)$, for $t \le t_n$. As $n \to \infty$, $\{\xi_n(\cdot)\}$ and $\{\xi_n^*(\cdot)\}$ will converge, uniformly, to shock-free minimal 1-separatrices (in the sense of Definition 10.3.3) $\chi(\cdot)$ and $\chi^*(\cdot)$, emanating from $(\bar{x},\bar{t})$, such that $\chi(t) \le \chi^*(t) \le p_r(t)$, for $t \le \bar{t}$. In particular, $\dot{\chi}(\bar{t}-) \ge \dot{\chi}^*(\bar{t}-)$ and so

(12.4.57) $$\lim_{t\uparrow\bar{t}} \lambda(z(\chi(t)\pm,t), w(\chi(t)\pm,t)) \ge \lim_{t\uparrow\bar{t}} \lambda(z(\chi^*(t)\pm,t), w(\chi^*(t)\pm,t)).$$

Applying (12.4.56) with $\chi(\cdot)$ and $\chi^*(\cdot)$ in the roles of $p_\ell(\cdot)$ and $p_r(\cdot)$ yields

(12.4.58) $$\lim_{t\uparrow\bar{t}} w(\chi(t)+,t) \ge \lim_{t\uparrow\bar{t}} w(\chi^*(t)-,t).$$

Since $\lambda_z < 0$ and $\lambda_w < 0$, (12.4.57) and (12.4.58) together imply

(12.4.59) $$\lim_{t\uparrow\bar{t}} z(\chi(t)\pm,t) \le \lim_{t\uparrow\bar{t}} z(\chi^*(t)\pm,t).$$

By virtue of Lemma 12.4.2, $z(\xi_n(t)-,t)$ and $z(\xi_n^*(t)-,t)$ are nonincreasing functions on $[0,t_n]$ and so

(12.4.60) $$\lim_{t\uparrow\bar{t}} z(\chi(t)\pm,t) \ge \lim_{t\uparrow\bar{t}} z(p_\ell(t)+,t),$$

$$\lim_{t\uparrow\bar{t}} z(\chi^*(t)\pm,t) \geq \lim_{t\uparrow\bar{t}} z(p_r(t)-,t). \tag{12.4.61}$$

Thus, to complete the proof of (12.4.55), one has to show

$$\lim_{t\uparrow\bar{t}} z(\chi^*(t)\pm,t) = \lim_{t\uparrow\bar{t}} z(p_r(t)-,t). \tag{12.4.62}$$

Since (12.4.62) is trivially true when $\chi^* \equiv p_r$, we take up the case $\chi^*(t) < p_r(t)$, for $t < \bar{t}$. We set $\mathscr{S} = \{(x,t) : 0 \leq t < \bar{t},\ \chi^*(t) < x < p_r(t)\}$. We shall verify (12.4.62) by constructing $t_0 < t_1 < \cdots < \bar{t}$ such that

$$\operatorname{osc}_{\mathscr{S}\cap\{t>t_m\}} z \leq (3b\theta)^m\delta, \qquad \operatorname{osc}_{\mathscr{S}\cap\{t>t_m\}} w \leq (3b\theta)^m\delta, \tag{12.4.63}$$

for $m = 0,1,2,\cdots$.

For $m = 0$, (12.4.63) is satisfied with $t_0 = 0$. Arguing by induction, we assume $t_0 < t_1 < \cdots < t_{k-1} < \bar{t}$ have already been fixed so that (12.4.63) holds for $m = 0,\cdots,k-1$, and proceed to determine t_k. We fix $\hat{t} \in (t_{k-1},\bar{t})$ with $\bar{t}-\hat{t}$ so small that the oscillation of $z(\chi^*(\tau)\pm,\tau)$ and $w(\chi^*(\tau)-,\tau)$ over the interval $[\hat{t},\bar{t})$ does not exceed $\frac{1}{3}(3b\theta)^k\delta$. Next we locate $\hat{x} \in (\chi^*(\hat{t}),p_r(\hat{t}))$ with $\hat{x}-\chi^*(\hat{t})$ so small that the oscillation of $z(y+,\hat{t})$ over the interval $[\chi^*(\hat{t}),\hat{x})$ is similarly bounded by $\frac{1}{3}(3b\theta)^k\delta$.

By the construction of $\chi^*(\cdot)$, if we fix $t_k \in (\hat{t},\bar{t})$ with $\bar{t}-t_k$ sufficiently small, then the minimal backward 1-characteristic $\xi(\cdot)$, emanating from any point (x,t) in $\mathscr{S}\cap\{t>t_k\}$, will intersect either the graph of $\chi^*(\cdot)$ at some time $t^* \in (\hat{t},\bar{t})$ or the $\hat{t}$-time line at some $x^* \in (\chi^*(\hat{t}),\hat{x})$; while the maximal backward 2-characteristic $\zeta(\cdot)$, emanating from (x,t), will intersect the graph of $\chi^*(\cdot)$ at some time $\tilde{t} \in (\hat{t},\bar{t})$.

Assume, for definiteness, that $\xi(\cdot)$ intersects the $\hat{t}$-time line. By virtue of Lemmas 12.4.2 and 12.4.3,

$$z(x-,t) \leq z(\xi(\hat{t})-,\hat{t}) = z(x^*-,\hat{t}) \leq z(x^*+,\hat{t}). \tag{12.4.64}$$

On account of (12.4.63), for $m = k-1$, the oscillation of $w(\xi(\tau)+,\tau)$ over the interval $[\hat{t},t]$ does not exceed $(3b\theta)^{k-1}\delta$. It then follows from $(12.4.12)_1$

$$z(x+,t) \geq z(\xi(\hat{t})+,\hat{t}) - b\theta(3b\theta)^{k-1}\delta = z(x^*+,\hat{t}) - \frac{1}{3}(3b\theta)^k\delta. \tag{12.4.65}$$

Recalling that the oscillation of $z(y+,\hat{t})$ over $[\chi^*(\hat{t}),\hat{x})$ and the oscillation of $z(\chi^*(\tau)+,\tau)$ over $[\hat{t},\bar{t})$ are bounded by $\frac{1}{3}(3b\theta)^k\delta$, (12.4.64) and (12.4.65) together imply the bound (12.4.63) on the oscillation of z, for $m = k$.

The argument for w is similar: On the one hand, Lemmas 12.4.2 and 12.4.3 give

$$w(x+,t) \leq w(\zeta(\tilde{t})+,\tilde{t}) = w(\chi^*(\tilde{t})+,\tilde{t}) \leq w(\chi^*(\tilde{t})-,\tilde{t}). \tag{12.4.66}$$

On the other hand, considering that the oscillation of $z(\zeta(\tau)-,\tau)$ over the interval $[\tilde{t},t]$ is bounded by $(3b\theta)^{k-1}\delta + \frac{1}{3}(3b\theta)^k\delta$, which in turn is smaller than $2(3b\theta)^{k-1}\delta$, $(12.4.12)_2$ yields

$$w(x-,t) \geq w(\zeta(\tilde{t})-,\tilde{t}) - 2b\theta(3b\theta)^{k-1}\delta = w(\chi^*(\tilde{t})-,\tilde{t}) - \frac{2}{3}(3b\theta)^k\delta. \tag{12.4.67}$$

Since the oscillation of $w(\chi^*(\tau)-,\tau)$ over $[\hat{t},\bar{t})$ does not exceed $\frac{1}{3}(3b\theta)^k\delta$, the inequalities (12.4.66) and (12.4.67) together imply the bound (12.4.63) on the oscillation of w, for $m=k$. The proof of the proposition is now complete.

In particular, one may apply Lemma 12.4.9 with $\xi(\cdot)$ and/or $\xi^*(\cdot)$ in the role of $p_\ell(\cdot)$ or $p_r(\cdot)$, so that, by virtue of Lemma 12.4.3, the inequalities $(12.3.6)_1$ and $(12.3.7)_1$ follow from (12.4.55) and (12.4.56). We have thus verified condition $(\mathrm{b})_1$ of Definition 12.3.2. Condition $(\mathrm{b})_2$ may be validated by a completely symmetrical argument.

It remains to check Condition (c) of Definition 12.3.2. It will suffice to verify $(\mathrm{c})_1$, because then $(\mathrm{c})_2$ will readily follow by a similar argument. In the shock case, $\phi_- \equiv \phi_+$, the required inequalities $z_W \le z_N$ and $w_W \ge w_N$ are immediate corollaries of Lemma 12.4.3. Thus, one need consider only the rarefaction wave case.

12.4.10 Lemma. *Let $\phi_-(t) < \phi_+(t)$, for $t > \bar{t}$. For θ sufficiently small, as (x,t) tends to $(\bar{x},\bar{t})$ in the region $\mathscr{W} = \{(x,t) : t > \bar{t},\ \phi_-(t) < x < \phi_+(t)\}$, $w(x\pm,t)$ tend to w_W. Furthermore,* $(12.3.8)_1$ *holds for any* 1*-characteristics $p_\ell(\cdot)$ and $p_r(\cdot)$, with $\phi_-(t) \le p_\ell(t) \le p_r(t) \le \phi_+(t)$, for $t > \bar{t}$.*

Proof. Consider points (x,t) that tend to $(\bar{x},\bar{t})$ through $\mathscr{W}$. The maximal backward 2-characteristic $\zeta(\cdot)$ emanating from (x,t) is intercepted by the $\bar{t}$-time line at $\zeta(\bar{t})$, which tends from below to $\bar{x}$. It then readily follows on account of Lemma 12.4.2 that $\limsup w(x\pm,t) \le w_W$. To verify the assertion of the proposition, one needs to show that $\liminf w(x\pm,t) = w_W$. The plan is to argue by contradiction, and so we make the hypothesis $\liminf w(x\pm,t) = w_W - \beta$, with $\beta > 0$.

We fix $\hat{t} > \bar{t}$ with $\hat{t} - \bar{t}$ so small that

$$w_W - 2\beta < w(x\pm,t) \le w_W + \beta, \quad \bar{t} < t < \hat{t},\ \phi_-(t) < x < \phi_+(t) \tag{12.4.68}$$

and, in addition, the oscillation of the functions $z(\phi_-(t)\pm,t)$ and $w(\phi_-(t)\pm,t)$ over the interval $(\bar{t},\hat{t})$ does not exceed $\frac{1}{2}\beta$.

We consider the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from any point $(\tilde{x},\tilde{t})$, with $\bar{t} < \tilde{t} < \hat{t}$, $\phi_-(\tilde{t}) < x < \phi_+(\tilde{t})$, and intersecting the graph of $\phi_-(\cdot)$ at time $t^* \in (\bar{t},\hat{t})$. We demonstrate that when θ is sufficiently small, independent of β, then

$$w(\zeta(t)-,t) - w(\tilde{x}-,\tilde{t}) \le \frac{\beta}{4}, \qquad t^* < t < \tilde{t}. \tag{12.4.69}$$

Indeed, if (12.4.69) were false, one may find t_1, t_2, with $t^* < t_1 < t_2 \le \tilde{t}$ and $t_2 - t_1$ arbitrarily small, such that

$$|z(\zeta(t_1)\pm,t_1) - z(\zeta(t_2)\pm,t_2)| > \frac{\beta}{4b\theta}\,. \tag{12.4.70}$$

In particular, if $\xi_1(\cdot)$ and $\xi_2(\cdot)$ denote the minimal backward 1-characteristics that emanate from the points $(\zeta(t_1),t_1)$ and $(\zeta(t_2),t_2)$ and thus necessarily pass through the point $(\bar{x},\bar{t})$, t_1 and t_2 may be fixed so close that

(12.4.71)
$$0 \le \int_0^{t_1} \lambda(z(\xi_2(t)-,t),w(\xi_2(t)-,t))dt - \int_0^{t_1} \lambda(z(\xi_1(t)-,t),w(\xi_1(t)-,t))dt \le \beta t_0 .$$

By virtue of (12.4.68), $|w(\xi_2(t)-,t) - w(\xi_1(t)-,t)| < 3\beta$, for all t in $(\bar{t},t_1)$. Also, on account of Lemma 12.4.2, $(12.4.12)_1$ and (12.4.68), we have

$$\begin{cases} z(\zeta(t_1)-,t_1) \le z(\xi_1(t)-,t) = z(\xi_1(t)+,t) \le z(\zeta(t_1)+,t_1) + 3\beta b\theta, \\ z(\zeta(t_2)-,t_2) \le z(\xi_2(t)-,t) = z(\xi_2(t)+,t) \le z(\zeta(t_2)+,t_2) + 3\beta b\theta, \end{cases} \tag{12.4.72}$$

for almost all t in $(\bar{t},t_1)$. It is now clear that, for θ sufficiently small, (12.4.72) renders the inequalities (12.4.70) and (12.4.71) incompatible. This provides the desired contradiction that verifies (12.4.69).

By Lemma 12.4.6, and the construction of $\hat{t}$,

$$\lim_{t\downarrow\bar{t}} z(\phi_-(t)-,t) = z_W\,, \qquad \lim_{t\downarrow\bar{t}} w(\phi_-(t)-,t) = w_W\,, \tag{12.4.73}$$

$$|z(\phi_-(t^*)-,t^*) - z_W| \le \frac{\beta}{2}\,, \qquad |w(\phi_-(t^*)-,t^*) - w_W| \le \frac{\beta}{2}\,. \tag{12.4.74}$$

The next step is to establish an estimate

$$|z(\phi_-(t^*)-,t^*) - \lim_{t\downarrow t^*} z(\zeta(t)-,t)| \le a\beta, \tag{12.4.75}$$

for some constant a independent of θ and β. Let

$$\lim_{t\downarrow\bar{t}} z(\phi_-(t)+,t) = z_W + \gamma, \tag{12.4.76}$$

with $\gamma \ge 0$. We fix $t_3 \in (t^*,\hat{t})$ and $x_3 \in (\phi_-(t_3),\phi_+(t_3))$, with $x_3 - \phi_-(t_3)$ so small that

$$|z(x_3\pm,t_3) - z_W - \gamma| \le \beta. \tag{12.4.77}$$

By also choosing $t_3 - t^*$ small, the minimal backward 1-characteristic $\xi(\cdot)$, emanating from the point (x_3,t_3), will intersect the graph of $\zeta(\cdot)$ at time t_4, arbitrarily close to t^*. By Lemma 12.4.2, $z(\zeta(t_4)-,t_4) \ge z(x_3-,t_3)$. On the other hand, by (12.4.68), Lemma 12.4.4 implies $z(\zeta(t_4)+,t_4) \le z(x_3+,t_3) + 3b\theta\beta$. Hence, for θ so small that $6b\theta \le 1$, we have $|z_W + \gamma - \lim_{t\downarrow t^*} z(\zeta(t)-,t)| \le \frac{3}{2}\beta$. In conjunction with (12.4.74), this yields

$$|z(\phi_-(t^*)-,t^*) - \lim_{t\downarrow t^*} z(\zeta(t)-,t)| \le 2\beta + \gamma. \tag{12.4.78}$$

Thus, to verify (12.4.75), one has to show $\gamma \le c\beta$.

The characteristic $\xi(\cdot)$ lies to the right of $\phi_-(\cdot)$ and passes through the point $(\bar{x},\bar{t})$, so $\dot{\phi}_-(\bar{t}+) \le \dot{\xi}(\bar{t}+)$. On account of (12.4.73), (12.4.76), (8.2.1), (8.2.2), (7.3.12), (8.2.3), and (12.1.2), we conclude

$$\dot{\phi}_-(\bar{t}+) = \lambda(z_W, w_W) + \tfrac{1}{2}\lambda_z(z_W, w_W)\gamma + O(\gamma^2). \tag{12.4.79}$$

To estimate $\dot{\xi}(\bar{t}+) = \lim_{t\downarrow\bar{t}} \lambda(z(\xi(t)-,t), w(\xi(t)-,t))$, we recall that $\lambda_z < 0$, $\lambda_w < 0$, $z(\xi(t)-,t) \ge z(x_3-,t_3) \ge z_W + \gamma - \beta$, $w(\xi(t)-,t) \ge w_W - 2\beta$, and so

(12.4.80)

$$\dot{\xi}(\bar{t}+) \le \lambda(z_W + \gamma - \beta, w_W - 2\beta) = \lambda(z_W, w_W) + \lambda_z(z_W, w_W)\gamma + O(\beta + \gamma^2).$$

Therefore, $\gamma = O(\beta)$ and (12.4.75) follows from (12.4.78).

By virtue of Lemma 12.4.4, (12.4.75) yields

$$w(\phi_-(t^*)-,t^*) - \lim_{t\downarrow t^*} w(\zeta(t)-,t) \le ab\theta\beta. \tag{12.4.81}$$

Hence, if $\theta \le (8ab)^{-1}$, then (12.4.69), (12.4.81) and (12.4.74) together imply that $w_W - w(\tilde{x}-,\tilde{t}) \le \frac{7}{8}\beta$, for all $(\tilde{x},\tilde{t})$ in $\mathscr{W} \cap \{t < \hat{t}\}$. This provides the desired contradiction to the hypothesis $\liminf w(x\pm,t) = w_W - \beta$, with $\beta > 0$, thus verifying the assertion that $w(x\pm,t)$ tend to w_W, as (x,t) tends to $(\bar{x},\bar{t})$ through $\mathscr{W}$.

We now focus attention on $\phi_+(\cdot)$. We already have $\lim_{t\downarrow\bar{t}} w(\phi_+(t)-,t) = w_W$, $\lim_{t\downarrow\bar{t}} z(\phi_+(t)+,t) = z_N$, $\lim_{t\downarrow\bar{t}} w(\phi_+(t)+,t) = w_N$. We set $z_0 = \lim_{t\downarrow\bar{t}} z(\phi_+(t)-,t)$. Then $\lambda(z_0, w_W) \ge \dot{\phi}_+(\bar{t}+) \ge \lambda(z_N, w_N)$. The aim is to show that $\dot{\phi}_+(\bar{t}+) = \lambda(z_0, w_W)$ so as to infer $z_N = z_0$, $w_N = w_W$. We consider the minimal backward 1-characteristic $\xi(\cdot)$ emanating from the point $(\phi_+(t_5), t_5)$, where $t_5 - \bar{t}$ is very small. The assertion $z_N = z_0$, $w_N = w_W$ is obviously true when $\xi \equiv \phi_+$, so let us assume that $\xi(t) < \phi_+(t)$ for $t \in (\bar{t}, t_5)$. Then $|w(\xi(t)+,t) - w_W|$ is very small on $(\bar{t}, t_5)$. Moreover, by Lemma 12.4.4, the oscillation of $z(\xi(t)+,t)$ over the interval $(\bar{t}, t_5)$ is very small, so this function takes values near z_0. Hence, $t_5 - \bar{t}$ sufficiently small renders $\dot{\xi}(\bar{t}+)$ arbitrarily close to $\lambda(z_0, w_W)$. Since $\dot{\xi}(\bar{t}+) \le \dot{\phi}_+(\bar{t}+)$, we conclude that $\dot{\phi}_+(\bar{t}+) \ge \lambda(z_0, w_W)$ and thus necessarily $\dot{\phi}_+(\bar{t}_+) = \lambda(z_0, w_W)$.

Consider now any forward 1-characteristic $\chi(\cdot)$ issuing from $(\bar{x},\bar{t})$, such that $\phi_-(t) \le \chi(t) \le \phi_+(t)$, for $t > \bar{t}$. Since $\lim_{t\downarrow\bar{t}} w(\chi(t)-,t)$ and $\lim_{t\downarrow\bar{t}} w(\chi(t)+,t)$ take the same value, namely w_W, $\lim_{t\downarrow\bar{t}} z(\chi(t)-,t)$ and $\lim_{t\downarrow\bar{t}} z(\chi(t)+,t)$ must also take the same value, say z_χ. In particular, $\dot{\chi}(\bar{t}+) = \lambda(z_\chi, w_W)$. Therefore, if $p_\ell(\cdot)$ and $p_r(\cdot)$ are any 1-characteristics with $\phi_-(t) \le p_\ell(t) \le p_r(t) \le \phi_+(t)$, for $t > \bar{t}$, then the inequalities $\dot{\phi}_-(\bar{t}+) \le \dot{p}_\ell(\bar{t}+) \le \dot{p}_r(\bar{t}+) \le \dot{\phi}_+(\bar{t}+)$, ordering the speeds of propagation at $\bar{t}$, together with $\lambda_z < 0$, imply $(12.3.8)_1$. The proof is complete.

We have now completed the proof of Theorem 12.3.3, on local regularity, as well as of Theorem 12.4.1, on the laws of propagation of Riemann invariants along extremal backward characteristics. These will serve as the principal tools for deriving a priori estimates leading to a description of the long-time behavior of solutions.

Henceforth, our solutions will be normalized on $(-\infty,\infty)\times(0,\infty)$ by defining $(z(x,t),w(x,t)) = (z_S,w_S)$, namely the "southern" limit at (x,t). The trace of the solution on any space-like curve is then defined as the restriction of the normalized (z,w) to this curve. In particular, this renders the trace of (z,w) along the minimal backward 1-characteristic and the maximal backward 2-characteristic, emanating from any point $(\bar{x},\bar{t})$, continuous from the left on $(0,\bar{t}]$.

12.5 Bounds on Solutions

We consider a solution, normalized as above, bounded by

$$|z(x,t)|+|w(x,t)| < 2\delta, \qquad -\infty < x < \infty,\ \ 0 < t < \infty, \tag{12.5.1}$$

where δ is a small positive constant. It is convenient to regard the initial data as multi-valued functions, allowing $(z(x,0),w(x,0))$ to take as values any state in the range of the solution of the Riemann problem with end-states $(z(x\pm,0),w(x\pm,0))$. The supremum and total variation are measured for the selection that maximizes these quantities. We then assume

$$\sup_{(-\infty,\infty)}|z(\cdot,0)| + \sup_{(-\infty,\infty)}|w(\cdot,0)| \leq \delta, \tag{12.5.2}$$

$$TV_{(-\infty,\infty)}z(\cdot,0) + TV_{(-\infty,\infty)}w(\cdot,0) < a\delta^{-1}, \tag{12.5.3}$$

where a is a small constant, to be fixed later, independently of δ. Thus, there is a tradeoff, allowing for arbitrarily large total variation at the expense of keeping the oscillation sufficiently small. The aim is to establish bounds on the solution. In what follows, c will stand for a generic constant that depends solely on F. The principal result is

12.5.1 Theorem. *Consider any space-like curve $t = t^*(x)$, $x_\ell \leq x \leq x_r$, in the upper half-plane, along which the trace of (z,w) is denoted by (z^*,w^*). Then*

$(12.5.4)_1$
$$TV_{[x_\ell,x_r]}z^*(\cdot) \leq TV_{[\xi_\ell(0),\xi_r(0)]}z(\cdot,0) + c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\},$$

$(12.5.4)_2$
$$TV_{[x_\ell,x_r]}w^*(\cdot) \leq TV_{[\zeta_\ell(0),\zeta_r(0)]}w(\cdot,0) + c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0) + TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\},$$

where $\xi_\ell(\cdot)$, $\xi_r(\cdot)$ are the minimal backward 1-characteristics and $\zeta_\ell(\cdot)$, $\zeta_r(\cdot)$ are the maximal backward 2-characteristics emanating from the endpoints (x_ℓ,t_ℓ) and (x_r,t_r) of the graph of $t^*(\cdot)$.

Since generalized characteristics are space-like curves, one may combine the above proposition with Theorem 12.4.1 and the assumptions (12.5.1), (12.5.3) to deduce the following corollary:

12.5.2 Theorem. *For any point (x,t) of the upper half-plane:*

$$\sup_{(-\infty,\infty)} z(\cdot,0) \geq z(x,t) \geq \inf_{(-\infty,\infty)} z(\cdot,0) - ca\delta, \tag{12.5.5$_1$}$$

$$\sup_{(-\infty,\infty)} w(\cdot,0) \geq w(x,t) \geq \inf_{(-\infty,\infty)} w(\cdot,0) - ca\delta. \tag{12.5.5$_2$}$$

Thus, on account of our assumption (12.5.2) and by selecting a sufficiently small, we ensure a posteriori that the solution will satisfy (12.5.1).

The task of proving Theorem 12.5.1 is quite laborious and will require extensive preparation. In the course of the proof we shall verify that certain quantities measuring the total amount of wave interaction are also bounded.

Consider a 1-shock joining the state (z_-,w_-), on the left, with the state (z_+,w_+), on the right. The jumps $\Delta z = z_+ - z_-$ and $\Delta w = w_+ - w_-$ are related through an equation

$$\Delta w = f(\Delta z; z_-, w_-) \tag{12.5.6$_1$}$$

resulting from the reparametrization of the 1-shock curve emanating from the state (z_-,w_-). In particular, f and its first two derivatives with respect to Δz vanish at $\Delta z = 0$ and hence f as well as $\partial f/\partial z_-$ and $\partial f/\partial w_-$ are $O(\Delta z^3)$ as $\Delta z \to 0$.

Similarly, the jumps $\Delta w = w_+ - w_-$ and $\Delta z = z_+ - z_-$ of the Riemann invariants across a 2-shock joining the state (z_-,w_-), on the left, with the state (z_+,w_+), on the right, are related through an equation

$$\Delta z = g(\Delta w; z_+, w_+) \tag{12.5.6$_2$}$$

resulting from the reparametrization of the backward 2-shock curve (see Section 9.3) that emanates from the state (z_+,w_+). Furthermore, g together with $\partial g/\partial z_+$ and $\partial g/\partial w_+$ are $O(\Delta w^3)$ as $\Delta w \to 0$.

For convenience, points of the upper half-plane will be labeled by single capital letters I, J, etc. With any point $I = (\bar{x},\bar{t})$ we associate the special characteristics $\phi^I_\pm, \psi^I_\pm, \xi^I_\pm, \zeta^I_\pm$ emanating from it, as discussed in Section 12.3 and depicted in Fig. 12.3.1, and identify the limits $(z^I_W, w^I_W), (z^I_E, w^I_E), (z^I_N, w^I_N), (z^I_S, w^I_S)$ as I is approached through the sectors $\mathscr{S}^I_W, \mathscr{S}^I_E, \mathscr{S}^I_N, \mathscr{S}^I_S$. From I emanate minimal 1-separatrices $p^I_\pm$ and maximal 2-separatrices $q^I_\pm$ constructed as follows: p^I_- (or q^I_+) is simply the minimal (or maximal) backward 1-characteristic ξ^I_- (or 2-characteristic ζ^I_+) emanating from I; while p^I_+ (or q^I_-) is the limit of a sequence of minimal (or maximal) backward 1-characteristics ξ_n (or 2-characteristics ζ_n) emanating from points (x_n,t_n) in $\mathscr{S}^I_E$ (or $\mathscr{S}^I_W$), where $(x_n,t_n) \to (\bar{x},\bar{t})$, as $n \to \infty$. We introduce the notation

$$\mathscr{F}_I = \{(x,t) : 0 \leq t < \bar{t},\ p^I_-(t) \leq x \leq p^I_+(t)\}, \tag{12.5.7$_1$}$$

$$\mathscr{G}_I = \{(x,t) : 0 \leq t < \bar{t},\ q^I_-(t) \leq x \leq q^I_+(t)\}. \tag{12.5.7$_2$}$$

By virtue of Theorems 12.3.3 and 12.4.1,

$$(12.5.8)_1 \qquad \lim_{t\uparrow\bar{t}} z(p_-^I(t),t) = z_S^I\,, \qquad \lim_{t\uparrow\bar{t}} z(p_+^I(t),t) = z_E^I\,,$$

$$(12.5.8)_2 \qquad \lim_{t\uparrow\bar{t}} w(q_-^I(t),t) = w_W^I\,, \qquad \lim_{t\uparrow\bar{t}} w(q_+^I(t),t) = w_S^I\,.$$

The cumulative strength of 1-waves and 2-waves, incoming at I, is respectively measured by

$$(12.5.9) \qquad \Delta z^I = z_E^I - z_S^I\,, \qquad \Delta w^I = w_S^I - w_W^I\,.$$

If the incoming 1-waves alone were allowed to interact, they would produce an outgoing 1-shock with w-amplitude

$$(12.5.10)_1 \qquad \Delta w_*^I = f(\Delta z^I; z_S^I, w_S^I),$$

together with an outgoing 2-rarefaction wave. Consequently, $|\Delta w_*^I|$ exceeds the cumulative w-strength $|w_E^I - w_S^I|$ of incoming 1-waves. Similarly, the interaction of incoming 2-waves alone would produce an outgoing 2-shock with z-amplitude

$$(12.5.10)_2 \qquad \Delta z_*^I = g(\Delta w^I; z_S^I, w_S^I),$$

exceeding their cumulative z-strength $z_S^I - z_W^I$. Note that if $z_S^I = z_W^I$, $w_S^I = w_W^I$ then $\Delta w_*^I = w_N^I - w_W^I$, while if $z_S^I = z_E^I$, $w_S^I = w_E^I$ then $\Delta z_*^I = z_E^I - z_N^I$.

We visualize the upper half-plane as a partially ordered set under the relation induced by the rule $I < J$ whenever J is confined between the graphs of the minimal 1-separatrices p_-^I and p_+^I emanating from I. In particular, when J lies strictly to the right of the graph of p_-^I then I lies on the graph of the 1-characteristic ϕ_-^J emanating from J. Thus $I < J$ implies that I always lies on the graph of a forward 1-characteristic issuing from J, that is either ϕ_-^J or p_-^I. This special characteristic will be denoted by χ_-^J.

We consider 1-*characteristic trees* $\mathscr{M}$ consisting of a finite set of points of the upper half-plane, called *nodes*, with the following properties: $\mathscr{M}$ contains a unique minimal node I_0, namely the *root* of the tree. Furthermore, if J and K are any two nodes, then the point I of confluence of the forward 1-characteristics χ_-^J and χ_-^K, which pass through the root I_0, is also a node of $\mathscr{M}$. In general, $\mathscr{M}$ will contain several maximal nodes (Fig. 12.5.1).

Every node $J \neq I_0$ is *consecutive* to some node I, namely, its strict greatest lower bound relative to $\mathscr{M}$. The set of nodes that are consecutive to a node I is denoted by $\mathscr{C}_I$. When J is consecutive to I, the pair (I,J) is called a *link*. A finite sequence $\{I_0, I_1, \cdots, I_m\}$ of nodes such that I_{j+1} is consecutive to I_j, for $j = 0, \cdots, m-1$, which connects the root I_0 with some maximal node I_m, constitutes a *chain* of $\mathscr{M}$.

If (I,J) is a link of $\mathscr{M}$, so that $I = (\chi_-^J(\bar{t}), \bar{t})$, we set

$$(12.5.11)_1 \qquad z_\pm^{IJ} = \lim_{t\uparrow\bar{t}} z(\chi_-^J(t)\pm, t), \qquad w_\pm^{IJ} = \lim_{t\uparrow\bar{t}} w(\chi_-^J(t)\pm, t),$$

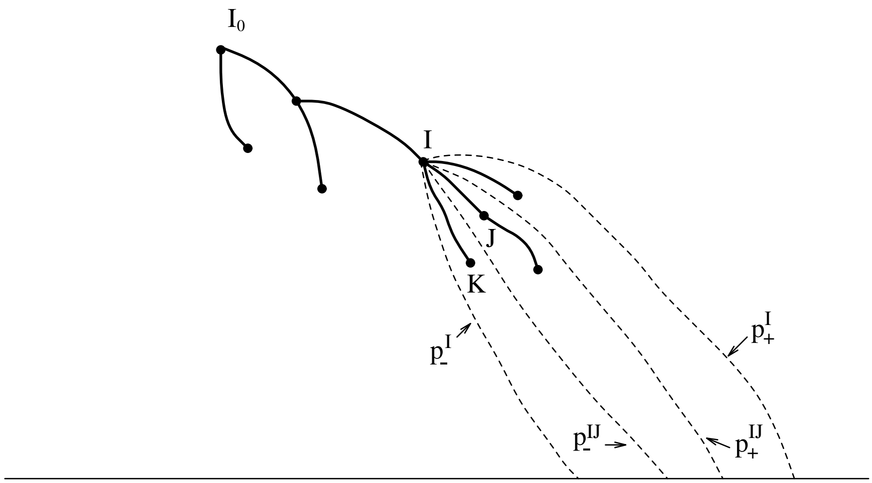

Fig. 12.5.1

$$(12.5.12)_1 \qquad \Delta z^{IJ} = z_+^{IJ} - z_-^{IJ}\,, \qquad \Delta w^{IJ} = w_+^{IJ} - w_-^{IJ}\,.$$

In particular,

$$(12.5.13)_1 \qquad \Delta w^{IJ} = f(\Delta z^{IJ}; z_-^{IJ}, w_-^{IJ}).$$

With (I,J) we associate minimal 1-separatrices $p_\pm^{IJ}$, emanating from I, constructed as follows: p_-^{IJ} is the $t \uparrow \bar{t}$ limit of the family ξ_t of minimal backward 1-characteristics emanating from the point $(\chi_-^J(t), t)$; while p_+^{IJ} is the limit of a sequence of minimal backward 1-characteristics ξ_n emanating from points (x_n, t_n) such that, as $n \to \infty$, we have $t_n \uparrow \bar{t}$, $x_n - \chi_-^J(t_n) \downarrow 0$ and $z(x_n-, t_n) \to z_+^{IJ}$, $w(x_n-, t_n) \to w_+^{IJ}$. Notice that the graphs of $p_\pm^{IJ}$ are confined between the graph of p_-^I and the graph of p_+^I; see Fig. 12.5.1. In turn, the graphs of $p_\pm^J$, as well as the graphs of $p_\pm^K$, for any $K > J$, are confined between the graph of p_-^{IJ} and the graph of p_+^{IJ}. Furthermore,

$$(12.5.14)_1 \qquad \lim_{t \uparrow \bar{t}} z(p_-^{IJ}(t), t) = z_-^{IJ}\,, \qquad \lim_{t \uparrow \bar{t}} z(p_+^{IJ}(t), t) = z_+^{IJ}\,.$$

Indeed, the first of the above two equations has already been established in the context of the proof of Lemma 12.4.9 (under different notation; see (12.4.62)); while the second may be verified by a similar argument.

We now set

$$(12.5.15)_1 \qquad \mathscr{P}_1(\mathscr{M}) = -\sum_{I \in \mathscr{M}} [\Delta w_*^I - \sum_{J \in \mathscr{C}_I} \Delta w^{IJ}]\,,$$

$$(12.5.16)_1 \qquad \mathscr{Q}_1(\mathscr{M}) = \sum_{I \in \mathscr{M}} \sum_{J \in \mathscr{C}_I} |\Delta w^{IJ} - \Delta w_*^J|\,.$$

By virtue of $(12.3.7)_1$,

$$(12.5.17)_1 \qquad \sum_{J\in\mathscr{C}_I} \Delta w^{IJ} \geq w_E^I - w_S^I \geq \Delta w_*^I,$$

so that both $\mathscr{P}_1$ and $\mathscr{Q}_1$ are nonnegative.

With subsets $\mathscr{F}$ of the upper half-plane, we associate functionals

$$(12.5.18)_1 \qquad \mathscr{P}_1(\mathscr{F}) = \sup_{\mathscr{J}} \sum_{\mathscr{M}\in\mathscr{J}} \mathscr{P}_1(\mathscr{M}),$$

$$(12.5.19)_1 \qquad \mathscr{Q}_1(\mathscr{F}) = \sup_{\mathscr{J}} \sum_{\mathscr{M}\in\mathscr{J}} \mathscr{Q}_1(\mathscr{M}),$$

where $\mathscr{J}$ denotes any (finite) collection of 1-characteristic trees $\mathscr{M}$ contained in $\mathscr{F}$, which are disjoint, in the sense that the roots of any pair of them are non-comparable. One may view $\mathscr{P}_1(\mathscr{F})$ as a measure of the amount of 1-wave interactions inside $\mathscr{F}$, and $\mathscr{Q}_1(\mathscr{F})$ as a measure of the strengthening of 1-shocks induced by interaction with 2-waves.

We introduce corresponding notions for the 2-characteristic family: $I < J$ whenever J is confined between the graphs of the maximal 2-separatrices q_-^I and q_+^I emanating from I. In that case, I lies on the graph of a forward 2-characteristic χ_+^J issuing from J, namely either ψ_+^J or q_+^I. One may then construct 2-*characteristic trees* $\mathscr{N}$, with *nodes*, *root*, *links* and *chains* defined as above. In the place of $(12.5.11)_1$, $(12.5.12)_1$ and $(12.5.13)_1$, we now have

$$(12.5.11)_2 \qquad z_\pm^{IJ} = \lim_{t\uparrow\bar{t}} z(\chi_+^J(t)\pm,t), \qquad w_\pm^{IJ} = \lim_{t\uparrow\bar{t}} w(\chi_+^J(t)\pm,t),$$

$$(12.5.12)_2 \qquad \Delta z^{IJ} = z_+^{IJ} - z_-^{IJ}, \qquad \Delta w^{IJ} = w_+^{IJ} - w_-^{IJ},$$

$$(12.5.13)_2 \qquad \Delta z^{IJ} = g(\Delta w^{IJ}; z_+^{IJ}, w_+^{IJ}).$$

With links (I,J) we associate maximal 2-separatrices $q_\pm^{IJ}$, emanating from I, in analogy to $p_\pm^{IJ}$. The graphs of $q_\pm^{IJ}$ are confined between the graphs of q_-^I and q_+^I. On the other hand, the graphs of $q_\pm^J$ are confined between the graphs of q_-^{IJ} and q_+^{IJ}. In the place of $(12.5.14)_1$,

$$(12.5.14)_2 \qquad \lim_{t\uparrow\bar{t}} w(q_-^{IJ}(t),t) = w_-^{IJ}, \qquad \lim_{t\uparrow\bar{t}} w(q_+^{IJ}(t),t) = w_+^{IJ}.$$

Analogs of $(12.5.15)_1$ and $(12.5.16)_1$ are also defined:

$$(12.5.15)_2 \qquad \mathscr{P}_2(\mathscr{N}) = \sum_{I\in\mathscr{N}} [\Delta z_*^I - \sum_{J\in\mathscr{C}_I} \Delta z^{IJ}],$$

$$(12.5.16)_2 \qquad \mathscr{Q}_2(\mathscr{N}) = \sum_{I\in\mathscr{N}} \sum_{J\in\mathscr{C}_I} |\Delta z^{IJ} - \Delta z_*^J|,$$

which are nonnegative since

$$(12.5.17)_2 \qquad \sum_{J\in\mathscr{C}_I} \Delta z^{IJ} \le z_S^I - z_W^I \le \Delta z_*^I .$$

This induces functionals analogous to $\mathscr{P}_1$ and $\mathscr{Q}_1$:

$$(12.5.18)_2 \qquad \mathscr{P}_2(\mathscr{F}) = \sup_{\mathscr{J}} \sum_{\mathscr{N}\in\mathscr{J}} \mathscr{P}_2(\mathscr{N}),$$

$$(12.5.19)_2 \qquad \mathscr{Q}_2(\mathscr{F}) = \sup_{\mathscr{J}} \sum_{\mathscr{N}\in\mathscr{J}} \mathscr{Q}_2(\mathscr{N}).$$

12.5.3 Lemma. *Let $\mathscr{F}_1,\cdots,\mathscr{F}_m$ be a collection of subsets of a set $\mathscr{F}$ contained in the upper half-plane. Suppose that for any $I\in\mathscr{F}_i$ and $J\in\mathscr{F}_j$ that are comparable, say $I<J$, the arc of the characteristic χ_-^J (or χ_+^J) which connects J to I is contained in $\mathscr{F}$. Then*

$$(12.5.20)_1 \qquad \sum_{i=1}^m \{\mathscr{P}_1(\mathscr{F}_i) + \mathscr{Q}_1(\mathscr{F}_i)\} \le k\{\mathscr{P}_1(\mathscr{F}) + \mathscr{Q}_1(\mathscr{F})\},$$

or

$$(12.5.20)_2 \qquad \sum_{i=1}^m \{\mathscr{P}_2(\mathscr{F}_i) + \mathscr{Q}_2(\mathscr{F}_i)\} \le k\{\mathscr{P}_2(\mathscr{F}) + \mathscr{Q}_2(\mathscr{F})\},$$

where k is the smallest positive integer with the property that any $k+1$ of $\mathscr{F}_1,\cdots,\mathscr{F}_m$ have empty intersection.

Proof. It will suffice to verify $(12.5.20)_1$. With each $i=1,\cdots,m$, we associate a family $\mathscr{J}_i$ of disjoint 1-characteristic trees $\mathscr{M}$ contained in $\mathscr{F}_i$. Clearly, by adjoining if necessary additional nodes contained in $\mathscr{F}$, one may extend the collection of the $\mathscr{J}_i$ into a single family $\mathscr{J}$ of disjoint trees contained in $\mathscr{F}$. The contribution of the additional nodes may only increase the value of $\mathscr{P}_1$ and $\mathscr{Q}_1$. Therefore,

$$(12.5.21) \qquad \sum_{i=1}^m \sum_{\mathscr{M}\in\mathscr{J}_i} \{\mathscr{P}_1(\mathscr{M}) + \mathscr{Q}_1(\mathscr{M})\} \le k \sum_{\mathscr{M}\in\mathscr{J}} \{\mathscr{P}_1(\mathscr{M}) + \mathscr{Q}_1(\mathscr{M})\},$$

where the factor k appears on the right-hand side because the same node or link may be counted up to k times on the left-hand side. Recalling $(12.5.18)_1$ and $(12.5.19)_1$, we arrive at $(12.5.20)_1$. The proof is complete.

12.5.4 Lemma. *Consider a space-like curve $t=\bar{t}(x)$, $\hat{x}\le x\le\tilde{x}$, in the upper half-plane. The trace of (z,w) along $\bar{t}$ is denoted by $(\bar{z},\bar{w})$. Let $\hat{p}(\cdot)$ and $\tilde{p}(\cdot)$ (or $\hat{q}(\cdot)$*

and $\tilde{q}(\cdot)$) be minimal (or maximal) 1-separatrices (or 2-separatrices) emanating from the left endpoint $(\hat{x},\hat{t})$ and the right endpoint $(\tilde{x},\tilde{t})$ of the graph of $\bar{t}$. The trace of z (or w) along $\hat{p}$ and $\tilde{p}$ (or $\hat{q}$ and $\tilde{q}$) is denoted by $\hat{z}$ and $\tilde{z}$ (or $\hat{w}$ and $\tilde{w}$). Let $\mathscr{F}$ (or $\mathscr{G}$) stand for the region bordered by the graphs of $\hat{p}$, $\tilde{p}$ (or $\hat{q}$, $\tilde{q}$), $\bar{t}$ and the x-axis. Then

$(12.5.22)_1$

$$|\tilde{z}(\tilde{t}-)-\hat{z}(\hat{t}-)| \le |\tilde{z}(0+)-\hat{z}(0+)| + c\delta^2 TV_{[\hat{x},\tilde{x}]}\bar{w}(\cdot) + \mathscr{P}_2(\mathscr{F}) + \mathscr{Q}_2(\mathscr{F}),$$

or

$(12.5.22)_2$

$$|\tilde{w}(\tilde{t}-)-\hat{w}(\hat{t}-)| \le |\tilde{w}(0+)-\hat{w}(0+)| + c\delta^2 TV_{[\hat{x},\tilde{x}]}\bar{z}(\cdot) + \mathscr{P}_1(\mathscr{G}) + \mathscr{Q}_1(\mathscr{G}).$$

Proof. It will suffice to verify $(12.5.22)_1$. We write

$$\tilde{z}(\tilde{t}-)-\hat{z}(\hat{t}-) = [\tilde{z}(0+)-\hat{z}(0+)] + [\tilde{z}(\tilde{t}-)-\tilde{z}(0+)] - [\hat{z}(\hat{t}-)-\hat{z}(0+)]. \tag{12.5.23}$$

By virtue of Theorem 12.4.1,

$$\begin{cases} \hat{z}(\hat{t}-)-\hat{z}(0+) = \sum[\hat{z}(\tau+)-\hat{z}(\tau-)], \\ \tilde{z}(\tilde{t}-)-\tilde{z}(0+) = \sum[\tilde{z}(\tau+)-\tilde{z}(\tau-)], \end{cases} \tag{12.5.24}$$

where the summations run over the countable set of jump discontinuities of $\hat{z}(\cdot)$ and $\tilde{z}(\cdot)$.

On account of Theorem 12.3.3, if $\bar{z}(\cdot)$ is the trace of z along any minimal 1-separatrix which passes through some point $K=(x,\tau)$, then

$$z_S^K - z_W^K \le \bar{z}(\tau-)-\bar{z}(\tau+) \le \Delta z_*^K. \tag{12.5.25}$$

Starting out from points K of jump discontinuity of $\hat{z}(\cdot)$ on the graph of $\hat{p}$, we construct the characteristic ψ_-^K until it intersects the graph of either $\tilde{p}$ or $\bar{t}$. This generates families of disjoint 2-characteristic trees $\mathscr{N}$, with maximal nodes, say $K_1=(x_1,\tau_1),\cdots,K_m=(x_m,\tau_m)$, lying on the graph of $\hat{p}$, and root $K_0=(x_0,\tau_0)$ lying on the graph of either $\tilde{p}$ or $\bar{t}$. In the former case, on account of (12.5.25), $(12.5.15)_2$ and $(12.5.16)_2$,

$$|\tilde{z}(\tau_0+)-\tilde{z}(\tau_0-) - \sum_{\ell=1}^{m}[\hat{z}(\tau_\ell+)-\hat{z}(\tau_\ell-)]| \le \mathscr{P}_2(\mathscr{N}) + \mathscr{Q}_2(\mathscr{N}). \tag{12.5.26}$$

On the other hand, if K_0 lies on the graph of $\bar{t}$,

$$\sum_{J\in\mathscr{C}_{K_0}} \Delta z^{K_0 J} \le z_S^{K_0} - z_W^{K_0} \le c\delta^2 |w_S^{K_0} - w_W^{K_0}|, \tag{12.5.27}$$

and so

(12.5.28) $$|-\sum_{\ell=1}^{m}[\hat{z}(\tau_\ell+)-\hat{z}(\tau_\ell-)]| \leq c\delta^2|w_S^{K_0}-w_W^{K_0}|+\mathscr{P}_2(\mathscr{N})+\mathscr{Q}_2(\mathscr{N}).$$

Suppose that on the graph of $\tilde{p}$ there still remain points K_0 of jump discontinuity of $\tilde{z}(\cdot)$ which cannot be realized as roots of trees with maximal nodes on the graph of $\hat{p}$. We then adjoin (trivial) 2-characteristic trees $\mathscr{N}$ that contain a single node, namely such a $K_0=(x_0,\tau_0)$, in which case

(12.5.29) $$|\tilde{z}(\tau_0+)-\tilde{z}(\tau_0-)| \leq \mathscr{P}_2(\mathscr{N})+\mathscr{Q}_2(\mathscr{N}).$$

Recalling (12.5.23) and tallying the jump discontinuities of $\bar{z}_1(\cdot)$ and $\bar{z}_2(\cdot)$, as indicated in (12.5.24), according to (12.5.26), (12.5.28) or (12.5.29), we arrive at $(12.5.22)_1$. The proof is complete.

12.5.5 Lemma. *Under the assumptions of Theorem* 12.5.1,

$(12.5.30)_1$

$$TV_{[x_\ell,x_r]}z^*(\cdot) \leq TV_{[\xi_\ell(0),\xi_r(0)]}z(\cdot,0)+c\delta^2 TV_{[x_\ell,x_r]}w^*(\cdot)+2\{\mathscr{P}_2(\mathscr{F})+\mathscr{Q}_2(\mathscr{F})\},$$

$(12.5.30)_2$

$$TV_{[x_\ell,x_r]}w^*(\cdot) \leq TV_{[\zeta_\ell(0),\zeta_r(0)]}w(\cdot,0)+c\delta^2 TV_{[x_\ell,x_r]}z^*(\cdot)+2\{\mathscr{P}_1(\mathscr{G})+\mathscr{Q}_1(\mathscr{G})\},$$

where $\mathscr{F}$ denotes the region bordered by the graphs of ξ_ℓ, ξ_r, t^*, and the x-axis, while $\mathscr{G}$ stands for the region bordered by the graphs of ζ_ℓ, ζ_r, t^*, and the x-axis.

Proof. It will suffice to establish $(12.5.30)_1$. We have to estimate

(12.5.31) $$TV_{[x_\ell,x_r]}z^*(\cdot)=\sup\sum_{i=1}^{m}|z_S^{L_i}-z_S^{L_{i-1}}|,$$

where the supremum is taken over all finite sequences $\{L_0,\cdots,L_m\}$ of points along t^* (Fig. 12.5.2).

We construct the minimal backward 1-characteristics ξ_i emanating from the point $L_i=(x_i,t_i)$, $i=0,\cdots,m$, and let $z_i(\cdot)$ denote the trace of z along $\xi_i(\cdot)$. We apply Lemma 12.5.4 with $\bar{t}$ the arc of t^* with endpoints L_{i-1} and L_i; $\hat{x}=x_{i-1}$; $\tilde{x}=x_i$; $\hat{p}=\xi_{i-1}$; $\tilde{p}=\xi_i$; and $\mathscr{F}=\mathscr{F}_i$, namely the region bordered by the graphs of ξ_{i-1}, ξ_i, t^*, and the x-axis. The estimate $(12.5.22)_1$ then yields

(12.5.32)

$$|z_S^{L_i}-z_S^{L_{i-1}}| \leq |z_i(0+)-z_{i-1}(0+)|+c\delta^2 TV_{[x_{i-1},x_i]}w^*(\cdot)+\mathscr{P}_2(\mathscr{F}_i)+\mathscr{Q}_2(\mathscr{F}_i).$$

Combining (12.5.31), (12.5.32) and Lemma 12.5.3, we arrive at $(12.5.30)_1$. The proof is complete.

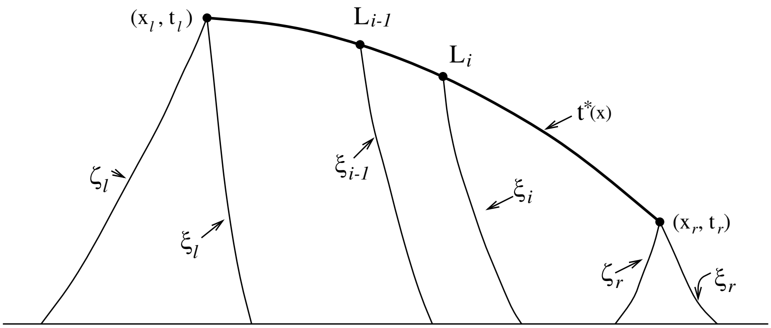

Fig. 12.5.2

12.5.6 Lemma. *Let $\mathscr{M}$ (or $\mathscr{N}$) be a* 1*-characteristic (or* 2*-characteristic) tree rooted at* I_0. *Then*

$(12.5.33)_1$

$$\mathscr{P}_1(\mathscr{M})+\mathscr{Q}_1(\mathscr{M})\le c\delta^2(1+V_{\mathscr{M}})\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0)+\mathscr{P}_2(\mathscr{F}_{I_0})+\mathscr{Q}_2(\mathscr{F}_{I_0})\},$$

or

$(12.5.33)_2$

$$\mathscr{P}_2(\mathscr{N})+\mathscr{Q}_2(\mathscr{N})\le c\delta^2(1+W_{\mathscr{N}})\{TV_{[q_-^{I_0}(0),q_+^{I_0}(0)]}w(\cdot,0)+\mathscr{P}_1(\mathscr{G}_{I_0})+\mathscr{Q}_1(\mathscr{G}_{I_0})\},$$

where $V_{\mathscr{M}}$ *(or* $W_{\mathscr{N}}$*) denotes the maximum of*

$$(12.5.34)_1 \qquad \sum_{i=0}^{m-1}\{|z_-^{I_iI_{i+1}}-z_S^{I_{i+1}}|+|w_-^{I_iI_{i+1}}-w_S^{I_{i+1}}|\}$$

or

$$(12.5.34)_2 \qquad \sum_{i=0}^{m-1}\{|z_+^{I_iI_{i+1}}-z_S^{I_{i+1}}|+|w_+^{I_iI_{i+1}}-w_S^{I_{i+1}}|\}$$

over all chains $\{I_0,\cdots,I_m\}$ *of* $\mathscr{M}$ *(or* $\mathscr{N}$*).*

Proof. It will suffice to validate $(12.5.33)_1$, the other case being completely analogous. By virtue of $(12.5.15)_1$ and $(12.5.16)_1$,

$$(12.5.35) \qquad \begin{aligned}\mathscr{P}_1(\mathscr{M})&\le-\sum_{I\in\mathscr{M}}[\Delta w_*^I-\sum_{J\in\mathscr{C}_I}\Delta w_*^J]+\mathscr{Q}_1(\mathscr{M})\\ &=\sum_{\substack{\text{maximal}\\ \text{nodes}}}\Delta w_*^K-\Delta w_*^{I_0}+\mathscr{Q}_1(\mathscr{M}).\end{aligned}$$

Since $\Delta w_*^K \leq 0$, to establish $(12.5.33)_1$ it is sufficient to show

$$-\Delta w_*^{I_0} \leq c\delta^2\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0)+\mathscr{P}_2(\mathscr{F}_{I_0})+\mathscr{Q}_2(\mathscr{F}_{I_0})\}, \tag{12.5.36}$$

$$\mathscr{Q}_1(\mathscr{M}) \leq c\delta^2(1+V_{\mathscr{M}})\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0)+\mathscr{P}_2(\mathscr{F}_{I_0})+\mathscr{Q}_2(\mathscr{F}_{I_0})\}. \tag{12.5.37}$$

To demonstrate (12.5.36), we first employ $(12.5.10)_1$ to get

$$-\Delta w_*^{I_0} = -f(\Delta z^{I_0}; z_S^{I_0}, w_S^{I_0}) \leq c\delta^2 \Delta z^{I_0}\,, \tag{12.5.38}$$

and then, to estimate Δz^{I_0}, we apply Lemma 12.5.4, with $(\hat{x},\hat{t}) = (\tilde{x},\tilde{t}) = I_0\,, \hat{p} = p_-^{I_0}$ and $\tilde{p} = p_+^{I_0}$.

We now turn to the proof of (12.5.37), recalling the definition $(12.5.16)_1$ of $\mathscr{Q}_1(\mathscr{M})$. For any nodes $I \in \mathscr{M}$ and $J \in \mathscr{C}_I$, we use $(12.5.10)_1$ and $(12.5.13)_1$ to get

$$\begin{aligned}\Delta w^{IJ} - \Delta w_*^J = {} & f(\Delta z^{IJ}; z_-^{IJ}, w_-^{IJ}) - f(\Delta z^{IJ}; z_S^J, w_S^J) \\ & + f(\Delta z^{IJ}; z_S^J, w_S^J) - f(\Delta z^J; z_S^J, w_S^J).\end{aligned} \tag{12.5.39}$$

On account of the properties of the function f,

$$|f(\Delta z^{IJ}; z_-^{IJ}, w_-^{IJ}) - f(\Delta z^{IJ}; z_S^J, w_S^J)| \leq c\delta^2 \Delta z^{IJ}\{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\}, \tag{12.5.40}$$

$$|f(\Delta z^{IJ}; z_S^J, w_S^J) - f(\Delta z^J; z_S^J, w_S^J)| \leq c\delta^2 |\Delta z^{IJ} - \Delta z^J|. \tag{12.5.41}$$

Thus, to verify (12.5.37) we have to show

$$\begin{aligned}&\sum_{I\in\mathscr{M}}\sum_{J\in\mathscr{C}_I} \Delta z^{IJ}\{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\} \\ &\qquad\qquad \leq V_{\mathscr{M}}\{\Delta z^{I_0} + \sum_{I\in\mathscr{M}}\sum_{J\in\mathscr{C}_I}|\Delta z^{IJ} - \Delta z^J|\}\,,\end{aligned} \tag{12.5.42}$$

$$\sum_{I\in\mathscr{M}}\sum_{J\in\mathscr{C}_I}|\Delta z^{IJ} - \Delta z^J| \leq c\{TV_{[p_-^{I_0}(0),p_+^{I_0}(0)]}z(\cdot,0)+\mathscr{P}_2(\mathscr{F}_{I_0})+\mathscr{Q}_2(\mathscr{F}_{I_0})\}. \tag{12.5.43}$$

We tackle (12.5.42) first. We perform the summation starting out from the maximal nodes and moving down towards the root of $\mathscr{M}$. For $L \in \mathscr{M}$, we let $\mathscr{M}_L$ denote the subtree of $\mathscr{M}$ which is rooted at L and contains all $I \in \mathscr{M}$ with $L < I$. For some $K \in \mathscr{M}$, assume

$$\sum_{I\in\mathscr{M}_L}\sum_{J\in\mathscr{C}_I} \Delta z^{IJ}\{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\} \leq V_{\mathscr{M}_L}\{\Delta z^L + \sum_{I\in\mathscr{M}_L}\sum_{J\in\mathscr{C}_I}|\Delta z^{IJ} - \Delta z^J|\} \tag{12.5.44}$$

holds for every $L \in \mathscr{C}_K$. Since $\Delta z^L \leq \Delta z^{KL} + |\Delta z^{KL} - \Delta z^L|$ and

(12.5.45)
$$\sum_{L \in \mathscr{C}_K} \Delta z^{KL} \leq \Delta z^K,$$

(12.5.44) implies

(12.5.46)
$$\begin{aligned}\sum_{I \in \mathscr{M}_K} \sum_{J \in \mathscr{C}_I} \Delta z^{IJ} \{|z_-^{IJ} - z_S^J| + |w_-^{IJ} - w_S^J|\} & \\ \leq \sum_{L \in \mathscr{C}_K} \Delta z^{KL} \{|z_-^{KL} - z_S^L| + |w_-^{KL} - w_S^L| + V_{\mathscr{M}_L}\} & \\ + \sum_{L \in \mathscr{C}_K} V_{\mathscr{M}_L} \{|\Delta z^{KL} - \Delta z^L| + \sum_{I \in \mathscr{M}_L} \sum_{J \in \mathscr{C}_I} |\Delta z^{IJ} - \Delta z^J|\} & \\ \leq V_{\mathscr{M}_K} \{\Delta z^K + \sum_{I \in \mathscr{M}_K} \sum_{J \in \mathscr{C}_I} |\Delta z^{IJ} - \Delta z^J|\}. &\end{aligned}$$

Thus, proceeding step by step, we arrive at (12.5.42).

It remains to show (12.5.43). We note that

(12.5.47)
$$\Delta z^{IJ} - \Delta z^J = [z_+^{IJ} - z_E^J] + [z_S^J - z_-^{IJ}].$$

We bound the right-hand side by applying Lemma 12.5.4 twice: First with $(\hat{x}, \hat{t}) = J$, $(\tilde{x}, \tilde{t}) = I$, $\hat{p} = p_+^J$, $\tilde{p} = p_+^{IJ}$, and then with $(\hat{x}, \hat{t}) = I$, $(\tilde{x}, \tilde{t}) = J$, $\hat{p} = p_-^{IJ}$, $\tilde{p} = p_-^J$. In either case, the arc of χ_J^- joining I to J serves as $\bar{t}$. We combine the derivation of $(12.5.22)_1$ for the two cases: The characteristic ϕ_-^K issuing from any point K on the graph of p_+^J is always intercepted by the graph of p_+^{IJ}; never by the graph of χ_J^-. On the other hand, ϕ_-^K issuing from points K on the graph of p_-^{IJ} and crossing the graph of χ_J^- may be prolonged until they intersect the graph of p_+^{IJ}. Consequently, the contribution of the common $\bar{t}$ drops out and we are left with the estimate

(12.5.48)
$$|\Delta z^{IJ} - \Delta z^J| \leq TV_{[p_-^{IJ}(0), p_-^J(0)]} z(\cdot, 0) + TV_{[p_+^J(0), p_+^{IJ}(0)]} z(\cdot, 0) + \mathscr{P}_2(\mathscr{F}_{IJ}) + \mathscr{Q}_2(\mathscr{F}_{IJ}),$$

with $\mathscr{F}_{IJ}$ defined through

(12.5.49)
$$\mathscr{F}_{IJ} = \{(x,t) : 0 \leq t < t_I,\ p_-^{IJ}(t) \leq x \leq p_+^{IJ}\} \cap \overline{\mathscr{F}_J^C}.$$

When (I,J) and (K,L) are any two distinct links (possibly with $I = K$), the intervals $(p_-^{IJ}(0), p_-^J(0))$, $(p_+^J(0), p_+^{IJ}(0))$, $(p_-^{KL}(0), p_-^L(0))$ and $(p_+^L(0), p_+^{KL}(0))$ are pairwise disjoint; likewise, the interiors of the sets $\mathscr{F}_{IJ}$ and $\mathscr{F}_{KL}$ are disjoint. Therefore, by virtue of Lemma 12.5.3, tallying (12.5.48) over $J \in \mathscr{C}_I$ and then over $I \in \mathscr{M}$ yields (12.5.43). The proof is complete.

12.5.7 Lemma. *Under the assumptions of Theorem* 12.5.1, *if* $\mathscr{H}$ *denotes the region bordered by the graphs of* ζ_ℓ, ξ_r, t^*, *and the x-axis, then*

(12.5.50)
$$\mathscr{P}_1(\mathscr{H})+\mathscr{Q}_1(\mathscr{H})+\mathscr{P}_2(\mathscr{H})+\mathscr{Q}_2(\mathscr{H}) \le c\delta^2\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0)+TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)\}.$$

Proof. Consider any family $\mathscr{J}$ of disjoint 1-characteristic trees $\mathscr{M}$ contained in $\mathscr{H}$. If I and J are the roots of any two trees in $\mathscr{J}$, $(p_-^I(0),p_+^I(0))$ and $(p_-^J(0),p_+^J(0))$ are disjoint intervals contained in $(\zeta_\ell(0),\xi_\ell(0))$; also $\mathscr{F}_I$ and $\mathscr{F}_J$ are subsets of $\mathscr{H}$ with disjoint interiors. Consequently, by combining Lemmas 12.5.3 and 12.5.6 we deduce

$(12.5.51)_1$
$$\mathscr{P}_1(\mathscr{H})+\mathscr{Q}_1(\mathscr{H}) \le c\delta^2(1+V_{\mathscr{H}})\{TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0)+\mathscr{P}_2(\mathscr{H})+\mathscr{Q}_2(\mathscr{H})\},$$

where $V_{\mathscr{H}}$ denotes the supremum of the total variation of the trace of (z,w) over all 1-characteristics with graph contained in $\mathscr{H}$.

Similarly,

$(12.5.51)_2$
$$\mathscr{P}_2(\mathscr{H})+\mathscr{Q}_2(\mathscr{H}) \le c\delta^2(1+W_{\mathscr{H}})\{TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0)+\mathscr{P}_1(\mathscr{H})+\mathscr{Q}_1(\mathscr{H})\},$$

where $W_{\mathscr{H}}$ stands for the supremum of the total variation of the trace of (z,w) over all 1-characteristics with graph contained in $\mathscr{H}$.

The constants in $(12.5.30)_1$ and $(12.5.30)_2$ do not depend on the particular t^*, so long as $\mathscr{H}$ remains fixed. In particular, we may apply these estimates taking as t^* any 1-characteristic or 2-characteristic, contained in $\mathscr{H}$. Therefore,

$(12.5.52)_1$
$$(1-c\delta^2)V_{\mathscr{H}} \le TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0)+TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0) +2\{\mathscr{P}_1(\mathscr{H})+\mathscr{Q}_1(\mathscr{H})+\mathscr{P}_2(\mathscr{H})+\mathscr{Q}_2(\mathscr{H})\},$$

$(12.5.52)_2$
$$(1-c\delta^2)W_{\mathscr{H}} \le TV_{[\zeta_\ell(0),\xi_r(0)]}z(\cdot,0)+TV_{[\zeta_\ell(0),\xi_r(0)]}w(\cdot,0) +2\{\mathscr{P}_1(\mathscr{H})+\mathscr{Q}_1(\mathscr{H})+\mathscr{P}_2(\mathscr{H})+\mathscr{Q}_2(\mathscr{H})\}.$$

Combining $(12.5.51)_1$, $(12.5.51)_2$, $(12.5.52)_1$, $(12.5.52)_2$ and recalling (12.5.3), we deduce (12.5.50), provided δ is sufficiently small. This completes the proof.

We now combine Lemmas 12.5.5 and 12.5.7. Since $\mathscr{F}$ and $\mathscr{G}$ are subsets of $\mathscr{H}$, $(12.5.30)_1$, $(12.5.30)_2$ and (12.5.50) together imply $(12.5.4)_1$ and $(12.5.4)_2$. The assertion of Theorem 12.5.1 has thus been established.

In addition to serving as a stepping stone in the proof of Theorem 12.5.1, Lemma 12.5.7 reveals that the amount of self-interaction of waves of the first and second characteristic family, measured by $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively, as well as the amount of mutual interaction of waves of opposite families, measured by $\mathscr{Q}_1$ and $\mathscr{Q}_2$, are bounded and controlled by the total variation of the initial data.

In our derivation of (12.5.4), the initial data were regarded as multi-valued and their total variation was evaluated for the "most unfavorable" selection of allowable values. According to this convention, the set of values of $z(x,0)$ is either confined between $z(x-,0)$ and $z(x+,0)$ or else it lies within $c|w(x+,0)-w(x-,0)|^3$ distance from $z(x+,0)$; and an analogous property holds for $w(x,0)$. Consequently, (12.5.4) will still hold, with readjusted constant c, when $(z(\cdot,0),w(\cdot,0))$ are renormalized to be single-valued, for example continuous from the right at $\xi_r(0)$ and at $\zeta_r(0)$ and continuous from the left at any other point.

12.6 Spreading of Rarefaction Waves

In Section 11.2 we saw that the spreading of rarefaction waves induces one-sided Lipschitz conditions on solutions of genuinely nonlinear scalar conservation laws. Here we shall encounter a similar effect in the context of our system (12.1.1) of two conservation laws. We shall see that the spreading of 1- (or 2-) rarefaction waves acts to reduce the falling (or rising) slope of the corresponding Riemann invariant z (or w). Because of intervening wave interactions, this mechanism is no longer capable of sustaining one-sided Lipschitz conditions, as in the scalar case; it still manages, however, to keep the total variation of solutions bounded, independently of the initial data.

Let us consider again the solution (z,w) discussed in the previous section, with small oscillation (12.5.1). The principal result is

12.6.1 Theorem. *For any* $-\infty < x < y < \infty$ *and* $t > 0$,

$$TV_{[x,y]}z(\cdot,t) + TV_{[x,y]}w(\cdot,t) \leq b\,\frac{y-x}{t} + \beta\delta, \tag{12.6.1}$$

where b *and* β *are constants that may depend on* F *but are independent of the initial data.*

The proof of the above theorem will be partitioned into several steps. The notation introduced in Section 12.5 will be used here freely. In particular, as before, c will stand for a generic constant that may depend on F but is independent of δ.

12.6.2 Lemma. *Fix* $\bar{t} > 0$ *and pick any* $-\infty < x_\ell < x_r < \infty$, *with* $x_r - x_\ell$ *small compared to* $\bar{t}$. *Construct the minimal (or maximal) backward* 1-(*or* 2-) *characteristics* $\xi_\ell(\cdot)$, $\xi_r(\cdot)$ (*or* $\zeta_\ell(\cdot)$, $\zeta_r(\cdot)$) *emanating from* $(x_\ell,\bar{t})$, $(x_r,\bar{t})$, *and let* $\mathscr{F}$ (*or* $\mathscr{G}$) *denote the region bordered by the graphs of* ξ_ℓ, ξ_r (*or* ζ_ℓ, ζ_r) *and the time lines* $t=\bar{t}$ *and* $t=\bar{t}/2$. *Then*

$$z(x_\ell,\bar{t}) - z(x_r,\bar{t}) \leq \hat{c}\exp(\bar{c}\delta\bar{V})\frac{x_r-x_\ell}{\bar{t}} + \mathscr{P}_2(\mathscr{F}) + \mathscr{Q}_2(\mathscr{F}), \tag{12.6.2$_1$}$$

or

$$w(x_r,\bar{t}) - w(x_\ell,\bar{t}) \leq \hat{c}\exp(\bar{c}\delta\bar{V})\frac{x_r-x_\ell}{\bar{t}} + \mathscr{P}_1(\mathscr{G}) + \mathscr{Q}_1(\mathscr{G}), \tag{12.6.2$_2$}$$

where $\bar{V}$ denotes the total variation of the trace of w (or z) along $\xi_\ell(\cdot)$ (or $\zeta_r(\cdot)$) over the interval $[\frac{1}{2}\bar{t},\bar{t}]$.

Proof. It will suffice to show $(12.6.2)_1$. Let $(z_\ell(\cdot),w_\ell(\cdot))$ and $(z_r(\cdot),w_r(\cdot))$ denote the trace of (z,w) along $\xi_\ell(\cdot)$ and $\xi_r(\cdot)$, respectively.

We consider the infimum $\tilde{\mu}$ and the supremum $\bar{\mu}$ of the characteristic speed $\mu(z,w)$ over the range of the solution. The straight lines with slope $\tilde{\mu}$ and $\bar{\mu}$ emanating from the point $(\xi_r(t),t)$, $t\in[\frac{1}{2}\bar{t},\bar{t}]$, are intercepted by $\xi_\ell(\cdot)$ at time $f(t)$ and $g(t)$, respectively. Both functions f and g are Lipschitz with slope $1+O(\delta)$, and

$$0\le g(t)-f(t)\le c_1\delta[\xi_r(t)-\xi_\ell(f(t))]. \tag{12.6.3}$$

The map that carries $(\xi_r(t),t)$ to $(\xi_\ell(f(t)),f(t))$ induces a pairing of points of the graphs of ξ_ℓ and ξ_r. From

$$\xi_r(t)-\xi_\ell(f(t))=\tilde{\mu}[t-f(t)], \tag{12.6.4}$$

we obtain

$$\dot{f}(t)=1-\frac{1}{\tilde{\mu}-\dot{\xi}_\ell(f(t))}[\dot{\xi}_r(t)-\dot{\xi}_\ell(f(t))], \tag{12.6.5}$$

$$\frac{d}{dt}[\xi_r(t)-\xi_\ell(f(t))]=\frac{\tilde{\mu}}{\tilde{\mu}-\dot{\xi}_\ell(f(t))}[\dot{\xi}_r(t)-\dot{\xi}_\ell(f(t))], \tag{12.6.6}$$

almost everywhere on $[\frac{1}{2}\bar{t},\bar{t}]$. In order to bound the right-hand side of (12.6.6) from below, we begin with

$$\begin{aligned}\dot{\xi}_r(t)-\dot{\xi}_\ell(f(t))&=\lambda(z_r(t),w_r(t))-\lambda(z_\ell(f(t)),w_\ell(f(t)))\\&=\bar{\lambda}_z[z_r(t)-z_\ell(f(t))]+\bar{\lambda}_w[w_r(t)-w_\ell(f(t))].\end{aligned} \tag{12.6.7}$$

By virtue of Theorem 12.4.1,

$$z_r(t)-z_\ell(f(t))\le z(x_r,\bar{t})-z(x_\ell,\bar{t})-\sum[z_r(\tau+)-z_r(\tau-)], \tag{12.6.8}$$

where the summation runs over the set of jump points of $z_r(\cdot)$ inside the interval $(t,\bar{t})$. As in the proof of Lemma 12.5.4, with each one of these jump points τ one may associate the trivial 2-characteristic tree $\mathscr{N}$ which consists of the single node $(\xi_r(\tau),\tau)$ so as to deduce

$$-\sum[z_r(\tau+)-z_r(\tau-)]\le\mathscr{P}_2(\mathscr{F})+\mathscr{Q}_2(\mathscr{F}). \tag{12.6.9}$$

For $t\in[\frac{1}{2}\bar{t},\bar{t}]$, we construct the maximal backward 2-characteristic emanating from $(\xi_r(t),t)$, which is intercepted by $\xi_\ell(\cdot)$ at time $h(t)$; $f(t)\le h(t)\le g(t)$. On account of Theorem 12.4.1, $w_\ell(h(t))\ge w_r(t)$ and so

$$w_r(t)-w_\ell(f(t))\le w_\ell(h(t))-w_\ell(f(t))\le V(f(t))-V(g(t)), \tag{12.6.10}$$

where $V(\tau)$ measures the total variation of $w_\ell(\cdot)$ over the interval $[\tau,\bar{t})$.

We now integrate (12.6.6) over the interval $(s,\bar{t})$. Recalling that $\bar{\lambda}_z < 0$, $\bar{\lambda}_w < 0$, upon combining (12.6.7), (12.6.8), (12.6.9) and (12.6.10), we deduce

$$\begin{aligned}(12.6.11)\qquad \xi_r(s) - \xi_\ell(f(s)) \leq \xi_r(\bar{t}) - \xi_\ell(f(\bar{t}))& \\ + c_2^{-1}(\bar{t}-s)[z(x_r,\bar{t}) - z(x_\ell,\bar{t}) + \mathscr{P}_2(\mathscr{F}) + \mathscr{Q}_2(\mathscr{F})]& \\ + c_3\int_s^{\bar{t}}[V(f(t)) - V(g(t))]dt.&\end{aligned}$$

By interchanging the order of integration,

$$\begin{aligned}(12.6.12)\qquad \int_s^{\bar{t}}[V(f(t)) - V(g(t))]dt &= -\int_s^{\bar{t}}\int_{f(t)}^{g(t)} dV(\tau)dt \\ &\leq -\int_{f(s)}^{f(\bar{t})}[f^{-1}(\tau) - g^{-1}(\tau)]dV(\tau) - \int_{f(\bar{t})}^{g(\bar{t})}[\bar{t} - g^{-1}(\tau)]dV(\tau) \\ &= -\int_s^{\bar{t}}[t - g^{-1}(f(t))]dV(f(t)) - \int_{f(\bar{t})}^{g(\bar{t})}[\bar{t} - g^{-1}(\tau)]dV(\tau).\end{aligned}$$

On account of (12.6.3),

$$(12.6.13)\qquad t - g^{-1}(f(t)) \leq c_4\delta[\xi_r(t) - \xi_\ell(f(t))], \qquad \frac{\bar{t}}{2} \leq t \leq \bar{t},$$

$$(12.6.14)\qquad \bar{t} - g^{-1}(\tau) \leq c_4\delta[\xi_r(\bar{t}) - \xi_\ell(f(\bar{t}))], \qquad f(\bar{t}) \leq \tau \leq g(\bar{t}),$$

and hence (12.6.11) yields

$$\begin{aligned}(12.6.15)\qquad \xi_r(s) - \xi_\ell(f(s)) \leq \exp(c_3c_4\delta\bar{V})[\xi_r(\bar{t}) - \xi_\ell(f(\bar{t}))]& \\ + c_2^{-1}(\bar{t}-s)[z(x_r,\bar{t}) - z(x_\ell,\bar{t}) + \mathscr{P}_2(\mathscr{F}) + \mathscr{Q}_2(\mathscr{F})]& \\ - c_3c_4\delta\int_s^{\bar{t}}[\xi_r(t) - \xi_\ell(f(t))]dV(f(t)),&\end{aligned}$$

for any $s \in [\frac{3}{4}\bar{t},\bar{t}]$. Integrating the above, Gronwall-type, inequality, we obtain

$$\begin{aligned}(12.6.16)\qquad \xi_r(s) - \xi_\ell(f(s)) \leq \exp(2c_3c_4\delta\bar{V})[\xi_r(\bar{t}) - \xi_\ell(f(\bar{t}))]& \\ +c_2^{-1}[\int_s^{\bar{t}}\exp\{c_3c_4\delta[V(f(s)) - V(f(\tau))]\}d\tau][z(x_r,\bar{t}) - z(x_\ell,\bar{t}) + \mathscr{P}_2(\mathscr{F}) + \mathscr{Q}_2(\mathscr{F})].&\end{aligned}$$

We apply (12.6.16) for $s = \frac{3}{4}\bar{t}$. The left-hand side of (12.6.16) is nonnegative. Also, $\xi_r(\bar{t}) - \xi_\ell(f(\bar{t})) \leq c_5(x_r - x_\ell)$. Therefore, (12.6.16) implies $(12.6.2)_1$ with constants $\bar{c} = 2c_3c_4$, $\hat{c} = 4c_2c_5$. The proof is complete.

In what follows, we shall be operating under the assumption that the constants $\bar{V}$ appearing in $(12.6.2)_1$ and $(12.6.2)_2$ satisfy

$$\bar{c}\delta\bar{V} \leq \log 2. \tag{12.6.17}$$

This will certainly be the case, by virtue of Theorem 12.5.1, when the initial data satisfy (12.5.3) with a sufficiently small. Furthermore, because of the finite domain of dependence property, (12.6.17) shall hold for $\bar{t}$ sufficiently small, even when the initial data have only locally bounded variation and satisfy (12.5.2) with δ sufficiently small. It will be shown below that (12.6.17) actually holds for any $\bar{t} > 0$, provided only that the initial data have sufficiently small oscillation, i.e., δ is small.

12.6.3 Lemma. *For any $-\infty < \bar{x} < \bar{y} < \infty$ and $\bar{t} > 0$,*

$$\begin{aligned} NV_{[\bar{x},\bar{y}]}z(\cdot,\bar{t}) + PV_{[\bar{x},\bar{y}]}w(\cdot,\bar{t}) &\leq 4\hat{c}\,\frac{\bar{y}-\bar{x}}{\bar{t}} \\ &+ c\delta^2\left\{TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}z\left(\cdot,\frac{\bar{t}}{2}\right) + TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}w\left(\cdot,\frac{\bar{t}}{2}\right)\right\}, \end{aligned} \tag{12.6.18}$$

$$\begin{aligned} TV_{[\bar{x},\bar{y}]}z(\cdot,\bar{t}) + TV_{[\bar{x},\bar{y}]}w(\cdot,\bar{t}) &\leq 8\hat{c}\,\frac{\bar{y}-\bar{x}}{\bar{t}} + 8\delta \\ &+ c\delta^2\left\{TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}z\left(\cdot,\frac{\bar{t}}{2}\right) + TV_{[\bar{x}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}w\left(\cdot,\frac{\bar{t}}{2}\right)\right\}, \end{aligned} \tag{12.6.19}$$

where $\bar{\lambda}$ is the infimum of $\lambda(z,w)$ and $\bar{\mu}$ is the supremum of $\mu(z,w)$ over the range of the solution.

Proof. By combining $(12.6.2)_1$, $(12.6.2)_2$, (12.6.17), and Lemma 12.5.3, we immediately infer

$$NV_{[\bar{x},\bar{y}]}z(\cdot,\bar{t}) + PV_{[\bar{x},\bar{y}]}w(\cdot,\bar{t}) \leq 4\hat{c}\,\frac{\bar{y}-\bar{x}}{\bar{t}} + 2[\mathscr{P}_1(\mathscr{H}) + \mathscr{Q}_1(\mathscr{H}) + \mathscr{P}_2(\mathscr{H}) + \mathscr{Q}_2(\mathscr{H})], \tag{12.6.20}$$

where $\mathscr{H}$ denotes the region bordered by the graph of the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(\bar{y},\bar{t})$, the graph of the maximal backward 2-characteristic $\zeta(\cdot)$ emanating from $(\bar{x},\bar{t})$, and the time lines $t = \bar{t}$ and $t = \bar{t}/2$.

We estimate $\mathscr{P}_1(\mathscr{H}) + \mathscr{Q}_1(\mathscr{H}) + \mathscr{P}_2(\mathscr{H}) + \mathscr{Q}_2(\mathscr{H})$ by applying Lemma 12.5.7, with the time origin shifted from $t = 0$ to $t = \bar{t}/2$. This yields (12.6.18).

Since total variation is the sum of negative variation and positive variation, while the difference of negative variation and positive variation is majorized by the oscillation, (12.6.18) together with (12.5.1) yield (12.6.19). The proof is complete.

Proof of Theorem 12.6.1. In order to establish (12.6.1), we first write (12.6.19) with $\bar{t} = t$, $\bar{x} = x$ and $\bar{y} = y$. To estimate the right-hand side of the resulting inequality, we reapply (12.6.19), for $\bar{t} = \frac{1}{2}t$, $\bar{x} = x - \frac{1}{2}\bar{\mu}t$ and $\bar{y} = y - \frac{1}{2}\bar{\lambda}t$. This yields

$$
\begin{aligned}
(12.6.21)\qquad & TV_{[x-\frac{1}{2}\bar{\mu}t,y-\frac{1}{2}\bar{\lambda}t]}z\left(\cdot,\frac{t}{2}\right)+TV_{[x-\frac{1}{2}\bar{\mu}t,y-\frac{1}{2}\bar{\lambda}t]}w\left(\cdot,\frac{t}{2}\right)\\
&\leq 16\hat{c}\,\frac{y-x}{t}+8\hat{c}(\bar{\mu}-\bar{\lambda})+8\delta\\
&+c\delta^2\left\{TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}z\left(\cdot,\frac{t}{4}\right)+TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}w\left(\cdot,\frac{t}{4}\right)\right\}.
\end{aligned}
$$

Similarly, in order to estimate the right-hand side of (12.6.21), we apply (12.6.19) with $\bar{t}=\frac{1}{4}t$, $\bar{x}=x-\frac{3}{4}\bar{\mu}t$ and $\bar{y}=y-\frac{3}{4}\bar{\lambda}t$. We thus obtain

$$
\begin{aligned}
(12.6.22)\qquad & TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}z\left(\cdot,\frac{t}{4}\right)+TV_{[x-\frac{3}{4}\bar{\mu}t,y-\frac{3}{4}\bar{\lambda}t]}w\left(\cdot,\frac{t}{4}\right)\\
&\leq 32\hat{c}\,\frac{y-x}{t}+24\hat{c}(\bar{\mu}-\bar{\lambda})+8\delta\\
&+c\delta^2\left\{TV_{[x-\frac{7}{8}\bar{\mu}t,y-\frac{7}{8}\bar{\lambda}t]}z\left(\cdot,\frac{t}{8}\right)+TV_{[x-\frac{7}{8}\bar{\mu}t,y-\frac{7}{8}\bar{\lambda}t]}w\left(\cdot,\frac{t}{8}\right)\right\}.
\end{aligned}
$$

Continuing on and passing to the limit, we arrive at (12.6.1) with

$$
(12.6.23)\qquad b=\frac{8\hat{c}}{1-2c\delta^2}\,,\qquad \beta=\frac{8}{1-c\delta^2}+\frac{8c\hat{c}\delta(\bar{\mu}-\bar{\lambda})}{(1-c\delta^2)(1-2c\delta^2)}\,.
$$

The above derivations hinge on the assumption that (12.6.17) holds; hence, in order to complete the proof, we now have to verify this condition. Recalling the definition of $\bar{V}$ in Lemma 12.6.2 and applying Theorem 12.5.1, with time origin shifted from 0 to $\frac{1}{2}t$, we deduce

$$
(12.6.24)\qquad \bar{V}\leq c\sup_{\bar{x}}\{TV_{[\bar{x}-\frac{1}{2}\bar{\mu}t,\bar{x}-\frac{1}{2}\bar{\lambda}t]}z(\cdot,\frac{t}{2})+TV_{[\bar{x}-\frac{1}{2}\bar{\mu}t,\bar{x}-\frac{1}{2}\bar{\lambda}t]}w(\cdot,\frac{t}{2})\}.
$$

We estimate the right-hand side of (12.6.24) by means of (12.6.1), which yields

$$
(12.6.25)\qquad \bar{V}\leq cb(\bar{\mu}-\bar{\lambda})+c\beta\delta,
$$

so that (12.6.17) is indeed satisfied, provided δ is sufficiently small. The proof is complete.

We now show that initial data of sufficiently small oscillation, but arbitrarily large total variation, induce the L^∞ bound (12.5.1), which has been assumed throughout this section.

12.6.4 Theorem. *There is a positive constant γ, depending solely on F, such that solutions generated by initial data with small oscillation*

$$
(12.6.26)\qquad |z(x,0)|+|w(x,0)|<\gamma\delta^2,\qquad -\infty<x<\infty,
$$

but unrestricted total variation, satisfy (12.5.1).

Proof. Assuming (12.6.26) holds, with γ sufficiently small, we will demonstrate that $-\delta < z(x,t) < \delta$ and $-\delta < w(x,t) < \delta$ on the upper half-plane. Arguing by contradiction, suppose any one of the above four inequalities is violated at some point, say for example $z(\bar{x},\bar{t}) \geq \delta$.

We determine $\bar{y}$ through $8\hat{c}(\bar{y}-\bar{x}) = \delta\bar{t}$, where $\hat{c}$ is the constant appearing in $(12.6.2)_1$, and apply (12.6.18). The first term on the right-hand side of (12.6.18) is here bounded by $\frac{1}{4}\delta$; the second term is bounded by $\tilde{c}\delta^2$, on account of (12.6.1). Consequently, for δ sufficiently small, the negative (decreasing) variation of $z(\cdot,\bar{t})$ over the interval $[\bar{x},\bar{y}]$ does not exceed $\frac{1}{2}\delta$. It follows that $z(x,\bar{t}) \geq \frac{1}{2}\delta$, for all $x \in [\bar{x},\bar{y}]$. In particular,

$$\int_{\bar{x}}^{\bar{y}} [|z(x,\bar{t})| + |w(x,\bar{t})|]dx \geq (\bar{y}-\bar{x})\frac{\delta}{2} = \frac{1}{16\hat{c}}\delta^2\bar{t}. \tag{12.6.27}$$

We now appeal to the L^1 estimate (12.8.3), which will be established in Section 12.8, Lemma 12.8.2, and combine it with (12.6.26) to deduce

$$\int_{\bar{x}}^{\bar{y}} [|z(x,\bar{t})| + |w(x,\bar{t}|]dx \leq 4[(\bar{y}-\bar{x}) + 2c\bar{t}]\gamma\delta^2 = \gamma[\frac{\delta}{2\hat{c}} + 8c]\delta^2\bar{t}. \tag{12.6.28}$$

It is clear that, for γ sufficiently small, (12.6.27) is inconsistent with (12.6.28), and this provides the desired contradiction. The proof is complete.

In conjunction with the compactness properties of BV functions, recounted in Section 1.7, the estimate (12.6.1) indicates that, starting out with solutions with initial data of locally bounded variation, one may construct, via completion, BV_{loc} solutions under initial data that are merely in L^∞, with sufficiently small oscillation. Thus, the solution operator of genuinely nonlinear systems of two conservation laws regularizes the initial data by the mechanism already encountered in the context of the genuinely nonlinear scalar conservation law (Theorem 11.2.2).

12.7 Regularity of Solutions

The information collected thus far paints the following picture for the regularity of solutions:

12.7.1 Theorem. *Let $U(x,t)$ be an admissible BV solution of the genuinely nonlinear system* (12.1.1) *of two conservation laws, with the properties recounted in the previous sections. Then*

(a) *Any point $(\bar{x},\bar{t})$ of approximate continuity is a point of continuity of U.*
(b) *Any point $(\bar{x},\bar{t})$ of approximate jump discontinuity is a point of (classical) jump discontinuity of U.*
(c) *Any irregular point $(\bar{x},\bar{t})$ is the focus of a centered compression wave of either, or both, characteristic families, and/or a point of interaction of shocks of the same or opposite characteristic families.*
(d) *The set of irregular points is (at most) countable.*

Proof. Assertions (a), (b) and (c) are corollaries of Theorem 12.3.3. In particular, $(\bar{x},\bar{t})$ is a point of approximate continuity if and only if $(z_W,w_W) = (z_E,w_E)$, in which case all four limits (z_W,w_W), (z_E,w_E), (z_N,w_N) and (z_S,w_S) coincide. When $(z_W,w_W) \neq (z_E,w_E)$, then $(\bar{x},\bar{t})$ is a point of approximate jump discontinuity in the 1-shock set if $(z_W,w_W) = (z_S,w_S)$, $(z_E,w_E) = (z_N,w_N)$; or a point of approximate jump discontinuity in the 2-shock set if $(z_W,w_W) = (z_N,w_N)$, $(z_E,w_E) = (z_S,w_S)$; and an irregular point in all other cases.

To verify assertion (d), assume the irregular point $I = (\bar{x},\bar{t})$ is a node of some 1-characteristic tree $\mathscr{M}$ or a 2-characteristic tree $\mathscr{N}$. If I is the focusing point of a centered 1-compression wave and/or point of interaction of 1-shocks, then, by virtue of $(12.5.15)_1$, I will register a positive contribution to $\mathscr{P}_1(\mathscr{M})$. Similarly, if I is the focusing point of a centered 2-compression wave and/or point of interaction of 2-shocks, then, on account of $(12.5.15)_2$, I will register a positive contribution to $\mathscr{P}_2(\mathscr{N})$. Finally, suppose I is a point of interaction of a 1-shock with a 2-shock. We adjoin to $\mathscr{M}$ an additional node K lying on the graph of χ_-^I very close to I. Then $|\Delta w^{KI} - \Delta w_*^I| > 0$ and so, by $(12.5.16)_1$, we get a positive contribution to $\mathscr{Q}_1(\mathscr{M})$. Since the total amount of wave interaction is bounded, by virtue of Lemma 12.5.7, we conclude that the set of irregular points is necessarily (at most) countable. This completes the proof.

An analog of Theorem 11.3.5 is also in force here:

12.7.2 Theorem. *Assume the set $\mathscr{C}$ of points of continuity of the solution U has nonempty interior $\mathscr{C}^0$. Then U is locally Lipschitz continuous on $\mathscr{C}^0$.*

Proof. We verify that z is locally Lipschitz continuous on $\mathscr{C}^0$. Assume $(\bar{x},\bar{t}) \in \mathscr{C}^0$ and $\mathscr{C}$ contains a rectangle $\{(x,t) : |x-\bar{x}| < kp,\ |t-\bar{t}| < p\}$, with $p > 0$ and k large compared to $|\lambda|$ and μ. By shifting the axes, we may assume, without loss of generality, that $\bar{t} = p$. We fix $\bar{y} > \bar{x}$, where $\bar{y}-\bar{x}$ is small compared to p, and apply $(12.6.2)_1$, with $x_\ell = \bar{x}$, $x_r = \bar{y}$. Since the solution is continuous in the rectangle, both $\mathscr{P}_2(\mathscr{F})$ and $\mathscr{Q}_2(\mathscr{F})$ vanish and so, recalling (12.6.17),

$$z(\bar{x},\bar{t}) - z(\bar{y},\bar{t}) \leq \frac{2\hat{c}}{p}(\bar{y}-\bar{x}). \tag{12.7.1}$$

The functions $(\hat{z},\hat{w})(x,t) = (z,w)(\bar{x}+\bar{y}-x, 2p-t)$ are Riemann invariants of another solution $\hat{U}$ which is continuous, and thereby admissible, on the rectangle $\{(x,t) : |x-\bar{y}| < kp,\ |t-\bar{t}| < p\}$. Applying (12.7.1) to $\hat{z}$ yields

$$z(\bar{y},\bar{t}) - z(\bar{x},\bar{t}) = \hat{z}(\bar{x},\bar{t}) - \hat{z}(\bar{y},\bar{t}) \leq \frac{2\hat{c}}{p}(\bar{y}-\bar{x}). \tag{12.7.2}$$

We now fix $\bar{s} > \bar{t}$, with $\bar{s}-\bar{t}$ small compared to p. We construct the minimal backward 1-characteristic ξ emanating from $(\bar{x},\bar{s})$, which is intercepted by the $\bar{t}$-time line at the point $\bar{y} = \xi(\bar{t})$, where $0 < \bar{y}-\bar{x} \leq -\bar{\lambda}(\bar{s}-\bar{t})$. By Theorem 12.4.1, $z(\bar{x},\bar{s}) = z(\bar{y},\bar{t})$ and so, by virtue of (12.7.1) and (12.7.2),

$$|z(\bar{x},\bar{s}) - z(\bar{x},\bar{t})| \leq \frac{2\hat{c}}{p}(\bar{y}-\bar{x}) \leq -\frac{2\bar{\lambda}\hat{c}}{p}(\bar{s}-\bar{t}). \tag{12.7.3}$$

Thus z is Lipschitz.

A similar argument shows that w is also Lipschitz in $\mathscr{C}^0$. This completes the proof.

12.8 Initial Data in L^1

Recall that, by virtue of Theorem 11.5.2, initial data in L^1 induce decay of solutions of genuinely nonlinear scalar conservation laws, as $t \to \infty$, at the rate $O(t^{-\frac{1}{2}})$. The aim here is to establish an analogous property for solutions of genuinely nonlinear systems of two conservation laws. Accordingly, we consider a solution $(z(x,t), w(x,t))$ of small oscillation (12.5.1), with initial values of unrestricted total variation lying in $L^1(-\infty,\infty)$:

$$L = \int_{-\infty}^{\infty} [|z(x,0)| + |w(x,0)|]dx \; < \infty. \tag{12.8.1}$$

The principal result is

12.8.1 Theorem. *As $t \to \infty$,*

$$(z(x,t), w(x,t)) = O(t^{-\frac{1}{2}}), \tag{12.8.2}$$

uniformly in x on $(-\infty,\infty)$.

The proof will be partitioned into several steps.

12.8.2 Lemma. *For any $\bar{t} \in [0,\infty)$, and $-\infty < \bar{x} < \bar{y} < \infty$,*

$$\int_{\bar{x}}^{\bar{y}} [|z(x,\bar{t})| + |w(x,\bar{t})|]dx \leq 4\int_{\bar{x}-c\bar{t}}^{\bar{y}+c\bar{t}} [|z(x,0)| + |w(x,0)|]dx. \tag{12.8.3}$$

In particular, $(z(\cdot,\bar{t}), w(\cdot,\bar{t}))$ are in $L^1(-\infty,\infty)$.

Proof. We construct a Lipschitz continuous entropy η by solving the Goursat problem for (12.2.2) with prescribed data

$$\begin{cases} \eta(z,0) = |z| + \alpha z^2, & -\infty < z < \infty, \\ \eta(0,w) = |w| + \alpha z^2, & -\infty < w < \infty, \end{cases} \tag{12.8.4}$$

where α is a positive constant. From (12.2.3) it follows that, for α sufficiently large, η is a convex function of U on some neighborhood of the origin containing the range of the solution.

Combining (12.2.2) and (12.8.4), one easily deduces, for δ small,

$$\frac{1}{2}(|z|+|w|) \le \eta(z,w) \le 2(|z|+|w|), \quad -2\delta < z < 2\delta, \quad -2\delta < w < 2\delta. \tag{12.8.5}$$

Furthermore, if q is the entropy flux associated with η, normalized by $q(0,0)=0$, (12.2.1) and (12.8.5) imply

$$|q(z,w)| \le c\eta(z,w), \qquad -2\delta < z < 2\delta, \quad -2\delta < w < 2\delta. \tag{12.8.6}$$

We now fix $\bar{t} > 0$, $-\infty < \bar{x} < \bar{y} < \infty$ and integrate (12.3.1), for the entropy-entropy flux pair (η, q) constructed above, over the trapezoidal region defined by $\{(x,t) : 0 < t < \bar{t}, \bar{x} - c(\bar{t}-t) < x < \bar{y} + c(\bar{t}-t)\}$. Upon using (12.8.6), this yields

$$\int_{\bar{x}}^{\bar{y}} \eta(z(x,\bar{t}), w(x,\bar{t}))dx \le \int_{\bar{x}-c\bar{t}}^{\bar{y}+c\bar{t}} \eta(z(x,0), w(x,0))dx. \tag{12.8.7}$$

By virtue of (12.8.5), (12.8.7) implies (12.8.3). The proof is complete.

12.8.3 Lemma. *Let* $(\bar{z}(\cdot), \bar{w}(\cdot))$ *denote the trace of* (z,w) *along the minimal (or maximal) backward* 1-*(or* 2-*) characteristic* $\xi(\cdot)$ *(or* $\zeta(\cdot)$*) emanating from any point* $(\bar{y}, \bar{t})$ *of the upper half-plane. Then*

$$\int_0^{\bar{t}} [\bar{z}^2(t) + |\bar{w}(t)|]dt \le \tilde{c}L, \tag{12.8.8$_1$}$$

or

$$\int_0^{\bar{t}} [|\bar{z}(t)| + \bar{w}^2(t)]dt \le \tilde{c}L. \tag{12.8.8$_2$}$$

Proof. It will suffice to verify $(12.8.8)_1$. Suppose η is any Lipschitz continuous convex entropy associated with entropy flux q, normalized so that $\eta(0,0) = 0$, $q(0,0) = 0$. We fix $\bar{x} < \bar{y}$ and integrate the inequality (12.3.1) over the region $\{(x,t) : 0 < t < \bar{t}, \bar{x} < x < \xi(t)\}$ to get

$$\begin{aligned} &\int_{\bar{x}}^{\bar{y}} \eta(z(x,\bar{t}), w(x,\bar{t}))dx - \int_{\bar{x}}^{\xi(0)} \eta(z(x,0), w(x,0))dx \\ &\qquad + \int_0^{\bar{t}} G(\bar{z}(t), \bar{w}(t))dt - \int_0^{\bar{t}} q(z(\bar{x}+,t), w(\bar{x}+,t))dt \le 0, \end{aligned} \tag{12.8.9}$$

where G is defined by

$$G(z,w) = q(z,w) - \lambda(z,w)\eta(z,w). \tag{12.8.10}$$

We seek an entropy-entropy flux pair that renders $G(z,w)$ positive definite on $(-2\delta, 2\delta) \times (-2\delta, 2\delta)$. On account of (12.2.1),

$$G_z = -\lambda_z \eta, \tag{12.8.11}$$

(12.8.12) $$G_w = [(\mu - \lambda)\eta]_w - \mu_w \eta,$$

which indicate that G decays fast, at least quadratically, as $z \to 0$, but it may decay more slowly, even linearly, as $w \to 0$.

We construct an entropy η by solving the Goursat problem for (12.2.2) with data

(12.8.13) $$\begin{cases} \eta(z,0) = 2z + \alpha z^2, & -\infty < z < \infty, \\ \eta(0,w) = |w| + \alpha w^2, & -\infty < w < \infty. \end{cases}$$

For α sufficiently large, it follows from (12.2.3) that η is a convex function of U on some neighborhood of the origin containing the range of the solution. From (12.8.12), (12.2.2) and (12.8.13) we deduce

(12.8.14) $$G(0,w) = [\mu(0,0) - \lambda(0,0)]|w| + O(w^2),$$

(12.8.15) $$\eta(z,w) = 2z + |w| + O(z^2 + w^2),$$

for (z,w) near the origin. Combining (12.8.14) with (12.8.11) and (12.8.15), we conclude

(12.8.16) $$G(z,w) = [\mu(0,0) - \lambda(0,0)]|w| - \lambda_z(0,0)z^2 + O(w^2 + |zw| + |z|^3).$$

We now return to (12.8.9). On account of Lemma 12.8.2, $(z(\cdot,t), w(\cdot,t))$ are in $L^1(-\infty,\infty)$, for all $t \in [0,\bar{t}]$, and hence

(12.8.17) $$\liminf_{\bar{x} \to -\infty} \left| \int_0^{\bar{t}} q(z(\bar{x}+,t), w(\bar{x}+,t))dt \right| = 0.$$

Therefore, (12.8.9), (12.8.17), (12.8.15), (12.8.3) and (12.8.1) together imply

(12.8.18) $$\int_0^{\bar{t}} G(\bar{z}(t), \bar{w}(t))dt \leq 12L,$$

provided (12.5.1) holds, with δ sufficiently small. The assertion $(12.8.8)_1$ now follows easily from (12.8.18), (12.8.16) and (12.1.3). This completes the proof.

Lemma 12.8.3 indicates that along minimal backward 1-characteristics z is $O(t^{-\frac{1}{2}})$ and w is $O(t^{-1})$, while along maximal backward 2-characteristics z is $O(t^{-1})$ and w is $O(t^{-\frac{1}{2}})$. In fact, recalling that $\bar{z}(\cdot)$ and $\bar{w}(\cdot)$ are nonincreasing along minimal and maximal backward 1- and 2-characteristics, respectively, we infer directly from $(12.8.8)_1$ and $(12.8.8)_2$ that the positive parts of $z(x,t)$ and $w(x,t)$ are $O(t^{-\frac{1}{2}})$, as $t \to \infty$. The proof of Theorem 12.8.1 will now be completed by establishing $O(t^{-\frac{1}{2}})$ decay on both sides:

12.8.4 Lemma. *For δ sufficiently small,*

$$z^2(x,t) \le \frac{8\tilde{c}L}{t}, \tag{12.8.19$_1$}$$

$$w^2(x,t) \le \frac{8\tilde{c}L}{t}, \tag{12.8.19$_2$}$$

hold, for all $-\infty < x < \infty, 0 < t < \infty$, *where* $\tilde{c}$ *is the constant in* $(12.8.8)_1$ *and* $(12.8.8)_2$.

Proof. Arguing by contradiction, suppose the assertion is false and let $\bar{t} > 0$ be the greatest lower bound of the set of points t on which $(12.8.19)_1$ and/or $(12.8.19)_2$ is violated for some x. According to Theorem 12.3.3, the continuation of the solution beyond $\bar{t}$ is initiated by solving Riemann problems along the $\bar{t}$-time line. Consequently, since $(12.8.19)_1$ and/or $(12.8.19)_2$ fail for $t > \bar{t}$, one can find $\bar{y} \in (-\infty, \infty)$ such that

$$z^2(\bar{y},\bar{t}) > \frac{4\tilde{c}L}{\bar{t}}, \tag{12.8.20$_1$}$$

and/or

$$w^2(\bar{y},\bar{t}) > \frac{4\tilde{c}L}{\bar{t}}. \tag{12.8.20$_2$}$$

For definiteness, assume $(12.8.20)_1$ holds.

Let $(\bar{z}(\cdot), \bar{w}(\cdot))$ denote the trace of (z,w) along the minimal backward 1-characteristic $\xi(\cdot)$ emanating from $(\bar{y},\bar{t})$. By applying Theorem 12.5.1, with the time origin shifted from $t = 0$ to $t = \bar{t}/2$, we deduce

$$TV_{[\frac{1}{2}\bar{t},\bar{t}]}\bar{w}(\cdot) \le \hat{c}\{TV_{[\bar{y}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}\, z(\cdot,\frac{\bar{t}}{2}) + TV_{[\bar{y}-\frac{1}{2}\bar{\mu}\bar{t},\bar{y}-\frac{1}{2}\bar{\lambda}\bar{t}]}\, w(\cdot,\frac{\bar{t}}{2})\}, \tag{12.8.21}$$

where $\bar{\lambda}$ stands for the infimum of $\lambda(z,w)$ and $\bar{\mu}$ denotes the supremum of $\mu(z,w)$ over the range of the solution. We estimate the right-hand side of (12.8.21) with the help of Theorem 12.6.1, thus obtaining

$$TV_{[\frac{1}{2}\bar{t},\bar{t}]}\bar{w}(\cdot) \le \hat{c}[b(\bar{\mu}-\bar{\lambda}) + \beta\delta]. \tag{12.8.22}$$

By hypothesis,

$$\bar{w}^2(t) \le \frac{16\tilde{c}L}{\bar{t}}, \qquad \frac{\bar{t}}{2} \le t < \bar{t}. \tag{12.8.23}$$

We also have $|\bar{z}(t)| \le 2\delta$. Therefore, by applying $(12.4.2)_1$ we deduce

$$\bar{z}^2(\bar{t}-) - \bar{z}^2(t) \le \bar{c}\delta\,\frac{4\tilde{c}L}{\bar{t}}, \tag{12.8.24}$$

with $\bar{c} = 64a\hat{c}[b(\bar{\mu}-\bar{\lambda}) + \beta\delta]$.

Since $\bar{z}(\bar{t}-) = z(\bar{y},\bar{t})$, combining $(12.8.20)_1$ with (12.8.24) yields

(12.8.25) $$\bar{z}^2(t) \geq \frac{4\tilde{c}L}{\bar{t}}(1-\bar{c}\delta), \qquad \frac{\bar{t}}{2} \leq t < \bar{t}.$$

From (12.8.25),

(12.8.26) $$\int_{\frac{\bar{t}}{2}}^{\bar{t}} \bar{z}^2(t)dt \geq 2\tilde{c}L(1-\bar{c}\delta),$$

which provides the desired contradiction to $(12.8.8)_1$, when δ is sufficiently small. The proof is complete.

12.9 Initial Data with Compact Support

Here we consider the large-time behavior of solutions, with small oscillation (12.5.1), to our genuinely nonlinear system (12.1.1) of two conservation laws under initial data $(z(x,0),w(x,0))$ that vanish outside a bounded interval $[-\ell,\ell]$. We already know, from Section 12.8, that $(z(x,t),w(x,t)) = O(t^{-\frac{1}{2}})$. The aim is to examine the asymptotics in finer scale, establishing the analog of Theorem 11.6.1 on the genuinely nonlinear scalar conservation law.

12.9.1 Theorem. *Employing the notation introduced in Section* 12.3, *consider the special forward characteristics* $\phi_-(\cdot),\psi_-(\cdot)$ *issuing from* $(-\ell,0)$ *and* $\phi_+(\cdot),\psi_+(\cdot)$ *issuing from* $(\ell,0)$. *Then*

(a) *For* t *large*, ϕ_-, ψ_-, ϕ_+ *and* ψ_+ *propagate according to*

$(12.9.1)_1$ $$\phi_-(t) = \lambda(0,0)t - (p_-t)^{\frac{1}{2}} + O(1),$$

$(12.9.1)_2$ $$\psi_+(t) = \mu(0,0)t + (q_+t)^{\frac{1}{2}} + O(1),$$

$(12.9.2)_1$ $$\phi_+(t) = \lambda(0,0)t + (p_+t)^{\frac{1}{2}} + O\left(t^{\frac{1}{4}}\right),$$

$(12.9.2)_2$ $$\psi_-(t) = \mu(0,0)t - (q_-t)^{\frac{1}{2}} + O\left(t^{\frac{1}{4}}\right),$$

where p_-, p_+, q_- *and* q_+ *are nonnegative constants.*

(b) *For* $t > 0$ *and either* $x < \phi_-(t)$ *or* $x > \psi_+(t)$,

(12.9.3) $$z(x,t) = 0, \qquad w(x,t) = 0.$$

(c) *For* t *large*,

(12.9.4) $$TV_{[\phi_-(t),\psi_+(t)]}\, z(\cdot,t) + TV_{[\phi_-(t),\psi_+(t)]}\, w(\cdot,t) = O\left(t^{-\frac{1}{2}}\right).$$

(d) *For t large and $\phi_-(t) < x < \phi_+(t)$,*

$$\lambda(z(x,t),0) = \frac{x}{t} + O\left(\frac{1}{t}\right), \tag{12.9.5$_1$}$$

while for $\psi_-(t) < x < \psi_+(t)$,

$$\mu(0,w(x,t)) = \frac{x}{t} + O\left(\frac{1}{t}\right). \tag{12.9.5$_2$}$$

(e) *For t large and $x > \phi_+(t)$, if $p_+ > 0$ then*

$$0 \leq -z(x,t) \leq c[x - \lambda(0,0)t]^{-\frac{3}{2}}, \tag{12.9.6$_1$}$$

while for $x < \psi_-(t)$, if $q_- > 0$ then

$$0 \leq -w(x,t) \leq c[\mu(0,0)t - x]^{-\frac{3}{2}}. \tag{12.9.6$_2$}$$

According to the above proposition, as $t \to \infty$ the two characteristic families decouple and each one develops an N-wave profile, of width $O(t^{\frac{1}{2}})$ and strength $O(t^{-\frac{1}{2}})$, which propagates into the rest state at characteristic speed. When one of p_-, p_+ (or q_-, q_+) vanishes, the 1- (or 2-) N-wave is one-sided, of triangular profile. If both p_-, p_+ (or q_-, q_+) vanish, the 1- (or 2-) N-wave is absent altogether. In the wake of the N-waves, the solution decays at the rate $O(t^{-\frac{3}{4}})$, so long as $p_+ > 0$ and $q_- > 0$. In cones properly contained in the wake, the decay is even faster, $O(t^{-\frac{3}{2}})$.

Statement (b) of Theorem 12.9.1 is an immediate corollary of Theorem 12.5.1. The remaining assertions will be established in several steps.

12.9.2 Lemma. *As $t \to \infty$, the total variation decays according to* (12.9.4).

Proof. We fix t large and construct the maximal forward 1-characteristic $\chi_-(\cdot)$ issuing from $(\psi_+(t^{\frac{1}{2}}), t^{\frac{1}{2}})$ and the minimal forward 2-characteristic $\chi_+(\cdot)$ issuing from $(\phi_-(t^{\frac{1}{2}}), t^{\frac{1}{2}})$.

In order to estimate the total variation over the interval $(\chi_-(t), \chi_+(t))$, we apply Theorem 12.5.1, shifting the time origin from 0 to $t^{\frac{1}{2}}$. The minimal backward 1-characteristics as well as the maximal backward 2-characteristics emanating from points (x,t) with $\chi_-(t) < x < \chi_+(t)$ are intercepted by the $t^{\frac{1}{2}}$-time line outside the support of the solution. Furthermore, the oscillation of (z,w) along the $t^{\frac{1}{2}}$-time line is $O(t^{-\frac{1}{4}})$ so that in (12.5.4)$_1$ and (12.5.4)$_2$ one may take $\delta = O(t^{-\frac{1}{4}})$. Therefore,

$$TV_{(\chi_-(t),\chi_+(t))}\, z(\cdot,t) + TV_{(\chi_-(t),\chi_+(t))}\, w(\cdot,t) = O(t^{-\frac{1}{2}}). \tag{12.9.7}$$

In order to estimate the total variation over the intervals $[\phi_-(t), \chi_-(t)]$ and $[\chi_+(t), \psi_+(t)]$, we apply Theorem 12.6.1, shifting the time origin from 0 to $\frac{1}{2}t$. The oscillation of (z,w) along the $\frac{1}{2}t$-time line is $O(t^{-\frac{1}{2}})$ so that in (12.6.1) we may take $\delta = O(t^{-\frac{1}{2}})$. Since $\chi_-(t) - \phi_-(t)$ and $\psi_+(t) - \chi_+(t)$ are $O(t^{\frac{1}{2}})$,

$$\text{(12.9.8)} \qquad \begin{cases} TV_{[\phi_-(t),\chi_-(t)]}\, z(\cdot,t) + TV_{[\phi_-(t),\chi_-(t)]}\, w(\cdot,t) = O(t^{-\frac{1}{2}}), \\[2ex] TV_{[\chi_+(t),\psi_+(t)]}\, z(\cdot,t) + TV_{[\chi_+(t),\psi_+(t)]}\, w(\cdot,t) = O(t^{-\frac{1}{2}}). \end{cases}$$

Combining (12.9.7) with (12.9.8), we arrive at (12.9.4). This completes the proof.

12.9.3 Lemma. *Let $\bar{\lambda}$ be any fixed strict upper bound of $\lambda(z,w)$ and $\bar{\mu}$ any fixed strict lower bound of $\mu(z,w)$, over the range of the solution. Then, for t large and $x > \bar{\lambda}t$,*

$$\text{(12.9.9)}_1 \qquad z(x,t) = O\big(t^{-\frac{3}{2}}\big),$$

while for $x < \bar{\mu}t$,

$$\text{(12.9.9)}_2 \qquad w(x,t) = O\big(t^{-\frac{3}{2}}\big).$$

Proof. We fix t large and $x > \bar{\lambda}t$. Since $\bar{\lambda}$ is a strict upper bound of $\lambda(z,w)$, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from (x,t) will be intercepted by the graph of ψ_+ at time $t_1 \geq \kappa t$, where κ is a positive constant depending solely on $\bar{\lambda}$. If $(\bar{z}(\cdot), \bar{w}(\cdot))$ denotes the trace of (z,w) along $\xi(\cdot)$, then the oscillation of $\bar{w}(\cdot)$ over $[t_1,t]$ is $O(t^{-\frac{1}{2}})$. Applying Theorem 12.5.1, with time origin shifted to t_1, and using Lemma 12.9.2, we deduce that the total variation of $\bar{w}(\cdot)$ over $[t_1,t]$ is likewise $O(t^{-\frac{1}{2}})$. It then follows from Theorem 12.4.1 that $\bar{z}(t-) = O(t^{-\frac{3}{2}})$. Since $z(x,t) = \bar{z}(t-)$, we arrive at $(12.9.9)_1$.

In a similar fashion, one establishes $(12.9.9)_2$, for $x < \bar{\mu}t$. The proof is complete.

12.9.4 Lemma. *Assertion* (d) *of Theorem* 12.9.1 *holds.*

Proof. By the construction of ϕ_- and ϕ_+, the minimal backward 1-characteristic $\xi(\cdot)$ emanating from any point (x,t) with $\phi_-(t) < x < \phi_+(t)$ will be intercepted by the x-axis on the interval $[-\ell, \ell]$. Therefore, if $(\bar{z}(\cdot), \bar{w}(\cdot))$ denotes the trace of (z,w) along $\xi(\cdot)$,

$$\text{(12.9.10)} \qquad x = \int_1^t \lambda(\bar{z}(\tau), \bar{w}(\tau))d\tau + \xi(1)$$

$$= t\lambda(z(x,t),0) + \int_1^t \{\bar{\lambda}_z[\bar{z}(\tau) - \bar{z}(t-)] + \bar{\lambda}_w \bar{w}(\tau)\}d\tau + O(1).$$

On account of Lemma 12.9.3, $\bar{w}(\tau) = O(\tau^{-\frac{3}{2}})$. Applying Theorem 12.5.1, with time origin shifted to τ, and using Lemma 12.9.2, we deduce that the total variation of $\bar{w}(\cdot)$ over $[\tau,t]$ is $O(\tau^{-\frac{1}{2}})$. It then follows from Theorem 12.4.1 that $\bar{z}(\tau)-\bar{z}(t-)$ is $O(\tau^{-\frac{7}{2}})$. In particular, the integral on the right-hand side of (12.9.10) is $O(1)$ and this establishes $(12.9.5)_1$.

A similar argument shows $(12.9.5)_2$. The proof is complete.

12.9.5 Lemma. *For t large, $\phi_-(t)$ and $\psi_+(t)$ satisfy* $(12.9.1)_1$ *and* $(12.9.1)_2$.

Proof. For t large, $\phi_-(t)$ joins the state $(z(\phi_-(t)-,t),w(\phi_-(t)-,t))=(0,0)$, on the left, to the state $(z(\phi_-(t)+,t),w(\phi_-(t)+,t))$, on the right, where $w(\phi_-(t)+,t)$ is $O(t^{-\frac{3}{2}})$, while $z(\phi_-(t)+,t)$ satisfies $(12.9.5)_1$ for $x=\phi_-(t)$. The jump across $\phi_-(t)$ is $O(t^{-\frac{1}{2}})$. Consequently, by use of (8.1.9) we infer

$$\dot{\phi}_-(t)=\frac{1}{2}\lambda(0,0)+\frac{1}{2t}\phi_-(t)+O\left(\frac{1}{t}\right), \tag{12.9.11}$$

almost everywhere.

We set $\phi_-(t)=\lambda(0,0)t-v(t)$. By the admissibility condition $\dot{\phi}_-(t)\le\lambda(0,0)$, we deduce that $\dot{v}(t)\ge 0$. Substituting into (12.9.11) yields

$$\dot{v}(t)=\frac{1}{2t}v(t)+O\left(\frac{1}{t}\right). \tag{12.9.12}$$

If $v(t)=O(1)$, as $t\to\infty$, we obtain $(12.9.1)_1$ with $p_-=0$. On the other hand, if $v(t)\uparrow\infty$, as $t\to\infty$, then (12.9.12) implies $v(t)=(p_-t)^{\frac{1}{2}}+O(1)$, which establishes $(12.9.1)_1$ with $p_->0$.

One validates $(12.9.1)_2$ by a similar argument. The proof is complete.

12.9.6 Lemma. *For t large, $\phi_+(t)$ and $\psi_-(t)$ satisfy* $(12.9.2)_1$ *and* $(12.9.2)_2$. *Furthermore, Assertion* (e) *of Theorem* 12.9.1 *holds.*

Proof. For t large, $\phi_+(t)$ joins the state $(z(\phi_+(t)-,t),w(\phi_+(t)-,t))$, on the left, to the state $(z(\phi_+(t)+,t),w(\phi_+(t)+,t))$, on the right, where both $w(\phi_+(t)\pm,t)$ are $O(t^{-\frac{3}{2}})$, while $z(\phi_+(t)-,t)$ satisfies $(12.9.5)_1$ for $x=\phi_+(t)$. The jump across $\phi_+(t)$ is $O(t^{-\frac{1}{2}})$. Hence, by use of (8.1.9) we obtain

$$\dot{\phi}_+(t)=\frac{1}{2}\lambda(z(\phi_+(t)+,t),0)+\frac{1}{2t}\phi_+(t)+O\left(\frac{1}{t}\right). \tag{12.9.13}$$

Since ϕ_+ is maximal, minimal backward 1-characteristics $\zeta(\cdot)$ emanating from points (x,t) with $x>\phi_+(t)$ stay strictly to the right of $\phi_+(\cdot)$ on $[0,t]$ and are thus intercepted by the x-axis at $\zeta(0)>\ell$. By virtue of Theorem 12.4.1, it follows that $z(\phi_+(t)+,t)\le 0$ and so $\lambda(z(\phi_+(t)+,t),0)\ge\lambda(0,0)$.

We now set $\phi_+(t)=\lambda(0,0)t+v(t)$, $\lambda(z(\phi_+(t)+,t),0)=\lambda(0,0)+g(t)$. As shown above, $g(t)\ge 0$. Furthermore, notice that the shock admissibility condition, namely $\dot{\phi}_+(t)\ge\lambda(z(\phi_+(t)+,t),w(\phi_+(t)+,t))$ implies $\dot{v}(t)\ge g(t)+O(t^{-\frac{3}{2}})$. When $v(t)$ is bounded, as $t \to \infty$, we obtain $(12.9.2)_1$, with $p_+ = 0$, corresponding to the case of one-sided N-wave. This case is delicate and will not be discussed here, so let us assume $v(t) \to \infty$, as $t \to \infty$.

Substituting $\phi_+(t)$ into (12.9.13), we obtain

$$\dot{v}(t) = \frac{1}{2t}v(t) + \frac{1}{2}g(t) + O(\frac{1}{t}). \tag{12.9.14}$$

Since $g(t) \geq 0$, (12.9.14) yields $v(t) \geq \alpha t^{\frac{1}{2}}$, with $\alpha > 0$. On the other hand, we know that $v(t) = O(t^{\frac{1}{2}})$ and so (12.9.14) implies

$$\frac{\dot{v}}{v} \geq \frac{1}{2t} + \beta g(t)t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}). \tag{12.9.15}$$

It is clear that (12.9.15) induces a contradiction to $v(t) = O(t^{\frac{1}{2}})$ unless

$$\int_1^\infty g(\tau)\tau^{-\frac{1}{2}}d\tau < \infty. \tag{12.9.16}$$

We now demonstrate that, in consequence of (12.9.16), there is $T > 0$ with the property that

$$\inf\{\tau^{\frac{1}{2}}g(\tau) : \frac{t}{2} \leq \tau \leq t\} < \frac{\alpha}{2}, \qquad \text{for all } t > T. \tag{12.9.17}$$

Indeed, if this assertion is false, we can find a sequence $\{t_m\}$, with $t_{m+1} \geq 2t_m$, for $m = 1, 2, \cdots$, along which (12.9.17) is violated. But then

$$\int_1^\infty g(\tau)\tau^{-\frac{1}{2}}d\tau \geq \tfrac{1}{2}\alpha \sum_m \int_{\frac{1}{2}t_m}^{t_m} \frac{dt}{t} = \infty, \tag{12.9.18}$$

in contradiction to (12.9.16).

Let us fix (x,t), with $t > T$ and $x > \phi_+(t)$. The minimal backward 1-characteristic $\zeta(\cdot)$ emanating from (x,t) stays strictly to the right of $\phi_+(\cdot)$. We locate $\bar{t} \in [\frac{1}{2}t, t]$ such that

$$\lambda(z(\phi_+(\bar{t})+, \bar{t}), 0) - \lambda(0,0) = g(\bar{t}) < \tfrac{1}{4}\alpha\bar{t}^{-\frac{1}{2}} \tag{12.9.19}$$

and consider the minimal backward 1-characteristic $\xi(\cdot)$ emanating from a point $(\bar{x}, \bar{t})$, where $\bar{x}$ lies between $\phi_+(\bar{t})$ and $\zeta(\bar{t})$ and is so close to $\phi_+(\bar{t})$ that

$$\lambda(z(\bar{x}, \bar{t}), 0) - \lambda(0,0) < \tfrac{1}{4}\alpha\bar{t}^{-\frac{1}{2}}. \tag{12.9.20}$$

Let $(\bar{z}(\cdot), \bar{w}(\cdot))$ denote the trace of (z,w) along $\xi(\cdot)$. By virtue of Theorem 12.4.1, $\bar{z}(\cdot)$ is a nonincreasing function on $(0, \bar{t})$ so that $\bar{z}(\tau) \leq \bar{z}(\bar{t}-) = z(\bar{x}, \bar{t})$. Consequently, on account of (12.9.20),

$$\begin{aligned}(12.9.21)\qquad \dot{\xi}(\tau) = \lambda(\bar{z}(\tau),\bar{w}(\tau)) &\leq \lambda(z(\bar{x},\bar{t}),\bar{w}(\tau)) \\ &\leq \lambda(0,0) + \tfrac{1}{2}\alpha\bar{t}^{-\frac{1}{2}} + \bar{c}|\bar{w}(\tau)|.\end{aligned}$$

The integral of $|\bar{w}(\cdot)|$ over $[0,\bar{t}]$ is $O(1)$, by virtue of Lemma 12.8.3. Moreover,

$$(12.9.22)\qquad \xi(\bar{t}) = \bar{x} > \phi_+(\bar{t}) \geq \lambda(0,0)\bar{t} + \alpha\bar{t}^{\frac{1}{2}}.$$

Therefore, integrating (12.9.21) over $[0,\bar{t}]$ yields

$$(12.9.23)\qquad \xi(0) \geq \tfrac{1}{2}\alpha\bar{t}^{\frac{1}{2}} + O(1) \geq \tfrac{\sqrt{2}}{4}\alpha t^{\frac{1}{2}} + O(1).$$

Since $\zeta(\cdot)$ stays to the right of $\xi(\cdot)$, (12.9.23) implies, in particular, that the graph of $\zeta(\cdot)$ will intersect the graph of $\psi_+(\cdot)$ at time $\hat{t} = O(t^{\frac{1}{2}})$.

Let $(\hat{z}(\cdot),\hat{w}(\cdot))$ denote the trace of (z,w) along $\zeta(\cdot)$. The oscillation of $\hat{w}(\cdot)$ over $[\hat{t},t)$ is $O(t^{-\frac{1}{4}})$. Furthermore, on account of Theorem 12.5.1, with time origin shifted to $\hat{t}$, and Lemma 12.9.2, we deduce that the total variation of $\hat{w}(\cdot)$ over $[\hat{t},t)$ is also $O(t^{-\frac{1}{4}})$. It then follows from Theorem 12.4.1 that $\hat{z}(t-) = O(t^{-\frac{3}{4}})$.

By virtue of the above result, (12.9.21) now implies

$$(12.9.24)\qquad \dot{\xi}(\tau) \leq \lambda(0,0) + O(t^{-\frac{3}{4}}) + \bar{c}|\bar{w}(\tau)|,$$

which, upon integrating over $[0,t]$, yields

$$(12.9.25)\qquad \xi(0) \geq x - \lambda(0,0)t + O(t^{\frac{1}{4}}) \geq \tfrac{1}{2}[x - \lambda(0,0)t].$$

Thus, $\hat{t} \geq c'[x - \lambda(0,0)t]$. But then the oscillation and total variation of $\hat{w}(\cdot)$ over $[\hat{t},t]$ is bounded by $\hat{c}[x - \lambda(0,0)t]^{-\frac{1}{2}}$, in which case $(12.9.6)_1$ follows from Theorem 12.4.1.

Finally, we return to (12.9.14). Since $z(\phi_+(t)+,t)$ is $O(t^{-\frac{3}{4}})$, we deduce that $g(t) = O(t^{-\frac{3}{4}})$, and this in turn yields $v(t) = (p_+t)^{\frac{1}{2}} + O(t^{\frac{1}{4}})$, with $p_+ > 0$. We have thus verified $(12.9.2)_1$.

A similar argument establishes $(12.9.6)_2$, for $x < \psi_-(t)$, and validates $(12.9.2)_2$. This completes the proof of Lemma 12.9.6 and thereby the proof of Theorem 12.9.1.

It is now easy to determine the large-time asymptotics of the solution $U(x,t)$ in $L^1(-\infty,\infty)$. Starting out from the (finite) Taylor expansion

$$(12.9.26)\qquad U(z,w) = zR(0,0) + wS(0,0) + O(z^2 + w^2),$$

and using Theorem 12.9.1, we conclude

12.9.7 Theorem. *Assume $p_+ > 0$ and $q_- > 0$. Then, as $t \to \infty$,*

(12.9.27)

$$\|U(x,t) - M(x,t;p_-,p_+)R(0,0) - N(x,t;q_-,q_+)S(0,0)\|_{L^1(-\infty,\infty)} = O(t^{-\frac{1}{4}}),$$

where M and N denote the N-wave profiles:

$(12.9.28)_1$

$$M(x,t;p_-,p_+) = \begin{cases} \dfrac{x-\lambda(0,0)t}{\lambda_z(0,0)t}, & for\ -(p_-t)^{\frac{1}{2}} \le x-\lambda(0,0)t \le (p_+t)^{\frac{1}{2}} \\ 0 & otherwise, \end{cases}$$

$(12.9.28)_2$

$$N(x,t;q_-,q_+) = \begin{cases} \dfrac{x-\mu(0,0)t}{\mu_w(0,0)t}, & for\ -(q_-t)^{\frac{1}{2}} \le x-\mu(0,0)t \le (q_+t)^{\frac{1}{2}} \\ 0 & otherwise. \end{cases}$$

12.10 Periodic Solutions

The study of genuinely nonlinear hyperbolic systems (12.1.1) of two conservation laws will be completed with a discussion of the large-time behavior of solutions with small oscillation that are periodic,

$$U(x+\ell,t) = U(x,t), \qquad -\infty < x < \infty,\ \ t > 0, \tag{12.10.1}$$

and have zero mean[2]:

$$\int_y^{y+\ell} U(x,t)dx = 0, \qquad -\infty < y < \infty,\ \ t > 0. \tag{12.10.2}$$

The confinement of waves resulting from periodicity induces active interactions and cancellation. As a result, the total variation per period decays at the rate $O(t^{-1})$:

12.10.1 Theorem. *For any $x \in (\infty,\infty)$, and $t > 0$,*

$$TV_{[x,x+\ell]}\, z(\cdot,t) + TV_{[x,x+\ell]}\, w(\cdot,t) \le \frac{b\ell}{t}. \tag{12.10.3}$$

Proof. Apply (12.6.1) with $y = x + n\ell$; then divide by n and let $n \to \infty$. This completes the proof.

[2] If the Cauchy problem has unique solution, initial data that are periodic with zero mean necessarily generate solutions with the same property.

We now resolve the asymptotics at the scale $O(t^{-1})$. The mechanism encountered in Section 11.7, in the context of genuinely nonlinear scalar conservation laws, namely the confinement of the intercepts of extremal backward characteristics in intervals of the x-axis of period length, is here in force as well and generates similar, serrated asymptotic profiles. The nodes of the profiles are again tracked by divides, in the sense of Definition 10.3.3.

12.10.2 Theorem. *The upper half-plane is partitioned by minimal (or maximal)* 1- (*or* 2-) *divides along which* z (*or* w) *decays rapidly to zero,* $O(t^{-2})$, *as* $t \to \infty$. *Let* $\chi_-(\cdot)$ *and* $\chi_+(\cdot)$ *be any two adjacent* 1- (*or* 2-) *divides, with* $\chi_-(t) < \chi_+(t)$. *Then* $\chi_+(t) - \chi_-(t)$ *approaches a constant at the rate* $O(t^{-1})$, *as* $t \to \infty$. *Furthermore, between* χ_- *and* χ_+ *lies a* 1- (*or* 2-) *characteristic* ψ *such that, as* $t \to \infty$,

$$\psi(t) = \tfrac{1}{2}[\chi_-(t) + \chi_+(t)] + o(1), \tag{12.10.4}$$

$$\lambda_z(0,0)z(x,t) = \begin{cases} \dfrac{x - \chi_-(t)}{t} + o\left(\dfrac{1}{t}\right), & \chi_-(t) < x < \psi(t), \\[2ex] \dfrac{x - \chi_+(t)}{t} + o\left(\dfrac{1}{t}\right), & \psi(t) \ < x < \chi_+(t), \end{cases} \tag{12.10.5$_1$}$$

or

$$\mu_w(0,0)w(x,t) = \begin{cases} \dfrac{x - \chi_-(t)}{t} + o\left(\dfrac{1}{t}\right), & \chi_-(t) < x < \psi(t), \\[2ex] \dfrac{x - \chi_+(t)}{t} + o\left(\dfrac{1}{t}\right), & \psi(t) \ < x < \chi_+(t). \end{cases} \tag{12.10.5$_2$}$$

The first step towards proving the above proposition is to investigate the large-time behavior of divides:

12.10.3 Lemma. *Along minimal (or maximal)* 1- (*or* 2-) *divides,* z (*or* w) *decays at the rate* $O(t^{-2})$, *as* $t \to \infty$. *Furthermore, if* $\chi_-(\cdot)$ *and* $\chi_+(\cdot)$ *are any two minimal (or maximal)* 1- (*or* 2-) *divides, then, as* $t \to \infty$,

$$\chi_+(t) - \chi_-(t) = h_\infty + O\left(\frac{1}{t}\right), \tag{12.10.6}$$

$$\int_{\chi_-(t)}^{\chi_+(t)} z(x,t)dx = O\left(\frac{1}{t^2}\right), \tag{12.10.7$_1$}$$

or

$$\int_{\chi_-(t)}^{\chi_+(t)} w(x,t)dx = O\left(\frac{1}{t^2}\right). \tag{12.10.7$_2$}$$

Proof. Assume $\chi(\cdot)$ is a minimal 1-divide, say the limit of a sequence $\{\xi_n(\cdot)\}$ of minimal backward 1-characteristics emanating from points $\{(x_n,t_n)\}$, with $t_n \to \infty$, as $n \to \infty$. Let $(z_n(\cdot), w_n(\cdot))$ denote the trace of (z,w) along $\xi_n(\cdot)$. Applying Theorem 12.5.1, with time origin shifted to τ, and using Theorem 12.10.1, we deduce that the total variation of $w_n(\cdot)$ over any interval $[\tau, \tau+1] \subset [0,t_n]$ is $O(\tau^{-1})$, uniformly in n. Therefore, by virtue of Theorem 12.4.1, $z_n(\cdot)$ is a nonincreasing function on $[0,t_n]$ whose oscillation over $[\tau,\tau+1]$ is $O(\tau^{-3})$, uniformly in n. It follows that the trace $\bar{z}(\cdot)$ of z along $\chi(\cdot)$ is likewise a nonincreasing function with $O(\tau^{-3})$ oscillation over $[\tau,\tau+1]$. By tallying the oscillation of $\bar{z}(\cdot)$ over intervals of unit length, from t to infinity, we verify the assertion $\bar{z}(t) = O(t^{-2})$.

A similar argument shows that the trace $\bar{w}(\cdot)$ of w along maximal 2-divides is likewise $O(t^{-2})$, as $t \to \infty$.

Let $\chi_-(\cdot)$ and $\chi_+(\cdot)$ be minimal 1-divides with $h(t) = \chi_+(t) - \chi_-(t) \geq 0$, for $0 \leq t < \infty$. Note that, because of periodicity, $h(0) < k\ell$, for some integer k, implies $h(t) \leq k\ell$, $0 \leq t < \infty$. Letting $(z_-(\cdot), w_-(\cdot))$ and $(z_+(\cdot), w_+(\cdot))$ denote the trace of (z,w) along $\chi_-(\cdot)$ and $\chi_+(\cdot)$, respectively, we have

$$\dot{h}(\tau) = \lambda(z_+(\tau), w_+(\tau)) - \lambda(z_-(\tau), w_-(\tau)), \tag{12.10.8}$$

for almost all τ in $[0,\infty)$.

The maximal backward 2-characteristic $\zeta_\tau(\cdot)$ emanating from the point $(\chi_+(\tau),\tau)$ is intercepted by the graph of $\chi_-(\cdot)$ at time $\tau - f(\tau)$. If $(\hat{z}(\cdot), \hat{w}(\cdot))$ denotes the trace of (z,w) along $\zeta_\tau(\cdot)$, Theorems 12.5.1 and 12.10.1 together imply that the total variation of $\hat{z}(\cdot)$ over the interval $[\tau - f(\tau), \tau]$ is $O(\tau^{-1})$, as $\tau \to \infty$. It then follows from Theorem 12.4.1 that the oscillation of $\hat{w}(\cdot)$ over $[\tau - f(\tau), \tau]$ is $O(\tau^{-3})$. Hence

$$w_+(\tau) = w_-(\tau - f(\tau)) + O(\tau^{-3}). \tag{12.10.9}$$

Since $z_\pm(\tau) = O(\tau^{-2})$, (12.10.8) yields

$$\dot{h}(\tau) = \lambda(0, w_-(\tau - f(\tau))) - \lambda(0, w_-(\tau)) + O(\frac{1}{\tau^2}). \tag{12.10.10}$$

From $\dot{h}(\tau) = O(\tau^{-1})$ and $\dot{\zeta}_\tau = \mu(0,0) + O(\tau^{-1})$, we infer that the oscillation of $f(\cdot)$ over the interval $[\tau,\tau+1]$ is $O(\tau^{-1})$. The total variation of $w_-(\cdot)$ over $[\tau,\tau+1]$ is likewise $O(\tau^{-1})$. Then, for any $t < t' < \infty$,

$$|\int_t^{t'} \{\lambda(0, w_-(\tau - f(\tau))) - \lambda(0, w_-(\tau))\} d\tau| \leq \frac{c}{t}. \tag{12.10.11}$$

Upon combining (12.10.10) with (12.10.11), one arrives at (12.10.6).

Let $U_-(\cdot)$ and $U_+(\cdot)$ denote the trace of U along $\chi_-(\cdot)$ and $\chi_+(\cdot)$, respectively. Integration of (12.1.1) over $\{(x,\tau) : t < \tau < \infty, \chi_-(\tau) < x < \chi_+(\tau)\}$ yields the equation 12.9

$$\int_{\chi_-(t)}^{\chi_+(t)} U(x,t)dx \tag{12.10.12}$$

$$= \int_t^\infty \{F(U_+(\tau)) - \lambda(U_+(\tau))U_+(\tau) - F(U_-(\tau)) + \lambda(U_-(\tau))U_-(\tau)\}d\tau.$$

We multiply (12.10.12), from the left, by the row vector $\mathrm{D}z(0)$. On account of (7.3.12), $U_z = R$ and $U_w = S$ so that, using (12.1.2), we deduce

$$\mathrm{D}z(0)U = z + O(z^2 + w^2), \tag{12.10.13}$$

$$\mathrm{D}z(0)[F(U) - \lambda(U)U] = \mathrm{D}z(0)F(0) + aw^2 + O(z^2 + |zw| + |w|^3), \tag{12.10.14}$$

where the constant a is the value of $\frac{1}{2}(\lambda - \mu)S^\top \mathrm{D}^2 zS$ at $U = 0$. Upon combining $z_\pm(\tau) = O(\tau^{-2})$, $w_\pm(\tau) = O(\tau^{-1})$ and (12.10.9), we conclude

$$\int_{\chi_-(t)}^{\chi_+(t)} z(x,t)dx = a\int_t^\infty [w_-^2(\tau - f(\tau)) - w_-^2(\tau)]d\tau + O\left(\frac{1}{t^2}\right). \tag{12.10.15}$$

As explained above, over the interval $[\tau, \tau+1]$ the oscillation of $f(\cdot)$ is $O(\tau^{-1})$ and the total variation of $w_-^2(\cdot)$ is $O(\tau^{-2})$. Then, the integral on the right-hand side of (12.10.15) is $O(t^{-2})$, as $t \to \infty$, which establishes $(12.10.7)_1$.

When $\chi_-(\cdot)$ and $\chi_+(\cdot)$ are maximal 2-divides, a similar argument verifies (12.10.6) and $(12.10.7)_2$. The proof is complete.

The remaining assertions of Theorem 12.10.2 will be established through the following

12.10.4 Lemma. *Consider any two adjacent minimal (or maximal)* 1- (*or* 2-) *divides* $\chi_-(\cdot), \chi_+(\cdot)$, *with* $\chi_-(t) < \chi_+(t)$, $0 \le t < \infty$. *The special forward* 1- (*or* 2-) *characteristic* $\phi_-(\cdot)$ (*or* $\psi_+(\cdot)$), *in the notation of Section* 12.3, *issuing from any fixed point* $(\bar{x}, 0)$, *where* $\chi_-(0) < \bar{x} < \chi_+(0)$, *is denoted by* $\psi(\cdot)$. *Then* $\psi(\cdot)$ *satisfies* (12.10.4). *Furthermore,* $(12.10.5)_1$ (*or* $(12.10.5)_2$) *holds.*

Proof. It will suffice to discuss the case where χ_- and χ_+ are 1-divides. We consider minimal backward 1-characteristics $\xi(\cdot)$ emanating from points (x,t), with $t > 0$ and $\chi_-(t) < x < \chi_+(t)$. Their graphs are trapped between the graphs of χ_- and χ_+. The intercepts $\xi(0)$ of such ξ, by the x-axis, cannot accumulate to any $\hat{x}$ in the open interval $(\chi_-(0), \chi_+(0))$, because in that case a minimal 1-divide would issue from the point $(\hat{x}, 0)$, contrary to our assumption that χ_- and χ_+ are adjacent. Therefore, by the construction of $\psi(\cdot)$ we infer that, as $t \to \infty$, $\xi(\tau) \to \chi_-(\tau)$, when x is in $(\chi_-(t), \psi(t)]$, or $\xi(\tau) \to \chi_+(\tau)$, when x is in $(\psi(t), \chi_+(t)]$, the convergence being uniform on compact subsets of $[0, \infty)$.

Let us now fix $\xi(\cdot)$ that emanates from some point (x,t), with $\chi_-(t) < x \le \psi(t)$, and set $h(\tau) = \xi(\tau) - \chi_-(\tau)$, $0 \le \tau \le t$. Then, for almost all $\tau \in [0,t]$ we have

$$\dot{h}(\tau) = \lambda(\bar{z}(\tau), \bar{w}(\tau)) - \lambda(z_-(\tau), w_-(\tau)), \tag{12.10.16}$$

where $(\bar{z}(\cdot),\bar{w}(\cdot))$ denotes the trace of (z,w) along $\xi(\cdot)$, while $(z_-(\cdot),w_-(\cdot))$ stands for the trace of (z,w) along $\chi_-(\cdot)$.

By virtue of Theorems 12.5.1 and 12.10.1, the total variation of $\bar{w}(\cdot)$ on any interval $[s,s+1]\subset[0,t]$ is $O(s^{-1})$. It then follows from Theorem 12.4.1 that the oscillation of $\bar{z}(\cdot)$ over $[s,s+1]$ is $O(s^{-3})$ and hence

$$\bar{z}(\tau)=z(x,t)+O\left(\frac{1}{\tau^2}\right). \tag{12.10.17}$$

Furthermore, by Lemma 12.10.3, $z_-(\tau)=O(\tau^{-2})$. Also, $z(x,t)=O(t^{-1})$ so, *a fortiori*, $z(x,t)=O(\tau^{-1})$. On account of these observations, (12.10.16) yields

$$\dot{h}(\tau)=\lambda_z(0,0)z(x,t)+\lambda(0,\bar{w}(\tau))-\lambda(0,w_-(\tau))+O\left(\frac{1}{\tau^2}\right). \tag{12.10.18}$$

For any fixed $\tau >> 0$, we consider the maximal backward 2-characteristic $\zeta_\tau(\cdot)$ emanating from the point $(\xi(\tau),\tau)$, which is intercepted by the graph of $\chi_-(\cdot)$ at time $\tau-f(\tau)$. If $(\hat{z}(\cdot),\hat{w}(\cdot))$ denotes the trace of (z,w) along $\zeta_\tau(\cdot)$, Theorems 12.5.1 and 12.10.1 together imply that the total variation of $\hat{z}(\cdot)$ over the interval $[\tau-f(\tau),\tau]$ is $O(\tau^{-1})$. It then follows from Theorem 12.4.1 that the oscillation of $\hat{w}(\cdot)$ over $[\tau-f(\tau),\tau]$ is $O(\tau^{-3})$. Hence

$$\bar{w}(\tau)=w_-(\tau-f(\tau))+O\left(\frac{1}{\tau^3}\right), \tag{12.10.19}$$

and so (12.10.18) implies

$$\dot{h}(\tau)=\lambda_z(0,0)z(x,t)+\lambda(0,w_-(\tau-f(\tau)))-\lambda(0,w_-(\tau))+O\left(\frac{1}{\tau^2}\right). \tag{12.10.20}$$

As in the proof of Lemma 12.10.3, on any interval $[\tau,\tau+1]\subset[0,t]$ the oscillation of $f(\cdot)$ is $O(\tau^{-1})$ and the total variation of $w_-(\cdot)$ is also $O(\tau^{-1})$. Therefore, upon integrating (12.10.20) over the interval $[s,t]$, $0<s<t$, we deduce

$$x-\chi_-(t)-\lambda_z(0,0)z(x,t)t=\xi(s)-\chi_-(s)+O\left(\frac{1}{s}\right)+sO\left(\frac{1}{t}\right). \tag{12.10.21}$$

With reference to the right-hand side of (12.10.21), given $\varepsilon>0$, we first fix s so large that $O(s^{-1})$ is less than $\frac{1}{3}\varepsilon$. With s thus fixed, we determine $\hat{t}$ such that, for $t\geq\hat{t}$, $sO(t^{-1})$ does not exceed $\frac{1}{3}\varepsilon$, while at the same time $\xi(s)-\chi_-(s)<\frac{1}{3}\varepsilon$, for all $x\in(\chi_-(t),\psi(t)]$. It is sufficient to check this last condition for $t=\hat{t}$, $x=\psi(\hat{t})$. We have thus verified that the left-hand side of (12.10.21) is $o(1)$, as $t\to\infty$, uniformly in x on $(\chi_-(t),\psi(t))$, which verifies the upper half of $(12.10.5)_1$. The lower half of $(12.10.5)_1$ is established by a similar argument. This completes the proof.

12.11 Notes

There is voluminous literature addressing various aspects of the theory of genuinely nonlinear systems of two conservation laws. The approach in this chapter, via the theory of generalized characteristics, was initiated by the author. Many of these results, occasionally under slightly stricter assumptions, were derived earlier in the framework of solutions constructed by the random choice method, which will be presented in Chapter XIII. The seminal contribution in that direction is Glimm and Lax [1], who introduced and employed the notions of "approximate characteristic" and "approximate conservation law". Approximate characteristics, namely concatenations of classical characteristics and shocks, are intimately related to generalized characteristics, in the sense of the present work. However, there is a fundamental distinction between the two approaches, in that Glimm and Lax follow approximate characteristics as they propagate forward in time, whereas here we view generalized characteristics retrospectively.

The Lax entropies, discussed in Section 12.2, were first introduced in Lax [4]. The hodograph transformation was discovered by Riemann [1]. For detailed discussions and applications to aerodynamics, see Courant and Friedrichs [1], and von Mises [1]. For applications to other areas of mathematical physics, see D. Fusco [1].

A somewhat stronger version of Theorem 12.3.3 was established by DiPerna [3], for solutions constructed by the random choice method. Theorem 12.4.1 improves a proposition in Dafermos [19].

Theorems 12.5.1, 12.6.1 and 12.6.4 were originally established in Glimm and Lax [1], for solutions constructed by the random choice method. The treatment here employs and refines methodology developed by Dafermos and Geng [1,2], for special systems, and Trivisa [1], for general systems, albeit when solutions are "countably regular." Trivisa [2] extends these results to genuine nonlinear systems of n conservation laws endowed with a coordinate system of Riemann invariants.

Tsikkou [1,2] treats by the same method the special system (4.1.11) of isentropic elasticity. She approximates $\sigma(u)$ by piecewise linear functions and conducts a thorough examination of the resulting pattern of characteristics, thus deriving sharp bounds on the total variation. She also shows that when $\sigma'(u)$ contains jump discontinuities the total variation of solutions may blow up in finite time.

The results of Section 12.7 were established earlier by DiPerna [3], for solutions constructed by the random choice method.

For solutions with initial data in L^1, Temple [5] derives decay at the rate $O(1/\sqrt{\log t})$. The $O(t^{-\frac{1}{2}})$ decay rate established in Theorem 12.8.1, which is taken from Dafermos [19], is sharp. Similarly, Lemma 12.8.2 improves an earlier result of Temple [8]. L^1 stability has now been established for general systems; see Chapter XIV.

The mechanism that generates N-wave profiles was understood quite early, through formal asymptotics (see Courant and Friedrichs [1]), even though a rigorous proof was lacking (Lax [2]). In a series of papers by DiPerna [4,6] and Tai-Ping Liu [8,9,22], decay to N-waves of solutions with initial data of compact support, constructed by the random choice method, was established at progressively sharper rates, not only for genuinely nonlinear systems of two conservation laws but even for systems of n conservation laws with characteristic families that are either genuinely nonlinear or linearly degenerate. For an alternative approach, which decouples the characteristic fields by a change of variable, see Ayad and Heibig [1,2]. The decay rates recorded in Theorem 12.9.1 are sharp. When the initial data do not have compact support but instead approach distinct limits U_L and U_R, as $x \to \pm\infty$, then the solution U converges, as $t \to \infty$, to the solution of the Riemann problem with initial data (9.1.12); see Tai-Ping Liu [6] and compare with the scalar case discussed in Section 11.5. Relatively little is known for systems that are not genuinely nonlinear; see Zumbrun [1,2].

Theorem 12.10.1 is due to Glimm and Lax [1], while Theorem 12.10.2 is taken from Dafermos [21]. For the stability of periodic solutions to the system of relativistic isentropic gas flow, see Calvo, Colombo and Frid [1]. Decay of solutions with periodic initial data may be peculiar to systems of two conservation laws. Indeed, the work of R. Young [3,4] and Temple and Young [3,4] indicates that, for the system of nonisentropic gas dynamics, solutions with periodic initial data remain bounded but do not necessarily decay. On the other hand, recent work of Qu and Xin [1] shows that the lifespan of such solutions is very long. On the other hand, systems that are, or may be transformed into, linearly degenerate, typically admit smooth time-periodic solutions. For an example, see M. Shearer [7].

For applications of the theory of characteristics to investigating uniqueness, regularity and large-time behavior of solutions of special systems with coinciding shock and rarefaction wave curves (Temple [3]), see Serre [7,11], Dafermos and Geng [1,2], Heibig [2], Heibig and Sahel [1], and Ostrov [1]. BV solutions for such systems have been constructed by the Godunov difference scheme (LeVeque and Temple [1]) as well as by the method of vanishing viscosity (Serre [1,11]). For blowing up in L^∞ of solutions to initial-boundary value problems, see Bourdarias, Gisclon and Junca [2].

There is extensive literature on systems of two conservation laws treated through the Glimm-Lax approach.For an extension of the Lax-Glimm theory to genuinely nonlinear systems of two conservation laws under fewer restrictions on the shock and rarefaction wave curves, see Bianchini, Colombo and Monti [1,2]. For an example of a system with inhomogeneous flux, see Frid, Risebro and Sande [1].

XIII

The Random Choice Method

This chapter introduces the celebrated random choice method, which has provided the earliest, but still very effective, scheme for constructing globally defined, admissible BV solutions to the Cauchy problem for strictly hyperbolic systems of conservation laws, under initial data with small total variation. The solution is obtained as the limit of a sequence of approximate solutions that do not smear shocks. Solutions to the Riemann problem, discussed at length in Chapter IX, serve as building blocks for constructing the approximate solutions to the Cauchy problem. Striving to preserve the sharpness of shocks may be in conflict with the requirement of consistency of the algorithm. The "randomness" feature of the method is employed in order to strike the delicate balance of safeguarding consistency without smearing the sharpness of propagating shock fronts. At the cost of delineating the global wave pattern, the device of wave tracing, which will be discussed here only briefly, renders the algorithm deterministic.

A detailed presentation of the random choice method will be given for systems with characteristic families that are either genuinely nonlinear or linearly degenerate. The case of more general systems, which involves substantial technical complication, will be touched on rather briefly here.

In Chapter XVI we shall see how the algorithm may be adapted for handling systems of balance laws, with inhomogeneities and source terms.

13.1 The Construction Scheme

We consider the initial value problem for a strictly hyperbolic system of conservation laws, defined on a ball $\mathscr{O}$ centered at the origin:

$$(13.1.1)\qquad \begin{cases} \partial_t U(x,t) + \partial_x F(U(x,t)) = 0, & -\infty < x < \infty,\ 0 \le t < \infty, \\ U(x,0) = U_0(x), & -\infty < x < \infty. \end{cases}$$

The initial data U_0 are functions of bounded variation on $(-\infty,\infty)$. The ultimate goal is to establish the following

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13.1.1 Theorem. *There are positive constants δ_0 and δ_1 such that if*

(13.1.2) $$\sup_{(-\infty,\infty)}|U_0(\cdot)| < \delta_0\,,$$

(13.1.3) $$TV_{(-\infty,\infty)}U_0(\cdot) < \delta_1\,,$$

then there exists a solution U of (13.1.1), *which is a function of locally bounded variation on $(-\infty,\infty)\times[0,\infty)$, taking values in $\mathscr{O}$. This solution satisfies the entropy admissibility criterion for any entropy-entropy flux pair (η,q) of the system, with $\eta(U)$ convex. Furthermore, for each fixed $t\in[0,\infty)$, $U(\cdot,t)$ is a function of bounded variation on $(-\infty,\infty)$ and*

(13.1.4) $$\sup_{(-\infty,\infty)}|U(\cdot,t)| \le c_0 \sup_{(-\infty,\infty)}|U_0(\cdot)|, \qquad 0\le t<\infty,$$

(13.1.5) $$TV_{(-\infty,\infty)}U(\cdot,t) \le c_1 TV_{(-\infty,\infty)}U_0(\cdot), \qquad 0\le t<\infty,$$

(13.1.6) $$\int_{-\infty}^{\infty}|U(x,t)-U(x,\tau)|dx \le c_2|t-\tau|TV_{(-\infty,\infty)}U_0(\cdot), \quad 0\le\tau<t<\infty,$$

where c_0, c_1 and c_2 are constants depending solely on F. When the system is endowed with a coordinate system of Riemann invariants, δ_1 in (13.1.3) *may be fixed arbitrarily large, so long as*

(13.1.7) $$(\sup_{(-\infty,\infty)}|U_0(\cdot)|)(TV_{(-\infty,\infty)}U_0(\cdot)) < \delta_2\,,$$

with δ_2 sufficiently small, depending on δ_1.

The proof of the above proposition is quite lengthy and shall occupy the entire chapter. Even though the assertion holds at the level of generality stated above, certain steps in the proof (Sections 13.3, 13.4, 13.5 and 13.6) will be carried out under the simplifying assumption that each characteristic family of the system is either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2). The case of general systems will be touched on in Sections 13.7 and 13.8.

The solution U will be attained as the $h\downarrow 0$ limit of a family of approximate solutions U_h constructed by the following process.

We fix a spatial mesh-length h, which will serve as parameter, and an associated temporal mesh-length $\lambda^{-1}h$, where λ is a fixed upper bound of the characteristic speeds $|\lambda_i(U)|$, for $U\in\mathscr{O}$ and $i=1,\cdots,n$. Setting $x_r = rh\,,\ r=0,\pm1,\pm2,\cdots$ and $t_s = s\lambda^{-1}h\,,\ s=0,1,2,\cdots$, we build the staggered grid of *mesh-points* (x_r,t_s), with $s=0,1,2,\cdots$, and $r+s$ even.

Assuming now U_h has been defined on $\{(x,t):\ -\infty<x<\infty,\ 0\le t<t_s\}$, we determine $U_h(\cdot,t_s)$ as a step function that is constant on intervals defined by neighboring mesh-points along the line $t=t_s$,

(13.1.8) $$U_h(x,t_s) = U_s^r\,, \qquad x_{r-1}<x<x_{r+1}\,,\ \ r+s \text{ odd},$$

and approximates the function $U_h(\cdot,t_s-)$. The major issue of selecting judiciously the constant states U_s^r will be addressed in Section 13.2.

Next we determine U_h on the strip $\{(x,t) : -\infty < x < \infty\,,\ t_s \le t < t_{s+1}\}$ as a solution of our system, namely,

$$\partial_t U_h(x,t) + \partial_x F(U_h(x,t)) = 0, \qquad -\infty < x < \infty\,,\ t_s \le t < t_{s+1}\,, \tag{13.1.9}$$

under the initial condition (13.1.8), along the line $t = t_s$. Notice that the solution of (13.1.9), (13.1.8) consists of centered wave fans emanating from the mesh-points lying on the t_s-time line (Fig. 13.1.1). The wave fan centered at the mesh point (x_r,t_s), $r+s$ even, is constructed by solving the Riemann problem for our system, with left state U_s^{r-1} and right state U_s^{r+1}. We employ admissible solutions, with shocks satisfying the viscous shock admissibility condition (cf. Chapter IX). The resulting outgoing waves from neighboring mesh-points do not interact on the time interval $[t_s,t_{s+1})$, because of our selection of the ratio λ of spatial and temporal mesh-lengths.

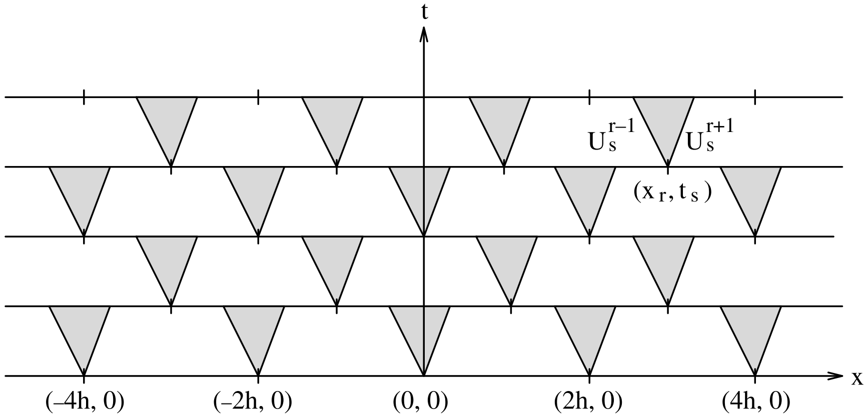

Fig. 13.1.1

To initiate the algorithm, at $s = 0$, we employ the initial data:

$$U_h(x,0-) = U_0(x), \qquad -\infty < x < \infty\,. \tag{13.1.10}$$

The construction of U_h may proceed for as long as one can solve the resulting Riemann problems. As we saw in Chapter IX, this can be performed, in general, so long as the jumps $|U_s^{r+1} - U_s^{r-1}|$ stay sufficiently small.

After considerable preparation, we shall demonstrate, in Sections 13.5 and 13.6, that the U_h satisfy estimates

$$\sup_{(-\infty,\infty)}|U_h(\cdot,t)| \le c_0 \sup_{(-\infty,\infty)}|U_0(\cdot)|, \qquad 0 \le t < \infty, \tag{13.1.11}$$

$$TV_{(-\infty,\infty)} U_h(\cdot,t) \le c_1 TV_{(-\infty,\infty)} U_0(\cdot), \qquad 0 \le t < \infty, \tag{13.1.12}$$

(13.1.13)

$$\int_{-\infty}^{\infty} |U_h(x,t) - U_h(x,\tau)|dx \le c_2(|t-\tau| + h)TV_{(-\infty,\infty)}U_0(\cdot), \qquad 0 \le \tau < t < \infty.$$

In particular, (13.1.11) guarantees that when (13.1.2) holds with δ_0 sufficiently small, U_h may be constructed on the entire upper half-plane.

13.2 Compactness and Consistency

Deferring the proof of (13.1.11), (13.1.12) and (13.1.13) to Sections 13.5 and 13.6, here we shall take these stability estimates for granted and will examine their implications. By virtue of (13.1.12), Helly's theorem and the Cantor diagonal process, there is a sequence $\{h_m\}$, with $h_m \to 0$ as $m \to \infty$, such that $\{U_{h_m}(\cdot,\tau)\}$ is Cauchy in $L^1_{loc}(-\infty,\infty)$, for each positive rational number τ. Since the rationals are dense in $[0,\infty)$, (13.1.13) implies that $\{U_{h_m}(\cdot,t)\}$ must be Cauchy in $L^1_{loc}(-\infty,\infty)$, for any $t \ge 0$. Thus

$$U_{h_m}(x,t) \to U(x,t), \quad \text{as } m \to \infty, \quad \text{in } L^1_{loc}((-\infty,\infty)\times[0,\infty)), \tag{13.2.1}$$

where, for each fixed $t \in [0,\infty)$, $U(\cdot,t)$ is a function of bounded variation on $(-\infty,\infty)$, which satisfies (13.1.4), (13.1.5) and (13.1.6). In particular, U is in BV_{loc}.

We now turn to the question of *consistency* of the algorithm, investigating whether U is a solution of the initial value problem (13.1.1). By its construction, U_h satisfies the system inside each strip $\{(x,t) : -\infty < x < \infty,\ t_s \le t < t_{s+1}\}$. Consequently, the errors are induced by the jumps of U_h across the dividing time lines $t = t_s$. To estimate the cumulative effect of these errors, we fix any C^∞ test function ϕ, with compact support on $(-\infty,\infty)\times[0,\infty)$, we apply the measure (13.1.9) to ϕ on the rectangle $\{(x,t) : x_{r-1} < x < x_{r+1}\, , t_s \le t < t_{s+1}\, , r+s \text{ odd}\}$ and sum over all such rectangles in the upper half-plane. After an integration by parts, and upon using (13.1.8) and (13.1.10), we obtain

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty [\partial_t\phi U_h + \partial_x\phi F(U_h)]dxdt + \int_{-\infty}^\infty \phi(x,0)U_0(x)dx \\ &\qquad = \sum_{s=0}^\infty \sum_{r+s \text{ odd}} \int_{x_{r-1}}^{x_{r+1}} \phi(x,t_s)[U_h(x,t_s-) - U_s^r]dx. \end{aligned} \tag{13.2.2}$$

Therefore, U will be a weak solution of (13.1.1), i.e., the algorithm will be consistent, if U_s^r approximates the function $U_h(\cdot,t_s-)$, over the interval $(x_{r-1}\, , x_{r+1})$, in such a manner that the right-hand side of (13.2.2) tends to zero, as $h \downarrow 0$.

One may attain consistency via the *Lax-Friedrichs scheme*:

$$U_s^r = \frac{1}{2h}\int_{x_{r-1}}^{x_{r+1}} U_h(x,t_s-)dx, \qquad r+s \text{ odd}. \tag{13.2.3}$$

Indeed, with that choice, each integral on the right-hand side of (13.2.2) is majorized by $h^2 \max|\partial_x\phi|\operatorname{osc}_{(x_{r-1},x_{r+1})}U_h(\cdot,t_s-)$. The sum of these integrals over r is then majorized by $h^2 \max|\partial_x \phi| TV_{(-\infty,\infty)} U_h(\cdot, t_s-)$, which, in turn, is bounded by $c_1 \delta_1 h^2 \max|\partial_x \phi|$, on account of (13.1.12) and (13.1.3). The summation over s, within the support of ϕ, involves $O(h^{-1})$ terms, and so finally the right-hand side of (13.2.2) is $O(h)$, as $h \downarrow 0$.

Even though it passes the test of consistency, the Lax-Friedrichs scheme stumbles on the issue of stability: it is at present unknown whether estimates (13.1.12) and (13.1.13) hold within its framework.[1] One of the drawbacks of this scheme is that it smears, through averaging, the shocks of the exact solution. This feature may be vividly illustrated in the context of the Riemann problem for the linear, scalar conservation law,

$$\text{(13.2.4)} \qquad \begin{cases} \partial_t u(x,t) + a\lambda \partial_x u(x,t) = 0, & -\infty < x < \infty, \ 0 \le t < \infty, \\ u(x,0) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 < x < \infty, \end{cases} \end{cases}$$

where a is a constant in $(-1,1)$ (recall that λ denotes the ratio of the spatial and temporal mesh-lengths). The solution of (13.2.4) comprises, of course, the constant states $u = 0$, on the left, and $u = 1$, on the right, joined by the shock $x = a\lambda t$. The first four steps of the construction of the approximate solution u_h according to the Lax-Friedrichs scheme are depicted in Fig. 13.2.1. The smearing of the shock is clearly visible.

In order to prevent the smearing of shocks, we try a different policy for evaluating the U_s^r. We start out with some sequence $\wp = \{a_0, a_1, a_2, \cdots\}$, where $a_s \in (-1,1)$, we set $y_s^r = x_r + a_s h$, and build, on the upper half-plane, another staggered grid of points (y_s^r, t_s), with $s = 0,1,2,\cdots$ and $r+s$ odd. We employ (y_s^r, t_s) as a *sampling point* for the interval (x_{r-1}, x_{r+1}), on the t_s-time line, by selecting

$$\text{(13.2.5)} \qquad U_s^r = \lim_{t \uparrow t_s} U_h(y_s^r -, t), \qquad r+s \text{ odd.}$$

To test this approach, we consider again the Riemann problem (13.2.4). The first few steps of the construction of the approximate solution U_h are depicted in Fig. 13.2.2. We observe that according to the rule (13.2.5), as one passes from $t = t_s$ to $t = t_{s+1}$, the shock is preserved but its location is shifted by h, to the left when $a_s > a$, or to the right when $a_s < a$. Consequently, in the limit $h \downarrow 0$ the shock will be thrown off course, unless the number m_- of indices $s \le m$ with $a_s < a$ and the number m_+ of indices $s \le m$ with $a_s > a$ are related through $m_- - m_+ \sim am$, as $m \to \infty$. Combining this with $m_- + m_+ = m$, we conclude that u_h will converge to the solution of (13.2.4) if and only if $m_-/m \to \frac{1}{2}(1+a)$ and $m_+/m \to \frac{1}{2}(1-a)$, as $m \to \infty$. For consistency of the algorithm, it will be necessary that the above

[1] In fact, it has been demonstrated, in the context of the closely related Godunov scheme, that selecting λ to be an irrational number, but very close to a rational, induces resonance generating spurious oscillations in the approximate solutions, which drives the total variation to infinity.

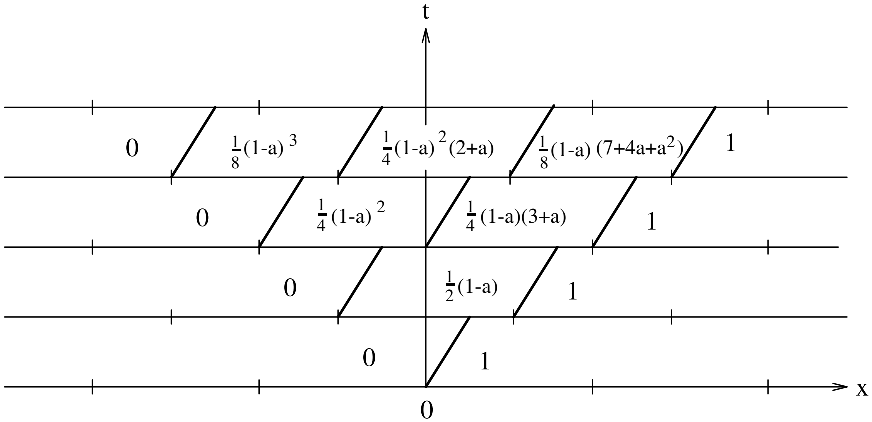

Fig. 13.2.1

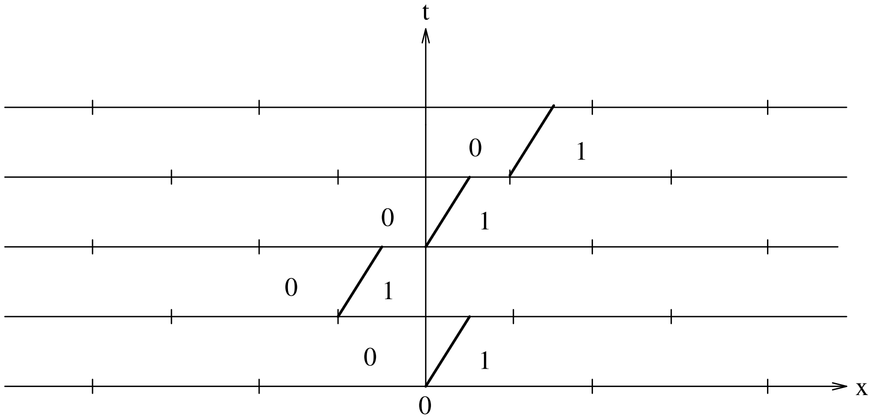

Fig. 13.2.2

condition hold for arbitrary $a \in (-1,1)$. Clearly, this will be the case only when the sequence $\wp$ is *equidistributed* on the interval $(-1,1)$, that is, for any subinterval $I \subset (-1,1)$ of length $\mu(I)$:

$$\lim_{m\to\infty} \frac{2}{m}[\text{number of indices } s \leq m \text{ with } a_s \in I] = \mu(I), \tag{13.2.6}$$

uniformly with respect to I.

Later on, in Section 13.8, we shall see that the algorithm based on (13.2.5), with any sequence $\wp$ which is equidistributed in $(-1,1)$, is indeed consistent, for the general initial value problem (13.1.1); but this may be established only by paying the price of tracking the global wave pattern. The objective here is to demonstrate a slightly weaker result, whose proof however relies solely on the stability estimate

(13.1.5). Roughly, it will be shown that if one picks the sequence $\wp$ at random, then the resulting algorithm will be consistent, with probability one. It is from this feature that the method derives its name: *random choice*.

We realize $\wp$ as a point in the Cartesian product space $\mathscr{A} = \prod_{s=0}^{\infty}(-1,1)$. Each factor $(-1,1)$ is regarded as a probability space, under Lebesgue measure rescaled by a factor $1/2$, and this induces a probability measure ν on $\mathscr{A}$ as well. In connection with our earlier discussions on consistency, it may be shown (references in Section 13.10) that almost all sequences $\wp \in \mathscr{A}$ are equidistributed in $(-1,1)$. The main result is

13.2.1 Theorem. *There is a null subset $\mathscr{N}$ of $\mathscr{A}$ with the property that the algorithm induced by any sequence $\wp \in \mathscr{A} \backslash \mathscr{N}$ is consistent. That is, when the U_s^r are evaluated through* (13.2.5), *with $y_s^r = x_r + a_s h$, then the limit U in* (13.2.1) *is a solution of the initial value problem* (13.1.1).

Proof. The right-hand side of (13.2.2) is completely determined by the spatial mesh-length h, the sequence $\wp$ and the test function ϕ, so it shall be denoted by $e(\wp;\phi,h)$. By virtue of (13.2.5),

$$e(\wp;\phi,h) = \sum_{s=0}^{\infty} e_s(\wp;\phi,h), \tag{13.2.7}$$

where

$$e_s(\wp;\phi,h) = \sum_{r+s\ \mathrm{odd}} \int_{x_{r-1}}^{x_{r+1}} \phi(x,t_s)[U_h(x,t_s-) - U_h(y_s^r,t_s-)]dx. \tag{13.2.8}$$

The integral on the right-hand side of (13.2.8) is bounded from above by $2h\max|\phi|\,\mathrm{osc}_{(x_{r-1},x_{r+1})}U_h(\cdot,t_s-)$ and hence $e_s(\wp;\phi,h)$ is in turn majorized by $2h\max|\phi|\,TV_{(-\infty,\infty)}U_h(\cdot,t_s-)$. By (13.1.12) and (13.1.3), we conclude

$$|e_s(\wp;\phi,h)| \leq 2c_1\delta_1 h\max|\phi|, \qquad s = 0,1,2,\cdots. \tag{13.2.9}$$

In the summation (13.2.7), the number of nonzero terms, lying inside the support of ϕ, is $O(h^{-1})$, and so the most one may generally extract from (13.2.9) is $e(\wp;\phi,h) = O(1)$, as $h \downarrow 0$. This again indicates that one should not expect consistency for an arbitrary sequence $\wp$. The success of the random choice method stems from the fact that, as $h \downarrow 0$, the average of $e_s(\wp;\phi,h)$ decays to zero faster than $e_s(\wp;\phi,h)$ itself. Indeed,

$$\begin{aligned} &\int_{-1}^{1}\int_{x_{r-1}}^{x_{r+1}} \phi(x,t_s)[U_h(x,t_s-) - U_h(y_s^r,t_s-)]dxda_s \\ &\qquad = \frac{1}{h}\int_{x_{r-1}}^{x_{r+1}}\int_{x_{r-1}}^{x_{r+1}} \phi(x,t_s)[U_h(x,t_s-) - U_h(y,t_s-)]dxdy \end{aligned} \tag{13.2.10}$$

is majorized by $2h^2\max|\partial_x\phi|\,\mathrm{osc}_{(x_{r-1}x_{r+1})}U_h(\cdot,t_s-)$. The sum over r of these integrals is then majorized by $2h^2\max|\partial_x\phi|\,TV_{(-\infty,\infty)}U_h(\cdot,t_s-)$. Recalling (13.1.12) and (13.1.3), we finally conclude

$$(13.2.11)\qquad |\int_{-1}^{1} e_s(\wp;\phi,h)da_s| \leq 2c_1\delta_1 h^2 \max|\partial_x\phi|, \qquad s=0,1,2,\cdots.$$

Next we demonstrate that, for $0 \leq s < \sigma < \infty$, $e_s(\wp;\phi,h)$ and $e_\sigma(\wp;\phi,h)$ are "weakly correlated" in that their inner product in $\mathscr{A}$ decays to zero very rapidly, $O(h^3)$, as $h \downarrow 0$. In the first place, $e_s(\wp;\phi,h)$ depends on $\wp$ solely through the first $s+1$ components $(a_0,\cdots,a_s)$ and, similarly, $e_\sigma(\wp;\phi,h)$ depends on $\wp$ only through $(a_0,\cdots,a_\sigma)$. Hence, upon using (13.2.9) and (13.2.11),

$$(13.2.12)\qquad |\int_{\mathscr{A}} e_s(\wp;\phi,h)e_\sigma(\wp;\phi,h)d\nu(\wp)|$$

$$= |2^{-\sigma-1}\int_{-1}^{1}\cdots\int_{-1}^{1} e_s(\int_{-1}^{1} e_\sigma da_\sigma)da_0\cdots da_{\sigma-1}|$$

$$\leq 2c_1^2\delta_1^2 h^3 \max|\phi|\max|\partial_x\phi|.$$

By virtue of (13.2.7),

$$(13.2.13)\qquad |e|^2 = \sum_{s=0}^{\infty}|e_s|^2 + 2\sum_{s=0}^{\infty}\sum_{\sigma=s+1}^{\infty} e_s e_\sigma .$$

Since ϕ has compact support, on the right-hand side of (13.2.13) the first summation contains $O(h^{-1})$ nonzero terms and the second summation contains $O(h^{-2})$ nonzero terms. Consequently, on account of (13.2.9) and (13.2.12),

$$(13.2.14)\qquad \int_{\mathscr{A}} |e(\wp;\phi,h)|^2 d\nu(\wp) = O(h), \qquad \text{as } h\downarrow 0.$$

Thus there exists a null subset $\mathscr{N}_\phi$ of $\mathscr{A}$ such that $e(\wp;\phi,h_m) \to 0$, as $m\to\infty$, for any $\wp \in \mathscr{A}\backslash\mathscr{N}_\phi$. If $\{\phi_k\}$ is any countable set of test functions, which is C^1-dense in the set of all test functions with compact support in $(-\infty,\infty)\times[0,\infty)$, the null subset $\mathscr{N} = \bigcup_k \mathscr{N}_{\phi_k}$ of $\mathscr{A}$ will obviously satisfy the assertion of the theorem. The proof is complete.

To conclude this section, we discuss the admissibility of the constructed solution.

13.2.2 Theorem. *Assume the system is endowed with an entropy-entropy flux pair (η,q), where $\eta(U)$ is convex in $\mathscr{O}$. Then there is a null subset $\mathscr{N}$ of $\mathscr{A}$ with the following property: When the U_s^r are evaluated via* (13.2.5), *with $y_s^r = x_r + a_s h$, then for any $\wp \in \mathscr{A}\backslash\mathscr{N}$, the limit U in* (13.2.1) *is a solution of* (13.1.1) *which satisfies the entropy admissibility criterion.*

Proof. Inside each strip $\{(x,t) : -\infty < x < \infty,\ t_s \leq t < t_{s+1}\}$, U_h is a solution of (13.1.9), with shocks that satisfy the viscous shock admissibility condition and thereby also the entropy shock admissibility criterion, relative to the entropy-entropy flux pair (η,q) (cf. Theorem 8.6.2). Therefore, we have

$$\partial_t \eta(U_h(x,t)) + \partial_x q(U_h(x,t)) \le 0, \quad -\infty < x < \infty,\ t_s \le t < t_{s+1}, \tag{13.2.15}$$

in the sense of measures.

Consider any nonnegative C^∞ test function ϕ with compact support on $(-\infty,\infty)\times[0,\infty)$. We apply the measure (13.2.15) to the function ϕ on the rectangle $\{(x,t) : x_{r-1} < x < x_{r+1}\,,\ t_s \le t < t_{s+1}\,,\ r+s \text{ odd}\}$ and sum over all such rectangles in the upper half-plane. After an integration by parts, and upon using (13.1.8) and (13.1.10), this yields

$$\int_0^\infty \int_{-\infty}^\infty [\partial_t \phi\, \eta(U_h) + \partial_x \phi\, q(U_h)] dxdt + \int_{-\infty}^\infty \phi(x,0)\eta(U_0(x))dx \tag{13.2.16}$$

$$\ge \sum_{s=0}^\infty \sum_{r+s \text{ odd}} \int_{x_{r-1}}^{x_{r+1}} \phi(x,t_s)[\eta(U_h(x,t_s-)) - \eta(U_s^r)]dx.$$

Retracing the steps of the proof of Theorem 13.2.1, we deduce that there is a null subset $\mathscr{N}_\phi$ of $\mathscr{A}$ with the property that, when $\wp \in \mathscr{A}\backslash\mathscr{N}_\phi$, the right-hand side of (13.2.16) tends to zero, along the sequence $\{h_m\}$, as $m \to \infty$. Consequently, the limit U in (13.2.1) satisfies the inequality

$$\int_0^\infty \int_{-\infty}^\infty [\partial_t \phi\, \eta(U) + \partial_x \phi\, q(U)] dxdt + \int_{-\infty}^\infty \phi(x,0)\eta(U_0(x))dx \ge 0. \tag{13.2.17}$$

We now consider any countable set $\{\phi_k\}$ of nonnegative test functions that is C^1-dense in the set of all nonnegative test functions with compact support in the upper half-plane $(-\infty,\infty)\times[0,\infty)$, and define $\mathscr{N} = \bigcup_k \mathscr{N}_{\phi_k}$. It is clear that if one selects any $\wp \in \mathscr{A}\backslash\mathscr{N}$, then (13.2.17) will hold for all nonnegative test functions ϕ and hence U will satisfy the entropy admissibility condition. This completes the proof.

In the absence of entropy-entropy flux pairs, or whenever the entropy admissibility criterion is not sufficiently selective to rule out all spurious solutions (cf. Chapter VIII), the question of admissibility of solutions constructed by the random choice method is subtle. It is plausible that the requisite shock admissibility conditions will hold at points of approximate jump discontinuity of the solution U, so long as they are satisfied by the shocks of the approximate solutions U_h. Proving this, however, requires a more refined treatment of the limit process that yields U from U_h which may be attained by the method of wave partitioning outlined in Section 13.8.

13.3 Wave Interactions in Genuinely Nonlinear Systems

We now embark on the long journey that will eventually lead to the stability estimates (13.1.11), (13.1.12) and (13.1.13). The first step is to estimate local changes in the total variation of the approximate solutions U_h. For simplicity, we limit the discussion to systems with characteristic families that are either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2). The general case is considerably more complicated and will be discussed briefly in Section 13.7.

According to the construction scheme, a portion of the wave fan emanating from the mesh-point $(x_{r-1}, t_{s-1}), r+s$ even, combines with a portion of the wave fan emanating from the mesh-point (x_{r+1}, t_{s-1}) to produce the wave fan that emanates from the mesh-point (x_r, t_s). This is conveniently illustrated by enclosing the mesh-point (x_r, t_s) in a diamond-shaped region Δ_s^r with vertices at the four surrounding sampling points, (y_s^{r-1}, t_s), (y_{s-1}^r, t_{s-1}), (y_s^{r+1}, t_s) and (y_{s+1}^r, t_{s+1}); see Fig. 13.3.1.

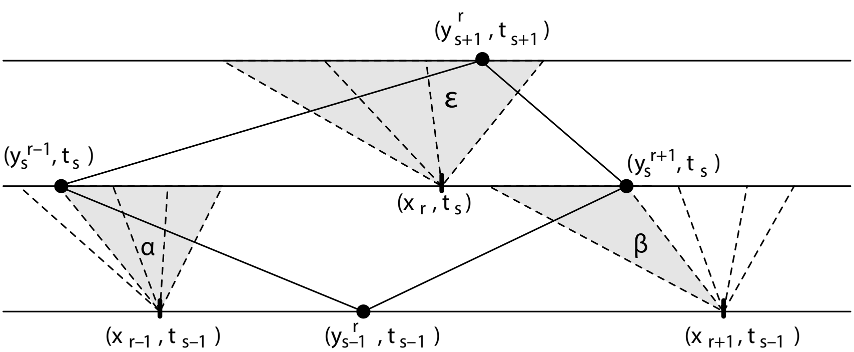

Fig. 13.3.1

A wave fan emanating from (x_{r-1}, t_{s-1}) and joining the state U_s^{r-1}, on the left, with the state U_{s-1}^r, on the right, enters Δ_s^r through its "southwestern" edge. It may be represented, as explained in Sections 9.3 and 9.9, by the n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of its wave amplitudes. A second wave fan, emanating from (x_{r+1}, t_{s-1}), joining the state U_{s-1}^r, on the left, with the state U_s^{r+1}, on the right, and similarly represented by the n-tuple $\beta = (\beta_1, \cdots, \beta_n)$ of its wave amplitudes, enters Δ_s^r through its "southeastern" edge.

The output from Δ_s^r consists of the full wave fan that emanates from (x_r, t_s), joins the state U_s^{r-1}, on the left, with the state U_s^{r+1}, on the right, and is represented by the n-tuple $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ of its wave amplitudes. A portion $\beta' = (\beta_1', \cdots, \beta_n')$ of ε exits through the "northwestern" edge of Δ_s^r and enters the diamond Δ_{s+1}^{r-1}, while the balance $\alpha' = (\alpha_1', \cdots, \alpha_n')$ exits through the "northeastern" edge of Δ_s^r and enters the diamond Δ_{s+1}^{r+1}. Clearly, $\varepsilon_i = \alpha_i' + \beta_i'$, $i = 1, \cdots, n$. As explained in Section 9.4, for genuinely nonlinear characteristic families, a positive amplitude indicates a rarefaction wave and a negative amplitude indicates a compressive shock. Needless to say, a zero amplitude indicates that the wave of that family is missing from the wave fan in question. In particular, we identify $j = 1, \cdots, n$ such that $\alpha_i' = 0$ for $i = 1, \cdots, j-1$ and $\beta_i' = 0$ for $i = j+1, \cdots, n$. Both α_j' and β_j' may be nonzero, but then both must be positive, associated with rarefaction waves.

If the incoming wave fans α and β were allowed to propagate freely, beyond the t_s-time line, the resulting wave interactions would generate a very intricate wave pattern. Nevertheless, following the discussion in Section 9.9, it should be expected that as $t \to \infty$ this wave pattern will reduce to a centered wave fan which is none other than ε. Thus the essence of our construction scheme is that it replaces actual, complex, wave patterns by their time-asymptotic, simpler, forms. In that connection, the role of "random choice" is to arrange the relative position of the wave fans in such a manner that "on the average" the law of "mass" conservation holds.

According to the terminology of Section 9.9, the wave fan ε shall be regarded as the result of the interaction of the wave fan α, on the left, with the wave fan β, on the right. It is convenient to realize ε, α and β as n-vectors normed by the ℓ_1^n norm, in which case Theorem 9.9.1 yields the estimate

$$|\varepsilon - (\alpha + \beta)| \leq [c_3 + c_4(|\alpha| + |\beta|)]\mathscr{D}(\Delta_s^r), \tag{13.3.1}$$

with c_3 and c_4 depending solely on F. In particular, $c_3 = 0$ when the system is endowed with a coordinate system of Riemann invariants. Here the symbol $\mathscr{D}(\Delta_s^r)$ is being used, in the place of $D(\alpha, \beta)$ in Section 9.9, to denote the *amount of wave interaction in the diamond* Δ_s^r, namely,

$$\mathscr{D}(\Delta_s^r) = \sum_{\text{app}} |\alpha_k||\beta_j|. \tag{13.3.2}$$

The summation runs over all pairs of approaching waves, i.e., over all (k, j) such that either $k > j$, or $k = j$ and at least one of α_j, β_j is negative, corresponding to a shock.

Formula (13.3.1) will serve as the vehicle for estimating how the total variation and the supremum of the approximate solutions U_h change with time, as a result of wave interactions. It will suffice for establishing the desired stability estimates and thereby the convergence of the algorithm and the existence of solutions to the Cauchy problem. However, in order to study finer properties of the solution it is necessary to look more closely at wave interactions, with an eye to potential cancellations.

By (13.3.1), when α_i and β_i have the same sign, the total strength $|\alpha_i'| + |\beta_i'|$ of i-waves leaving the diamond Δ_s^r is nearly equal to the total strength $|\alpha_i| + |\beta_i|$ of entering i-waves. However, when α_i and β_i have opposite signs, cancellation of i-waves takes place. To account for this phenomenon, which greatly affects the behavior of solutions, certain notions will now be introduced.

The *amount of i-wave cancellation in the diamond* Δ_s^r is conveniently measured by the quantity

$$\mathscr{C}_i(\Delta_s^r) = \tfrac{1}{2}(|\alpha_i| + |\beta_i| - |\alpha_i + \beta_i|). \tag{13.3.3}$$

In order to account separately for shocks and rarefaction waves, we rewrite (13.3.1) in the form

$$\varepsilon_i^{\pm} = \alpha_i^{\pm} + \beta_i^{\pm} - \mathscr{C}_i(\Delta_s^r) + [c_3 O(1) + O(\tau)]\mathscr{D}(\Delta_s^r), \tag{13.3.4}$$

where the superscript plus or minus denotes positive or negative part of the amplitude, and τ is the oscillation of U_h.

Upon summing (13.3.4) over any collection of diamonds, whose union forms a domain Λ in the upper half-plane, we end up with equations

$$L_i^{\pm}(\Lambda) = E_i^{\pm}(\Lambda) - \mathscr{C}_i(\Lambda) + [c_3 O(1) + O(\tau)]\mathscr{D}(\Lambda), \tag{13.3.5}$$

where E_i^- (or E_i^+) denotes the total amount of i-shock (or i-rarefaction wave) that enters Λ, L_i^- (or L_i^+) denotes the total amount of i-shock (or i-rarefaction wave) that leaves Λ, $\mathscr{C}_i(\Lambda)$ is the amount of i-wave cancellation inside Λ, and $\mathscr{D}(\Lambda)$ is the amount of wave interaction inside Λ. The equations (13.3.5) express the balance of i-waves relative to Λ and, accordingly, are called *approximate conservation laws* for i-shocks (with minus sign) or i-rarefaction waves (with plus sign).

The total amount of wave cancellation in the diamond Δ_s^r is naturally measured by

$$\mathscr{C}(\Delta_s^r) = \sum_{i=1}^{n} \mathscr{C}_i(\Delta_s^r). \tag{13.3.6}$$

Notice that (13.3.1) implies

$$|\alpha'| + |\beta'| = |\varepsilon| \leq |\alpha| + |\beta| - 2\mathscr{C}(\Delta_s^r) + [c_3 + c_4(|\alpha| + |\beta|)]\mathscr{D}(\Delta_s^r). \tag{13.3.7}$$

13.4 The Glimm Functional for Genuinely Nonlinear Systems

The aim here is to establish bounds on the total variation of approximate solutions U_h along certain curves. We are still operating under the assumption that each characteristic family is either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2).

A *mesh curve*, associated with U_h, is a polygonal graph with vertices that form a finite sequence of sample points $(y_{s_1}^{r_1}, t_{s_1}), \cdots, (y_{s_m}^{r_m}, t_{s_m})$, where $r_{\ell+1} = r_\ell + 1$ and $s_{\ell+1} = s_\ell - 1$ or $s_{\ell+1} = s_\ell + 1$ (Fig. 13.4.1). Thus the edges of any mesh curve I are also edges of diamond-shaped regions considered in the previous section. Any wave entering into a diamond through an edge shared with the mesh curve I is said to be *crossing* I.

A mesh curve J is called an *immediate successor* of the mesh curve I when $J \backslash I$ is the upper (i.e., "northwestern" and "northeastern") boundary of some diamond, say Δ_s^r, and $I \backslash J$ is the lower (i.e., "southwestern" and "southeastern") boundary of Δ_s^r. Thus J has the same vertices as I, save for one, (y_{s-1}^r, t_{s-1}), which is replaced by (y_{s+1}^r, t_{s+1}). This induces a natural partial ordering in the family of mesh curves: J is a *successor* of I, denoted $I < J$, whenever there is a finite sequence, namely $I = I_0, I_1, \cdots, I_m = J$, of mesh curves such that I_ℓ is an immediate successor of $I_{\ell-1}$, for $\ell = 1, \cdots, m$.

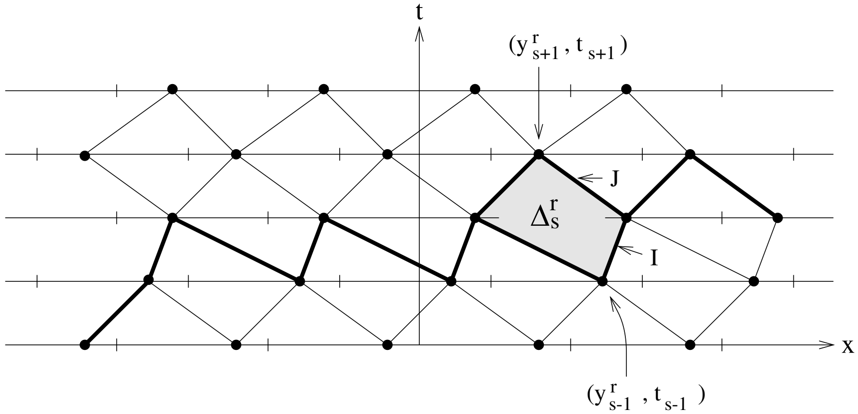

Fig. 13.4.1

With mesh curves I we associate the functionals

$$\mathscr{S}(I) = \max |\xi|, \tag{13.4.1}$$

$$\mathscr{L}(I) = \sum |\xi|, \tag{13.4.2}$$

where both the maximum and the summation are taken over the amplitudes ξ of all waves that are crossing I. Clearly, $\mathscr{S}(I)$ measures the oscillation and $\mathscr{L}(I)$ measures the total variation of U_h along the curve I. We shall estimate the supremum and total variation of U_h by monitoring how $\mathscr{S}$ and $\mathscr{L}$ change as one passes from I to its successors.

Assume J is an immediate successor of I, as depicted in Fig. 13.4.1. Wave fans $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ enter the diamond Δ_s^r through its "southwestern" and "southeastern" edge, respectively, and interact to generate, as discussed in Section 13.3, the wave fan $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, which exits Δ_s^r through its "northwestern" and "northeastern" edge. By virtue of (13.3.1) we deduce

$$\mathscr{S}(J) \leq \mathscr{S}(I) + [c_3 + c_4 \mathscr{S}(I)]\mathscr{D}(\Delta_s^r), \tag{13.4.3}$$

$$\mathscr{L}(J) \leq \mathscr{L}(I) + [c_3 + c_4 \mathscr{S}(I)]\mathscr{D}(\Delta_s^r), \tag{13.4.4}$$

where c_3 and c_4 are the constants that appear also in (13.3.1). In particular, when the system is endowed with a coordinate system of Riemann invariants, $c_3 = 0$. Clearly, $\mathscr{S}$ and $\mathscr{L}$ may increase as one passes from I to J, and thus (13.4.3), (13.4.4) alone are insufficient to render the desired bounds (13.1.11), (13.1.12).

What saves the day is the realization that $\mathscr{L}$ may increase only as a result of interaction among approaching waves, which, after crossing paths, separate and move away from each other, never to meet again. Consequently, the potential for future interactions is embodied in the initial arrangement of waves and may thus be anticipated and estimated in advance. To formalize the above heuristic arguments, we shall associate with mesh curves I a functional $\mathscr{Q}(I)$ which measures the *potential for future interactions of waves that are crossing I*.

An i-wave and a j-wave, crossing the mesh curve I, are said to be *approaching* if (a) $i > j$ and the i-wave is crossing on the left of the j-wave; or (b) $i < j$ and the i-wave is crossing on the right of the j-wave; or (c) $i = j$, the i-characteristic family is genuinely nonlinear and at least one of the waves is a shock. The reader should note the analogy with the notion of approaching waves in two interacting wave fans, introduced in Section 9.9. After this preparation, we set

$$\mathscr{Q}(I) = \sum_{\text{app}} |\zeta||\xi|, \tag{13.4.5}$$

where the summation runs over all pairs of approaching waves that are crossing I and ζ, ξ are their amplitudes. Clearly,

$$\mathscr{Q}(I) \leq \tfrac{1}{2}[\mathscr{L}(I)]^2. \tag{13.4.6}$$

The change in the potential of future wave interactions as one passes from the mesh curve I to its immediate successor J, depicted in Fig. 13.4.1, is controlled by the estimate

$$\mathscr{Q}(J) - \mathscr{Q}(I) \leq \{[c_3 + c_4\mathscr{S}(I)]\mathscr{L}(I) - 1\}\mathscr{D}(\Delta_s^r), \tag{13.4.7}$$

where c_3 and c_4 are the same constants appearing in (13.4.3) and (13.4.4).

To verify (13.4.7), we shall distinguish between *peripheral waves,* which are crossing both I and J on the left of (y_s^{r-1}, t_s) or on the right of (y_s^{r+1}, t_s), and *principal waves,* that is, constituents of the wave fans α, β or ε, which enter or exit Δ_s^r by crossing I or J between (y_s^{r-1}, t_s) and (y_s^{r+1}, t_s).

We first observe that pairs of principal waves from the incoming wave fans α and β interact to contribute the amount $\mathscr{D}(\Delta_s^r)$ to $\mathscr{Q}(I)$. By contrast, no pair of principal waves from the outgoing wave fan ε is approaching so as to make a contribution to $\mathscr{Q}(J)$.

Next we note that pairs of peripheral waves contribute equally to $\mathscr{Q}(I)$ and to $\mathscr{Q}(J)$; hence their net contribution to $\mathscr{Q}(J) - \mathscr{Q}(I)$ is nil.

It remains to discuss the pairing of peripheral with principal waves. Let us examine the contributions to $\mathscr{Q}(I)$ and to $\mathscr{Q}(J)$ from the pairing of some fixed peripheral i-wave, of amplitude ζ, with the j-waves of α, β and ε. One must distinguish the following cases: (i) $j > i$ and the peripheral i-wave is crossing I on the left of (y_s^{r-1}, t_s); (ii) $j < i$ and the peripheral i-wave is crossing I on the right of (y_s^{r+1}, t_s); (iii) $j = i$ and the i-characteristic family is linearly degenerate; (iv) $j > i$ and the peripheral i-wave is crossing I on the right of (y_s^{r+1}, t_s); (v) $j < i$ and the peripheral i-wave is crossing I on the left of (y_s^{r-1}, t_s); (vi) $j = i$, the i-characteristic family is genuinely nonlinear, and the peripheral wave is an i-shock, $\zeta < 0$; and (vii) $j = i$, the i-characteristic family is genuinely nonlinear, and the peripheral wave is an i-rarefaction, $\zeta > 0$.

In cases (i), (ii) and (iii), the peripheral i-wave is not approaching any of the j-waves of $\alpha, \beta, \varepsilon$; hence the contribution to both $\mathscr{Q}(I)$ and $\mathscr{Q}(J)$ is nil. By contrast, in cases (iv), (v) and (vi), the peripheral i-wave is approaching all three of the j-waves of $\alpha, \beta, \varepsilon$; thus the contribution to $\mathscr{Q}(I)$ and $\mathscr{Q}(J)$ is $|\zeta|(|\alpha_j|+|\beta_j|)$ and $|\zeta||\varepsilon_j|$, respectively.

In the remaining case (vii), depending on the signs of α_i, β_i and ε_i, the peripheral i-wave may be approaching all, some, or none of the i-waves of α, β and ε; the contribution to $\mathscr{Q}(I)$ and $\mathscr{Q}(J)$ is $\zeta(\alpha_i^- + \beta_i^-)$ and $\zeta\varepsilon_i^-$, respectively, where the superscript "minus" denotes "negative part."

From the above and (13.3.1), the total contribution to $\mathscr{Q}(J)-\mathscr{Q}(I)$ from the pairing of any peripheral wave of amplitude ζ with all principal waves cannot exceed the amount $|\zeta|[c_3+c_4\mathscr{S}(I)]\mathscr{D}(\Delta_s^r)$. Therefore we conclude that the overall contribution to $\mathscr{Q}(J)-\mathscr{Q}(I)$ from such interactions is bounded by $[c_3+c_4\mathscr{S}(I)]\mathscr{L}(I)\mathscr{D}(\Delta_s^r)$. This establishes (13.4.7).

The key consequence of (13.4.7) is that when $\mathscr{L}(I)$ is sufficiently small the potential $\mathscr{Q}$ for future wave interactions will decrease as one passes from the mesh curve I to its immediate successor J. We shall exploit this property to compensate for the possibility that $\mathscr{S}$ and $\mathscr{L}$ may be increasing, to the extent allowed by (13.4.3) and (13.4.4). For that purpose, we associate with mesh curves I the *Glimm functional*

(13.4.8)
$$\mathscr{G}(I) = \mathscr{L}(I) + 2\kappa\mathscr{Q}(I),$$

where κ is some fixed upper bound of $c_3+c_4\mathscr{S}(I)$, independent of I and h. Even though $\mathscr{G}$ majorizes $\mathscr{L}$, it is actually equivalent to $\mathscr{L}$ on account of (13.4.6).

13.4.1 Theorem. *Let I be a mesh curve with $2\kappa\mathscr{L}(I)\leq 1$. Then, for any mesh curve J that is a successor of I,*

(13.4.9)
$$\mathscr{G}(J) \leq \mathscr{G}(I),$$

(13.4.10)
$$\mathscr{L}(J) \leq 2\mathscr{L}(I).$$

Furthermore, the amount of wave interaction and the amount of wave cancellation in the diamonds confined between the curves I and J are bounded:

(13.4.11)
$$\sum \mathscr{D}(\Delta_s^r) \leq [\mathscr{L}(I)]^2,$$

(13.4.12)
$$\sum \mathscr{C}(\Delta_s^r) \leq \mathscr{L}(I).$$

Proof. Assume first that J is the immediate successor of I depicted in Fig. 13.4.1. Upon combining (13.4.8) with (13.4.4) and (13.4.7), we deduce

(13.4.13)
$$\mathscr{G}(J) \leq \mathscr{G}(I) + \kappa[2\kappa\mathscr{L}(I)-1]\mathscr{D}(\Delta_s^r).$$

Since $2\kappa\mathscr{L}(I)\leq 1$, (13.4.13) yields (13.4.9). Furthermore, by virtue of (13.4.8), (13.4.6) and $2\kappa\mathscr{L}(I)\leq 1$, we obtain

$$\mathscr{G}(I) \le 2\mathscr{L}(I). \tag{13.4.14}$$

Assume now that J is any successor of I. Iterating the above argument, we establish (13.4.9) for that case as well. Since $\mathscr{L}(J) \le \mathscr{G}(J)$, (13.4.10) follows from (13.4.9) and (13.4.14). Summing (13.4.7) over all diamonds confined between the curves I and J and using (13.4.10), we obtain

$$\tfrac{1}{2}\sum \mathscr{D}(\Delta_s^r) \le \mathscr{Q}(I) - \mathscr{Q}(J), \tag{13.4.15}$$

which yields (13.4.11), by virtue of (13.4.6).

We sum (13.3.7) over all the diamonds confined between the curves I and J, to get

$$2\sum \mathscr{C}(\Delta_s^r) \le \mathscr{L}(I) - \mathscr{L}(J) + \kappa \sum \mathscr{D}(\Delta_s^r). \tag{13.4.16}$$

Combining (13.4.16) with (13.4.10) and (13.4.11) we arrive at (13.4.12). This completes the proof.

The above theorem is of fundamental importance. In particular, the estimates (13.4.9) and (13.4.10) provide the desired bounds on the total variation, while (13.4.11) and (13.4.12) embody the dissipative effects of nonlinearity and have significant implications for regularity and large-time behavior of solutions.

The assumption $2\kappa\mathscr{L}(I) \le 1$ in the above theorem means that $\mathscr{L}(I)$ itself should be sufficiently small, for general systems. However, in systems endowed with a coordinate system of Riemann invariants, where $c_3 = 0$, it would suffice that $(\sup U_h)\mathscr{L}(I)$ be sufficiently small. For this special class of systems, $\sup U_h$ will be estimated with the help of

13.4.2 Theorem. *Assume that the system is endowed with a coordinate system of Riemann invariants. Let I be a mesh curve with $2\kappa\mathscr{L}(I) \le 1$. Then, for any mesh curve J that is a successor of I,*

$$\mathscr{S}(J) \le \exp[c_4 \mathscr{L}(I)^2]\,\mathscr{S}(I). \tag{13.4.17}$$

Proof. Assume first that J is the immediate successor of I depicted in Fig. 13.4.1. Since $c_3 = 0$, (13.4.3) yields

$$\mathscr{S}(J) \le [1 + c_4 \mathscr{D}(\Delta_s^r)]\,\mathscr{S}(I). \tag{13.4.18}$$

Iterating the above argument, we deduce that if J is any successor of I, then

$$\mathscr{S}(J) \le \prod [1 + c_4 \mathscr{D}(\Delta_s^r)]\,\mathscr{S}(I), \tag{13.4.19}$$

where the product runs over all the diamonds confined between the curves I and J. Combining (13.4.19) with (13.4.11), we arrive at (13.4.17). This completes the proof.

For systems endowed with a coordinate system of Riemann invariants, it is expedient to measure wave strength by the jump of the corresponding Riemann invariant across the wave. In particular, for systems with coinciding shock and rarefaction wave curves (see Section 8.2), this policy renders $\mathscr{L}$ itself nonincreasing, as one passes from a mesh curve to its successor, and thus allows us to estimate the total variation of the solution without any restriction on the size of the total variation of the initial data. There is another, very special, class of systems of two conservation laws in which a suitable measurement of wave strength yields a nonincreasing $\mathscr{L}$, and thereby existence of solutions to the Cauchy problem under initial data with large total variation. An interesting representative of that class is the system

$$(13.4.20)\qquad \begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x (u^{-1}) = 0, \end{cases}$$

namely the special case of (7.1.11) with $\sigma(u) = -u^{-1}$. In particular, this system governs the isothermal flow of an ideal gas, in Lagrangian coordinates.

13.5 Bounds on the Total Variation for Genuinely Nonlinear Systems

Here we prove the estimates (13.1.12) and (13.1.13), always operating under the assumption that the oscillation of U_h is bounded, uniformly in h. The vehicle will be the following corollary of Theorem 13.4.1:

13.5.1 Theorem. *Fix $0 \le \tau < t < \infty$ and $-\infty < a < b < \infty$. Assume that κ times the total variation of $U_h(\cdot,t)$ over the interval $[a - \lambda(t-\tau) - 6h\, ,\, b + \lambda(t-\tau) + 6h]$ is sufficiently small.[2] Then*

$$(13.5.1)\qquad TV_{[a,b]} U_h(\cdot,t) \le c_1 TV_{[a-\lambda(t-\tau)-6h,\, b+\lambda(t-\tau)+6h]} U_h(\cdot,\tau),$$

where c_1 depends solely on F. Furthermore, if x is a point of continuity of both $U_h(\cdot,\tau)$ and $U_h(\cdot,t)$, and κ times the total variation of $U_h(\cdot,t)$ over the interval $[x - \lambda(t-\tau) - 6h\, ,\, x + \lambda(t-\tau) + 6h]$ is sufficiently small, then

$$(13.5.2)\qquad |U_h(x,t) - U_h(x,\tau)| \le c_5 TV_{[x-\lambda(t-\tau)-6h, x+\lambda(t-\tau)+6h]} U_h(\cdot,\tau),$$

where c_5 depends solely on F.

Proof. First we determine nonnegative integers σ and s such that $t_\sigma \le \tau < t_{\sigma+1}$ and $t_s \le t < t_{s+1}$. Next we identify integers r_1 and r_2 such that $y_{s+1}^{r_1+1} < a \le y_{s+1}^{r_1+3}$ and $y_{s+1}^{r_2-3} \le b < y_{s+1}^{r_2-1}$. We then set $r_3 = r_1 - (s-\sigma)$ and $r_4 = r_2 + (s-\sigma)$.

We now construct two mesh curves I and J, as depicted in Fig. 13.5.1, by the following procedure: I originates at the sampling point $(y_\sigma^{r_3}, t_\sigma)$ and zig-zags between t_σ and $t_{\sigma+1}$ until it reaches the sampling point $(y_\sigma^{r_4}, t_\sigma)$, where it terminates. J also

[2] As before, λ here denotes the ratio of spatial and temporal mesh-lengths.

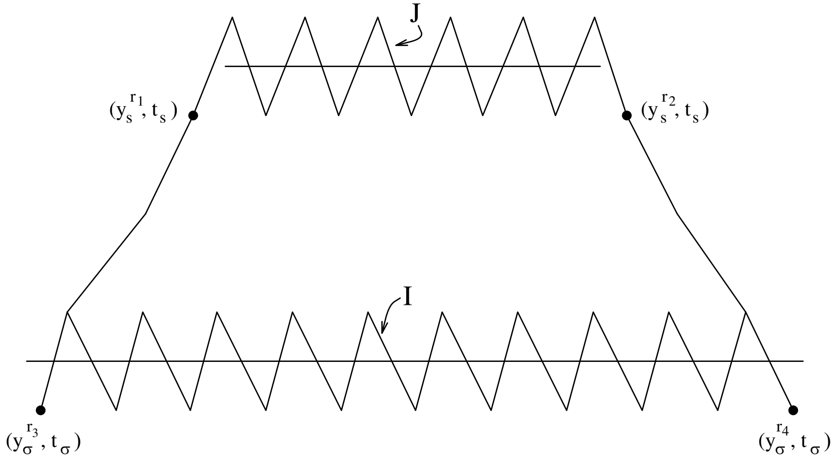

Fig. 13.5.1

originates at $(y_\sigma^{r_3}, t_\sigma)$, takes $s-\sigma$ steps to the "northeast," reaching the sampling point $(y_s^{r_1}, t_s)$, then zig-zags between t_s and t_{s+1} until it arrives at the sampling point $(y_s^{r_2}, t_s)$, and finally takes $s-\sigma$ steps to the "southeast" terminating at $(y_\sigma^{r_4}, t_\sigma)$.

Clearly,

$$TV_{[a,b]}U_h(\cdot,t) \le c_6\mathscr{L}(J). \tag{13.5.3}$$

It is easy to see that $y_\sigma^{r_3} \ge a - \lambda(t-\tau) - 6h$ and $y_\sigma^{r_4} \le b + \lambda(t-\tau) + 6h$. Therefore,

$$\mathscr{L}(I) \le c_7 TV_{[a-\lambda(t-\tau)-6h,\, b+\lambda(t-\tau)+6h]}U_h(\cdot,\tau). \tag{13.5.4}$$

Also, J is a successor of I and hence, if $2\kappa\mathscr{L}(I) \le 1$, Theorem 13.4.1 implies $\mathscr{L}(J) \le 2\mathscr{L}(I)$. Combining this with (13.5.3) and (13.5.4), we arrive at (13.5.1), with $c_1 = 2c_6c_7$.

Given x, we repeat the above construction of I and J with $a = b = x$. We can identify a point (y', τ') on I with $U_h(y', \tau') = U_h(x, \tau)$ as well as a point (x', t') on J with $U_h(x', t') = U_h(x, t)$. Hence

$$|U_h(x,t) - U_h(x,\tau)| \le c_8[\mathscr{L}(I) + \mathscr{L}(J)] \le 3c_8\mathscr{L}(I). \tag{13.5.5}$$

From (13.5.5) and (13.5.4), with $a = b = x$, we deduce (13.5.2) with $c_5 = 3c_7c_8$. This completes the proof.

Applying (13.5.1) for $\tau = 0$, $a \to -\infty$, $b \to \infty$, and taking into account that $TV_{(-\infty,\infty)}U_h(\cdot,0) \le TV_{(-\infty,\infty)}U_0(\cdot)$, we verify (13.1.12).

Finally, we integrate (13.5.2) over $(-\infty,\infty)$, apply Fubini's theorem, and use (13.1.12) to get

(13.5.6)

$$\begin{aligned}\int_{-\infty}^{\infty}|U_h(x,t)-U_h(x,\tau)|dx &\leq c_5\int_{-\infty}^{\infty}TV_{[x-\lambda(t-\tau)-6h,x+\lambda(t-\tau)+6h]}U_h(\cdot,\tau)dx\\ &= 2c_5[\lambda(t-\tau)+6h]TV_{(-\infty,\infty)}U_h(\cdot,\tau)\\ &\leq 2c_1c_5[\lambda(t-\tau)+6h]TV_{(-\infty,\infty)}U_0(\cdot),\end{aligned}$$

which establishes (13.1.13).

13.6 Bounds on the Supremum for Genuinely Nonlinear Systems

One may readily obtain a bound on the L^∞ norm of U_h from (13.5.2), with $\tau = 0$:

$$\sup_{(-\infty,\infty)}|U_h(\cdot,t)| \leq \sup_{(-\infty,\infty)}|U_0(\cdot)| + c_5 TV_{(-\infty,\infty)}U_0(\cdot). \tag{13.6.1}$$

This estimate is not as strong as the asserted (13.1.11), because, in addition to the supremum, it involves the total variation of the initial data. Even so, combining (13.6.1) with the estimates (13.1.12) and (13.1.13), established in Section 13.5, allows us to invoke the results of Section 13.2 and thus infer the existence of a solution U to the initial value problem (13.1.1), which is the limit of a sequence of approximate solutions; cf. (13.2.1). Clearly, U satisfies (13.1.5) and (13.1.6), by virtue of (13.1.12) and (13.1.13). We have thus verified all the assertions of Theorem 13.1.1, save (13.1.4). This estimate is of interest, as a statement of stability. Furthermore, it plays a useful role in the special situations where one can handle initial data with large total variation, but still needs small oscillation for solving the Riemann problems and for applying the wave interaction estimates. It is thus important to establish the estimate (13.1.11), which yields (13.1.4).

We first note that for systems endowed with a coordinate system of Riemann invariants, (13.1.11) is an immediate corollary of Theorem 13.4.2 and thus δ_1 in (13.1.3) need not be small, so long as δ_2 in (13.1.7) is. The proof in this case is so simple because terms of quadratic order are missing in the interaction estimate (13.3.1), i.e., $c_3 = 0$. By contrast, in systems devoid of this special structure, the interaction terms of quadratic order complicate the situation. The proof of (13.1.11) hinges on the special form of the quadratic terms, which, as seen in (9.9.13), involve the Lie brackets of the eigenvectors of DF. The analysis is too laborious to be reproduced here in its entirety, so only an outline of the main ideas shall be presented. The reader may find the details in the references cited in Section 13.9.

The general strategy of the proof is motivated by the ideas expounded in Section 13.4, which culminated in the proof of Theorems 13.4.1 and 13.4.2. Two functionals, $\mathscr{R}$ and $\mathscr{P}$, will be associated with mesh curves I, where $\mathscr{R}(I)$ measures the oscillation of U_h over I while $\mathscr{P}(I)$ provides an estimate on how the oscillation may be affected by future wave interactions. For measuring the oscillation with accuracy, it becomes necessary to account for the mutual cancellation of shocks and rarefaction waves of the same characteristic family. We thus have to tally amplitudes, rather than strengths of waves. Accordingly, with any (finite) sequence of, say, M waves with amplitudes $\xi = (\xi_1, \dots, \xi_M)$, we associate the number

$$|\xi| = \sum_{j=1}^{n} |\sum_{j-\text{waves}} \xi_L|, \tag{13.6.2}$$

where the second summation runs over the indices $L = 1, \cdots, M$ for which the L-th wave in the sequence is a j-wave. We then define

$$\mathscr{R}(I) = \sup |\xi|, \tag{13.6.3}$$

where the supremum is taken over all sequences of waves crossing I that are *consecutive*, in the sense that any two of them occupying consecutive places in the sequence are separated by a constant state of U_h. After a little reflection, one sees that, as long as $\mathscr{L}(I)$ is sufficiently small, $\mathscr{R}(I)$ measures the oscillation of U_h over I.

As one passes from I to its successors, the value of $\mathscr{R}$ changes for two reasons: First, as waves travel at different speeds, crossings occur and wave sequences are reordered (notice, however, that the relative order of waves of the same characteristic family is necessarily preserved). Secondly, the amplitude of waves changes in result of wave interactions, as indicated in (9.9.13). It turns out that the effect of wave interactions of third or higher order in wave strength may be estimated grossly, as in the proof of Theorem 13.4.2. However, the effect of wave interactions of quadratic order in wave strength is more significant and thus must be estimated with higher precision. This may be accomplished in an effective manner by realizing the quadratic terms in (9.9.13) as new *virtual waves* which should be accounted for, along with the actual waves.

The aforementioned functional $\mathscr{P}$, which will help us estimate the effect of future wave interactions, is constructed by the following procedure. With any sequence of consecutive waves crossing the mesh curve I, one associates a family of sequences of waves, which are regarded as its "descendents". A descendent sequence of waves is derived from its "parental" one by the following two operations: (a) Admissible reorderings of the waves in the parental sequence j, e.g., a k-wave occupying the K-th place and an ℓ-wave occupying the L-th place in the parental sequence, exchange places if $k > \ell$ and $K < L$. (b) Insertion of any virtual waves that may be generated from interactions of waves in the parental sequence. The precise construction of descendent sequences entails a major technical endeavor, which shall not be undertaken here, but can be found in the references cited in Section 13.10. For any mesh curve I, we set

$$\mathscr{P}(I) = \sup |\xi|, \tag{13.6.4}$$

where the supremum is now taken over the union of the descendent families of all sequences of consecutive waves that are crossing I.

As long as the total variation is small, $\mathscr{P}$ is actually equivalent to $\mathscr{R}$:

$$\mathscr{R}(I) \leq \mathscr{P}(I) \leq [1 + c_9 \mathscr{L}(I)]\mathscr{R}(I). \tag{13.6.5}$$

The idea of the proof of (13.6.5) is as follows. Recall that the principal difference between $\mathscr{L}(I)$ and $\mathscr{R}(I)$ is that in the former we tally the (positive) strengths of crossing waves while in the latter we sum the (signed) amplitudes of crossing waves, thus allowing for cancellation between waves in the same characteristic family but of opposite signs (i.e., shocks and rarefaction waves). Consider the interaction of a single j-wave, with amplitude ζ, with a number of k-waves. Since waves in the same characteristic family preserve their relative order, the interactions of the k-waves with the j-wave will occur consecutively and so the resulting virtual waves will also appear in the same order. Furthermore, whenever the amplitudes of the k-waves alternate in sign, then so do the corresponding Lie bracket terms. Consequently, the virtual waves undergo the same cancellation as their parent waves and thus the contribution to $\mathscr{P}(I)$ by the interaction of the j-wave with the k-waves will be of the order $O(1)|\zeta|\mathscr{S}(I)$. Thus the total contribution to $\mathscr{P}(I)$ from such interactions will be $O(1)\mathscr{L}(I)\mathscr{S}(I)$, whence (13.6.5) follows. The detailed proof is quite lengthy and may be found in the references.

The next step is to show that if J is the immediate successor of the mesh curve I depicted in Fig. 13.4.1, then

$$\mathscr{P}(J) \leq \mathscr{P}(I) + c_{10}\mathscr{R}(I)\mathscr{D}(\Delta_s^r). \tag{13.6.6}$$

The idea of the proof is as follows. Sequences of waves crossing J are reorderings of sequences that cross I, with the waves that enter the diamond Δ_s^r through its "southwestern" and "southeastern" edges exchanging their relative positions as they exit Δ_s^r. Furthermore, as one passes from I to J the virtual waves produced by the interaction of the waves that enter Δ_s^r are converted into actual waves, embodied in the waves that exit Δ_s^r. Again, the detailed proof is quite lengthy and should be sought in the references.

By virtue of (13.6.5), we may substitute $\mathscr{P}(I)$ for $\mathscr{R}(I)$ on the right-hand side of (13.6.6), without violating the inequality. Therefore, upon iterating the argument, we conclude that if J is any successor of I, then

$$\mathscr{P}(J) \leq \prod[1 + c_{10}\mathscr{D}(\Delta_s^r)]\mathscr{P}(I), \tag{13.6.7}$$

where the product runs over all the diamonds Δ_s^r confined between the curves I and J.

We now assume $4\kappa\mathscr{L}(I) \leq 1$ and appeal to Theorem 13.4.1. Combining (13.6.7), (13.4.11) and (13.6.5) yields

$$\mathscr{R}(J) \leq \exp[c_9\mathscr{L}(I) + c_{10}\mathscr{L}(I)^2]\mathscr{R}(I), \tag{13.6.8}$$

whence the desired estimate (13.1.11) readily follows.

13.7 General Systems

In this section we discuss briefly how to obtain bounds on the total variation of approximate solutions U_h along mesh curves, for systems with characteristic families that are merely piecewise genuinely nonlinear. These bounds will be derived by the procedure used in Section 13.4 for genuinely nonlinear systems, except that the functional measuring the potential for future wave interactions shall be modified, as wave interactions are here governed by Theorem 9.9.2 (rather than 9.9.1). Thus, if we consider the diamond Δ_s^r, with incoming wave fans α and β, entering through the "southwest" and the "southeast" edge, respectively, and outgoing wave fan ε, (9.9.14) yields

$$|\varepsilon - (\alpha + \beta)| \leq c_{11} \mathscr{D}(\Delta_s^r), \tag{13.7.1}$$

where

$$\mathscr{D}(\Delta_s^r) = \sum \theta |\gamma||\delta|. \tag{13.7.2}$$

Recall that the above summation runs over all pairs of elementary i-waves, with amplitude γ, and j-waves, with amplitude δ, entering Δ_s^r through its "southwestern" and "southeastern" edge, respectively. The weighting factor θ is determined as follows: $\theta = 0$ if $i < j$; $\theta = 1$ if either $i > j$ or $i = j$ and $\gamma\delta < 0$; finally, θ is given by (9.9.16) if $i = j$ and $\gamma\delta > 0$.

As in Section 13.4, with any mesh curve I we associate the functional $\mathscr{L}(I)$, defined by (13.4.2). Assuming J is the immediate successor of I depicted in Fig. 13.4.1, (13.7.1) yields

$$\mathscr{L}(J) \leq \mathscr{L}(I) + c_{11} \mathscr{D}(\Delta_s^r). \tag{13.7.3}$$

The increase in $\mathscr{L}$ allowed by (13.7.3) will be offset by the decrease in a functional $\mathscr{Q}$, which monitors the potential for future wave interactions and is here defined by

$$\mathscr{Q}(I) = \sum \theta |\zeta||\xi|. \tag{13.7.4}$$

The above summation runs over all pairs of elementary i-waves and j-waves, with respective amplitudes ζ and ξ, that are crossing the mesh curve I. When the i-wave is crossing I on the left of the j-wave, then $\theta = 0$ if $i < j$ and $\theta = 1$ if $i > j$. When $i = j$ and $\zeta\xi < 0$, then $\theta = 1$. Finally, if $i = j$ and $\zeta\xi > 0$, then θ is determined by (9.9.16); and in particular by $(9.9.16)_1$ when the wave on the left is an i-shock with speed σ_L and the wave on the right is an i-shock with speed σ_R; or by $(9.9.16)_2$ when the wave on the left is an i-shock with speed σ_L, while the wave on the right is an i-rarefaction, joining U_R with $V_i(\tau_R; U_R)$; or by $(9.9.16)_3$ when the wave on the left is an i-rarefaction, joining U_L with $V_i(\tau_L; U_L)$, while the wave on the right is an i-shock with speed σ_R; or by $(9.9.16)_4$ when the wave on the left is a rarefaction, joining U_L with $V_i(\tau_L; U_L)$, and the wave on the right is also a rarefaction, joining U_R with $V_i(\tau_R; U_R)$.

The aim is to demonstrate the analog of (13.4.7), namely that if J is the immediate successor of I depicted in Fig. 13.4.1, then

$$\mathscr{Q}(J) - \mathscr{Q}(I) \leq [c_{12} \mathscr{L}(I) - 1] \mathscr{D}(\Delta_s^r). \tag{13.7.5}$$

Once (13.7.5) is established, one considers, as in Section 13.4, the Glimm functional $\mathscr{G}$, defined by (13.4.8), and shows that if κ is selected sufficiently large and $\mathscr{L}(I)$ is small, then $\mathscr{G}(J) \leq \mathscr{G}(I)$. This in turn yields the desired estimates (13.1.12) and (13.1.13), by the arguments employed in Section 13.5.

To verify (13.7.5), let us retrace the steps in the proof of (13.4.7), making the necessary adjustments. We shall use again the terms "peripheral" and "principal" waves, to distinguish the elementary waves that are crossing both I and J from those that enter or exit Δ_s^r, thus crossing only I or only J.

To begin with, the interaction among principal waves of the two incoming wave fans α and β contributes the amount $\mathscr{D}(\Delta_s^r)$ to $\mathscr{Q}(I)$. By contrast, pairs of principal waves from the outgoing wave fan ε make no contribution to $\mathscr{Q}(J)$.

The next observation is that pairs of peripheral waves contribute equally to $\mathscr{Q}(I)$ and $\mathscr{Q}(J)$; hence their net contribution to $\mathscr{Q}(J) - \mathscr{Q}(I)$ is nil.

It remains to examine the pairing of peripheral waves with principal waves. Let us estimate the contribution to $\mathscr{Q}(I)$ and to $\mathscr{Q}(J)$ from the pairing of some fixed peripheral i-wave, of amplitude ζ, with the elementary j-waves of α, β and ε. As in the genuinely nonlinear situation, we must consider a number of cases: (i) $j > i$ and the peripheral i-wave is crossing I on the left of (y_s^{r-1}, t_s); (ii) $j < i$ and the peripheral i-wave is crossing I on the right of (y_s^{r+1}, t_s); (iii) $j > i$ and the peripheral i-wave is crossing I on the right of (y_s^{r+1}, t_s); (iv) $j < i$ and the peripheral i-wave is crossing I on the left of (y_s^{r-1}, t_s); (v) $j = i$, $\alpha_i \beta_i > 0$ and $\zeta(\alpha_i + \beta_i) < 0$; (vi) $j = i$, $\alpha_i \beta_i < 0$ and $\zeta(\alpha_i + \beta_i) < 0$; (vii) $j = i$, $\alpha_i \beta_i > 0$ and $\zeta(\alpha_i + \beta_i) > 0$; and (viii) $j = i$, $\alpha_i \beta_i < 0$ and $\zeta(\alpha_i + \beta_i) > 0$.

In cases (i) and (iii), the contribution to both $\mathscr{Q}(I)$ and $\mathscr{Q}(J)$ is obviously nil. By contrast, in cases (iii) and (iv), the contribution to $\mathscr{Q}(I)$ and $\mathscr{Q}(J)$ is $|\zeta|(|\alpha_j| + |\beta_j|)$ and $|\zeta||\varepsilon_j|$, respectively.

In case (v), the contribution to $\mathscr{Q}(I)$ is $|\zeta|(|\alpha_i| + |\beta_i|)$. The contribution to $\mathscr{Q}(J)$ depends on the sign of $\zeta \varepsilon_i$, but under any circumstance may not exceed the amount $|\zeta||\varepsilon_i|$. Similarly, in case (vi) the contribution to $\mathscr{Q}(I)$ is at least $|\zeta| \max\{|\alpha_i|, |\beta_i|\}$, while the contribution to $\mathscr{Q}(J)$ is at most $|\zeta||\varepsilon_i|$.

From the above and (13.7.1) it follows that the total contribution to $\mathscr{Q}(J) - \mathscr{Q}(I)$ from the pairing of the peripheral i-wave with all the principal waves that fall under one of cases (i) through (vi) cannot exceed the amount $c_{11}|\zeta|\mathscr{D}(\Delta_s^r)$.

The remaining cases (vii) and (viii) require a more delicate treatment. In fact, it is at this point that the difference between genuinely nonlinear systems and general systems comes to the fore. For orientation, let us examine the special, albeit representative, situation considered in the proof of Theorem 9.9.2: The incoming wave fans α and β consist of a single i-shock each, with respective amplitudes γ and δ and respective speeds σ_L and σ_R, $\sigma_R \leq \sigma_L$. The i-th wave fan of the outgoing wave fan ε also consists of a single i-shock, with amplitude ε_i and speed σ. For definiteness, it will be further assumed that the peripheral i-wave is likewise an i-shock, with amplitude ζ and speed $\sigma_0 < \sigma_R$, which is crossing I on the right of (y_s^{r+1}, t_s). In accordance with case (vii), above, let $\gamma, \delta, \varepsilon_i$ and ζ be all positive. Then the contribution to $\mathscr{Q}(J) - \mathscr{Q}(I)$ is

(13.7.6) $$\zeta\{(\sigma-\sigma_0)^+\varepsilon_i-(\sigma_L-\sigma_0)^+\gamma-(\sigma_R-\sigma_0)^+\delta\},$$

which is $O(1)\zeta\theta\gamma\delta$, by virtue of (9.9.30) and (9.9.31). The proof in the general case, where α and β are arbitrary incoming wave fans, requires lengthy and technical analysis, but follows the same pattern, with (9.9.14) and (9.9.33) playing the role of (9.9.30) and (9.9.31); see the references cited in Section 13.9. The final conclusion is that the total contribution to $\mathscr{Q}(J)-\mathscr{Q}(I)$ from the pairing of any peripheral wave of amplitude ζ with all the principal waves cannot exceed an amount $c_{12}|\zeta|\mathscr{D}(\Delta_s^r)$. Therefore, the overall contribution to $\mathscr{Q}(J)-\mathscr{Q}(I)$ from such interactions is bounded by $c_{12}\mathscr{L}(I)\mathscr{D}(\Delta_s^r)$. This establishes (13.7.5) and thereby the bounds on the total variation of U_h.

In the literature cited in Section 13.9, it is shown that the above estimates may even be extended to the more general class of strictly hyperbolic systems of conservation laws that can be approximated "uniformly" by systems with piecewise genuinely nonlinear characteristic families. This broader class encompasses, for example, the system (7.1.11) of isentropic elastodynamics, for arbitrary smooth, strictly increasing stress-strain curve.

13.8 Wave Tracing

The aim here is to track the waves of approximate solutions U_h and monitor the evolution of their strength and speed of propagation. This is not an easy task, as wave interactions may induce the fusion or demise of colliding waves of the same characteristic family, while giving birth to new waves of other characteristic families.

For orientation, let us consider wave interactions in a diamond for the simple case of the Burgers equation $\partial_t u+\frac{1}{2}\partial_x u^2=0$, (4.2.1). The wave interaction estimate (13.3.1) now reduces to $\varepsilon=\alpha+\beta$.

In one typical situation, shocks with (negative) amplitudes α and β, and respective speeds σ_L and σ_R, enter the diamond through its "southwestern" and "southeastern" edge, respectively, and fuse into a single shock of amplitude $\varepsilon=\alpha+\beta$ and speed σ. It is instructive to regard the outgoing shock as a composite of two "virtual waves", with respective amplitudes α and β, so that the two incoming shocks continue on beyond the collision, with the same amplitude but altered speeds. Since $\sigma\varepsilon=\sigma_L\alpha+\sigma_R\beta$, we easily deduce

(13.8.1) $$|\sigma-\sigma_L||\alpha|=|\sigma-\sigma_R||\beta|=\tfrac{1}{2}\alpha\beta.$$

Recall that $\alpha\beta$ represents the amount of wave interaction in the diamond.

In the dual situation, rarefaction waves with (positive) amplitudes α and β enter the diamond through its "southwestern" and "southeastern" edge, respectively, and combine into a single rarefaction with amplitude $\varepsilon=\alpha+\beta$, which in turn splits into new rarefactions with amplitudes α' and β', exiting the diamond through its "northwestern" and "northeastern" edge, respectively. Assuming, for instance, that $\alpha'<\alpha$, we visualize the left incoming wave as a composite of two rarefactions, with respective amplitudes α' and $\alpha-\alpha'$, and the right outgoing wave as a composite of two rarefactions, with respective amplitudes $\alpha - \alpha'$ and β. This way, all three incoming waves continue beyond the interaction with unchanged amplitudes, albeit with altered speeds.

Still another case arises when a shock of (negative) amplitude α and speed σ_L enters the diamond through its "southwestern" edge and interacts with a rarefaction of (positive) amplitude β entering through the "southeastern" edge. Assuming, for instance, that $|\alpha| > |\beta|$, the outgoing wave will be a shock with amplitude $\varepsilon = \alpha + \beta$ and speed σ. As before, we shall regard the incoming shock as a composite of two "virtual waves", with respective amplitudes $\alpha + \beta$ and $-\beta$. Then, as a result of the interaction, the second incoming virtual wave and the incoming rarefaction cancel each other out, while the first virtual wave continues on with unchanged amplitude, but with altered speed. A simple calculation shows that the change in speed is

$$|\sigma - \sigma_L| = \tfrac{1}{2}\beta. \tag{13.8.2}$$

Notice that β represents the amount of wave cancellation in the diamond.

The waves exiting the above diamond will get involved in future collisions, in the context of which they may have to be partitioned further into finer virtual waves. These partitions should be then carried backwards in time and applied retroactively to every ancestor of the wave in question. The end result of this laborious process is that, in any specified time zone, each wave is partitioned into a number of virtual waves which fall into one of the following two categories: those that survive all collisions, within the specified time interval, and those that are eventually extinguished by cancellation.

The situation is similar for systems of hyperbolic conservation laws, except that now one should bear in mind that collisions of any two waves generally give birth to new waves of every characteristic family. In a strictly hyperbolic system with piecewise genuinely nonlinear or linearly degenerate characteristic families, waves are partitioned into virtual waves by the following procedure.

A partitioning of an i-shock joining the state U_-, on the left, with the state U_+, on the right, is performed by some sequence of states $U_- = U^0, U^1, \cdots, U^\nu = U_+$, such that, for $\mu = 1, \cdots, \nu$, U^μ lies on the i-shock curve emanating from U_-, and $\lambda_i(U^\mu) \leq \lambda_i(U^{\mu-1})$. Even though $U^{\mu-1}$ and U^μ are not generally joined by a shock, we regard the pair $(U^{\mu-1}, U^\mu)$ as a virtual wave, with amplitude $V_i^\mu = U^\mu - U^{\mu-1}$ and speed λ_i^μ, equal to the speed of the shock (U_-, U_+).

A partitioning of an i-rarefaction wave joining the state U_-, on the left, with the state U_+, on the right, is similarly performed by a finite sequence of states, namely $U_- = U^0, U^1, \cdots, U^\nu = U_+$, such that, for $\mu = 1, \cdots, \nu$, U^μ lies on the i-rarefaction curve emanating from U_- and $\lambda_i(U^\mu) > \lambda_i(U^{\mu-1})$. Even though $U^{\mu-1}$ and U^μ can now be joined by an actual i-rarefaction wave, $(U^{\mu-1}, U^\mu)$ will still be regarded as a virtual wave with amplitude $V_i^\mu = U^\mu - U^{\mu-1}$ and speed $\lambda_i^\mu = \lambda_i(U^{\mu-1})$.

A partitioning of a general i-wave, joining a state U_-, on the left, with a state U_+, on the right, by a finite sequence of i-shocks and i-rarefaction waves, is performed by combining, in an obvious way, the pure shock with the pure rarefaction case, described above.

By a laborious construction, found in the references cited in Section 13.9, the waves of the approximate solution U_h, over a specified time zone Λ which is defined by $\{(x,t) : -\infty < x < \infty, s_1\lambda^{-1}h \le t \le s_2\lambda^{-1}h\}$, can be partitioned into virtual waves belonging to one of the following three classes:
I. Waves that enter Λ at $t = s_1\lambda^{-1}h$ with positive strength, survive over the time interval $[s_1\lambda^{-1}h, s_2\lambda^{-1}h]$, and exit Λ at $t = s_2\lambda^{-1}h$ with positive strength.
II. Waves that enter Λ at $t = s_1\lambda^{-1}h$ with positive strength, but are extinguished inside Λ by mutual cancellations.
III. Waves that are generated inside Λ, through wave interactions.

If $\mathscr{W}$ denotes the typical virtual wave in any one of the above three classes, the objective is to estimate its maximum strength, denoted by $|\mathscr{W}|$, the total variation of its amplitude, denoted by $[\mathscr{W}]$, and the total variation of its speed, denoted by $[\sigma(\mathscr{W})]$, over its life span inside Λ. The seeds for such estimations lie in the simple estimates (13.8.1) and (13.8.2), obtained in the scalar case, in conjunction with the wave interaction estimates derived in earlier sections.

For systems with genuinely nonlinear characteristic families, the requisite estimates read

$$\sum_{\mathscr{W} \in \mathrm{I}} \{[\mathscr{W}] + |\mathscr{W}|[\sigma(\mathscr{W})]\} = O(1)\mathscr{D}(\Lambda), \tag{13.8.3}$$

$$\sum_{\mathscr{W} \in \mathrm{II}} \{[\mathscr{W}] + |\mathscr{W}|\} = O(1)\mathscr{C}(\Lambda) + O(1)\mathscr{D}(\Lambda), \tag{13.8.4}$$

$$\sum_{\mathscr{W} \in \mathrm{III}} \{[\mathscr{W}] + |\mathscr{W}|\} = O(1)\mathscr{D}(\Lambda), \tag{13.8.5}$$

where $\mathscr{D}(\Lambda)$ and $\mathscr{C}(\Lambda)$ denote the total amount of wave interaction and wave cancellation inside Λ, namely

$$\mathscr{D}(\Lambda) = \sum \mathscr{D}(\Delta_s^r), \qquad \mathscr{C}(\Lambda) = \sum \mathscr{C}(\Delta_s^r), \tag{13.8.6}$$

with the summation running over all diamonds Δ_s^r contained in Λ, and $\mathscr{D}(\Delta_s^r)$, $\mathscr{C}(\Delta_s^r)$ defined by (13.3.2), (13.3.6).

For systems with characteristic families that are merely piecewise genuinely nonlinear, the analogs of the estimates (13.8.3), (13.8.4) and (13.8.5) are considerably more complicated. The difference stems from the fact that the amount of wave interaction $\mathscr{D}(\Delta_s^r)$ is of quadratic order, (13.3.2), in the genuinely nonlinear case, but merely of cubic order, (13.7.2), in the general case. Details are given in the references cited in Section 13.9.

It is now possible to establish the following proposition, which improves Theorem 13.2.1 by removing the "randomness" hypothesis in the selection of the sequence $\wp$:

13.8.1 Theorem. *The algorithm induced by any sequence* $\wp = \{a_0, a_1, \cdots\}$, *which is equidistributed on the interval* $(-1,1)$ *in the sense of* (13.2.6), *is consistent.*

In the proof, which may be found in the references cited in Section 13.10, one expresses the right-hand side of (13.2.2) in terms of the virtual waves that partition U_h and proceeds to show that it tends to zero, as $h \downarrow 0$, whenever the sequence $\wp$ is equidistributed. This happens for the following reason. Recall that in Section 13.2 we did verify the consistency of the algorithm, for any equidistributed sequence $\wp$, in the context of the linear conservation law $\partial_t u + a\lambda\partial_x u = 0$, by employing the property that every wave propagates with constant amplitude and at constant speed. The partitioning of waves performed above demonstrates that even nonlinear systems have this property, albeit in an approximate sense, and this makes it possible to extend the argument for consistency to that case as well.

Though somewhat cumbersome to use, wave partitioning is an effective tool for obtaining precise information on local structure, large-time behavior, and other qualitative properties of solutions; and in particular it is indispensable for deriving properties that hinge on the global wave pattern.

13.9 Notes

The random choice method was developed in the fundamental paper of Glimm [1]. It is in that work that the ideas of consistency (Section 13.2), wave interactions (Section 13.3), and the Glimm functional (Section 13.4) were originally introduced, and Theorem 13.1.1 was first established, for genuinely nonlinear systems. As we shall see in the following chapter, it is Glimm-type functionals that provide the key estimates for compactness in other solution approximation schemes as well. Furthermore, the Glimm functional can be defined, and profitably employed, even in the context of general BV solutions; see Section 14.11.

The construction of solutions with large variation for the special system (13.4.20) of isothermal gas dynamics is due to Nishida [1]. It is based on identifying a time-decreasing functional of Riemann invariants, inducing bounds on the variation of solutions. Related constructions of solutions with large, or at least moderately large, initial data are found in Bakhvarov [1], DiPerna [1,2], Nishida and Smoller [1], Cai Zhong Li [1], Luskin and Temple [1], Serre [11], Ying and Wang [1], Amadori and Guerra [2], Asakura and Corli [1], and Holden, Risebro and Sande [1]. On the other hand, Chen and Jenssen [1] show that the class of systems endowed with such structure is meager.

The implications of wave cancellation, introduced in Section 13.3, on the existence and long time behavior of solutions were first demonstrated in the seminal memoir by Glimm and Lax [1], already cited in Section 12.11. It is that work that revealed the effectiveness of the random choice method and played a decisive role in its dissemination.

The derivation of bounds on the supremum, outlined in Section 13.6, is taken from the thesis of R. Young [1], where the reader may find the technical details. In fact, this work introduces a new length scale for the Cauchy problem, which, under special circumstances, may be used in order to relax the requirement of small total variation on the initial data, for certain systems of more than two conservation laws. In that direction, see Temple [7], Temple and Young [1,2], and Cheverry [3]. Local or global solutions under initial data with large total variation are also constructed by Alber [1] and Schochet [3,4].

The Glimm functional was adapted to systems that are not genuinely nonlinear by Tai-Ping Liu [15], who was first to realize the important role played by the incidence angle between approaching waves of the same characteristic family. The outline presented here, in Section 13.7, follows the more recent work by Iguchi and LeFloch [1], and Tai-Ping Liu and Tong Yang [6].

The method of wave partitioning is developed in Tai-Ping Liu [7], for genuinely nonlinear systems, and in Tai-Ping Liu [15], where it is used for establishing the deterministic consistency of the algorithm for equidistributed sequences (Theorem 13.8.1). Furhermore, following the work of DiPerna [3] on genuinely nonlinear systems of two conservation laws, referenced in Section 12.11, Tai-Ping Liu [15] derives the local structure of solutions to general systems with characteristic families that are either genuinely nonlinear or linearly degenerate, constructed by the Glimm scheme. For a parallel treatment of the local structure of solutions constructed via the alternative, front tracking, algorithm, see Section 14.11.

For early work on the rate of convergence of the random choice scheme, see Hoff and Smoller [1], Tong Yang [3], Ye and Lin [1], and Hua and Yang [1]. For "well equidistributed" sequences, the sharper rate $o(h^{1/2}|\log h|)$ of convergence in L^1 was established by Bressan and Marson [3], for genuinely nonlinear systems.

The more recent work by Hua, Jiang and Yang [1], Hua and Yang [2], Ancona and Marson [8,9], Bianchini and Modena [1,2,3], Modena and Bianchini [1], and Modena [1], based on "quadratic Glimm functionals", provides a very powerful tool for probing the rate of convergence of the scheme and the local structure of resulting solutions.

Still another Glimm functional, proposed by Caravenna [1], which is based on maximal entropy production, is effective, at least for scalar conservation laws.

As $t \to \infty$, solutions of (13.1.1) approach the solution of the Riemann problem with data (9.1.12), where $U_L = U_0(-\infty)$ and $U_R = U_0(+\infty)$; cf. Tai-Ping Liu [9,11,15]. For a more recent exposition see Tai-Ping Liu [28], and Tai-Ping Liu and Tong Yang [6].

There is voluminous literature on extensions and applications of the random choice method. For systems of mixed type, see Pego and Serre [1], LeFloch [3], and Corli and Sablé-Tougeron [3]. For initial-boundary value problems, cf. Tai-Ping Liu [11], Luskin and Temple [1], Nishida and Smoller [2], Dubroca and Gallice [1], Sablé-Tougeron [1], and Frid [1]. For solutions involving strong shocks, see Sablé-Tougeron [2], Corli and Sablé-Tougeron [1,2], Asakura [1], Corli [2], and Schochet [3,4]. For periodic solutions, see Frid [4], and Frid and Perepelitsa [1]. For applications to gas dynamics, see Tai-Ping Liu [4,5,12,16,17], Temple [1], and Tong Yang [2]. For the effects of vacuum in gas dynamics, see Liu and Smoller [1], and Long-Wei Lin [1,2]. For applications to the theory of relativity, see Barnes, LeFloch, Schmidt and Stewart [1]. For systems that are not in conservation form, see LeFloch [2], and LeFloch and Liu [1]. Weak L^p stability is established by Temple [6]. Additional references are found in the books by Smoller [3], Serre [11], and LeFloch [5].

XIV

The Front Tracking Method and Standard Riemann Semigroups

A method is described in this chapter for constructing solutions of the initial value problem for hyperbolic systems of conservation laws by tracking the waves and monitoring their interactions as they collide. Interactions between shocks are easily resolved by solving Riemann problems; this is not the case, however, with interactions involving rarefaction waves. The random choice method, expounded in Chapter XIII, side-steps this difficulty by stopping the clock before the onset of wave collisions and reapproximating the solution by step functions. In contrast, the front tracking approach circumvents the obstacle by disposing of rarefaction waves altogether and resolving all Riemann problems in terms of shocks only. Such solutions generally violate the admissibility criteria. Nevertheless, considering the close local proximity between shock and rarefaction wave curves in state space, any rarefaction wave may be approximated arbitrarily close by fans of (inadmissible) shocks of very small strength. The expectation is that in the limit, as this approximation becomes finer, one recovers admissible solutions.

The implementation of the front tracking algorithm, with proof that it converges, will be presented here, first for scalar conservation laws and then in the context of genuinely nonlinear strictly hyperbolic systems of conservation laws of any size.

By a contraction argument with respect to a suitably weighted L^1 distance, it will be demonstrated that solutions of genuinely nonlinear systems, constructed by the front tracking method, may be realized as orbits of the *Standard Riemann Semigroup*, which is defined on the set of functions with small total variation and is Lipschitz continuous in L^1. It will further be shown that any BV solution that satisfies reasonable stability conditions is also identifiable with the orbit of the Standard Riemann Semigroup issuing from its initial data. This establishes, in particular, uniqueness for the initial value problem within a broad class of BV solutions, including those constructed by the random choice method, as well as those whose trace along space-like curves has bounded variation, encountered in earlier chapters.

The chapter will close with a discussion of the structural stability of the wave pattern under perturbations of the initial data.

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Grundlehren der mathematischen Wissenschaften 325,
DOI 10.1007/978-3-662-49451-6_14

14.1 Front Tracking for Scalar Conservation Laws

This section discusses the construction of the admissible solution to the initial value problem for scalar conservation laws by a front tracking scheme that aims at eliminating rarefaction waves. The building blocks will be composite waves consisting of constant states, admissible "compressive" shocks, and inadmissible "rarefaction" shocks of small strength.

The admissible solution of the Riemann problem for the scalar conservation law $\partial_t u + \partial_x f(u) = 0$, with C^1 flux f, was constructed in Section 9.5: the left end-state u_l and the right end-state u_r are joined by the wave fan

$$u(x,t) = [g']^{-1}\left(\frac{x}{t}\right), \tag{14.1.1}$$

where g is the convex envelope of f over $[u_l, u_r]$, when $u_l < u_r$, or the concave envelope of f over $[u_r, u_l]$, when $u_l > u_r$. Intervals on which g' is constant yield shocks, while intervals over which g' is strictly monotone generate rarefaction waves. The same construction applies even when f is merely Lipschitz, except that now, in addition to shocks and rarefaction waves, the ensuing composite wave may contain intermediate constant states, namely, the jump points of g'. In particular, when f, and thereby g, are piecewise linear, the composite wave does not contain any rarefaction waves but consists of shocks and constant states only (Fig. 14.1.1).

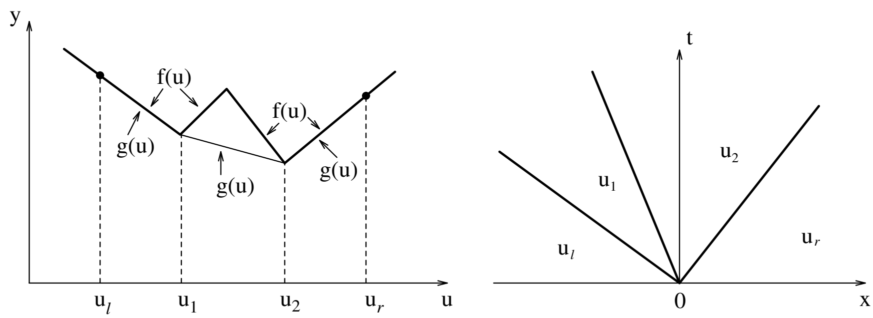

Fig. 14.1.1

We now consider the Cauchy problem

$$\begin{cases} \partial_t u(x,t) + \partial_x f(u(x,t)) = 0, & -\infty < x < \infty,\ 0 \le t < \infty, \\ u(x,0) = u_0(x), & -\infty < x < \infty, \end{cases} \tag{14.1.2}$$

for a scalar conservation law, where the flux f is Lipschitz continuous on $(-\infty, \infty)$ and the initial datum u_0 takes values in a bounded interval $[-M, M]$ and has bounded total variation over $(-\infty, \infty)$.

To solve (14.1.2), one first approximates the flux f by a sequence $\{f_m\}$ of piecewise linear functions, such that the graph of f_m is a polygonal line inscribed in the graph of f, with vertices at the points $(\frac{k}{m}, f(\frac{k}{m})), k \in \mathbb{Z}$. Next, one realizes the initial datum u_0 as the a.e. limit of a sequence $\{u_{0m}\}$ of step functions, where u_{0m} takes values in the set $\mathscr{U}_m = \{\frac{k}{m} : k \in \mathbb{Z}, |k| \leq mM\}$, and its total variation does not exceed the total variation of u_0 over $(-\infty, \infty)$. Finally, one solves the initial value problem

$$(14.1.3) \qquad \begin{cases} \partial_t u(x,t) + \partial_x f_m(u(x,t)) = 0, & -\infty < x < \infty,\ 0 \leq t < \infty, \\ u(x,0) = u_{0m}(x), & -\infty < x < \infty, \end{cases}$$

for $m = 1, 2, \cdots$. The aim is to show that the admissible solution u_m of (14.1.3) is a piecewise constant function, taking values in $\mathscr{U}_m$, which is constructed by solving a finite number of Riemann problems for the conservation law $(14.1.3)_1$; and that the sequence $\{u_m\}$ converges to the admissible solution u of (14.1.2).

The construction of u_m is initiated by solving the Riemann problems that resolve the jump discontinuities of u_{0m} into shocks and constant states in $\mathscr{U}_m$. In turn, wave interactions induced by shock collisions are similarly resolved, in the order in which they occur, into shocks and constant states in $\mathscr{U}_m$, resulting from the solution of Riemann problems. It should be noted that the admissible solution of the Riemann problem for $(14.1.3)_1$, with end-states in $\mathscr{U}_m$, is also a solution of $(14.1.2)_1$, albeit not necessarily an admissible one, because in that context some of the jump discontinuities may be rarefaction shocks. Thus, in addition to being the admissible solution of (14.1.3), u_m is a (generally inadmissible) solution of $(14.1.2)_1$.

We demonstrate that the number of shock collisions that may be encountered in the implementation of the above algorithm is a priori bounded, and hence u_m is constructed on the entire upper half-plane in finite steps. The reason is that each shock interaction simplifies the wave pattern by lowering either the number of shocks, measured by the number $j_m(t)$ of points of jump discontinuity of the step function $u_m(\cdot,t)$, or the number of "oscillations," counted by the lap number $\ell_m(t)$ of $u_m(\cdot,t)$, which is defined as follows.

For the case of a step function $v(\cdot)$ on $(-\infty,\infty)$, the *lap number* ℓ is set equal to 0 when $v(\cdot)$ is monotone, while when $v(\cdot)$ is nonmonotone it is defined as the largest positive integer such that there exist $\ell + 2$ points $-\infty < x_0 < \cdots < x_{\ell+1} < \infty$ of continuity of $v(\cdot)$, with $[v(x_{i+1}) - v(x_i)][v(x_i) - v(x_{i-1})] < 0$, $i = 1, \cdots, \ell$.

Clearly, both $j_m(t)$ and $\ell_m(t)$ stay constant along the open time intervals between consecutive shock collisions; they may change only across $t = 0$ and as shocks collide. When k shocks, joining (left, right) states $(u_0, u_1), \cdots, (u_{k-1}, u_k)$, collide at one point, the ensuing interaction is called *monotone* if the finite sequence $\{u_0, u_1, \cdots, u_k\}$ is monotone. Such an interaction produces a single shock joining the state u_0, on the left, with the state u_k, on the right. In particular, monotone interactions leave $\ell_m(t)$ unchanged, while lowering the value of $j_m(t)$ by at least one. In contrast, across nonmonotone interactions $\ell_m(t)$ decreases by at least one, while the value of $j_m(t)$ may change in either direction, but in any case it cannot increase by more than $s_m - 1$, s_m being the number of jump points of f_m' over the interval $(-M, M)$; thus $s_m - 1 < 2Mm$. It follows that the integer-valued function

$p_m(t) = j_m(t) + s_m \ell_m(t)$ stays constant along the open time intervals between consecutive shock collisions, while decreasing by at least one across any monotone or nonmonotone shock collision. Across the axis $t = 0$, we have $\ell_m(0+) = \ell_m(0)$ and $j_m(0+) \leq (s_m + 1) j_m(0)$. Therefore, $(s_m + 1)[j_m(0) + \ell_m(0)]$ provides an upper bound for the total number of shock collisions involved in the construction of u_m.

As a function of t, the total variation of $u_m(\cdot, t)$ over $(-\infty, \infty)$ stays constant along time intervals between consecutive shock collisions; it does not change across monotone shock collisions; and it decreases across nonmonotone shock collisions. Hence,

$$TV_{(-\infty,\infty)} u_m(\cdot,t) \leq TV_{(-\infty,\infty)} u_{m0}(\cdot) \leq TV_{(-\infty,\infty)} u_0(\cdot), \quad 0 \leq t < \infty. \tag{14.1.4}$$

Since the speed of any shock of u_m cannot exceed the Lipschitz constant c of f over $[-M, M]$, (14.1.4) implies

$$\int_{-\infty}^{\infty} |u_m(x,t) - u_m(x,\tau)| dx \leq c|t - \tau| TV_{(-\infty,\infty)} u_0(\cdot), \qquad 0 \leq \tau < t < \infty. \tag{14.1.5}$$

By virtue of (14.1.4), Helly's theorem, and the Cantor diagonal process, one finds a subsequence $\{u_{m_k}\}$ such that $\{u_{m_k}(\cdot, t)\}$ is convergent in $L^1_{\mathrm{loc}}(-\infty, \infty)$, for any rational $t \in [0, \infty)$. Then, (14.1.5) implies that $\{u_{m_k}(\cdot, t)\}$ is Cauchy in $L^1_{\mathrm{loc}}(-\infty, \infty)$ for all $t \in [0, \infty)$, and hence $\{u_{m_k}\}$ converges in L^1_{loc} to some function u of locally bounded variation on $(-\infty, \infty) \times [0, \infty)$.

As discussed in Chapter VI, since u_m is the admissible solution of (14.1.3),

$$\int_0^{\infty} \int_{-\infty}^{\infty} [\partial_t \psi \eta(u_m) + \partial_x \psi q_m(u_m)] dx dt + \int_{-\infty}^{\infty} \psi(x,0) \eta(u_{0m}(x)) dx \geq 0, \tag{14.1.6}$$

for any convex entropy η, with associated entropy flux $q_m = \int \eta' df_m$, and all nonnegative Lipschitz test functions ψ on $(-\infty, \infty) \times [0, \infty)$, with compact support. As $m \to \infty$, $\{u_{0m}\}$ converges, a.e. on $(-\infty, \infty)$, to u_0, and $\{q_m\}$ converges, uniformly on $[-M, M]$, to the function $q = \int \eta' df$, namely, the entropy flux associated with the entropy η in the conservation law $(14.1.2)_1$. Upon passing to the limit in (14.1.6), along the subsequence $\{m_k\}$, we deduce

$$\int_0^{\infty} \int_{-\infty}^{\infty} [\partial_t \psi \eta(u) + \partial_x \psi q(u)] dx dt + \int_{-\infty}^{\infty} \psi(x,0) \eta(u_0(x)) dx \geq 0, \tag{14.1.7}$$

which in turn implies that u is the admissible solution of (14.1.2). By uniqueness, we infer that the entire sequence $\{u_m\}$ converges to u.

14.2 Front Tracking for Genuinely Nonlinear Systems of Conservation Laws

Consider a system of conservation laws, in canonical form

$$\partial_t U + \partial_x F(U) = 0, \tag{14.2.1}$$

which is strictly hyperbolic (7.2.8), and each characteristic family is either genuinely nonlinear (7.6.13) or linearly degenerate (7.5.2). The object of this section is to introduce a front tracking algorithm that solves the initial value problem (13.1.1), under initial data U_0 with small total variation, and provides, in particular, an alternative proof of the existence Theorem 13.1.1.

The instrument of the algorithm will be special Riemann solvers, which will be employed to resolve jump discontinuities into centered wave fans composed of jump discontinuities and constant states, approximating the admissible solution of the Riemann problem. In implementing the algorithm, the initial data are approximated by step functions whose jump discontinuities are then resolved into wave fans. Interactions induced by the collision of jump discontinuities are in turn resolved, in the order in which they occur, into similar wave fans. It will suffice to consider the generic situation, in which no more than two jump discontinuities may collide simultaneously. The expectation is that such a construction will produce an approximate solution of the initial value problem in the class of piecewise constant functions.

The first item on the agenda is how to design suitable Riemann solvers. The experience with the scalar conservation law, in Section 14.1, suggests that one should synthesize the centered wave fans by a combination of constant states, admissible shocks, and inadmissible rarefaction shocks with small strength.

In an *admissible i-shock*, the right state U_+ lies on the i-th shock curve through the left state U_-, that is, in the notation of Section 9.3, $U_+ = \Phi_i(\tau; U_-)$, with $\tau < 0$ when the i-th characteristic family is genuinely nonlinear (compressive shock) or with $\tau \lessgtr 0$ when the i-th characteristic family is linearly degenerate (contact discontinuity). The amplitude is τ, the strength is measured by $|\tau|$, and the speed s is set by the Rankine-Hugoniot jump condition (8.1.2).

Instead of actual rarefaction shocks, it is more convenient to employ "rarefaction fronts," namely jump discontinuities that join states lying on a rarefaction wave curve and propagate with characteristic speed. Thus, in an *i-rarefaction front* (which may arise only when the i-th characteristic family is genuinely nonlinear) the right state U_+ lies on the i-th rarefaction wave curve through the left state U_-, i.e., $U_+ = \Phi_i(\tau; U_-)$, with $\tau > 0$. Both amplitude and strength are measured by τ, and the speed is set equal to $\lambda_i(U_+)$. Clearly, these fronts violate not only the entropy admissibility criterion but even the Rankine-Hugoniot jump condition, albeit only slightly when their strength is small.

Centered rarefaction waves may be approximated by composite waves consisting of constant states and rarefaction fronts with strength not exceeding some prescribed magnitude $\delta > 0$. Consider some i-rarefaction wave, centered, for definiteness, at the origin, which joins the state U_-, on the left, with the state U_+, on the right. Thus, U_+ lies on the i-rarefaction curve through U_-, say $U_+ = \Phi_i(\tau; U_-)$, for some $\tau > 0$. If ν is the smallest integer that is larger than τ/δ, we set $U^0 = U_-$, $U^\nu = U_+$, define $U^\mu = \Phi_i(\mu\delta; U_-)$, $\mu = 1, \cdots, \nu - 1$, and approximate the rarefaction wave, inside the sector $\lambda_i(U_-) < \frac{x}{t} < \lambda_i(U_+)$, by the wave fan

$$U(x,t) = U^{\mu}, \quad \lambda_i(U^{\mu-1}) < \frac{x}{t} < \lambda_i(U^{\mu}), \qquad \mu = 1, \cdots, \nu. \tag{14.2.2}$$

We are thus naturally lead to an *Approximate Riemann Solver*, which resolves the jump discontinuity between a state U_l, on the left, and U_r, on the right, into a wave fan composed of constant states, admissible shocks, and rarefaction fronts, by the following procedure: The starting point is the admissible solution of the Riemann problem, consisting of $n+1$ constant states $U_l = U_0, U_1, \cdots, U_n = U_r$, where U_{i-1} is joined to U_i by an admissible i-shock or an i-rarefaction wave. To pass to the approximation, the domain and values of the constant states, and thereby all shocks, are retained, whereas, as described above, any rarefaction wave is replaced, within its sector, by a fan of constant states and rarefaction fronts of the same family, with strength not exceeding δ (Fig. 14.2.1).

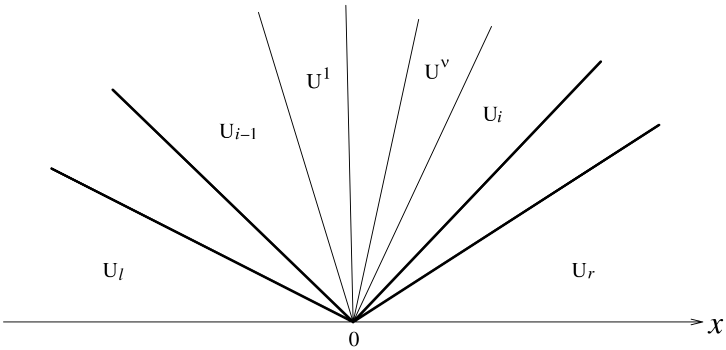

Fig. 14.2.1

Our earlier success with the scalar case may raise expectations that a front tracking algorithm, in which all shock interactions are resolved via the above approximate, though relatively accurate, Riemann solver, will produce an approximate solution of our system, converging to an admissible solution of the initial value problem, as the allowable strength δ of rarefaction fronts shrinks to zero. Unfortunately, such an approach would generally fail, for the following reason: by contrast to the case for scalar conservation laws, wave interactions in systems tend to increase the complexity of the wave pattern so that collisions become progressively more frequent and the algorithm may grind to a halt in finite time. As a remedy, in order to prevent the proliferation of waves, only shocks and rarefaction fronts of substantial strength shall be tracked with relative accuracy. The rest shall not be totally disregarded but shall be treated with less accuracy: they will be lumped together to form jump discontinuities, dubbed "pseudoshocks," which propagate with artificial, supersonic speed.

A *pseudoshock* is allowed to join arbitrary states U_- and U_+. Its strength is measured by $|U_+ - U_-|$ and its assigned speed is a fixed upper bound λ_{n+1} of $\lambda_n(U)$, for U in the range of the solution. Clearly, pseudoshocks are more serious violators of the Rankine-Hugoniot jump condition than rarefaction fronts, and may thus wreak havoc on the approximate solution, unless their combined strength is kept very small.

To streamline the exposition, i-rarefaction fronts and i-shocks (compression or contact discontinuities) together will be dubbed *i-fronts*. Fronts and pseudoshocks will be collectively called *waves*. Thus an i-front will be an *i-wave* and a pseudoshock will be termed $(n+1)$-*wave*. As in earlier chapters, the amplitudes of waves will be denoted by Greek letters $\alpha, \beta, \gamma, \dots$ with corresponding strengths $|\alpha|, |\beta|, |\gamma|, \cdots$.

Under circumstances to be specified below, the jump discontinuity generated by the collision of two waves shall be resolved via a *Simplified Riemann Solver*, which allows fronts to pass through the point of interaction without affecting their strength, while introducing an outgoing pseudoshock in order to bridge the resulting mismatch in the states. The following cases may arise.

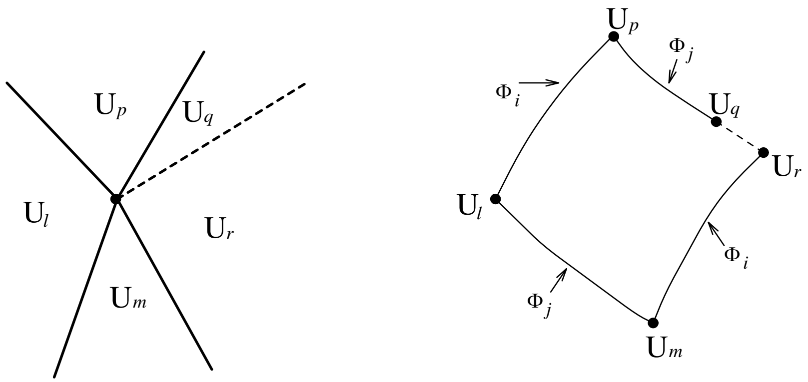

Fig. 14.2.2

Suppose that, for $i < j$, a j-front, joining the states U_l and U_m, collides with an i-front, joining the states U_m and U_r; see Fig. 14.2.2. Thus $U_m = \Phi_j(\tau_l; U_l)$ and $U_r = \Phi_i(\tau_r; U_m)$. To implement the Simplified Riemann Solver, one determines the states $U_p = \Phi_i(\tau_r; U_l)$ and $U_q = \Phi_j(\tau_l; U_p)$. Then, the outgoing wave fan will be composed of the i-front, joining the states U_l and U_p, the j-front, joining the states U_p and U_q, plus the pseudoshock that joins U_q with U_r.

Suppose next that an i-front, joining the states U_l and U_m, collides with another i-front, joining the states U_m and U_r (no such collision may occur unless at least one of these fronts is a compressive shock); see Fig. 14.2.3.

Thus $U_m = \Phi_i(\tau_l; U_l)$ and $U_r = \Phi_i(\tau_r; U_m)$. If $U_q = \Phi_i(\tau_l + \tau_r; U_l)$, the outgoing wave fan will be composed of the i-front, joining the states U_l and U_q, plus the pseudoshock that joins U_q with U_r.

Finally, suppose a pseudoshock, joining the states U_l and U_m, collides with an i-front, joining the states U_m and U_r; see Fig. 14.2.4. Hence, $U_r = \Phi_i(\tau_m; U_m)$. We determine $U_q = \Phi_i(\tau_m; U_l)$. The outgoing wave fan will be composed of the i-front, joining the states U_l and U_q, plus the pseudoshock that joins U_q with U_r.

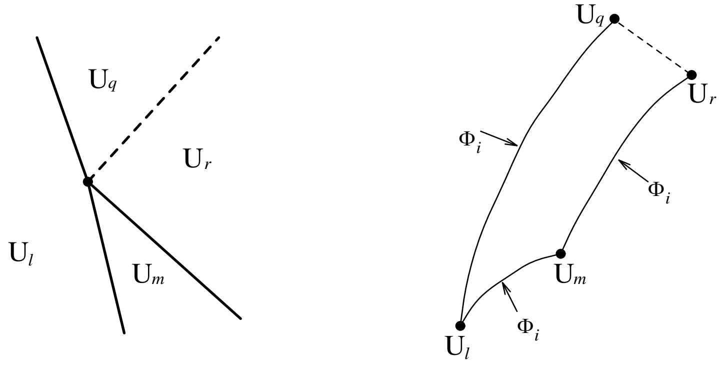

Fig. 14.2.3

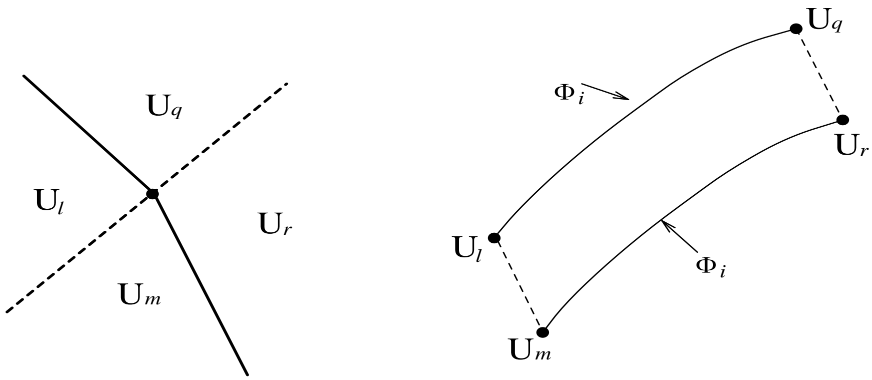

Fig. 14.2.4

In implementing the front tracking algorithm, one fixes, at the outset, the supersonic speed λ_{n+1} of pseudoshocks, sets the delimiter δ for the strength of rarefaction fronts, and also specifies a third parameter $\sigma > 0$, which rules how jump discontinuities are to be resolved:

- Jump discontinuities resulting from the collision of two fronts, with respective amplitudes α and β, must be resolved via the Approximate Riemann Solver if $|\alpha||\beta| > \sigma$, or via the Simplified Riemann Solver if $|\alpha||\beta| \leq \sigma$.
- Jump discontinuities resulting from the collision of a pseudoshock with any front must be resolved via the Simplified Riemann Solver.
- Jump discontinuities of the step function approximating the initial data are to be resolved via the Approximate Riemann Solver.

14.3 The Global Wave Pattern

Starting out from some fixed initial step function, the front tracking algorithm described in the previous section will produce a piecewise constant function U on a maximal time interval $[0,T)$. In principle, T may turn out to be finite, if the number of collisions grows without bound as $t \uparrow T$, so the task is to show that this will not happen.

To understand the structure of U, one has to untangle the complex wave pattern. Towards that end, waves must be tracked not just between consecutive collisions but globally, from birth to extinction or in perpetuity. The waves are granted global identity through the following convention: an i-wave involved in a collision does not necessarily terminate there, but generally continues on as the outgoing i-wave from that point of wave interaction. Any ambiguities that may arise in applying the above rule will be addressed and resolved below.

Pseudoshocks are generated by the collision of two fronts, resolved via the Simplified Riemann Solver, as depicted in Figs. 14.2.2 or 14.2.3. On the other hand, i-fronts may be generated either at $t = 0$, from the resolution of some jump discontinuity of the initial step function, or at $t > 0$, by the collision of a j-front with a k-front, where $j \neq i \neq k$, that is resolved via the Approximate Riemann Solver.

Every wave carries throughout its life span a number μ, identifying its *generation order*, that is the maximum number of collisions predating its birth. Thus, fronts originating at $t = 0$ are assigned generation order $\mu = 0$. Any other new wave, which is necessarily generated by the collision of two waves, with respective generation orders say μ_1 and μ_2, is assigned generation order $\mu = \max\{\mu_1, \mu_2\} + 1$.

As postulated above, waves retain their generation order as they traverse points of interaction. Ambiguity may arise when, in a collision of an i-rarefaction front with a j-front, resolved via the Approximate Riemann Solver, the outgoing i-wave fan contains two i-rarefaction fronts. In that case, the stronger of these fronts, with strength δ, is designated as the prolongation of the incoming i-front, while the other i-front, with strength $< \delta$, is regarded as a new front and is assigned a higher generation order, in accordance with the standard rule. Ambiguity may also arise when two fronts of the same family collide, since the outgoing wave fan may include (at most) one front of that family. In that situation, the convention is that the front with the lower generation order is designated the survivor, while the other one is terminated. In case both fronts are of the same generation order, either one, arbitrarily, may be designated as the survivor. Of course, both fronts may be terminated upon colliding, as depicted in Fig. 14.2.3, in the (nongeneric) case where one of them is a compression shock, the other is a rarefaction front of the same characteristic family, and both have the same strength. Pseudoshocks may also be extinguished in finite time by colliding with a front, as depicted in Fig. 14.2.4, in the (nongeneric) case $U_q = U_r$.

We now introduce the following notions, which will establish a connection with the approach pursued in Chapters X-XII.

For $i = 1, \cdots, n$, an *i-characteristic* associated with U is a Lipschitz, polygonal line $x = \xi(t)$ which traverses constant states, say $\bar{U}$, at classical i-characteristic speed, $\dot{\xi} = \lambda_i(\bar{U})$, but upon impinging on an i-front, or a generation point thereof, it adheres to that front, following it throughout its lifespan. Thus, in particular, any i-front is an i-characteristic. By analogy, $(n+1)$-*characteristics* are defined as straight lines with slope λ_{n+1}. Thus, pseudoshocks are $(n+1)$-characteristics.

Consider now an oriented Lipschitz curve with graph $\mathscr{C}$, which divides the upper half-plane into its "positive" and "negative" side. We say $\mathscr{C}$ is *nonresonant* if the set $\{1,\cdots,n,n+1\}$ can be partitioned into three, pairwise disjoint, possibly empty, subsets $\mathscr{N}_-$, $\mathscr{N}_0$ and $\mathscr{N}_+$, with the following properties: each of the subsets $\mathscr{N}_-$ and $\mathscr{N}_+$ consists of up to $n+1$ consecutive integers, while $\mathscr{N}_0$ may contain at most one member. For $i \in \mathscr{N}_-$ (or $i \in \mathscr{N}_+$), any i-characteristic impinging on $\mathscr{C}$ crosses from the positive to the negative (or from the negative to the positive) side. On the other hand, if $i \in \mathscr{N}_0$, any i-characteristic impinging upon $\mathscr{C}$, from either its positive or its negative side, is absorbed by $\mathscr{C}$, i.e., $\mathscr{C}$ itself is an i-characteristic.

Noteworthy examples of nonresonant curves include:

(a) Any i-characteristic, in particular any i-wave, in which case the partition is $\mathscr{N}_- = \{1,\cdots,i-1\}$, $\mathscr{N}_0 = \{i\}$ and $\mathscr{N}_+ = \{i+1,\cdots,n+1\}$.
(b) Any *space-like curve*. Assuming $\lambda_1(U) < 0 < \lambda_{n+1}$, these may be represented by Lipschitz functions $t = \hat{t}(x)$, such that $1/\lambda_1 < d\hat{t}/dx < 1/\lambda_{n+1}$, a.e. In that case, $\mathscr{N}_+ = \{1,\cdots,n+1\}$ while both $\mathscr{N}_-$ and $\mathscr{N}_0$ are empty.

The relevance of the above will become clear in the next section.

14.4 Approximate Solutions

The following definition collects all the requirements on a piecewise constant function, of the type produced by the front tracking algorithm, so as to qualify as a reasonable approximation to the solution of our Cauchy problem:

14.4.1 Definition. For $\delta > 0$, a δ-*approximate solution* of the hyperbolic system of conservation laws (14.2.1) is a piecewise constant function U, defined on $(-\infty,\infty)\times[0,\infty)$ and satisfying the following conditions: The domains of the constant states are bordered by jump discontinuities, called waves, each propagating with constant speed along a straight line segment $x = y(t)$. Any wave may originate either at a point of the x-axis, $t = 0$, or at a point of collision of other waves, and generally terminates upon colliding with another wave, unless no such collision occurs in which case it propagates all the way to infinity. Only two incoming waves may collide simultaneously, but any (finite) number of outgoing waves may originate at a point of collision. There is a finite number of points of collision, waves and constant states. The waves are of three types:

(a) *Shocks.* An (approximate) i-shock $x = y(t)$ borders constant states U_-, on the left, and U_+, on the right, which can be joined by an admissible i-shock, i.e., $U_+ = W_i(\tau;U_-)$, with $\tau < 0$ when the i-characteristic family is genuinely nonlinear, or $\tau \lessgtr 0$ when the i-characteristic family is linearly degenerate, and propagates approximately at the shock speed $s = s_i(\tau;U_-)$:

$$(14.4.1) \qquad |\dot{y}(\cdot) - s| \leq \delta.$$

(b) *Rarefaction Fronts.* An (approximate) i-rarefaction front $x = y(t)$ borders constant states U_-, on the left, and U_+, on the right, which can be joined by an i-rarefaction wave with strength $\leq \delta$, i.e., $U_+ = V_i(\tau; U_-)$, with $0 < \tau \leq \delta$, and propagates approximately at characteristic speed:

$$(14.4.2) \qquad |\dot{y}(\cdot) - \lambda_i(U_+)| \leq \delta.$$

(c) *Pseudoshocks.* A pseudoshock $x = y(t)$ may border arbitrary states U_- and U_+ and propagates at the specified supersonic speed:

$$(14.4.3) \qquad \dot{y}(\cdot) = \lambda_{n+1}\,.$$

The combined strength of pseudoshocks does not exceed δ:

$$(14.4.4) \qquad \sum |U(y(t)+,t) - U(y(t)-,t)| \leq \delta, \qquad 0 < t < \infty,$$

where for each t the summation runs over all pseudoshocks $x = y(\cdot)$ which cross the t-time line.

If, in addition, the step function $U(\cdot,0)$ approximates the initial data U_0 in L^1, within distance δ,

$$(14.4.5) \qquad \int_{-\infty}^{\infty} |U(x,0) - U_0(x)|dx \leq \delta,$$

then U is called a δ-*approximate solution of the Cauchy problem* (13.1.1).

The extra latitude afforded by the above definition in allowing the speed of (approximate) shocks and rarefaction fronts to (slightly) deviate from their more accurate values granted by the front tracking algorithm provides some flexibility which may be put to good use for ensuring that no more than two fronts may collide simultaneously.

The effectiveness of front tracking will be demonstrated through the following

14.4.2 Theorem. *Assume $U_0 \in BV(-\infty,\infty)$, with $TV_{(-\infty,\infty)}U_0(\cdot) \leq a << 1$. Fix any small positive δ, and approximate U_0 by some step function $U_{0\delta}$ such that $TV_{(-\infty,\infty)}U_{0\delta}(\cdot) \leq TV_{(-\infty,\infty)}U_0(\cdot)$ and $\|U_{0\delta}(\cdot) - U_0(\cdot)\|_{L^1(-\infty,\infty)} \leq \delta$. Then the front tracking algorithm with initial data $U_{0\delta}$, fixed supersonic speed λ_{n+1} for pseudoshocks, delimiter δ for the strength of rarefaction fronts, and sufficiently small parameter σ (depending on δ and on the number of jump points of $U_{0\delta}$) generates a δ-approximate solution U_δ of the initial value problem* (13.1.1). *Any sequence of δ's converging to zero contains a subsequence $\{\delta_k\}$ such that $\{U_{\delta_k}\}$ converges, a.e. on $(-\infty,\infty) \times [0,\infty)$, to a BV solution U of* (13.1.1), *which satisfies the entropy admissibility condition for any convex entropy-entropy flux pair (η, q) of the system* (14.2.1), *together with the estimates* (13.1.5) *and* (13.1.6). *Furthermore, the trace of U on any Lipschitz graph on the upper half-plane that is nonresonant relative to all U_δ has bounded variation.*

The above proposition reestablishes the assertions of Theorem 13.1.1. The property that the trace of U along nonresonant curves has bounded variation establishes a connection with the class of solutions discussed in Chapter XII.

The demonstration of Theorem 14.4.2 is quite lengthy and will be presented, in installments, in the next three sections. However, the following road map may prove useful at this juncture.

As already noted in Section 14.3, once the step function $U_{0\delta}$ has been designated, the front tracking algorithm will produce U_δ, at least on a time interval $[0,T)$, which as we shall see later is $[0,\infty)$. We shall be assuming throughout that the range of U_δ is contained in a ball of small radius in state space, a condition that must be verified a posteriori. The constants $c_1, c_2, \cdots, \kappa, \cdots$, which will appear in the course of the proof, all depend solely on bounds of F and its derivatives in that ball.

The first step will be to establish an estimate

$$TV_{(-\infty,\infty)}U_\delta(\cdot,t) \leq c_1 TV_{(-\infty,\infty)}U_0(\cdot), \qquad 0 \leq t < T, \tag{14.4.6}$$

on the total variation, together with a bound on the total amount of wave interaction. On account of the construction of U_δ, (14.4.6) will immediately imply

$$\int_{-\infty}^{\infty} |U_\delta(x,t) - U_\delta(x,\tau)|dx \leq c_2|t-\tau| TV_{(-\infty,\infty)}U_0(\cdot), \quad 0 \leq \tau < t < T, \tag{14.4.7}$$

with $c_2 = cc_1$, where c is any upper bound of the wave speeds; for instance c is the maximum of λ_{n+1} and $-\inf \lambda_1(U)$. The usefulness of these estimates is twofold: first, they will assist in the task of verifying that U_δ meets the requirements set by Definition 14.4.1; secondly, they will induce compactness that makes it possible to pass to the $\delta \downarrow 0$ limit.

In verifying that U_δ is a δ-approximate solution, the requirements (14.4.1), (14.4.2) and (14.4.3), on the speed of shocks, rarefaction fronts and pseudoshocks, are patently met, because of the specifications of the construction. Moreover, the selection of the delimiter entails that the strengths of rarefaction fronts will be bounded by δ. The remaining requirements, namely that the combined strength of pseudoshocks is also bounded by δ, as in (14.4.4), and that the number of collisions is finite, will be established by insightful analysis of the wave pattern. In particular, this will furnish the warranty that U_δ is generated, in finite steps, on the entire upper half-plane, i.e., $T = \infty$.

The final step in the proof will complete the construction of the solution to (13.1.1) by passing to the $\delta \downarrow 0$ limit in U_δ, via a compactness argument relying on the estimates (14.4.6) and (14.4.7).

14.5 Bounds on the Total Variation

As in Section 13.4, $TV_{(-\infty,\infty)}U_\delta(\cdot,t)$ will be measured through

$$L(t) = \sum |\gamma|, \tag{14.5.1}$$

namely by the sum of the strengths of all jump discontinuities that cross the t-time line. Clearly, $L(\cdot)$ stays constant along time intervals between consecutive collisions of fronts and changes only across points of wave interaction. To estimate these changes, we have to investigate the various types of collisions.

Suppose a j-front of amplitude α collides with an i-front of amplitude β. When $|\alpha||\beta| \geq \sigma$, so that the resulting jump discontinuity is resolved, via the Approximate Riemann Solver, into a full wave fan $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$, then, by virtue of Theorem 9.9.1[1],

$$|\varepsilon_j - \alpha| + |\varepsilon_i - \beta| + \sum_{k \neq i,j} |\varepsilon_k| = O(1)|\alpha||\beta|, \tag{14.5.2}$$

if $i < j$, or

$$|\varepsilon_i - \alpha - \beta| + \sum_{k \neq i} |\varepsilon_k| = O(1)|\alpha||\beta|, \tag{14.5.3}$$

if $i = j$. On the other hand, when $|\alpha||\beta| < \sigma$, in which case the resulting jump discontinuity is resolved via the Simplified Riemann Solver as shown in Fig. 14.2.2 or Fig. 14.2.3, the amplitude of the colliding fronts is conserved. The strength of the generated outgoing pseudoshock is easily estimated from the wave diagrams in state space:

$$|U_R - U_Q| = O(1)|\alpha||\beta|. \tag{14.5.4}$$

Consider next the case depicted in Fig. 14.2.4, where a pseudoshock collides with an i-front of amplitude β. Since the amplitude of the i-front is conserved across the collision, analysis of the wave diagram in state space, Fig. 14.2.4, yields that the strength of the outgoing pseudoshock is related to the strength of the incoming pseudoshock by

$$|U_R - U_Q| = |U_M - U_L| + O(1)|\beta||U_M - U_L|. \tag{14.5.5}$$

Let I denote the set of $t \in (0,T)$ where collisions occur. We let Δ denote the "jump" operator from $t-$ to $t+$, for $t \in I$. In light of the analysis of wave interactions, above, we infer

$$\Delta L(t) \leq \kappa|\alpha||\beta|, \qquad t \in I, \tag{14.5.6}$$

where $|\alpha|$ and $|\beta|$ are the strengths of the waves that collide at t.

Our strategy for keeping $TV_{(-\infty,\infty)}U_\delta(\cdot,t)$ under control is to show that any increase of $L(\cdot)$ allowed by (14.5.6) is offset by the simultaneous decrease in the amount of potential wave interaction.

A j-wave and an i-wave, with the former crossing the t-time line to the left of the latter, are called *approaching* when either $i < j$, or $i = j$ and at least one of these waves is a compression shock.

[1] If the outgoing k-wave is a fan of k-rarefaction fronts, ε_k denotes the cumulative amplitude and $|\varepsilon_k|$ stands for the cumulative strength of these fronts.

The potential for wave interaction at $t \in (0,T)\backslash I$ will be measured by

$$Q(t) = \sum |\zeta||\xi|, \qquad t \in (0,T)\backslash I, \tag{14.5.7}$$

where the summation runs over all pairs of approaching waves, with strengths, say, $|\zeta|$ and $|\xi|$, which cross the t-time line. In particular,

$$Q(t) \leq \tfrac{1}{2}L(t)^2\,, \qquad t \in (0,T)\backslash I. \tag{14.5.8}$$

Clearly, $Q(\cdot)$ stays constant along time intervals between consecutive collisions. On the other hand, at any $t \in I$ where waves with strength $|\alpha|$ and $|\beta|$ collide, our analysis of wave interactions implies

$$\Delta Q(t) \leq -|\alpha||\beta| + \kappa|\alpha||\beta|L(t-), \qquad t \in I. \tag{14.5.9}$$

In analogy to the Glimm functional (13.4.8), we set

$$G(t) = L(t) + 2\kappa Q(t), \qquad t \in (0,T)\backslash I. \tag{14.5.10}$$

Combining (14.5.10) with (14.5.6) and (14.5.9) yields

$$\Delta G(t) \leq \kappa[2\kappa G(t-) - 1]|\alpha||\beta|, \qquad t \in (0,T)\backslash I. \tag{14.5.11}$$

Assume the total variation of the initial data is so small that $4\kappa L(0+) \leq 1$. Then, on account of (14.5.10) and (14.5.8), $G(0+) \leq 2L(0+) \leq (2\kappa)^{-1}$. This together with (14.5.11) and a simple induction argument yields $\Delta G(t) \leq 0, t \in I$, i.e., $G(\cdot)$ is nonincreasing. Hence

$$L(t) \leq G(t) \leq G(0+) \leq 2L(0+), \qquad t \in (0,T)\backslash I, \tag{14.5.12}$$

which establishes the desired estimate (14.4.6).

Next we estimate the total amount of wave interaction. Since $\kappa L(t-) \leq \frac{1}{2}$, (14.5.9) yields

$$\Delta Q(t) \leq -\tfrac{1}{2}|\alpha||\beta|, \qquad t \in I. \tag{14.5.13}$$

By summing (14.5.13) over all $t \in I$, and upon using (14.5.8),

$$\sum |\alpha||\beta| \leq L(0+)^2\,, \tag{14.5.14}$$

where the summation runs over the set of collisions in $(-\infty,\infty) \times (0,T)$.

Let us now consider any Lipschitz graph $\mathscr{C}$ in $(-\infty,\infty) \times [0,T)$ that is nonresonant relative to U_δ, as defined in Section 14.3. The aim is to estimate the total variation of the trace of U_δ on $\mathscr{C}$, measured by the sum $L_{\mathscr{C}} = \sum |\gamma|$ of the strengths of all waves that impinge on $\mathscr{C}$.

Let J stand for the set of $t \in (0,T)$ where some wave impinges on $\mathscr{C}$. For t in $(0,T)\backslash(I \bigcup J)$ we set

$$M(t) = \sum\nolimits_{-} |\gamma| + \sum\nolimits_{+} |\gamma| + \sum\nolimits_{0} |\gamma|, \tag{14.5.15}$$

where the summation Σ_- (or Σ_+) runs over the i-waves, with $i \in \mathscr{N}_-$ (or $\mathscr{N}_+$), that cross the t-time line on the positive (or negative) side of $\mathscr{C}$; while Σ_0 runs over all i-waves, with $i \in \mathscr{N}_0$, that cross the t-time line on either side of $\mathscr{C}$. Clearly,

(14.5.16) $$\Delta M(t) = -|\gamma|, \qquad t \in J \backslash I,$$

(14.5.17) $$\Delta M(t) \leq \kappa|\alpha||\beta|, \qquad t \in I \backslash J,$$

(14.5.18) $$\Delta M(t) \leq -|\gamma| + \kappa|\alpha||\beta|, \qquad t \in I \cap J,$$

where $|\alpha|$ and $|\beta|$ are the strengths of the waves colliding at $t \in I$ and $|\gamma|$ is the strength of the wave that impinges on $\mathscr{C}$ at $t \in J$. Summing the above inequalities over all $t \in I \bigcup J$ and using (14.5.14) together with $4\kappa L(0+) \leq 1$, we conclude

(14.5.19) $$L_{\mathscr{C}} \leq M(0+) + \kappa \sum |\alpha||\beta| \leq 2L(0+).$$

Another important implication of the boundedness of the amount of wave interaction is that the total number of collisions is finite and bounded, independently of T. Indeed, recall that the Approximate Riemann Solver is employed to resolve collisions only when the product of the strengths of the two incoming fronts exceeds σ. By virtue of (14.5.14), the number of such collisions is bounded by $L(0+)^2/\sigma$. Fronts are generated exclusively by the application of the Approximate Riemann Solver to resolve jump discontinuities of $U_{0\delta}$ or collisions of fronts. Therefore, the number of fronts is bounded. Any two fronts may collide at most once in their lifetime, so the number of collisions between fronts is also bounded. Since all pseudoshocks are generated by collisions of fronts, the number of pseudoshocks is likewise bounded. But then, even the number of collisions between fronts and pseudoshocks must be bounded. To summarize, the total number of collisions is finite, bounded solely in terms of δ, σ, and the number of jump points of $U_{0\delta}$. Consequently, the front tracking algorithm generates U_δ, in finite steps, on the entire upper half-plane. In particular, the estimates (14.4.6) and (14.4.7) will hold for $0 \leq t < \infty$ and $0 \leq \tau < t < \infty$, respectively.

14.6 Bounds on the Combined Strength of Pseudoshocks

The final task for verifying that U_δ is a δ-approximate solution of (14.2.1) is to establish requirement (14.4.4). The notion of generation order was introduced in Section 14.3. Waves of high generation order are produced after a large number of collisions and so it should be expected that their strength is small. Indeed, the first step in our argument is to show that the combined strength of all waves, and thus in particular of all pseudoshocks, of sufficiently high generation order is arbitrarily small. To that end, one refines the analysis of Section 14.5 by sorting out and monitoring the waves separately according to their generation order.

We know by now that the total number of collisions is bounded, and hence the generation order of all waves lies in a finite range, $0 \le \mu \le \nu$. Note, however, that the magnitude of ν depends penultimately on δ, and should be expected to grow without bounds as $\delta \downarrow 0$. For $\mu = 0, 1, \cdots, \nu$ and $t \in [0,\infty)\backslash I$, we let $L_\mu(t)$ denote the sum of the strengths of all waves with generation order $\ge \mu$ that cross the t-time line; and $Q_\mu(t)$ stand for the sum of the products of the strengths of all couples of approaching waves that cross the t-time line and have generation order μ_1, μ_2 with $\max\{\mu_1, \mu_2\} \ge \mu$. Thus, in particular, $L_0(t) = L(t)$ and $Q_0(t) = Q(t)$. Finally, we identify the set I_μ of times $t \in I$ in which a wave of generation order μ collides with a wave of generation order $\le \mu$.

Collisions between waves of generation order $\le \mu - 2$ cannot affect waves of generation order $\ge \mu$, and so

$$\Delta L_\mu(t) = 0, \qquad t \in I_0 \cup \cdots \cup I_{\mu-2}\,. \tag{14.6.1}$$

Any change in $L_\mu(\cdot)$ at $t \in I$ must be induced by the collision of two waves, of which at least one is of generation order $\ge \mu - 1$. These colliding waves, with strengths say $|\alpha|$ and $|\beta|$, are contributing $|\alpha||\beta|$ to $Q_{\mu-1}(t-)$ but nothing to $Q_{\mu-1}(t+)$. As in Section 14.5, the resulting drop in $Q_{\mu-1}(\cdot)$ can be used to offset the potential increment of $L_\mu(\cdot)$, which is bounded by $\kappa|\alpha||\beta|$:

$$\Delta L_\mu(t) + 2\kappa \Delta Q_{\mu-1}(t) \le 0, \qquad t \in I_{\mu-1} \cup \cdots \cup I_\nu\,. \tag{14.6.2}$$

By similar arguments one verifies the inequalities

$$\Delta Q_\mu(t) + 2\kappa \Delta Q(t) L_\mu(t-) \le 0, \qquad t \in I_0 \cup \cdots \cup I_{\mu-2}\,, \tag{14.6.3}$$

$$\Delta Q_\mu(t) + 2\kappa \Delta Q_{\mu-1}(t) L(t-) \le 0, \qquad t \in I_{\mu-1}\,, \tag{14.6.4}$$

$$\Delta Q_\mu(t) \le 0, \qquad t \in I_\mu \cup \cdots \cup I_\nu\,, \tag{14.6.5}$$

which govern the change of $Q_\mu(\cdot)$ across collisions of various orders.

A superscript $+$ or $-$ will be employed below to indicate "positive" or "negative" part: $w^+ = \max\{w, 0\}, w^- = \max\{-w, 0\}$. The aim is to monitor the quantities

$$\hat{L}_\mu = \sup_t L_\mu(t), \qquad \hat{Q}_\mu = \sum_{t \in I} [\Delta Q_\mu(t)]^+\,, \tag{14.6.6}$$

for $\mu = 1, \cdots, \nu$, and show

$$\hat{L}_\mu \le 2^{-\mu} c_3 a\,, \qquad \hat{Q}_\mu \le 2^{-\mu+3} c_3^2 a^2\,, \tag{14.6.7}$$

where a is the bound on $TV_{(-\infty,\infty)} U_0(\cdot)$.

From (14.6.1), (14.6.2) and the "initial condition" $L_\mu(0+) = 0$, $\mu = 1, \cdots, \nu$, it follows that

$$\hat{L}_\mu \le 2\kappa \sum_{t \in I} [\Delta Q_{\mu-1}(t)]^-\,, \qquad \mu = 1, \cdots, \nu. \tag{14.6.8}$$

Next we focus on (14.6.3), (14.6.4) and (14.6.5), with $Q_\mu(0+) = 0$, as " initial condition". Recalling (14.5.8), (14.5.12) and using

$$\sum_{t\in I}[\Delta Q(t)]^- = Q(0+) - Q(\infty) \le \tfrac{1}{2}L(0+)^2\,, \tag{14.6.9}$$

we deduce

$$\hat{Q}_\mu \le \kappa L(0+)^2\hat{L}_\mu + 4\kappa L(0+)\sum_{t\in I}[\Delta Q_{\mu-1}(t)]^-, \qquad \mu = 1,\cdots,\nu. \tag{14.6.10}$$

We combine (14.6.8) with (14.6.10). Assuming the total variation of the initial data is so small that $10\kappa L(0+) \le 1$, we deduce

$$\hat{Q}_\mu \le \frac{1}{2}\sum_{t\in I}[\Delta Q_{\mu-1}(t)]^-\ , \qquad \mu = 1,\cdots,\nu. \tag{14.6.11}$$

In particular, for $\mu = 1$ and on account of (14.6.9), we infer $\hat{Q}_1 \le \frac{1}{4}L(0+)^2$.

We finally notice that, for $\mu = 1,\cdots,\nu$, since $Q_\mu(0+) = 0$,

$$\sum_{t\in I}[\Delta Q_\mu(t)]^- = \sum_{t\in I}[\Delta Q_\mu(t)]^+ - Q_\mu(\infty) \le \hat{Q}_\mu\,. \tag{14.6.12}$$

Therefore, (14.6.11) yields $\hat{Q}_\mu \le \frac{1}{2}\hat{Q}_{\mu-1}$, $\mu = 2,\cdots,\nu$, which in turn implies that $\hat{Q}_\mu \le 2^{-\mu-1}L(0+)^2$. This together with (14.6.9) and (14.6.10) yields the estimate $\hat{L}_\mu \le 2^{-\mu-2}L(0+)$. We have thus established (14.6.7).

It is now clear that one can fix μ_0 sufficiently large so that the combined strength of all waves of generation order $\ge \mu_0$, which is majorized by $\hat{L}_{\mu_0}$, does not exceed $\frac{1}{2}\delta$.

In order to estimate the combined strength of pseudoshocks of generation order $< \mu_0$, the first step is to estimate their number. For $\mu = 0,\cdots,\nu$, let K_μ denote the number of waves of generation order $\le \mu$. A crude upper bound for K_μ may be derived by the following argument. The number of outgoing waves produced by resolving a jump discontinuity, via either of the two Riemann solvers, is bounded by a number b/δ. Thus, $K_0 \le \frac{b}{\delta}N$, where N is the number of jump points of $U_{0\delta}$. Since any two waves may collide at most once in their lifetime, the number of collisions that may generate waves of generation order μ is bounded by $\frac{1}{2}K_{\mu-1}^2$. Therefore,

$$K_\mu \le K_{\mu-1} + \frac{b}{2\delta}K_{\mu-1}^2 \le \frac{b}{\delta}K_{\mu-1}^2\,, \tag{14.6.13}$$

whence one readily deduces

$$K_\mu \le (\frac{b}{\delta})^{2^{\mu+1}}N^{2^\mu}. \tag{14.6.14}$$

Next we estimate the strength of individual pseudoshocks. Any pseudoshock is generated by the collision of two fronts, with strengths $|\alpha|$ and $|\beta|$ such that $|\alpha||\beta| \le \sigma$, which is thus resolved via the Simplified Riemann Solver, as depicted in Figs. 14.2.2 and 14.2.3. It then follows from the corresponding interaction estimate (14.5.4) that the strength of any pseudoshock at birth does not exceed $c_4\sigma$. On account of (14.5.5), the collision of a pseudoshock with a front of strength $|\beta|$, as depicted in Fig. 14.2.4, may increase its strength at most by a factor $1+\kappa|\beta|$. Consequently, the strength of a pseudoshock may ultimately grow at most by the factor $\prod(1+\kappa|\gamma|)$, where the product runs over all fronts with which the pseudoshock collides during its life span. Since pseudoshocks are nonresonant, the estimate (14.5.19) here applies and implies $\sum|\gamma| \leq 2L(0+)$. Assuming $4\kappa L(0+) \leq 1$, we thus conclude that the strength of each pseudoshock, at any time, does not exceed $3c_4\sigma$.

It is now clear that by employing the upper bound for K_{μ_0-1} provided by (14.6.14), and upon selecting σ sufficiently small, one guarantees that the combined strength of pseudoshocks of generation order $< \mu_0$ is bounded by $\frac{1}{2}\delta$. In conjunction with our earlier estimate on the total strength of pseudoshocks of generation order $\geq \mu_0$, this establishes (14.4.4).

14.7 Compactness and Consistency

In this section, the proof of Theorem 14.4.2 will be completed by passing to the $\delta \downarrow 0$ limit. Here we will just be assuming that $\{U_\delta\}$ is any family of δ-approximate solutions, in the sense of Definition 14.4.1, with δ positive and small, that satisfy estimates (14.4.6) and (14.4.7). Thus, we shall not require the special features of the particular δ-approximate solutions constructed via the front tracking algorithm, for instance that shocks propagate with the correct shock speed.

Let us fix any test function ϕ, with compact support in $(-\infty,\infty)\times[0,T)$. By applying Green's theorem,

$$
\begin{aligned}
(14.7.1)\qquad &\int_0^\infty\int_{-\infty}^\infty [\partial_t\phi U_\delta + \partial_x\phi F(U_\delta)]dxdt + \int_{-\infty}^\infty \phi(x,0)U_\delta(x,0)dx \\
&= -\int_0^\infty \sum \phi(y(t),t)\{F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t)) \\
&\qquad - \dot y(t)[U_\delta(y(t)+,t) - U_\delta(y(t)-,t)]\}dt,
\end{aligned}
$$

where for each t the summation runs over all jump discontinuities $x = y(\cdot)$ that cross the t-time line.

When the jump discontinuity $x = y(\cdot)$ is an (approximate) shock, then by virtue of (14.4.1),

$$
\begin{aligned}
(14.7.2)\quad &|F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t)) - \dot y(t)[U_\delta(y(t)+,t) - U_\delta|y(t)-,t)]| \\
&\leq \delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.
\end{aligned}
$$

Similarly, when $x = y(\cdot)$ is an (approximate) rarefaction front, with strength $\leq \delta$, then on account of the proximity between shock and rarefaction wave curves, and (14.4.2),

$$(14.7.3)\quad |F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t)) - \dot{y}(t)[U_\delta(y(t)+,t) - U_\delta(y(t)-,t)]| \le c_5\delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

Finally, when $x = y(\cdot)$ is a pseudoshock,

$$(14.7.4)\quad |F(U_\delta(y(t)+,t)) - F(U_\delta(y(t)-,t))| \le c_6|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

By combining (14.7.2), (14.7.3), (14.7.4) with (14.4.6) and (14.4.4), we deduce that, for any fixed test function ϕ, the right-hand side of (14.7.1) is bounded by $C_\phi[TV_{(-\infty,\infty)}U_0(\cdot)+1]\delta$ and thus tends to zero as $\delta \downarrow 0$.

By virtue of (14.4.6), (14.4.7) and Theorem 1.7.3, any sequence of δ's converging to zero contains a subsequence $\{\delta_k\}$ such that $\{U_{\delta_k}\}$ converges a.e. to some U in BV_{loc}. Passing to the limit in (14.7.1) along the sequence $\{\delta_k\}$, and using (14.4.5), we conclude that U is indeed a weak solution of (13.1.1).

By passing to the $\delta \downarrow 0$ limit in (14.4.6) and (14.4.7), one verifies that U satisfies (13.1.5) and (13.1.6). Furthermore, if $\mathscr{C}$ is any Lipschitz graph that is nonresonant relative to U_δ, for all δ, then, as shown in Section 14.5, the trace of U_δ on $\mathscr{C}$ has bounded variation, uniformly in δ, and thus by passing to the $\delta \downarrow 0$ limit, we deduce that the trace of U on $\mathscr{C}$ will have the same property.

To conclude the proof, assume (η, q) is an entropy-entropy flux pair for the system (14.2.1), with $\eta(U)$ convex. Let ϕ be any nonnegative test function, with compact support in $(-\infty,\infty)\times[0,T)$. By Green's theorem,

$$(14.7.5)\quad \int_0^\infty\int_{-\infty}^\infty [\partial_t\phi\,\eta(U_\delta) + \partial_x\phi\, q(U_\delta)]dxdt + \int_{-\infty}^\infty \phi(x,0)\eta(U_\delta(x,0))dx$$
$$= -\int_0^\infty \sum \phi(y(t),t)\{q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t))$$
$$-\dot{y}(t)[\eta(U_\delta(y(t)+,t)) - \eta(U_\delta(y(t)-,t))]\}dt,$$

where, as in (14.7.1), for each t the summation runs over all jump discontinuities $x = y(\cdot)$ that cross the t-time line.

When $x = y(\cdot)$ is an (approximate) shock, the entropy inequality (8.5.1) together with (14.4.1) imply

(14.7.6)

$$q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot{y}(t)[\eta(U_\delta(y(t)+,t)) - \eta(U_\delta(y(t)-,t))]$$
$$\le c_7\delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

When $x = y(\cdot)$ is an (approximate) rarefaction front, with strength $\le \delta$, Theorem 8.5.1 together with (14.4.2) yield

(14.7.7)

$$|q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot{y}(t)[\eta(U_\delta(y(t)+,t)) - \eta(U_\delta(y(t)-,t))]|$$
$$\le c_8\delta|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

Finally, when $x = y(\cdot)$ is a pseudoshock,

(14.7.8)

$$|q(U_\delta(y(t)+,t)) - q(U_\delta(y(t)-,t)) - \dot{y}(t)[\eta(U_\delta(y(t)+,t) - \eta(U_\delta(y(t)-,t))]|$$
$$\leq c_9|U_\delta(y(t)+,t) - U_\delta(y(t)-,t)|.$$

By combining (14.7.6), (14.7.7), (14.7.8) with (14.4.6) and (14.4.4), we deduce that, for fixed test function ϕ, the right-hand side of (14.7.5) is bounded from below by $-C_\phi[TV_{(-\infty,\infty)}U_0(\cdot)+1]\delta$. Therefore, passing to the limit along the $\{\delta_k\}$ sequence, we conclude that the solution U satisfies the inequality (13.2.17), which expresses the entropy admissibility condition. The proof of Theorem 14.4.2 is now complete.

14.8 Continuous Dependence on Initial Data

The remainder of this chapter will address the issue of uniqueness and stability of solutions to the initial value problem (13.1.1). The existence proofs via Theorems 13.1.1 an 14.4.2, which rely on compactness arguments, offer no clue to that question. We will approach the subject via the approximate solutions generated by the front tracking algorithm. By monitoring the time evolution of a certain functional, we will demonstrate that δ-approximate solutions depend continuously on their initial data, modulo corrections of order δ. This will induce stability for solutions obtained by passing to the $\delta \downarrow 0$ limit.

Our earlier experiences with the scalar conservation law strongly suggest that the L^1 topology should provide the proper setting for continuous dependence. However, the L^1 distance shall not be measured via the standard L^1 metric but through a functional ρ, specially designed for the task at hand.

Let us consider two δ-approximate solutions U and $\bar{U}$ of (14.2.1). Fixing any point (x,t) of continuity for both U and $\bar{U}$, we shall measure the distance between the vectors $U(x,t)$ and $\bar{U}(x,t)$ in the special curvilinear coordinate system whose coordinate curves are the shock curves, with both the admissible and the nonadmissible branches retained. To that end, the vector $\bar{U}(x,t) - U(x,t)$ is represented by curvilinear "coordinates" $p_1(x,t), \cdots, p_n(x,t)$, obtained by means of the following process: One envisages a "virtual" jump discontinuity with left state $U(x,t)$ and right state $\bar{U}(x,t)$, and resolves it into a wave fan composed of $n+1$ constant states joined exclusively by (admissible or nonadmissible) virtual shocks. For $|U(x,t) - \bar{U}(x,t)|$ sufficiently small, this resolution is unique and can be achieved, via the implicit function theorem, by retracing the steps of the admissible solution to the Riemann problem, in Section 9.3, with the wave fan curves Φ_i here replaced by the shock curves W_i. We denote the amplitude of the resulting virtual i-shock by $p_i(x,t)$ and its speed by $s_i(x,t)$. The distance between $U(x,t)$ and $\bar{U}(x,t)$ will now be measured by the suitably weighted sum $\sum g_i(x,t)|p_i(x,t)|$ of the strengths of the n virtual shocks, and accordingly the distance between the two approximate solutions at time t will be measured through the functional

$$\rho(U(\cdot,t),\bar{U}(\cdot,t)) = \sum_{i=1}^{n}\int_{-\infty}^{\infty} g_i(x,t)|p_i(x,t)|dx. \tag{14.8.1}$$

We proceed to introduce suitable weights g_i. Let I and $\bar{I}$ denote the sets of collision times for U and $\bar{U}$, and consider the corresponding potentials for wave interaction $Q(t)$ and $\bar{Q}(t)$, defined through (14.5.7), for $t \in (0,\infty)\backslash I$ and $t \in (0,\infty)\backslash \bar{I}$, respectively. For $t \in (0,\infty)\backslash(I \bigcup \bar{I})$ and any point of continuity x of both $U(\cdot,t)$ and $\bar{U}(\cdot,t)$, we define

$$g_i(x,t) = 1 + \kappa[Q(t) + \bar{Q}(t)] + \nu A_i(x,t), \tag{14.8.2}$$

where κ and ν are sufficiently large positive constants, to be fixed later, and

$$A_i(x,t) = \Sigma_-|\gamma| + \bar{\Sigma}_-|\gamma| + \Sigma_+|\gamma| + \bar{\Sigma}_+|\gamma| + \Sigma_0|\gamma| + \bar{\Sigma}_0|\gamma|. \tag{14.8.3}$$

In (14.8.3), Σ_- (or $\bar{\Sigma}_-$) sums the strengths of all j-fronts of U (or $\bar{U}$), for those $j = i+1,\cdots,n$, that cross the t-time line to the left of the point x; Σ_+ (or $\bar{\Sigma}_+$) sums the strengths of all j-fronts of U (or $\bar{U}$), for $j = 1,\cdots,i-1$, which cross the t-time line to the right of the point x; Σ_0 (or $\bar{\Sigma}_0$) sums the strengths of all i-fronts of U (or $\bar{U}$) that cross the t-time line to the left (or right) of the point x, when $p_i(x,t) < 0$, or to the right (or left) of the point x, when $p_i(x,t) > 0$. Thus, one may justifiably say that $A_i(x,t)$ represents the total strength of the fronts of U and $\bar{U}$ that cross the t-time line and approach the virtual i-shock at (x,t).

Once κ and ν have been fixed, the total variation of the initial data shall be restricted to be so small that $\frac{1}{2} \le g_i(x,t) \le 2$. Then, $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ will be equivalent to the L^1 distance of $U(\cdot,t)$ and $\bar{U}(\cdot,t)$:

$$\begin{aligned}\frac{1}{C}\|U(\cdot,t)-\bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} &\le \rho(U(\cdot,t),\bar{U}(\cdot,t)) \\ &\le C\|U(\cdot,t)-\bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)}.\end{aligned} \tag{14.8.4}$$

It is easily seen that in the scalar case, $n = 1$, the functional ρ introduced by (14.8.1) is closely related to the functional ρ, defined by (11.8.11), when the latter is restricted to step functions.

The aim is to show that $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ is nonincreasing, modulo corrections of order δ:

$$\rho(U(\cdot,t),\bar{U}(\cdot,t)) - \rho(U(\cdot,\tau),\bar{U}(\cdot,\tau)) \le \omega\delta(t-\tau), \qquad 0 < \tau < t < \infty. \tag{14.8.5}$$

Notice that across points of I or $\bar{I}$, $Q(t)$ or $\bar{Q}(t)$ decreases by an amount approximately equal to the product of the strengths of the two colliding waves, while $A_i(x,t)$ may increase at most by a quantity of the same order of magnitude. Therefore, upon fixing κ/ν sufficiently large, $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ will be decreasing across points of I or $\bar{I}$. Between consecutive points of $I \bigcup \bar{I}$, $\rho(U(\cdot,t),\bar{U}(\cdot,t))$ is continuously differentiable; hence to establish (14.8.5) it will suffice to show

$$\frac{d}{dt}\rho(U(\cdot,t),\bar{U}(\cdot,t)) \le \omega\delta. \tag{14.8.6}$$

From (14.8.1),

$$\frac{d}{dt}\rho(U(\cdot,t),\bar{U}(\cdot,t)) = \sum_y \sum_{i=1}^n \{g_i^-|p_i^-| - g_i^+|p_i^+|\}\dot{y}, \tag{14.8.7}$$

where $\sum_y$ runs over all waves $x = y(\cdot)$ of U and $\bar{U}$ that cross the t-time line, and $\dot{y}$, $g_i^\pm$ and $p_i^\pm$ stand for $\dot{y}(t)$, $g_i(y(t)\pm,t)$ and $p_i(y(t)\pm,t)$. By adding and subtracting, appropriately, the speed $s_i^\pm = s_i(y(t)\pm,t)$ of the virtual i-shocks, one may recast (14.8.7) in the form

$$\frac{d}{dt}\rho(U(\cdot,t),\bar{U}(\cdot,t)) = \sum_y \sum_{i=1}^n E_i(y(\cdot),t), \tag{14.8.8}$$

where

$$\begin{aligned} E_i(y(\cdot),t) &= g_i^+(s_i^+ - \dot{y})|p_i^+| - g_i^-(s_i^- - \dot{y})|p_i^-| \\ &= (g_i^+ - g_i^-)(s_i^+ - \dot{y})|p_i^-| + g_i^-(s_i^+ - s_i^-)|p_i^-| + g_i^+(s_i^+ - \dot{y})(|p_i^+| - |p_i^-|). \end{aligned} \tag{14.8.9}$$

Suppose first $x = y(\cdot)$ is a pseudoshock, say of U. Then $g_i^+ = g_i^-$ and (14.8.9) yields

$$\sum_{i=1}^n E_i(y(\cdot),t) \le c_{10}|U(y(t)+,t) - U(y(t)-,t)|. \tag{14.8.10}$$

Thus, by virtue of (14.4.4), the portion of the sum on the right-hand side of (14.8.8) that runs over all pseudoshocks of U is bounded by $c_{10}\delta$. Of course, this applies equally to the portion of the sum that runs over all pseudoshocks of $\bar{U}$.

We now turn to the case $x = y(\cdot)$ is a j-front of U or $\bar{U}$, with amplitude γ. To complete the proof of (14.8.6), one has to show that

$$\sum_{i=1}^n E_i(y(\cdot),t) \le c_{11}\delta|\gamma|. \tag{14.8.11}$$

What follows is a road map to the proof of (14.8.11), which will expose the main ideas and, in particular, will explain why the weight function $g_i(x,t)$ was designed according to (14.8.2). The detailed proof, which is quite laborious, is found in the references cited in Section 14.13.

Let us first examine the three terms on the right-hand side of (14.8.9), for $i \neq j$. By virtue of (14.8.2), $g_i^+ - g_i^-$ equals $\nu|\gamma|$ when $j > i$, or $-\nu|\gamma|$ when $j < i$. In either case, the first term

$$(g_i^+ - g_i^-)(s_i^+ - \dot{y})|p_i^-| \cong -\nu|\lambda_i - \lambda_j||p_i^-||\gamma| \tag{14.8.12}$$

is strongly negative and the idea is that this dominates the other two terms, rendering the desired inequality (14.8.6). Indeed, the second term is majorized by $c_{12}|p_i^-||\gamma|$, which is clearly dominated by (14.8.12), when ν is sufficiently large. One estimates the remaining term by the following argument. The amplitudes $(p_1^-,\cdots,p_n^-)$ or $(p_1^+,\cdots,p_n^+)$ of the virtual shocks result respectively from the resolution of the jump discontinuity between U^- and $\bar{U}^-$ or U^+ and $\bar{U}^+$, where $U^\pm = U(y(t)\pm,t)$ and $\bar{U}^\pm = \bar{U}(y(t)\pm,t)$.

Assuming, for definiteness, that $x = y(\cdot)$ is a front of U, we have $\bar{U}^- = \bar{U}^+$, while the states U^- and U^+ are connected, in state space, by a j-wave curve. Consequently, to leading order, $p_j^+ \cong p_j^- - \gamma$ while, for any $k \neq j$, $p_k^+ \cong p_k^-$. Indeed, a study of the wave curves easily yields the estimate

$$(14.8.13)\quad |p_j^+ - p_j^- + \gamma| + \sum_{k\neq j} |p_k^+ - p_k^-| = O(1)[\delta + |p_j^-|(|p_j^-| + |\gamma|) + \sum_{k\neq j} |p_k^-|]|\gamma|\,,$$

which in turn implies

$$(14.8.14)\qquad E_i(y(\cdot),t) \leq -a\nu|p_i^-||\gamma| + c_{12}[\delta + |p_j^-|(|p_j^-| + |\gamma|) + \sum_{k\neq j} |p_k^-|]|\gamma|,$$

with $a > 0$.

For $i = j$, the estimation of $E_i(y(\cdot),t)$ is more delicate, as the j-front may resonate with the virtual i-shock. The same difficulty naturally arises, and has to be addressed, even for the scalar conservation law. In fact, the scalar case has already been treated, in Section 11.8, albeit under a different guise. For the system, one has to examine separately a number of cases, depending on whether $x = y(\cdot)$ is a shock or a rarefaction front, in conjunction with the signs of p_j^- and p_j^+. The resulting estimates, which vary slightly from case to case but are essentially equivalent, are derived in the references. For example, when either $x = y(\cdot)$ is a j-rarefaction front and $0 < p_j^- < p_j^+$ or $x = y(\cdot)$ is a j-shock and $p_j^+ < p_j^- < 0$,

$$(14.8.15)$$
$$E_j(y(\cdot),t) \leq -b\nu|p_j^-||\gamma|(|p_j^-| + |\gamma|) + c_{13}[\delta + |p_j^-|(|p_j^-| + |\gamma|) + \sum_{k\neq j} |p_k^-|]|\gamma|,$$

where $b > 0$.

We now sum the inequalities (14.8.14), for $i \neq j$, together with the inequality (14.8.15). Upon selecting ν sufficiently large to offset the possibly positive terms, we arrive at (14.8.11). As noted earlier, this implies (14.8.6), which in turn yields (14.8.5). Recalling (14.8.4), we conclude

$$(14.8.16)\qquad \|U(\cdot,t) - \bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} \leq C^2\|U(\cdot,0) - \bar{U}(\cdot,0)\|_{L^1(-\infty,\infty)} + C\omega\delta t,$$

which establishes that δ-approximate solutions depend continuously on their initial data, modulo δ. The implications for actual solutions, obtained as $\delta \downarrow 0$, will be discussed in the following section.

14.9 The Standard Riemann Semigroup

As a corollary of the stability properties of approximate solutions, established in the previous section, it will be shown here that any solution to our system constructed as the $\delta \downarrow 0$ limit of some sequence of δ-approximate solutions is uniquely determined by its initial data and may be identified with a trajectory of an L^1-Lipschitz semigroup, defined on a closed subset of $L^1(-\infty,\infty)$.

The first step in our investigation is to locate the domain of the semigroup. This must be a set that is positively invariant for solutions. Motivated by the analysis in Section 14.5, with any step function $W(\cdot)$, of compact support and small total variation over $(-\infty,\infty)$, we associate a number $H(W(\cdot))$ determined by the following procedure. The jump discontinuities of $W(\cdot)$ are resolved into fans of admissible shocks and rarefaction waves, by solving classical Riemann problems. Before any wave collisions may occur, one measures the total strength L and the potential for wave interaction Q of these outgoing waves and then sets $H(W(\cdot)) = L + 2\kappa Q$, where κ is a sufficiently large positive constant. Suppose a δ-approximate solution U, with initial data W, is constructed by the front tracking algorithm of Section 14.2. By the rules of the construction, all jump discontinuities of W will be resolved via the Approximate Riemann Solver and so, for any $\delta > 0$, $H(W(\cdot))$ will coincide with the initial value $G(0+)$ of the Glimm-type function $G(t)$ defined through (14.5.10). At a later time, as the Simplified Riemann Solver comes into play, $G(t)$ and $H(U(\cdot,t))$ may part from each other. In particular, by contrast to $G(t)$, $H(U(\cdot,t))$ will not necessarily be nonincreasing with t. Nevertheless, when κ is sufficiently large, $H(U(\cdot,t)) \leq H(U(\cdot,t-))$ and $H(U(\cdot,t+)) \leq H(U(\cdot,t-))$. Hence $H(U(\cdot,t)) \leq H(W(\cdot))$ for any $t \geq 0$ and so all sets of step functions $\{W(\cdot) : H(W(\cdot)) < r\}$ are positively invariant for δ-approximate solutions constructed by the front tracking algorithm. Following this preparation, we define the set that will serve as the domain of the semigroup by

$$\mathscr{D} = cl\{\text{step functions } W(\cdot) \text{ with compact support} : H(W(\cdot)) < r\}, \tag{14.9.1}$$

where cl denotes closure in $L^1(-\infty,\infty)$. By virtue of Theorem 1.7.3, the members of $\mathscr{D}$ are functions of bounded variation over $(-\infty,\infty)$, with total variation bounded by cr. The main result is

14.9.1 Theorem. *For r sufficiently small, there is a family of maps $S_t : \mathscr{D} \to \mathscr{D}$*, for $t \in [0,\infty)$, *with the following properties.*

(a) L^1*-Lipschitz continuity on* $\mathscr{D} \times [0,\infty)$: *For any* $W, \bar{W}$ *in* $\mathscr{D}$ *and* t, τ *in* $[0,\infty)$,

$$\|S_t W(\cdot) - S_\tau \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} \leq \kappa\{\|W(\cdot) - \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} + |t-\tau|\}. \tag{14.9.2}$$

(b) $\{S_t : t \in [0,\infty)\}$ *has the semigroup property, namely*

$$S_0 = \text{identity}, \tag{14.9.3}$$

(14.9.4)
$$S_{t+\tau} = S_t S_\tau, \qquad t, \tau \in [0, \infty).$$

(c) *If U is any solution of* (13.1.1), *with initial data $U_0 \in \mathscr{D}$, which is the $\delta \downarrow 0$ limit of some sequence of δ-approximate solutions, then*

(14.9.5)
$$U(\cdot, t) = S_t U_0(\cdot), \qquad t \in [0, \infty).$$

Proof. Let U and $\bar{U}$ be two solutions of (13.1.1), with initial data U_0 and $\bar{U}_0$, which are $\delta \downarrow 0$ limits of sequences of δ-approximate solutions $\{U_{\delta_n}\}$ and $\{\bar{U}_{\bar{\delta}_n}\}$, respectively. No assumption is made that these approximate solutions have necessarily been constructed by the front tracking algorithm. So long as the total variation is sufficiently small to meet the requirements of Section 14.8, we may apply (14.8.16) to get

(14.9.6)
$$\|U_{\delta_n}(\cdot, t) - \bar{U}_{\bar{\delta}_n}(\cdot, t)\|_{L^1(-\infty,\infty)}$$
$$\leq C^2 \|U_{\delta_n}(\cdot, 0) - \bar{U}_{\bar{\delta}_n}(\cdot, 0)\|_{L^1(-\infty,\infty)} + C\omega \max\{\delta_n, \bar{\delta}_n\} t.$$

Passing to the limit, $n \to \infty$, we deduce

(14.9.7)
$$\|U(\cdot, t) - \bar{U}(\cdot, t)\|_{L^1(-\infty,\infty)} \leq C^2 \|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^1(-\infty,\infty)}.$$

When r is sufficiently small, Theorem 14.4.2 asserts that for any $U_0 \in \mathscr{D}$ one can generate solutions U of (13.1.1) as limits of sequences $\{U_{\delta_n}\}$ of δ-approximate solutions constructed by the front tracking algorithm. Moreover, the initial values of U_δ may be selected so that $H(U_\delta(\cdot, 0)) < r$, in which case, as noted above, $H(U_\delta(\cdot, t)) < r$ and thereby $U(\cdot, t) \in \mathscr{D}$, for any $t \in [0, \infty)$. By virtue of (14.9.7), all these solutions must coincide so that U is uniquely defined. In fact, (14.9.7) further implies that U must even coincide with any solution, with initial value U_0, that is derived as the $\delta \downarrow 0$ limit of any sequence of δ-approximate solutions, regardless of whether they were constructed by the front tracking algorithm.

Once U has thus been identified, we define S_t through (14.9.5). The Lipschitz continuity property (14.9.2) follows by combining (14.9.7) with (13.1.6), and (14.9.3) is obvious. To verify (14.9.4), it suffices to notice that for any fixed $\tau > 0$, $U(\cdot, \tau + \cdot)$ is a solution of (13.1.1), with initial data $U(\cdot, \tau)$, which is derived as the $\delta \downarrow 0$ limit of δ-approximate solutions and thus, by uniqueness, must coincide with $S_t U(\cdot, \tau)$. The proof is complete.

The term Standard Riemann Semigroup is commonly used for S_t, as a reminder that its building block is the solution of the Riemann problem. The question of whether this semigroup also encompasses solutions derived via alternative methods will be addressed in the next section.

14.10 Uniqueness of Solutions

Uniqueness for the Cauchy problem (13.1.1) shall be established here by demonstrating that any solution in a reasonable function class can be identified with the trajectory of the Standard Riemann Semigroup that emanates from the initial data. As shown in Section 14.9, this is indeed the case for solutions constructed by front tracking.

For fair comparison one should, at the outset, limit the investigation to solutions U for which $U(\cdot,t)$ resides in the domain $\mathscr{D}$ of the Standard Riemann Semigroup, defined through (14.9.1). As noted earlier, this implies, in particular, that $U(\cdot,t)$ has bounded variation over $(-\infty,\infty)$:

$$TV_{(-\infty,\infty)}U(\cdot,t) \leq cr. \tag{14.10.1}$$

It then follows from Theorem 4.3.1 that $t \mapsto U(\cdot,t)$ is L^1-Lipschitz,

$$\int_{-\infty}^{\infty} |U(x,t) - U(x,\tau)|dx \leq c'r|t-\tau|, \qquad 0 \leq \tau < t < \infty, \tag{14.10.2}$$

and U is in BV_{loc} on $(-\infty,\infty)\times[0,\infty)$. Hence, as pointed out in Section 10.1, there is $\mathscr{N} \subset [0,\infty)$, of measure zero, such that any (x,t) with $t \notin \mathscr{N}$ and $U(x-,t) = U(x+,t)$ is a point of approximate continuity of U while any (x,t) with $t \notin \mathscr{N}$ and $U(x-,t) \neq U(x+,t)$ is a point of approximate jump discontinuity of U, with one-sided approximate limits $U_{\pm} = U(x\pm,t)$ and associated shock speeds determined through the Rankine-Hugoniot jump condition (8.1.2).

It is presently unknown whether uniqueness prevails within the above class of solutions. Accordingly, one should endow solutions with additional structure. Here we will experiment with the

14.10.1 Tame Oscillation Condition: There are positive constants λ and β such that

$$|U(x\pm,t+h) - U(x\pm,t)| \leq \beta\, TV_{(x-\lambda h,x+\lambda h)}U(\cdot,t), \tag{14.10.3}$$

for all $x \in (-\infty,\infty)$, $t \in [0,\infty)$ and any $h > 0$.

Clearly, solutions constructed by either the random choice method or the front tracking algorithm satisfy this condition, and so also do the solutions to systems of two conservation laws considered in Chapter XII.

The Tame Oscillation Condition induces uniqueness:

14.10.2 Theorem. *Any BV solution U of the Cauchy problem* (13.1.1), *with $U(\cdot,t)$ in $\mathscr{D}$ for all $t \in [0,\infty)$, that satisfies the Lax E-condition, at any point of approximate jump discontinuity, together with the Tame Oscillation Condition* (14.10.3), *coincides with the trajectory of the Standard Riemann Semigroup S_t, emanating from the initial data:*

$$U(\cdot,t) = S_t U_0(\cdot), \qquad t \in [0,\infty). \tag{14.10.4}$$

In particular, U is uniquely determined by its initial data.

Proof. The demonstration will be quite lengthy. The first step is to show that at every $\tau \notin \mathscr{N}$, $U(\cdot,t)$ is tangential to the trajectory of S_t emanating from $U(\cdot,\tau)$:

$$\limsup_{h\downarrow 0}\frac{1}{h}\|U(\cdot,\tau+h)-S_hU(\cdot,\tau)\|_{L^1(-\infty,\infty)}=0. \tag{14.10.5}$$

Then we shall verify that (14.10.5), in turn, implies (14.10.4).

Fixing $\tau\notin\mathscr{N}$, we will establish (14.10.5) by the following procedure. For any fixed bounded interval $[a,b]$ and $\varepsilon>0$, arbitrarily small, we will construct some function U^* on a rectangle $[a,b]\times[\tau,\tau+\delta]$ such that

$$\limsup_{h\downarrow 0}\frac{1}{h}\|U(\cdot,\tau+h)-U^*(\cdot,h)\|_{L^1(a,b)}\le c_{14}r\varepsilon, \tag{14.10.6}$$

$$\limsup_{h\downarrow 0}\frac{1}{h}\|S_hU(\cdot,\tau)-U^*(\cdot,h)\|_{L^1(a,b)}\le c_{14}r\varepsilon. \tag{14.10.7}$$

Naturally, such a U^* will provide a local approximation to the solution of (13.1.1) with initial data $U_0(\cdot)=U(\cdot,\tau)$, and will be constructed accordingly by patching together local approximate solutions of two types, one fit for points of strong jump discontinuity, the other suitable for regions with small local oscillation.

We begin by fixing λ which is larger than the absolute value of all characteristic speeds and also sufficiently large for the Tame Oscillation Condition (14.10.3) to apply.

With any point (y,τ) of jump discontinuity for U, with limits $U_\pm=U(y\pm,\tau)$ and shock speed s, we associate the sector $\mathscr{K}=\{(x,\sigma):\sigma>0,\ |x-y|\le\lambda\sigma\}$, on which we consider the solution $U^\sharp=U^\sharp_{(y,\tau)}$ defined by

$$U^\sharp(x,\sigma)=\begin{cases}U_-\,, & \text{for } x<y+s\sigma\\[2ex] U_+\,, & \text{for } x>y+s\sigma.\end{cases} \tag{14.10.8}$$

We prove that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{y-\lambda h}^{y+\lambda h}|U(x,\tau+h)-U^\sharp(x,h)|dx=0. \tag{14.10.9}$$

Indeed, for $0\le\sigma\le h$, let us set

$$\phi_h(\sigma)=\frac{1}{h}\int_{y-\lambda h}^{y+\lambda h}|U(x,\tau+\sigma)-U^\sharp(x,\sigma)|dx. \tag{14.10.10}$$

Suppose $\phi_h(h)>0$. Since $\sigma\mapsto U(\cdot,\tau+\sigma)-U^\sharp(\cdot,\sigma)$ is L^1-Lipschitz, with constant γ, we infer that, for $h<<1$, $\phi_h(h)<2\gamma$ and $\phi_h(\sigma)\ge\frac{1}{2}\phi_h(h)$, for any σ with $h-\sigma\le\frac{h}{2\gamma}\phi_h(h)$. Then

(14.10.11)

$$\frac{1}{h^2}\int_0^h\int_{y-\lambda h}^{y+\lambda h}|U(x,\tau+\sigma)-U^\sharp(x,\sigma)|dxd\sigma=\frac{1}{h}\int_0^h\phi_h(\sigma)d\sigma\ge\frac{1}{4\gamma}\phi_h^2(h).$$

As $h \downarrow 0$, the left-hand side of (14.10.11) tends to zero, by virtue of Theorem 1.7.4, and this verifies (14.10.9).

Next we fix any interval (ζ, ξ), with midpoint say z. On the triangular domain $\mathscr{T} = \{(x,\sigma) : \sigma > 0,\, \zeta + \lambda\sigma < x < \xi - \lambda\sigma\}$, we construct the solution $U^\flat = U^\flat_{(z,\tau)}$ of the linear Cauchy problem

$$\partial_t U^\flat + A^\flat \partial_x U^\flat = 0, \tag{14.10.12}$$

$$U^\flat(x,0) = U(x,\tau), \tag{14.10.13}$$

where $A^\flat$ is the constant matrix $\mathrm{D}F(U(z,\tau))$. The aim is to establish the estimate

$$\begin{aligned} &\int_{\zeta+\lambda h}^{\xi-\lambda h} |U(x,\tau+h) - U^\flat(x,h)|dx \\ &\qquad \le c_{15}[TV_{(\zeta,\xi)}U(\cdot,\tau)] \int_0^h TV_{(\zeta+\lambda\sigma,\xi-\lambda\sigma)}U(\cdot,\tau+\sigma)d\sigma. \end{aligned} \tag{14.10.14}$$

Integrating (14.10.12) along characteristic directions and using (14.10.13) yields

$$L_i^\flat U^\flat(x,h) = L_i^\flat U(x - \lambda_i^\flat h, \tau), \qquad i = 1,\cdots,n, \tag{14.10.15}$$

where $L_i^\flat = L_i(U(z,\tau))$ is a left eigenvector of $A^\flat$ associated with the eigenvalue $\lambda_i^\flat = \lambda_i(U(z,\tau))$. For fixed i, we may assume without loss of generality that $\lambda_i^\flat = 0$, since we may change variables $x \mapsto x - \lambda_i^\flat t$, $F(U) \mapsto F(U) - \lambda_i^\flat U$. In that case, since U satisfies (14.2.1) in the sense of distributions,

$$\begin{aligned} &\int_{\zeta+\lambda h}^{\xi-\lambda h} \phi(x) L_i^\flat [U(x,\tau+h) - U^\flat(x,h)]dx \\ &\qquad = \int_{\zeta+\lambda h}^{\xi-\lambda h} \phi(x) L_i^\flat [U(x,\tau+h) - U(x,\tau)]dx \\ &\qquad = \int_0^h \int_{\zeta+\lambda h}^{\xi-\lambda h} \partial_x\phi(x) L_i^\flat F(U(x,\tau+\sigma))dxd\sigma, \end{aligned} \tag{14.10.16}$$

for any test function $\phi \in C_0^\infty(\zeta+\lambda h, \xi-\lambda h)$. Taking the supremum over all such ϕ, with $|\phi(x)| \le 1$, yields

(14.10.17)

$$\int_{\zeta+\lambda h}^{\xi-\lambda h} |L_i^\flat [U(x,\tau+h) - U^\flat(x,h)]|dx \le \int_0^h TV_{(\zeta+\lambda h,\xi-\lambda h)} L_i^\flat F(U(\cdot,\tau+\sigma))d\sigma.$$

Given $\zeta + \lambda h < x < y < \xi - \lambda h$, let us set, for brevity, $V = U(x,\tau+\sigma)$ and $W = U(y,\tau+\sigma)$. Recalling the notation (8.1.4), one may write

(14.10.18)

$$F(V)-F(W)=A(V,W)(V-W)=A^\flat(V-W)+[A(V,W)-A^\flat](V-W).$$

We now note that $L_i^\flat A^\flat = 0$. Furthermore, $A(V,W)-A^\flat$ is bounded in terms of the oscillation of U inside the triangle $\mathscr{T}$, which is in turn bounded in terms of the total variation of $U(\cdot,\tau)$ over (ζ,ξ), by virtue of the Tame Oscillation Condition (14.10.3). Therefore, (14.10.17) yields the estimate

(14.10.19)
$$\int_{\zeta+\lambda h}^{\xi-\lambda h} |L_i^\flat[U(x,\tau+h)-U^\flat(x,h)]|dx$$
$$\leq c_{16}[TV_{(\zeta,\xi)}U(\cdot,\tau)]\int_0^h TV_{(\zeta+\lambda\sigma,\xi-\lambda\sigma)}U(\cdot,\tau+\sigma)d\sigma.$$

Since (14.10.19) holds for $i=1,\cdots,n$, (14.10.14) readily follows.

We have now laid the groundwork for synthesizing a function U^* that satisfies (14.10.6). We begin by identifying a finite collection of open intervals $(\zeta_j\,,\,\xi_j)$, for $j=1,\cdots,J$, with the following properties:

(i) $[a,b]\subset \bigcup_{j=1}^{J}[\zeta_j\,,\,\xi_j]$.

(ii) The intersection of any three of these intervals is empty.

(iii) $TV_{(\zeta_j,\xi_j)}U(\cdot,\tau)<\varepsilon$, for $j=1,\cdots,J$.

With each $(\zeta_j\,,\,\xi_j)$, we associate, as above, the triangle $\mathscr{T}_j$ and the approximate solution $U^\flat_{(z_j,\tau)}$ relative to the midpoint z_j. We also consider $[a,b]\backslash\bigcup_{j=1}^{J}(\zeta_j\,,\,\xi_j)$, which is a finite set $\{y_1,\cdots,y_K\}$ containing the points where strong shocks cross the τ-time line between a and b. With each y_k we associate the sector $\mathscr{K}_k$ and the corresponding approximate solution $U^\sharp_{(y_k,\tau)}$ (see Fig. 14.10.1). We then set

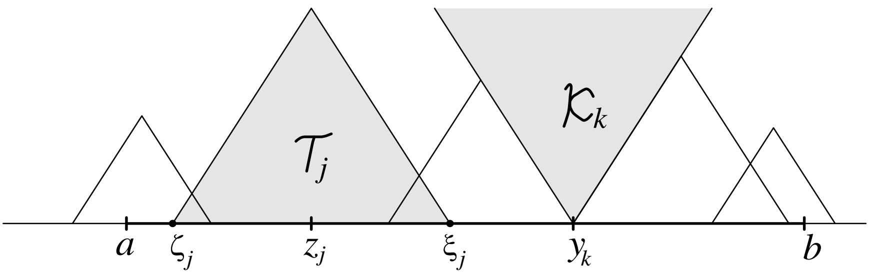

Fig. 14.10.1

$$U^*(x,h) = \begin{cases} U^{\sharp}_{(y_k,\tau)}(x,h), & \text{for } (x,h) \in \mathscr{K}_k \setminus \bigcup_{\ell=1}^{k-1} \mathscr{K}_\ell \\ \\ U^{\flat}_{(z_j,\tau)}(x,h), & \text{for } (x,h) \in \mathscr{T}_j \setminus \bigcup_{\ell=1}^{j-1} \mathscr{T}_\ell . \end{cases} \tag{14.10.20}$$

Clearly, for h sufficiently small, $U^*(\cdot,h)$ is defined for all $x \in [a,b]$ and

$$\begin{aligned} &\int_a^b |U(x,\tau+h) - U^*(x,h)|dx \\ &\qquad \leq \sum_{k=1}^{K} \int_{y_k-\lambda h}^{y_k+\lambda h} |U(x,\tau+h) - U^{\sharp}_{(y_k,\tau)}(x,h)|dx \\ &\qquad\quad + \sum_{j=1}^{J} \int_{\zeta_j+\lambda h}^{\xi_j-\lambda h} |U(x,\tau+h) - U^{\flat}_{(z_j,\tau)}(x,h)|dx. \end{aligned} \tag{14.10.21}$$

Upon combining (14.10.21), (14.10.9), (14.10.14), and (14.10.1), we arrive at (14.10.6), with $c_{14} = 2cc_{15}$.

We now note that $S_{t-\tau}U(\cdot,\tau)$ defines, for $t \geq \tau$, another solution of (14.2.1) which has the same properties, complies with the same bounds, and has identical restriction to $t = \tau$ with U. Therefore, this solution must equally satisfy the analog of (14.10.6), namely (14.10.7). Finally, (14.10.6) and (14.10.7) together yield (14.10.5).

It remains to show that (14.10.5) implies (14.10.4). To that end, we fix $t > 0$ and any, arbitrarily small, $\varepsilon > 0$. By virtue of (14.10.5) and the Vitali covering theorem, there is a finite collection of pairwise disjoint closed subintervals $[\tau_k, \tau_k + h_k]$, $k = 1,\cdots,K$, of $[0,t]$, with $0 \leq \tau_1 < \cdots < \tau_K < t$, such that $\tau_k \notin \mathscr{N}$ and

$$0 \leq t - \sum_{k=1}^{K} h_k < \varepsilon, \tag{14.10.22}$$

$$\|U(\cdot,\tau_k+h_k) - S_{h_k}U(\cdot,\tau_k)\|_{L^1(-\infty,\infty)} < \varepsilon h_k, \qquad k = 1,\cdots,K. \tag{14.10.23}$$

By the triangle inequality,

$$\begin{aligned} &\|U(\cdot,t) - S_t U_0(\cdot)\|_{L^1(-\infty,\infty)} \\ &\qquad \leq \sum_{k=0}^{K} \|S_{t-\tau_{k+1}}U(\cdot,\tau_{k+1}) - S_{t-\tau_k-h_k}U(\cdot,\tau_k+h_k)\|_{L^1(-\infty,\infty)} \\ &\qquad\quad + \sum_{k=1}^{K} \|S_{t-\tau_k-h_k}U(\cdot,\tau_k+h_k) - S_{t-\tau_k}U(\cdot,\tau_k)\|_{L^1(-\infty,\infty)}. \end{aligned} \tag{14.10.24}$$

In the first summation on the right-hand side of (14.10.24), $\tau_0 + h_0$ is to be interpreted as 0, and τ_{K+1} is to be interpreted as t. The general term in this summation is bounded by $\kappa(1+c'r)(\tau_{k+1}-\tau_k-h_k)$, on account of (14.9.2) and (14.10.2). Hence the first sum is bounded by $\kappa(1+c'r)\varepsilon$, because of (14.10.22). Turning now to the second summation, since $S_{t-\tau_k}=S_{t-\tau_k-h_k}S_{h_k}$,

$$\|S_{t-\tau_k-h_k}U(\cdot,\tau_k+h_k)-S_{t-\tau_k}U(\cdot,\tau_k)\|_{L^1(-\infty,\infty)}\leq\kappa\varepsilon h_k\,, \tag{14.10.25}$$

by virtue of (14.9.2) and (14.10.23). Therefore, the second sum is bounded by $\kappa t\varepsilon$. Thus the right-hand side of (14.10.24) can be made arbitrarily small and this establishes (14.10.4). The proof is complete.

14.11 Continuous Glimm Functionals, Spreading of Rarefaction Waves, and Structure of Solutions

In earlier chapters we studied in great detail the structure of BV solutions for scalar conservation laws as well as for systems of two conservation laws. The front tracking method, by its simplicity and explicitness, provides an appropriate vehicle for extending the investigation to genuinely nonlinear systems of arbitrary size. The aim of the study is to determine what features of piecewise constant solutions are inherited by the BV solutions that are generated via the limit process. In addition to providing a fairly detailed picture of local structure and regularity, this approach exposes various stability characteristics of solutions and elucidates the issue of structural stability of the wave pattern. A sample of results will be stated below, without proofs. The reader may find a detailed exposition in the literature cited in Section 14.13.

The first step towards developing a qualitative theory is to realize within the framework of BV solutions the key functionals that measure total wave strength and wave interaction potential, which were introduced earlier in the context of piecewise constant approximate solutions generated by front tracking. This will be effected by the following procedure.

Let V be a function of bounded variation on $(-\infty,\infty)$ taking values in $\mathbb{R}^n$, and normalized by $V(x)=\frac{1}{2}[V(x-)+V(x+)]$. The distributional derivative ∂_xV induces a signed vector-valued measure μ on $(-\infty,\infty)$, with continuous part μ^c and atomic part μ^a. We represent μ by means of its "projections" μ_i, $i=1,\dots,n$, on the characteristic directions, defined as follows.

The continuous part μ_i^c of μ_i is the Radon measure defined through

$$\int_{-\infty}^{\infty}\varphi(x)d\mu_i^c(x)=\int_{-\infty}^{\infty}\varphi(x)L_i(V(x))d\mu^c(x), \tag{14.11.1}$$

for all continuous functions φ with compact support on $(-\infty,\infty)$.

The atomic part μ_i^a of μ_i is concentrated on the countable set of points of jump discontinuity of V. If x is such a point, we set $\mu_i^a(x)=\varepsilon_i$, where ε_i is the amplitude of the i-wave in the wave fan that solves the Riemann problem (9.1.12) with

$U_L = V(x-)$, $U_R = V(x+)$. We have $\varepsilon_i = L_i(V(x))\big[U_R - U_L\big] + O(1)|U_R - U_L|^2$. Therefore, the measure $\mu_i = \mu_i^c + \mu_i^a$ can be characterized through

$$\int_{-\infty}^{\infty} \varphi(x) d\mu_i(x) = \int_{-\infty}^{\infty} \varphi(x) \tilde{L}_i(x) d\mu(x), \tag{14.11.2}$$

where $\tilde{L}_i(x) = L_i(V(x)) + O(1)|V(x+) - V(x-)|$.

We introduce the positive part μ_i^+ and the negative part μ_i^- of the measure μ_i, so that $\mu_i = \mu_i^+ - \mu_i^-$, $|\mu_i| = \mu_i^+ + \mu_i^-$; and we define the functionals

$$\mathscr{L}[V] = \sum_{i=1}^{n} |\mu_i|(\mathbb{R}), \tag{14.11.3}$$

(14.11.4)

$$\mathscr{Q}[V] = \sum_{i<j} (|\mu_j| \times |\mu_i|)(\{(x,y) : x < y\}) + \sum_{i \in GN} (\mu_i^- \times |\mu_i|)(\{(x,y) : x \neq y\}),$$

$$\mathscr{G}[V] = \mathscr{L}[V] + 2\kappa \mathscr{Q}[V], \tag{14.11.5}$$

where GN denotes the collection of genuinely nonlinear characteristic families of (14.2.1) and κ is a positive constant to be specified below. These functionals enjoy the following useful semicontinuity property:

14.11.1 Lemma. *For $\kappa > 0$, sufficiently large, and $r > 0$, sufficiently small, the functionals $\mathscr{Q}$ and $\mathscr{G}$ are lower semicontinuous on the set*

$$\mathscr{D} = \{V \in L^1(\mathbb{R};\mathbb{R}^n) : \mathscr{G}[V] \leq r\}, \tag{14.11.6}$$

equipped with the topology of L^1.

It should be noted that even though $\mathscr{L}[V]$ is equivalent to the total variation of $V(\cdot)$, $\mathscr{L}$ is not necessarily lower semicontinuous on $\mathscr{D}$, and that $\mathscr{G}$ may fail to be lower semicontinuous if r in (14.11.6) is large.

When U is the solution of a Cauchy problem for (14.2.1), constructed by the front tracking algorithm, we identify its restriction $U(\cdot,t)$, to some fixed time t, with the function $V(\cdot)$, above. In that case, the measure μ_i encodes the i-waves crossing the t-time line, and in particular μ_i^+ represents the i-rarefaction waves while μ_i^- represents the i-compression waves, including the i-shocks. Accordingly, this μ_i shall be dubbed the *i-wave measure at time t*. Moreover, $\mathscr{L}[U(\cdot,t)]$ and $\mathscr{Q}[U(\cdot,t)]$ respectively measure the total strength and interaction potential of all waves crossing the t-time line. In the particular situation where $U(\cdot,t)$ is piecewise constant on $(-\infty,\infty)$, $\mathscr{L}[U(\cdot,t)]$, $\mathscr{Q}[U(\cdot,t)]$ and $\mathscr{G}[U(\cdot,t)]$ reduce to $L(t)$, $Q(t)$ and $G(t)$ defined by (14.5.1), (14.5.7) and (14.5.10).

One may derive qualitative properties of solutions U by first identifying them in the context of piecewise constant approximate solutions generated by the front tracking algorithm and then passing to the limit, taking advantage of the lower semicontinuity property of $\mathscr{Q}$ and $\mathscr{G}$ asserted by Lemma 14.11.1. In that direction, the following proposition establishes the spreading of rarefaction waves, extending to genuinely nonlinear systems of n conservation laws what has already been demonstrated for scalar conservation laws and for systems of two conservation laws, in Sections 11.2 and 12.6.

14.11.2 Theorem. *With each genuinely nonlinear i-th characteristic family of the system* (14.2.1) *are associated positive numbers c and C with the following property. Let U be the solution of the Cauchy problem for* (14.2.1), *with initial data* U_0, *constructed by the front tracking algorithm. Fix any* $t > 0$ *and consider the i-wave measure* μ_i *at time t. Then*

$$\mu_i^+(a,b) \leq c\,\frac{b-a}{t} + C\{\mathscr{Q}[U_0(\cdot)] - \mathscr{Q}[U(\cdot,t)]\} \tag{14.11.7}$$

holds for any interval $(a,b) \subset (-\infty,\infty)$.

In the proof, one employs the notion of generalized characteristics, introduced in Chapter X, in order to establish the corresponding estimate in the context of the piecewise constant approximate solutions that generate U, and then passes to the limit.

The next proposition describes the local structure of BV solutions. It should be compared to Theorem 12.3.3, for systems of two conservation laws.

14.11.3 Theorem. *Let U be the solution of a Cauchy problem for* (14.2.1), *constructed through the front tracking algorithm. Fix any point* $(\bar{x},\bar{t})$ *on the open upper half-plane and consider the rescaled function*

$$U_\alpha(x,t) = U(\bar{x}+\alpha x, \bar{t}+\alpha t), \qquad \alpha > 0. \tag{14.11.8}$$

Then, for any $t \in (-\infty,\infty)$, *as* $\alpha \downarrow 0$, $U_\alpha(\cdot,t)$ *converges in* L^1_{loc} *to* $\bar{U}(\cdot,t)$, *where* $\bar{U}$ *is a self-similar solution of* (14.2.1). *On the upper half-plane,* $t \geq 0$, $\bar{U}$ *coincides with the admissible solution of the Riemann problem* (9.1.1),(9.1.12), *with end-states* $U_L = U(\bar{x}-,\bar{t})$, $U_R = U(\bar{x}+,\bar{t})$. *On the lower half-plane,* $t < 0$, $\bar{U}$ *contains only admissible shocks and/or centered compression waves. Furthermore, as* $\alpha \downarrow 0$, *the i-wave measures* $\mu_i^\pm$ *for* $U_\alpha(\cdot,t)$ *converge, in the weak topology of measures, to the corresponding i-wave measures* $\bar{\mu}_i^\pm$ *for* $U(\cdot,\bar{t})$.

The final proposition of this section provides a description of the global wave pattern, showing that admissible BV solutions are more regular than general BV functions. This should also be compared with the corresponding properties of solutions to scalar conservation laws and to systems of two conservation laws expounded in Sections 11.3 and 12.7.

14.11.4 Theorem. *Let U be the solution to a Cauchy problem for* (14.2.1), *constructed through the front tracking algorithm. Then the upper half-plane is partitioned into the union $\mathscr{C} \cup \mathscr{J} \cup \mathscr{I}$ of three subsets with the following properties:*

(a) *Any $(\bar{x},\bar{t}) \in \mathscr{C}$ is a point of continuity of U.*
(b) *$\mathscr{I}$ is (at most) countable.*
(c) *$\mathscr{J}$ is the countable union of Lipschitz arcs $\{(x,t) : t \in (a_m,b_m), x = y_m(t)\}$,* for $m = 1,2,\ldots$. *When $\bar{x} = y_m(\bar{t})$ and $(\bar{x},\bar{t}) \notin \mathscr{I}$*, then $(\bar{x},\bar{t})$ *is a point of continuity of U relative to $\{(x,t) : t \in (a_m,b_m), x < y_m(t)\}$ and also relative to $\{(x,t) : t \in (a_m,b_m), x > y_m(t)\}$, with distinct corresponding limits U_- and U_+. Furthermore, $y_m(\cdot)$ is differentiable at $\bar{t}$, with derivative $s = \dot{y}_m(\bar{t})$, and U_- ,U_+ and s satisfy the Rankine-Hugoniot jump condition* (8.1.2).

The proof of the above two theorems again proceeds by examining the structure of piecewise constant approximate solutions that generate U, in terms of their wave measures, and then passing to the limit.

14.12 Stability of Strong Waves

The example of blowing up of solutions exhibited in Section 9.10 demonstrates the futility of seeking a global existence theorem for solutions to the Cauchy problem in the general class of systems considered in this chapter, under arbitrary initial data with large total variation. This raises the issue of identifying the special class of systems for which solutions with large initial data exist, and the hope that the systems of importance in continuum physics will turn out to be members. The first test for admission to membership in the above class should be that particular solutions containing waves of large amplitude, which may be explicitly known, are stable under small perturbations of their initial values. This has been achieved for the case of self-similar solutions to genuinely nonlinear systems, with strong shocks and/or strong rarefaction waves:

14.12.1 Theorem. *Consider the strictly hyperbolic system of conservation laws* (14.2.1) *with characteristic families that are either genuinely nonlinear or linearly degenerate. Assume $\bar{U}(x,t) = V(x/t)$ is a self-similar solution, with strong compressive shocks, contact discontinuities and/or rarefaction waves, that satisfies an appropriate stability condition. For $\delta > 0$, define*

(14.12.1)

$$\mathscr{D}_\delta = \{W \in C(\mathbb{R};\mathbb{R}^n) : \|W(\varphi(\cdot)) - V(\cdot)\|_{L^\infty(-\infty,\infty)} + TV_{(-\infty,\infty)}[W(\varphi(\cdot)) - V(\cdot)] < \delta,$$

$$for\, some\, increasing\, C^1(\mathbb{R})\, function\, \varphi.\}$$

Then there exists a closed set $\mathscr{D}$ in $L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, containing $\mathscr{D}_\delta$ for δ sufficiently small, together with a family of maps $S_t : \mathscr{D} \to \mathscr{D}, t \in [0,\infty)$, having the following properties.

(a) L^1-Lipschitz continuity on $\mathscr{D} \times [0,\infty)$: For any $W, \bar{W}$ in $\mathscr{D}$ and t, τ in $[0,\infty)$,

$$\|S_t W(\cdot) - S_\tau \overline{W}(\cdot)\|_{L^1(-\infty,\infty)} \leq \kappa\{\|W(\cdot) - \bar{W}(\cdot)\|_{L^1(-\infty,\infty)} + |t-\tau|\}. \tag{14.12.2}$$

(b) $\{S_t : t \in [0,\infty)\}$ has the semigroup property, namely

$$S_0 = \text{identity}, \tag{14.12.3}$$

$$S_{t+\tau} = S_t S_\tau\,, \qquad t,\tau \in [0,\infty). \tag{14.12.4}$$

(c) For any $U_0 \in \mathscr{D}$, $U(\cdot,t) = S_t U_0(\cdot)$ is an admissible solution of the system (14.2.1) *with initial value U_0.*

In establishing the above proposition, a major issue is the formulation of the appropriate stability condition on the self-similar solution V. Such a condition must ensure (a) that each elementary wave in the wave fan V, whether compressive shock, contact discontinuity or rarefaction, is individually stable; and (b) that the collision of weak waves with the strong waves of V does not generate resonance that may lead to the breakdown of solutions exhibited in Section 9.10. Alternative, albeit equivalent, versions of stability conditions are recorded in the literature cited in Section 14.13, motivated either from analysis of wave interactions or through linearization of (14.2.1) about $V(x/t)$. Unfortunately, the statement of these conditions is complicated, technical and opaque.

To get a taste, let us consider the relatively simple special case where the wave fan $V(\cdot)$ comprises $m+1$ constant states $V_0, \dots, V_m$ connected by m compressive shocks belonging to characteristic families $i_1 < \cdots < i_m$ and propagating with speeds $s_1 < \cdots < s_m$. Each one of these shocks will be individually stable provided that the conditions (8.3.5) and (8.3.6), introduced in Section 8.3, hold, namely, for $\ell = 1, \cdots, m$,

$$\lambda_{i_{\ell-1}}(V_{\ell-1}) < s_\ell < \lambda_{i_\ell}(V_{\ell-1}), \tag{14.12.5}$$

$$\lambda_{i_\ell}(V_\ell) < s_\ell < \lambda_{i_\ell+1}(V_\ell), \tag{14.12.6}$$

$$\det[R_1(V_{\ell-1}), \dots, R_{i_\ell-1}(V_{\ell-1}), V_\ell - V_{\ell-1}, R_{i_\ell+1}(V_\ell), \dots, R_n(V_\ell)] \neq 0. \tag{14.12.7}$$

In addition, one has to ensure that the collision of the above strong shocks with any weak waves produces outgoing weak waves whose weighted strength does not exceed the weighted strength of the incoming weak waves. Suppose that the strong i_ℓ-shock is hit from the left by weak k-waves, $k = i_\ell, \dots, n$ with amplitude α_k and speed $\mu_k \sim \lambda_k(V_{\ell-1})$, and from the right by weak k-waves, $k = 1, \dots, i_\ell$, with amplitude β_k and speed $\nu_k \sim \lambda_k(V_\ell)$. These collisions will produce an outgoing strong i_ℓ-shock together with outgoing weak j-waves, $j \neq i_\ell$, with amplitude ε_j and speed $\zeta_j \sim \lambda_j(V_{\ell-1})$, for $j = 1, \dots, i_\ell - 1$, and $\xi_j \sim \lambda_j(V_\ell)$ for $j = i_\ell + 1, \cdots, n$.

Clearly, the $\mu_k, \nu_k, \varepsilon_j, \zeta_j$ and ξ_j are all smooth functions of the $n+1$ variables $(\beta_1, \dots, \beta_{i_\ell}, \alpha_{i_\ell}, \dots, \alpha_n)$. The wave stability condition will be satisfied if there exist positive weights ω_j^ℓ, $\ell = 0, \dots, m$, $j = 1, \dots, n$, so that, for any $\ell = 1, \dots, m$,

(14.12.8)

$$\sum_{j=1}^{i_\ell - 1} \omega_j^{\ell-1} \left| \frac{\partial}{\partial \alpha_k} \left[\frac{\varepsilon_j(\zeta_j - s_\ell)}{\nu_j - s_\ell} \right] \right| + \sum_{j=i_\ell+1}^{n} \omega_j^\ell \left| \frac{\partial}{\partial \alpha_k} \left[\frac{\varepsilon_j(\xi_j - s_\ell)}{\mu_j - s_\ell} \right] \right| < \omega_k^\ell, \;\; k = i_\ell, \dots, n,$$

(14.12.9)

$$\sum_{j=1}^{i_\ell - 1} \omega_j^{\ell-1} \left| \frac{\partial}{\partial \beta_k} \left[\frac{\varepsilon_j(\zeta_j - s_\ell)}{\nu_j - s_\ell} \right] \right| + \sum_{j=i_\ell+1}^{n} \omega_j^\ell \left| \frac{\partial}{\partial \beta_k} \left[\frac{\varepsilon_j(\xi_j - s_\ell)}{\mu_j - s_\ell} \right] \right| < \omega_k^{\ell-1}, \;\; k = 1, \dots, i_\ell,$$

where all partial derivatives are evaluated at $\beta_i = 0$, for $i = 1, \dots, i_\ell$ and $\alpha_i = 0$, for $i = i_\ell, \dots, n$.

To summarize, for wave fans V containing only compressive shocks, Theorem 14.12.1 applies, provided that (14.12.5), (14.12.6), (14.12.7), (14.12.8) and (14.12.9) are satisfied. The proof employs the methodology developed in earlier sections of this chapter and is quite technical. For more general wave fans V, which may also contain contact discontinuities and/or rarefactions, Theorem 14.12.1 holds under assumptions that are similar to, but more complicated than, (14.2.8) and (14.2.9). The hope is that these conditions shall be automatically satisfied for the systems arising in continuum physics. Indeed, it has been shown that the wave stability conditions hold identically for the wave fans of the isentropic elasticity system (7.1.11), in the genuinely nonlinear case $\sigma''(u) \neq 0$. On the other hand, for the system of nonisentropic gas dynamics, for an ideal gas (2.5.20), with adiabatic exponent γ, the wave stability conditions are verified only in the range $1.056 < \gamma < 8.757$.

14.13 Notes

Detailed, systematic presentation of most of the topics discussed in this chapter can be found in the texts by Bressan [9], and Holden and Risebro [5], as well as in the survey article by Bressan [12].

The front tracking method for scalar conservation laws was introduced by Dafermos [2] and is developed in Hedstrom [1], Holden, Holden and Høegh-Krohn [1], Holden and Holden [1], Holden and Risebro [1], Risebro and Tveito [2], Gimse and Risebro [1], Gimse [1], and Pan and Lin [1]. It has been employed, especially by the Norwegian School, as a computational tool. In fact, a similar approach had already been used for computations in the 1960's, by L.M. Barker [1]. For a detailed exposition, with applications, see Holden and Risebro [5]. For a proof of the equivalence between viscosity solutions for the Hamilton-Jacobi equation and admissible solutions for scalar conservation laws, based on the front tracking approach, see Karlsen and Risebro [1].

The front tracking method was extended to genuinely nonlinear systems of two conservation laws by DiPerna [5] and then to genuinely nonlinear systems of any size, independently, by Bressan [2] and Risebro [1]. In Bressan's algorithm, the Approximate Riemann Solver employs pseudoshocks, while in Risebro's approach all new waves are attached to one of the two main fronts involved in the interaction. Yet another possibility, proposed by Schochet [6], is to eliminate pseudoshocks altogether, by assigning to them infinite speed, with the sacrifice of finite speed of propagation in the algorithm. The presentation here, in Sections 14.2-14.7, follows the approach of Bressan and employs a technical simplification due to Baiti and Jenssen [2]. For a detailed treatment, see Bressan [9,12]. The notion of nonresonant curve is introduced here for the first time.

For early applications to special systems see Alber [2], Long-Wei Lin [1], Risebro and Tveito [1], and Wendroff [2].

Ancona and Marson [3,6] have extended the front tracking method, first to systems that are merely piecewise genuinely nonlinear and then to general strictly hyperbolic systems. A crucial role in the latter case is played by Bianchini's [6] solution of the Riemann problem; see Sections 9.12 and 15.9. For an alternative construction in the same spirit, see Glass and LeFloch [1].

The Standard Riemann Semigroup was originally constructed by means of a very technical procedure, based on linearization, in Bressan [1,3,5], for special systems, Bressan and Colombo [1], for genuinely nonlinear systems of two conservation laws, and finally in Bressan, Crasta and Piccoli [1], for systems of n conservation laws, with characteristic families that are either genuinely nonlinear or linearly degenerate. For systems with coinciding shock and rarefaction wave curves, the semigroup is defined for data with arbitrarily large total variation and may even be extended to the class of data that are merely in L^∞; see Baiti and Bressan [1], Bressan and Goatin [2], Bianchini [2,3], and Colombo and Corli [2]. Similarly, for the system of isothermal gas dynamics Colombo and Risebro [1] construct the semigroup for data with arbitrarily large total variation. This approach was extended, by Ancona and Marson [4,5], to systems of two conservation laws that are merely piecewise genuinely nonlinear. The presentation in Sections 14.8-14.9 follows the alternative, simpler approach of Bressan, Liu and Yang [1], in which the basic estimate is derived by means of the functional ρ introduced by Tai-Ping Liu and Tong Yang [2,3,4,5]. A detailed discussion is found in Bressan [7,9,12]. See also Bressan [8,10]. An alternative method, of Haar-Holmgren type, for proving continuous dependence of solutions in L^1 was devised, at about the same time, by Hu and LeFloch [1]. It has been extended to general, not necessarily genuinely nonlinear, systems by LeFloch [7]. See also Goatin and LeFloch [1,2]. Furthermore, in the context of the Euler equations, Goatin and LeFloch [3] discuss L^1 continuous dependence for solutions with large total variation. Finally, L^1 stability has been established, by Tai-Ping Liu and Tong Yang [5], even via the Glimm scheme. It should be noted that, in contrast to the scalar case, there is no standard L^1-contractive metric for systems (Temple [4]). The rate of decrease in the distance between two solutions (recall Theorem 11.8.3 for the scalar case) is estimated by Goatin and LeFloch [2]; see the presentation in the book by LeFloch [5].

Bianchini and Colombo [1] show that solutions to the Cauchy problem depend continuously on the flux function. For recent developments in that direction see Chen, Christoforou and Zhang [1,2]. The issue of "shift differentiability" of the flow generated by conservation laws, which is relevant to stability considerations, is discussed in Bressan and Guerra [1], and Bianchini [1].

Uniqueness under the Tame Oscillation Condition was established by Bressan and Goatin [1], improving an earlier theorem by Bressan and LeFloch [1], which required a Tame Variation Condition. Uniqueness also prevails when the Tame Oscillation Condition is replaced by the assumption that the trace of solutions along space-like curves has bounded variation; see Bressan and Lewicka [1]. The impetus for the above research was provided by Bressan [4], which established the unique limit of the Glimm scheme. For an alternative approach, based on Haar's method, see Hu and LeFloch [1]. Uniqueness is also discussed in Oleinik [3], Tai-Ping Liu [3], DiPerna [5], Dafermos and Geng [2], Heibig [1], LeFloch and Xin [1], Chen and Frid [7], and Chen, Frid and Li [1].

A detailed treatment of the topics outlined in Section 14.11 is found in Bressan [9]. Continuous Glimm functionals were first introduced by Schatzman [1], in the context of piecewise Lipschitz solutions. The extension of the notion to *BV* solutions, for genuinely nonlinear systems, and the proof of the lower semicontinuity property (Lemma 14.11.1) are due to Bressan and Colombo [2], and Baiti and Bressan [2]. Further extension, to systems that are not genuinely nonlinear, was made by LeFloch and Trivisa [1]. See also Bianchini [5].

As already noted in Sections 11.12 and 12.11, the decay of positive waves at the rate $O(1/t)$ was first discussed, for convex scalar conservation laws and genuinely nonlinear systems of two conservation laws, by Oleinik [2] and Glimm and Lax [1], respectively. The version presented here, Theorem 14.11.2, for genuinely nonlinear systems of n conservation laws is taken from Bressan and Colombo [3] and Bressan [9]. A sharp decay estimate is found in Bressan and Yang [2]. See also Bressan and Coclite [1], and Bressan and Goatin [2], for special systems. An analogous property for piecewise genuinely nonlinear systems, originally demonstrated by Tai-Ping Liu [15], has been reestablished, by use of continuous Glimm functionals, in LeFloch and Trivisa [1]. For implications on uniqueness, see Bressan and Goatin [1] and Goatin [1].

The local structure of *BV* solutions was first described by DiPerna [3], for genuinely nonlinear systems, and by Tai-Ping Liu [15], for piecewise genuinely nonlinear systems. The approach outlined here, culminating in Theorems 14.11.3 and 14.11.4, is due to Bressan and LeFloch [2]; see Bressan [9], for a detailed treatment. For systems that are merely piecewise genuinely nonlinear, see Bianchini and Yu [2].

For the *SBV* property of solutions, see Biancini [10], Bianchini and Caravenna [1], and Bianchini and Yu [1].

For early work on solutions that are small perturbations of a given, self-similar wave fan, with large shocks and/or rarefaction waves, see Chern [1], and Asakura [1,2], for the case of a single shock, Schochet [4], who established local existence for genuinely nonlinear systems of arbitrary size, and by Bressan and Colombo [2], who first demonstrated stability (i.e., continuous dependence in L^1) for genuinely nonlinear systems of two conservation laws. See also Bressan and Marson [2]. The combined treatment of existence and stability, for genuinely nonlinear systems of arbitrary size, outlined in Section 14.12, is based on the work of Lewicka and Trivisa [1], and Lewicka [2,3,4,5,6]. Solutions to a number of particular problems of this type, formulated as Goursat problems on the positive quadrant, are found in Shao [1,2,3,4,5,6].

As shown in Bressan, Chen and Zhang [1], the front tracking algorithm applied to the p-system under initial data with large total variation may produce approximate solutions with variation that blows up in finite time. It is not clear at the present time whether this manifests breaking of the solution itself or just indicates inadequacy of the front tracking algorithm to handle "large" data.

A great deal of experience has been amassed on the random choice scheme and the front tracking algorithm, for constructing solutions, as well as on the linearization technique, the Liu-Yang functional and Haar's method, for establishing uniqueness and L^1 stability. Accordingly, the above methods have been adapted and have been employed, interchangeably or in combination, in the study of Cauchy problems for (inhomogeneous) systems of balance laws (Amadori and Guerra [1,2,3], Amadori, Gosse and Guerra [1], Crasta and Piccoli [1], Karlsen, Risebro and Towers [1], Colombo and Corli [4], Colombo and Guerra [1,2]), as well as initial-boundary value problems for systems of conservation laws (Amadori [1], Amadori and Colombo [1,2], Donadello and Marson [1]); also for nonclassical solutions, with shocks satisfying admissibility conditions dictated by some kinetic relation, possibly even for systems not in conservation form (Crasta and LeFloch [1], Baiti, LeFloch and Piccoli [1,2], Amadori, Baiti, LeFloch and Piccoli [1], Laforest and LeFloch [1], Colombo and Corli [1,3], Baiti [1]); for certain Cauchy problems with large data (Holden, Risebro and Sande [2], Risebro and Weber [1], Sever [14], Amadori and Corli [1], Asakura and Corli [2]); and finally for problems in control theory (Ancona and Marson [1], Bressan and Coclite [1], Ancona and Coclite [1], Ancona and Goatin [1], and Glass [1,3]). For recent developments, see Caravenna and Spinolo [1,2].

Estimates of the rate of convergence of the front tracking algorithm have been derived by Lucier [1], in the scalar case, and by Bressan [9], for systems.

XV

Construction of *BV* Solutions by the Vanishing Viscosity Method

Admissible *BV* solutions to the Cauchy problem for general strictly hyperbolic systems of conservation laws, under initial data with small total variation, will be constructed by the vanishing viscosity method. It will be shown that these solutions may be realized as trajectories of an L^1-Lipschitz semigroup, which reduces to the standard Riemann semigroup, introduced in Chapter XIV, when the system is genuinely nonlinear.

15.1 The Main Result

Consider the Cauchy problem

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \qquad -\infty < x < \infty, \ \ 0 < t < \infty, \tag{15.1.1}$$

$$U(x,0) = U_0(x), \qquad -\infty < x < \infty, \tag{15.1.2}$$

for a system of conservation laws that is strictly hyperbolic in a ball $\mathscr{O}$ in $\mathbb{R}^n$, centered at a certain state U^*, and initial data U_0 of bounded variation on $(-\infty,\infty)$ such that $U_0(-\infty) = U^*$.

The aim is to construct *BV* solutions to (15.1.1), (15.1.2) as the $\mu \downarrow 0$ limit of solutions to the parabolic system

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = \mu \partial_x^2 U(x,t), \qquad -\infty < x < \infty, \ \ 0 < t < \infty, \tag{15.1.3}$$

under the same initial condition (15.1.2). This will be accomplished through the following:

15.1.1 Theorem. *There is* $\delta > 0$ *such that if*

$$TV_{(-\infty,\infty)} U_0(\cdot) < \delta, \tag{15.1.4}$$

then the following hold, for some positive constants a and b:

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Grundlehren der mathematischen Wissenschaften 325,
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(a) *For any* $\mu > 0$ *there exists a classical solution* U_μ *to* (15.1.3), (15.1.2) *and*

$$TV_{(-\infty,\infty)}U_\mu(\cdot,t) \le a\, TV_{(-\infty,\infty)}U_0(\cdot), \qquad t \in (0,\infty), \tag{15.1.5}$$

$$\|U_\mu(\cdot,t) - U_\mu(\cdot,\tau)\|_{L^1(-\infty,\infty)} \le b\left(|t-\tau| + |\sqrt{\mu t} - \sqrt{\mu\tau}|\right), \quad \tau, t \in (0,\infty). \tag{15.1.6}$$

(b) *If* $\bar{U}_\mu$ *denotes the solution of* (15.1.3) *with initial value* $\bar{U}_0$ *such that* $TV_{(-\infty,\infty)}\bar{U}_0(\cdot) < \delta$, $\bar{U}_0(-\infty) = U^*$ *and* $U_0 - \bar{U}_0$ *is in* $L^1(-\infty,\infty)$*, then*

$$\|U_\mu(\cdot,t) - \bar{U}_\mu(\cdot,t)\|_{L^1(-\infty,\infty)} \le a\,\|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^1(-\infty,\infty)}\,, \quad t \in (0,\infty). \tag{15.1.7}$$

(c) *As* $\mu \downarrow 0$, $\{U_\mu\}$ *converges in* L^1_{loc} *to a BV solution U of* (15.1.1), (15.1.2) *which inherits the stability properties* (15.1.5), (15.1.6) *and* (15.1.7)*, namely*

$$TV_{(-\infty,\infty)}U(\cdot,t) \le a\, TV_{(-\infty,\infty)}U_0(\cdot), \qquad t \in (0,\infty), \tag{15.1.8}$$

$$\|U(\cdot,t) - U(\cdot,\tau)\|_{L^1(-\infty,\infty)} \le b\,|t-\tau|, \qquad \tau, t \in (0,\infty), \tag{15.1.9}$$

$$\|U(\cdot,t) - \bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} \le a\,\|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^1(-\infty,\infty)}, \quad t \in (0,\infty). \tag{15.1.10}$$

The shocks of the solution U satisfy the viscosity shock admissibility criterion, and thereby all implied admissibility conditions, as described in Chapter VIII. When all characteristic families of (15.1.1) *are either genuinely nonlinear or linearly degenerate, U coincides with the solution of* (15.1.1), (15.1.2) *constructed by the random choice method of Chapter XIII or by the front tracking algorithm of Chapter XIV.*

The proof of the above proposition, which combines diverse ideas and techniques, will occupy the remainder of this chapter. For orientation, Section 15.2 will provide a road map.

It should be noted that the derivation of the estimates (15.1.5), (15.1.6) and (15.1.7) does not depend in an essential manner on the assumption that (15.1.3) is in conservative form but applies equally well to more general systems

$$\partial_t U(x,t) + A(U(x,t))\partial_x U(x,t) = \mu\partial_x^2 U(x,t), \quad x \in (-\infty,\infty),\ t \in (0,\infty), \tag{15.1.11}$$

provided only that $A(U)$ has real distinct eigenvalues. The $\mu \downarrow 0$ limit U of the family $\{U_\mu\}$ of solutions of (15.1.11), (15.1.2) may be interpreted as a "weak" solution of

$$\partial_t U + A(U)\partial_x U = 0, \tag{15.1.12}$$

even though it does not necessarily satisfy this system in the sense of distributions.

15.2 Road Map to the Proof of Theorem 15.1.1

Henceforth we employ the notation $A(U) = \mathrm{D}F(U)$, with eigenvalues $\lambda_i(U)$ and right and left eigenvectors $R_i(U)$ and $L_i(U)$ normalized by $|R_i(U)| = 1$ and (7.2.3). In particular, we set $A(U^*) = A^*$, $\lambda_i(U^*) = \lambda_i^*$, $R_i(U^*) = R_i^*$ and $L_i(U^*) = L_i^*$.

The first step is to eliminate the small parameter μ from (15.1.3) by rescaling the coordinates, $(x,t) \mapsto (\mu x, \mu t)$. Indeed, if U_μ is a solution of the Cauchy problem (15.1.3), (15.1.2), then $U(x,t) = U_\mu(\mu x, \mu t)$ satisfies

(15.2.1)

$$\partial_t U(x,t) + A(U(x,t))\partial_x U(x,t) = \partial_x^2 U(x,t), \qquad -\infty < x < \infty, \ \ 0 < t < \infty,$$

with initial conditions

(15.2.2)
$$U(x,0) = U_{0\mu}(x) = U_0(\mu x), \qquad -\infty < x < \infty.$$

As $TV_{(-\infty,\infty)}U_{0\mu}(\cdot) = TV_{(-\infty,\infty)}U_0(\cdot)$, $TV_{(-\infty,\infty)}U_\mu(\cdot,t) = TV_{(-\infty,\infty)}U(\cdot,\mu^{-1}t)$, it will suffice to estimate the total variation of solutions U of (15.2.1), in which the viscosity coefficient has been scaled to value one. The key estimate is

(15.2.3)
$$TV_{(-\infty,\infty)}U(\cdot,t) = \|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} < \delta_0 \,,$$

for $t \in (0,\infty)$, where δ_0 is some small positive number.

The above bound results from the synergy between the parabolic and the hyperbolic structure of (15.2.1), in the following way:

(a) There are positive constants α and κ such that when $TV_{(-\infty,\infty)}U_0(\cdot) < \frac{1}{2}\kappa\delta_0$ the diffusion induces (15.2.3) for t in some interval $(0,\bar{t}]$ of length $\bar{t} = (\alpha\kappa\delta_0)^{-2}$. Moreover, when (15.2.3) holds on a longer time interval $(0,T)$, with $T > \bar{t}$, then

(15.2.4)
$$\begin{cases} \|\partial_x^2 U(\cdot,t)\|_{L^1(-\infty,\infty)} < 2\alpha\delta_0^2 \,, \\ \|\partial_x^3 U(\cdot,t)\|_{L^1(-\infty,\infty)} < 5\alpha^2\delta_0^3 \,, \\ \|\partial_x^3 U(\cdot,t)\|_{L^\infty(-\infty,\infty)} < 16\alpha^3\delta_0^4 \,, \end{cases}$$

for any $t \in [\bar{t},T)$. This will be established in Section 15.3.

(b) For $t > \bar{t}$, the hyperbolic structure of (15.2.1) takes charge and induces (15.2.3) for t in any, bounded or unbounded, time interval $[\bar{t},T)$ on which (15.2.4) holds. Thus (b) in conjunction with (a) establish (15.2.3) for all $t \in (0,\infty)$.

The assertion in part (b) is verified in several steps. Section 15.4 explains how one employs the superposition of n (viscous) traveling waves of (15.2.1) that best fits the profile of the solution U in the vicinity of any point (x,t) in the domain $(-\infty,\infty) \times (\bar{t},\infty)$ so as to express $\partial_x U$ and $\partial_t U$ in a system of local coordinates

$$\partial_x U = \sum_{j=1}^{n} w_j S_j\,, \qquad \partial_t U = \sum_{j=1}^{n} \omega_j S_j\,, \tag{15.2.5}$$

with components w_j and ω_j that satisfy scalar parabolic equations of the form

$$\begin{cases} \partial_t w_j + \partial_x(\sigma_j w_j) - \partial_x^2 w_j = \varphi_j \\ \\ \partial_t \omega_j + \partial_x(\sigma_j \omega_j) - \partial_x^2 \omega_j = \psi_j\,. \end{cases} \tag{15.2.6}$$

The next step, carried out in Sections 15.5, 15.6 and 15.7, is to demonstrate that when (15.2.3) is satisfied on a time interval $(0,T)$, with $T > \bar{t}$, and at the same time

$$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} (|\varphi_j(x,t)| + |\phi_j(x,t)|)\,dx\,dt < \delta_0\,, \qquad j = 1,\cdots,n, \tag{15.2.7}$$

then the sharper bound

$$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} (|\varphi_j(x,t)| + |\psi_j(x,t)|)\,dx\,dt < c\delta_0^2\,, \qquad j = 1,\cdots,n, \tag{15.2.8}$$

holds, for some c independent of T.

The final ingredient is the standard estimate

$$\int_{-\infty}^{\infty} |w_j(x,t)|dx \leq \int_{-\infty}^{\infty} |w_j(x,\bar{t})|dx + \int_{\bar{t}}^{t} \int_{-\infty}^{\infty} |\varphi_j(x,\tau)|dx\,d\tau, \tag{15.2.9}$$

for solutions of (15.2.6) and $t > \bar{t}$.

One may now establish that when δ_0 is sufficiently small, (15.2.3) holds for any $t \in (0,\infty)$, by means of the following argument. Assume $TV_{(-\infty,\infty)}U_0(\cdot) < \frac{1}{4}\kappa\delta_0$. Then, by (a) above, $\|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} < \frac{1}{2}\delta_0$, for any $t \in (0,\bar{t}]$. Suppose now that (15.2.3) holds on a bounded interval $[\bar{t},\bar{T})$ but is violated at $t = \bar{T}$. For $c\delta_0 < 1$, as T increases from $\bar{t}$ to $\bar{T}$, the left-hand side of (15.2.7) cannot assume the value δ_0 unless it has already assumed the value $c\delta_0^2 < \delta_0$ at an earlier time T. However, this would be incompatible with the assertion, above, that (15.2.7) implies (15.2.8). Hence (15.2.7), and thereby (15.2.8), must hold for all $t \in [\bar{t},\bar{T})$. By applying (15.2.9) for $t = \bar{T}$ and using (15.2.5), we infer that $\|\partial_x U(\cdot,\bar{T})\|_{L^1(-\infty,\infty)} < \frac{1}{2}\delta_0 + c_1\delta_0^2$, which is smaller than δ_0 when $2c_1\delta_0 < 1$. We have thus arrived at a contradiction to the hypothesis that (15.2.3) is violated at $t = \bar{T}$.

The stability estimates (15.1.5), (15.1.6) and (15.1.7) will be derived in Section 15.8, with the help of (15.2.3). Clearly, once these estimates have been established, one may pass to the limit along sequences $\{\mu_k\}$, with $\mu_k \to 0$ as $k \to \infty$, and obtain solutions U of (15.1.1), (15.1.2) possessing the stability properties (15.1.8), (15.1.9) and (15.1.10). It will then be shown that any solution $U = \lim_{k\to\infty} U_{\mu_k}$ satisfies the Tame Oscillation Condition 14.10.1. In turn, by virtue of Theorem 14.10.2, this will imply that when all characteristic families are either genuinely nonlinear or linearly degenerate, then U must coincide with the unique solution constructed by the random choice method. Thus, for such systems, the entire family $\{U_\mu\}$ must converge to the same solution U, as $\mu \downarrow 0$.

For systems with general characteristic families, the issue of uniqueness has been settled in the literature cited in Section 15.9, by the following procedure.

As a first step, it is shown that, for special initial data (9.1.12), the entire family $\{U_\mu\}$ converges to the solution $V(x,t;U_L,U_R)$ of the Riemann Problem constructed by use of the wave curves identified in Section 9.8.

Next one demonstrates that any solution $U = \lim_{k\to\infty} U_{\mu_k}$ of (15.1.1), (15.1.2) is properly approximated, in the vicinity of every fixed point $(\bar{x},\bar{t})$ of the upper half-plane, by either of the following:
(a) the solution $V(x-\bar{x},t-\bar{t};U_L,U_R)$ of the Riemann problem for (15.1.1), with end-states $U_L = U(\bar{x}-,\bar{t})$, $U_R = U(\bar{x}+,\bar{t})$, in the sense that for any $\beta > 0$,

$$(15.2.10)\qquad \lim_{h\downarrow 0} \frac{1}{h} \int_{\bar{x}-\beta h}^{\bar{x}+\beta h} |U(x,\bar{t}+h) - V(x-\bar{x},h;U_L,U_R)|dx = 0;$$

(b) the solution $W(x-\bar{x},t-\bar{t})$ of the Cauchy problem for the linearized system

$$(15.2.11)\qquad \begin{cases} \partial_t W(x,t) + A(U(\bar{x},\bar{t}))\partial_x W(x,t) = 0 \\ W(x,0) = U(x-\bar{x},\bar{t}), \end{cases}$$

in the sense that there exist positive constants c and β such that, for any $y < \bar{x} < z$,

$$(15.2.12)\qquad \limsup_{h\downarrow 0} \int_{y+\beta h}^{z-\beta h} |U(x,\bar{t}+h) - W(x-\bar{x},h)|dx \le c[TV_{(y,z)}U(\cdot,\bar{t})]^2 .$$

It turns out that the above two conditions, (15.2.10) and (15.2.12), uniquely identify the solution and thus the entire family $\{U_\mu\}$ must converge to U, as $\mu \downarrow 0$.

15.3 The Effects of Diffusion

As noted in Section 15.2, the role of viscosity will be to sustain (15.2.3) on some interval $(0,\bar{t}]$, of length $\bar{t} = O(\delta_0^{-2})$, while at the same time reducing the size of the L^1 norms of spatial derivatives of higher order, as indicated in (15.2.4). Both objectives are met by virtue of

15.3.1 Lemma. *Let U be the solution of* (15.2.1), (15.2.2). *There are positive constants α and κ such that if* (15.2.3) *holds, for any fixed positive small δ_0, on the interval* $(0,\bar{t}]$ *of length* $\bar{t} = (\alpha\kappa\delta_0)^{-2}$, *then*

(15.3.1)
$$\|\partial_x^2 U(\cdot,t)\|_{L^1(-\infty,\infty)} < \frac{2\delta_0}{\kappa\sqrt{t}}, \qquad t \in (0,\bar{t}],$$

(15.3.2)
$$\|\partial_x^3 U(\cdot,t)\|_{L^1(-\infty,\infty)} < \frac{5\delta_0}{\kappa^2 t}, \qquad t \in (0,\bar{t}],$$

(15.3.3)
$$\|\partial_x^3 U(\cdot,t)\|_{L^\infty(-\infty,\infty)} < \frac{16\delta_0}{\kappa^3 t\sqrt{t}}, \qquad t \in (0,\bar{t}].$$

Moreover, when (15.2.3) *is satisfied on a longer interval* $(0,T)$, $\bar{t} < T \leq \infty$, *then* (15.2.4) *will hold for any* $t \in [\bar{t},T)$. *Finally,*

(15.3.4)
$$TV_{(-\infty,\infty)} U_0(\cdot) < \tfrac{1}{2}\kappa\delta_0$$

implies (15.2.3) *for all* $t \in (0,\bar{t}]$.

Sketch of Proof. We rewrite (15.2.1) in the form

(15.3.5)
$$\partial_t U + A^* \partial_x U - \partial_x^2 U = [A^* - A(U)]\partial_x U.$$

The ($n \times n$ matrix-valued) Green kernel $G(x,t)$ of the linear parabolic operator on the left-hand side of (15.3.5) can be written in closed form as follows. Multiplying (15.3.5), from the left, by L_i^*, decouples the left-hand side of this system into scalar equations with operator $\partial_t + \lambda_i^* \partial_x - \partial_x^2$, whose Green function reads

(15.3.6)
$$g_i(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left[-\frac{(x-\lambda_i^* t)^2}{4t}\right].$$

Therefore,

(15.3.7)
$$G(x,t) = \sum_{i=1}^{n} g_i(x,t) R_i^* L_i^* .$$

A simple calculation yields

(15.3.8)
$$\|G(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \frac{1}{\kappa}, \qquad \|\partial_x G(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \frac{1}{\kappa\sqrt{t}},$$

for some constant κ.

It will suffice to establish the desired estimates under the additional assumption $U_0 \in C^\infty$, because the general case will then follow by completion.

Differentiating (15.3.5) with respect to x and applying Duhamel's principle to the resulting equation yields

(15.3.9)
$$\partial_x U(\cdot,t) = G(\cdot,t) * \partial_x U_0(\cdot) + \int_0^t G(\cdot,t-\tau) * P(\cdot,\tau)\, d\tau,$$

where

(15.3.10)
$$P(x,\tau) = [A^* - A(U(x,\tau))]\partial_x^2 U(x,t) - \partial_x U^\top(x,\tau)\mathrm{DA}(U(x,\tau))\partial_x U(x,\tau)$$

and $*$ denotes convolution on $(-\infty,\infty)$ with respect to the x-variable.

Since $\|U - U^*\|_{L^\infty} \le \|\partial_x U\|_{L^1}$ and $\|\partial_x U\|_{L^\infty} \le \|\partial_x^2 U\|_{L^1}$,

$$\|P(\cdot,\tau)\|_{L^1(-\infty,\infty)} \le \beta \|\partial_x U(\cdot,\tau)\|_{L^1(-\infty,\infty)} \|\partial_x^2 U(\cdot,\tau)\|_{L^1(-\infty,\infty)}, \tag{15.3.11}$$

where β depends solely on $\sup|\mathrm{D}^2 F(U)|$, for $U \in \mathscr{O}$.

We now assume (15.2.3) holds on an interval $[0,\bar t]$ of length $\bar t = (\alpha\kappa\delta_0)^{-2}$, with $\alpha > 2\pi\beta\kappa^{-2}$, and proceed to verify (15.3.1) on $(0,\bar t]$. Differentiating (15.3.9) with respect to x,

$$\partial_x^2 U(\cdot,t) = \partial_x G(\cdot,t) * \partial_x U_0(\cdot) + \int_0^t \partial_x G(\cdot,t-\tau) * P(\cdot,\tau)\,d\tau. \tag{15.3.12}$$

Together with (15.3.8), (15.3.11) and (15.2.3), this implies

$$\|\partial_x^2 U(\cdot,t)\|_{L^1(-\infty,\infty)} \le \frac{\delta_0}{\kappa\sqrt{t}} + \frac{\beta\delta_0}{\kappa}\int_0^t \frac{1}{\sqrt{t-\tau}} \|\partial_x^2 U(\cdot,\tau)\|_{L^1(-\infty,\infty)} d\tau. \tag{15.3.13}$$

Suppose (15.3.1) is false, and let t be the earliest time in $(0,\bar t]$ where it fails. Then (15.3.13) yields

(15.3.14)
$$\frac{2\delta_0}{\kappa\sqrt{t}} \le \frac{\delta_0}{\kappa\sqrt{t}} + \frac{2\beta\delta_0^2}{\kappa^2}\int_0^t \frac{1}{\sqrt{\tau(t-\tau)}} d\tau = \frac{\delta_0}{\kappa\sqrt{t}} + \frac{2\pi\beta\delta_0^2}{\kappa^2} < \frac{\delta_0}{\kappa\sqrt{t}} + \frac{\delta_0}{\kappa\sqrt{\bar t}},$$

which is in contradiction to $t \le \bar t$.

The estimates (15.3.2) and (15.3.3) are established by similar arguments. The reader may find the details in the references cited in Section 15.9.

Because of (15.3.1), (15.3.2) and (15.3.3), (15.2.4) holds at $t = \bar t = (\alpha\kappa\delta_0)^{-2}$. When (15.2.3) is satisfied on a longer interval $(0,T)$, $\bar t < T \le \infty$, then (15.2.4) will hold for any $t \in [\bar t, T)$, because the time origin may be shifted to the point $t - \bar t$.

Finally, assume (15.3.4) and suppose (15.3.1) holds on some time interval $(0,\hat t)$. Then, for any $t \in (0,\hat t]$, (15.3.9), (15.3.8) and (15.3.11) together imply

$$\|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} \le \frac{\delta_0}{2} + \frac{2\beta\delta_0}{\kappa^2}\int_0^t \frac{1}{\sqrt{\tau}} \|\partial_x U(\cdot,\tau)\|_{L^1(-\infty,\infty)} d\tau. \tag{15.3.15}$$

By Gronwall's lemma,

(15.3.16) $$\|\partial_x U(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \frac{\delta_0}{2}\exp\left[\frac{4\beta\delta_0\sqrt{t}}{\kappa^2}\right], \qquad t \in [0,\hat{t}].$$

Consequently, both (15.3.1) and (15.2.3) will be satisfied on an interval $(0,\bar{t}]$ with $\bar{t} = (\alpha\kappa\delta_0)^{-2}$, provided that α is sufficiently small, but independent of δ_0. The proof is complete.

15.4 Decomposition into Viscous Traveling Waves

In Section 7.8 we saw that by expressing solutions as the superposition (7.8.1) of simple waves, the hyperbolic system (15.1.1) reduces to the system (7.8.6) of weakly coupled scalar equations. Here it is shown that the analog for the parabolic system (15.2.1) is the decomposition (15.2.5) of solutions into a sum of viscous traveling waves.

A viscous wave traveling with speed s is a solution U of (15.2.1) in the special form $U(x,t) = V(x-st)$. The function V must satisfy the second-order ordinary differential equation

(15.4.1) $$\ddot{V} = [A(V) - sI]\dot{V},$$

which may be recast as the first order system

(15.4.2) $$\begin{cases} \dot{V} = W \\ \dot{W} = [A(V) - sI]W \\ \dot{s} = 0. \end{cases}$$

Clearly there exists a $(2n+1)$-parameter family of viscous traveling waves, parametrized by their speed and the initial values of V and W. However, the only ones that may serve our present purposes are those for which $|W|$ is bounded and small on $\mathbb{R}$ and s is close to one of the characteristic speeds λ_j^*. These will be dubbed *viscous j-waves*.

For the system

(15.4.3) $$\begin{cases} \dot{V} = W \\ \dot{W} = [A^* - \lambda_j^* I]W \\ \dot{s} = 0, \end{cases}$$

resulting from linearizing (15.4.2) about $(V = U^*, W = 0, s = \lambda_j^*)$, orbits of solutions with W bounded span the $(n+2)$-dimensional hyperplane

(15.4.4) $$\mathscr{P}_j = \left\{(V,W,s) : V \in \mathbb{R}^n,\ W = wR_j^*,\ w \in \mathbb{R},\ s \in \mathbb{R}\right\}$$

embedded in $\mathbb{R}^{2n+1}$. It then follows from the center manifold theorem that the orbits of viscous j-waves span a smooth $(n+2)$-dimensional manifold $\mathscr{M}_j$ embedded in $\mathbb{R}^{2n+1}$, which is tangent to $\mathscr{P}_j$ at the point $(U^*,0,\lambda_j^*)$. Furthermore, this *center manifold* may be parametrized locally by (V,w,s), $V\in\mathbb{R}^n$, $w\in\mathbb{R}$, $s\in\mathbb{R}$, as

$$\mathscr{M}_j=\left\{(V,W,s):|V-U^*|<\varepsilon,\,W=wS_j(V,w,s),\,|w|<\varepsilon,\,|s-\lambda_j^*|<\varepsilon\right\},\tag{15.4.5}$$

where S_j is a smooth unit vector field such that $S_j(U^*,0,\lambda_j^*)=R_j^*$.

We proceed to show that along trajectories of (15.4.2) on the invariant manifold $\mathscr{M}_j$, w satisfies a differential equation. To begin with, since $|S_j|=1$,

$$S_j^\top S_j=1,\qquad S_j^\top\dot S_j=0,\qquad S_j^\top\ddot S_j=-\dot S_j^\top\dot S_j.\tag{15.4.6}$$

As $W=wS_j$ satisfies $(15.4.2)_2$,

$$\dot wS_j+w\dot S_j=w[A-sI]S_j.\tag{15.4.7}$$

Multiplying (15.4.7), from the left, by $S_j^\top$ and using (15.4.6) yields

$$\dot w=(\sigma_j-s)w,\tag{15.4.8}$$

where

$$\sigma_j(V,w,s)=S_j^\top(V,w,s)A(V)S_j(V,w,s).\tag{15.4.9}$$

Combining (15.4.8) with (15.4.7) and using $(15.4.2)_1$,

$$[A-\sigma_jI]S_j=\dot S_j=\mathrm{D}S_j\dot V+\dot w\partial_wS_j=w[\mathrm{D}S_jS_j+(\sigma_j-s)\partial_wS_j].\tag{15.4.10}$$

Letting $w\to0$ in (15.4.10), we conclude that

$$\sigma_j(V,0,s)=\lambda_j(V),\qquad S_j(V,0,s)=R_j(V).\tag{15.4.11}$$

Differentiating (15.4.8),

$$\ddot w=(\sigma_jw)^{\cdot}-s\dot w.\tag{15.4.12}$$

To summarize, when U is a viscous j-wave, so that $U(x,t)=V(x-st)$ with s near λ_j^*, then

$$\begin{cases}\partial_xU=w_jS_j(U,w_j,s_j)\\[2ex]\partial_tU=\omega_jS_j(U,w_j,s_j),\end{cases}\tag{15.4.13}$$

where $s_j(x,t)=s$, $w_j(x,t)=w(x-st)$, $\omega_j(x,t)=-s_j(x,t)w_j(x,t)$. Notice that (15.4.2) and (15.4.13) together imply

$$\partial_x^2U=w_j[A(U)-s_jI]S_j(U,w_j,s_j).\tag{15.4.14}$$

Furthermore, by virtue of (15.4.12),

$$\begin{cases} \partial_t w_j + \partial_x(\sigma_j w_j) - \partial_x^2 w_j = 0 \\ \\ \partial_t \omega_j + \partial_x(\sigma_j \omega_j) - \partial_x^2 \omega_j = 0. \end{cases} \tag{15.4.15}$$

We now consider any solution U of (15.2.1) and visualize it as a superposition of viscous waves, as follows. For any point (x,t) of the upper half-plane we seek numbers $w_j = w_j(x,t)$, $\omega_j = \omega_j(x,t)$, and $s_j = s_j(x,t)$, $j = 1, \cdots, n$, with $|w_j| < \varepsilon$, $|\omega_j| < \varepsilon$, $|s_j - \lambda_j^*| < \varepsilon$, and such that

$$\begin{cases} \partial_x U = \displaystyle\sum_{j=1}^n w_j S_j(U, w_j, s_j) \\ \\ \partial_t U = \displaystyle\sum_{j=1}^n \omega_j S_j(U, w_j, s_j). \end{cases} \tag{15.4.16}$$

In fact, assuming $|\partial_x U|$, $|\partial_t U|$ are sufficiently small and for any fixed $s_j \in \mathscr{B}_\varepsilon(\lambda_j^*)$, (15.4.11) and the implicit function theorem together imply the existence of unique w_j, ω_j for which (15.4.16) holds. Recalling (15.4.13), we infer that the j-th terms in the above summations are first order approximations of a viscous j-wave. Next we investigate whether it is possible to achieve a tighter fit by selecting judiciously the s_j. Combining (15.4.16) with (15.2.1) yields

$$\partial_x^2 U = \sum_{j=1}^n w_j \left[A(U) + \frac{\omega_j}{w_j} I \right] S_j(U, w_j, s_j). \tag{15.4.17}$$

Comparing (15.4.17) with (15.4.14) we deduce that second-order fit would be attained by selecting $s_j = -\omega_j / w_j$. The difficulty is that this may conflict with the requirement $|s_j - \lambda_j^*| < \varepsilon$. As a compromise, we employ

$$s_j = \lambda_j^* - \theta\left(\lambda_j^* + \frac{\omega_j}{w_j} \right), \tag{15.4.18}$$

where θ is a smooth "cutoff" function such that

$$\theta(r) = \begin{cases} r \ \text{ if } \ |r| \le \delta_1 \\ \\ 0 \ \text{ if } \ |r| > 3\delta_1 \end{cases} \qquad |\theta'(r)| \le 1, \qquad |\theta''(r)| \le \frac{4}{\delta_1}, \tag{15.4.19}$$

for some small positive constant δ_1. Thus, $s_j = -\omega_j / w_j$ whenever $-\omega_j / w_j$ takes values near λ_j^*. On the other hand, when $-\omega_j / w_j$ is far from λ_j^*, s_j is chosen constant, equal to λ_j^*.

After laborious analysis, which relies on the properties of the functions S_j and is found in the literature cited in Section 15.9, one shows that as long as

$|U-U^*|$, $|\partial_x U|$, $|\partial_x^2 U|$, and thereby also $|\partial_t U|$, are sufficiently small, for each (x,t) there exists a unique set of (w_j,ω_j), $j=1,\cdots,n$, which satisfies (15.4.16) together with (15.4.18). Moreover, the functions $(w_j(x,t),\omega_j(x,t))$, $j=1,\cdots,n$, are $C^{1,1}$ smooth. Furthermore, with reference to the setting and notation of Lemma 15.3.1, when (15.2.3) is satisfied on an interval $(0,T)$, with $\bar{t}<T\le\infty$, in which case (15.2.4) hold for any $t\in[\bar{t},T)$, then

$$
\left\{
\begin{aligned}
&\sum_{j=1}^{n}\left\{\|w_j(\cdot,t)\|_{L^1(-\infty,\infty)}+\|\omega_j(\cdot,t)\|_{L^1(-\infty,\infty)}\right\}=O(\delta_0),\\
&\sum_{j=1}^{n}\left\{\|w_j(\cdot,t)\|_{L^\infty(-\infty,\infty)}+\|\omega_j(\cdot,t)\|_{L^\infty(-\infty,\infty)}\right\}=O(\delta_0^2),\\
&\sum_{j=1}^{n}\left\{\|\partial_x w_j(\cdot,t)\|_{L^1(-\infty,\infty)}+\|\partial_x\omega_j(\cdot,t)\|_{L^1(-\infty,\infty)}\right\}=O(\delta_0^2),\\
&\sum_{j=1}^{n}\left\{\|\partial_x w_j(\cdot,t)\|_{L^\infty(-\infty,\infty)}+\|\partial_x\omega_j(\cdot,t)\|_{L^\infty(-\infty,\infty)}\right\}=O(\delta_0^3),
\end{aligned}
\right.
\tag{15.4.20}
$$

uniformly on $[\bar{t},T)$.

As we saw above, when U is just a viscous j-wave, w_j and ω_j satisfy (15.4.15). For general solutions U, we have, instead, Equations (15.2.6), with source terms φ_j and ψ_j. The expectation is that the approximation of U by viscous waves will be sufficiently tight to render φ_j and ψ_j "small".

After a lengthy and laborious calculation, which is found in the references cited in Section 15.9, one shows that

$$
\begin{aligned}
(\varphi_j,\psi_j)=O(1)&\sum_{i\neq k}\big(|w_iw_k|+|\omega_i\omega_k|+|w_i\omega_k|+|w_i\partial_x w_k|+|w_i\partial_x\omega_k|+|\omega_i\partial_x w_k|\big)\\
&+O(1)\sum_i|\omega_i\partial_x w_i-w_i\partial_x\omega_i|\\
&+O(1)\sum_i\left|w_i\partial_x\left(\frac{\omega_i}{w_i}\right)\right|^2\chi_{\{|\lambda_i^*+\omega_i/w_i|<3\delta_1\}}\\
&+O(1)\sum_i\big(|\partial_x w_i|+|\partial_x\omega_i|\big)|\omega_i+s_iw_i|.
\end{aligned}
\tag{15.4.21}
$$

The four terms on the right-hand side of (15.4.21) estimate the "deviation" of (15.2.6) from the single viscous j-wave case (15.4.15), arising for the following reasons:

(a) The first term accounts for transversal wave interactions: viscous waves belonging to distinct characteristic families, and thus propagating with distinct speeds, interact and make a contribution of quadratic order to φ_j and ψ_j.

(b) The second and third terms account for interactions of waves from the same characteristic family: the viscous i-waves approximating the profile $U(\cdot,t)$ at two

different points, say x and y, are propagating with distinct speeds $s_i(x,t)$ and $s_i(y,t)$ and may thus interact. The key factor in the estimate is $\partial_x s_i$ which monitors the rate of change of s_i.
(c) The fourth term accounts for the "error" committed by selecting s_i through (15.4.18) instead of $-\omega_i/w_i$, as would have been the case for a viscous i-wave. Indeed, notice that this term vanishes whenever $s_i = -\omega_i/w_i$.

In the following three sections we will estimate the right-hand side of (15.4.21). The aim is to verify the assertion made in Section 15.2, namely that if (15.2.3) holds on $(0,T)$, then (15.2.7) implies (15.2.8).

15.5 Transversal Wave Interactions

The aim here is to estimate the first term on the right-hand side of (15.4.21), which accounts for the interaction between viscous waves of distinct families. Under the assumption that (15.2.3) holds for $t \in (0,T)$, which in turn yields (15.4.20) for $t \in [\bar{t},T)$, it will be shown that (15.2.7) implies

(15.5.1)
$$\int_{\bar{t}}^{T}\int_{-\infty}^{\infty}\sum_{i\neq k}\big(|w_iw_k|+|\omega_i\omega_k|+|w_i\omega_k|+|w_i\partial_x w_k|+|w_i\partial_x\omega_k|+|\omega_i\partial_x w_k|\big)dxdt = O(\delta_0^2).$$

Towards that goal we shall compare the solutions of two parabolic equations

$$\begin{cases} \partial_t u^\flat(x,t)+\partial_x\big[\sigma^\flat(x,t)u^\flat(x,t)\big]-\partial_x^2 u^\flat(x,t)=p^\flat(x,t) \\ \partial_t u^\sharp(x,t)+\partial_x\big[\sigma^\sharp(x,t)u^\sharp(x,t)\big]-\partial_x^2 u^\sharp(x,t)=p^\sharp(x,t) \end{cases} \tag{15.5.2}$$

with strictly separated drifts:

$$\inf\sigma^\sharp - \sup\sigma^\flat \geq r > 0. \tag{15.5.3}$$

15.5.1 Lemma. *If* $(u^\flat,u^\sharp)$ *are solutions of* (15.5.2) *on* $(-\infty,\infty)\times[0,T)$,

$$\int_0^T\int_{-\infty}^{\infty}\big|u^\flat(x,t)\big|\big|u^\sharp(x,t)\big|dxdt \tag{15.5.4}$$

$$\leq \frac{1}{r}\left\{\int_{-\infty}^{\infty}|u^\flat(x,0)|dx+\int_0^T\int_{-\infty}^{\infty}|p^\flat(x,t)|dxdt\right\}\left\{\int_{-\infty}^{\infty}|u^\sharp(x,0)|dx+\int_0^T\int_{-\infty}^{\infty}|p^\sharp(x,t)|dxdt\right\}.$$

Proof. We consider first the homogeneous case, $p^\flat = p^\sharp = 0$. We introduce the interaction potential

$$q(v^\flat, v^\sharp) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x-y)|v^\flat(x)||v^\sharp(y)|dxdy, \tag{15.5.5}$$

for any pair of functions $v^\flat$ and $v^\sharp$ in $L^1(-\infty,\infty)$, where

$$k(z) = \begin{cases} r^{-1} & \text{if } z \geq 0 \\ r^{-1}\exp\left(\frac{1}{2}rz\right) & \text{if } z < 0. \end{cases} \tag{15.5.6}$$

Notice that $rk' - 2k''$ is the Dirac mass at the origin. We now have

$$\begin{aligned} \frac{d}{dt}q\big(u^\flat(\cdot,t), u^\sharp(\cdot,t)\big) &= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x-y)\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x-y)\big\{\big[\partial_x^2 u^\flat - \partial_x\big(\sigma^\flat u^\flat\big)\big]\operatorname{sgn} u^\flat\big\}(x,t)\,|u^\sharp(y,t)|\,dxdy \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x-y)\big\{\big[\partial_x^2 u^\sharp - \partial_x\big(\sigma^\sharp u^\sharp\big)\big]\operatorname{sgn} u^\sharp\big\}(y,t)\,|u^\flat(x,t)|\,dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k'(x-y)\,\big[\sigma^\flat(x,t) - \sigma^\sharp(y,t)\big]\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2k''(x-y)\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy \\ &\leq -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (rk' - 2k'')(x-y)\,|u^\flat(x,t)|\,|u^\sharp(y,t)|\,dxdy \\ &\leq -\int_{-\infty}^{\infty} |u^\flat(x,t)|\,|u^\sharp(x,t)|\,dx. \end{aligned} \tag{15.5.7}$$

Integrating (15.5.7) over $(0,T)$ and recalling (15.5.5) and (15.5.6), we deduce

$$\int_0^T\int_{-\infty}^{\infty} |u^\flat(x,t)|\,|u^\sharp(x,t)|\,dxdt \leq \frac{1}{r}\int_{-\infty}^{\infty} |u^\flat(x,0)|dx \int_{-\infty}^{\infty} |u^\sharp(x,0)|dx, \tag{15.5.8}$$

namely (15.5.4) for the special case $p^\flat = p^\sharp = 0$. In particular, if $\Gamma^\flat(x,t;y,\tau)$ and $\Gamma^\sharp(x,t;y,\tau)$ denote the Green functions for the (homogeneous form of the) equations (15.5.2),

$$\int_{\max\{\tau,\tau'\}}^{T}\int_{-\infty}^{\infty} \Gamma^\flat(x,t;y,\tau)\Gamma^\sharp(x,t;y',\tau')dxdt \leq \frac{1}{r}, \tag{15.5.9}$$

for any couple of initial points (y,τ) and (y',τ').

The solutions of (15.5.2) may now be written as

(15.5.10)
$$\begin{cases} u^\flat(x,t) = \int_{-\infty}^{\infty} \Gamma^\flat(x,t;y,0)u^\flat(y,0)dy + \int_0^t \int_{-\infty}^{\infty} \Gamma^\flat(x,t;y,\tau)p^\flat(y,\tau)dyd\tau \\ u^\sharp(x,t) = \int_{-\infty}^{\infty} \Gamma^\sharp(x,t;y,0)u^\sharp(y,0)dy + \int_0^t \int_{-\infty}^{\infty} \Gamma^\sharp(x,t;y,\tau)p^\sharp(y,\tau)dyd\tau. \end{cases}$$

Combining (15.5.9) with (15.5.10), we arrive at (15.5.4). The proof is complete.

15.5.2 Lemma. *Assume that*

(15.5.11)
$$\int_0^T \int_{-\infty}^{\infty} |p^\flat(x,t)|dxdt \leq \delta_0\,, \qquad \int_0^T \int_{-\infty}^{\infty} |p^\sharp(x,t)|dxdt \leq \delta_0\,,$$

(15.5.12)
$$\|\sigma^\flat(\cdot,t)\|_{L^\infty(-\infty,\infty)} \leq c\delta_0\,, \qquad \|\partial_x\sigma^\flat(\cdot,t)\|_{L^\infty(-\infty,\infty)} \leq c\delta_0\,.$$

Let $u^\flat, u^\sharp$ be solutions of (15.5.2) *such that*

(15.5.13)
$$\|u^\flat(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \delta_0\,, \qquad \|u^\sharp(\cdot,t)\|_{L^1(-\infty,\infty)} \leq \delta_0\,,$$

(15.5.14)
$$\|\partial_x u^\flat(\cdot,t)\|_{L^1(-\infty,\infty)} \leq c\delta_0^2\,, \qquad \|u^\sharp(\cdot,t)\|_{L^\infty(-\infty,\infty)} \leq c\delta_0^2\,,$$

for all $t \in [0,T)$. Then

(15.5.15)
$$\int_0^T \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t)|\,|u^\sharp(x,t)|\,dxdt = O(\delta_0^2).$$

Proof. The left-hand side of (15.5.15) is bounded by

(15.5.16)
$$\mathscr{I}(T) = \sup \int_0^{T-\tau} \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t)u^\sharp(x+y,t+\tau)|\,dxdt,$$

where the supremum is taken over all $(y,\tau) \in (-\infty,\infty) \times [0,T)$.

On account of (15.5.14),

(15.5.17)
$$\sup \int_0^1 \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t)u^\sharp(x+y,t+\tau)|\,dxdt \leq c^2\delta_0^4\,.$$

For $t > 1$, we write $\partial_x u^\flat$ in the form

$$\partial_x u^\flat(x,t) = \int_{-\infty}^{\infty} \partial_x g(z,1) u^\flat(x-z,t-1)dz + \int_0^1 \int_{-\infty}^{\infty} \partial_x g(z,s)\,[p^\flat - \partial_x(\sigma^\flat u^\flat)](x-z,t-s)\,dzds, \tag{15.5.18}$$

where $g(x,t) = (4\pi t)^{-\frac{1}{2}} \exp[-x^2/4t]$ is the standard heat kernel. Hence

$$\begin{aligned} &\int_1^{T-\tau} \int_{-\infty}^{\infty} |\partial_x u^\flat(x,t) u^\sharp(x+y,t+\tau)|\,dxdt \\ &\le \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_x g(z,1) u^\flat(x-z,t-1) u^\sharp(x+y,t+\tau)|\,dzdxdt \\ &+ \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \|\partial_x \sigma^\flat\|_{L^\infty} |\partial_x g(z,s) u^\flat(x-z,t-s) u^\sharp(x+y,t+\tau)|\,dzdsdxdt \\ &+ \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_0^1 \int_{-\infty}^{\infty} \|\sigma^\flat\|_{L^\infty} |\partial_x g(z,s) \partial_x u^\flat(x-z,t-s) u^\sharp(x+y,t+\tau)|\,dzdsdxdt \\ &+ \int_1^{T-\tau} \int_{-\infty}^{\infty} \int_{t-1}^{t} \int_{-\infty}^{\infty} |\partial_x g(x-z,t-s) p^\flat(z,s) u^\sharp(x+y,t+\tau)|\,dzdsdxdt. \end{aligned} \tag{15.5.19}$$

Upon combining (15.5.16), (15.5.17), (15.5.19), (15.5.4), (15.5.11), (15.5.12), (15.5.13), and (15.5.14), one obtains

$$\mathscr{J}(T) \le c^2 \delta_0^4 + \frac{4\delta_0^2}{\sqrt{\pi} r} + \frac{8c\delta_0^3}{\sqrt{\pi} r} + \frac{2c\delta_0}{\sqrt{\pi}} \mathscr{J}(T) + \frac{2c\delta_0^3}{\sqrt{\pi}}. \tag{15.5.20}$$

For δ_0 sufficiently small, (15.5.20) yields $\mathscr{J}(T) = O(\delta_0^2)$ and thence (15.5.15). This completes the proof.

We have now laid the groundwork for establishing (15.5.1). Recalling (15.2.6), we apply Lemma 15.5.1 with $u^\flat = w_i$, $\sigma^\flat = \sigma_i$, $p^\flat = \varphi_i$, $u^\sharp = w_k$, $\sigma^\sharp = \sigma_k$, and $p^\sharp = \varphi_k$, shifting the origin from $t = 0$ to $t = \bar{t}$. Using (15.2.7) and (15.4.22), we deduce that the integral of $|w_i w_k|$ over $(-\infty,\infty) \times (\bar{t},T)$ is $O(\delta_0^2)$. The integrals of $|w_i \omega_k|$ and $|\omega_i \omega_k|$ are treated by the same argument. To estimate the integral of $|w_i \partial_x w_k|$, we apply Lemma 15.5.2 with $u^\flat = w_k$, $\sigma^\flat = \sigma_k$, $p^\flat = \varphi_k$, $u^\sharp = w_i$, $\sigma^\sharp = \sigma_i$, and $p^\sharp = \varphi_i$. In order to meet the requirement $(15.5.12)_1$, we perform the change of variable $x \mapsto x - \lambda_k^* t$ so that the drift coefficient σ_k is replaced by $\sigma_k - \lambda_k^*$ which is $O(\delta_0)$. The integrals of the remaining terms $|w_i \partial_x \omega_k|$ and $|\omega_i \partial_k w_k|$ are handled by the same method.

15.6 Interaction of Waves of the Same Family

This section provides estimates for the second and third terms on the right-hand side of (15.4.21), which are induced by the interaction of viscous waves of the same family. The aim is to show that when (15.2.7) and (15.4.20) hold, for $t \in [\bar{t}, T)$, then

$$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} |\omega_i \partial_x w_i - w_i \partial_x \omega_i| \, dxdt = O(\delta_0^2), \tag{15.6.1}$$

$$\int_{\bar{t}}^{T} \int_{-\infty}^{\infty} \left| w_i \partial_x \left(\frac{\omega_i}{w_i} \right) \right|^2 \chi_{\{|\lambda_i^* + \omega_i / w_i| < 3\delta_1\}} dxdt = O(\delta_0^3). \tag{15.6.2}$$

This objective will be attained by monitoring the time evolution of two functionals of the solutions with very interesting geometric interpretation.

We consider solutions (w, ω) of the equations

$$\begin{cases} \partial_t w(x,t) + \partial_x[\sigma(x,t) w(x,t)] - \partial_x^2 w(x,t) = \varphi(x,t) \\ \\ \partial_t \omega(x,t) + \partial_x[\sigma(x,t) \omega(x,t)] - \partial_x^2 \omega(x,t) = \psi(x,t), \end{cases} \tag{15.6.3}$$

on $[\bar{t}, T)$, where φ, ψ and σ are given, smooth functions, with $\varphi(\cdot, t)$ and $\psi(\cdot, t)$ in $L^1(-\infty, \infty)$. Hence $w(\cdot, t)$ and $\omega(\cdot, t)$ will also lie in $L^1(-\infty, \infty)$, so that one may define the functionals

$$\mathscr{L}(t) = \int_{-\infty}^{\infty} [w^2(x,t) + \omega^2(x,t)]^{1/2} dx, \tag{15.6.4}$$

$$\mathscr{A}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{x<y} |w(x,t)\omega(y,t) - \omega(x,t) w(y,t)| \, dxdy. \tag{15.6.5}$$

We introduce the vector field

$$Z(x,t) = \left(\int_{-\infty}^{x} w(y,t) dy \ , \ \int_{-\infty}^{x} \omega(y,t) dy \right). \tag{15.6.6}$$

For fixed $t \in [\bar{t}, T)$, $Z(\cdot, t)$ defines a curve on $\mathbb{R}^2$, parametrized by x, and thus Z represents a moving curve on $\mathbb{R}^2$. We will use a prime to denote differentiation with respect to the parameter x along this curve, $' = \partial_x$. Notice that $\mathscr{L}(t)$ is the length of the curve at time t. Furthermore,

$$\frac{1}{2} \int_{-\infty}^{\infty} Z(y,t) \wedge Z'(y,t) \, dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{x<y} Z'(x,t) \wedge Z'(y,t) \, dxdy \tag{15.6.7}$$

yields the sum of the areas of the regions enclosed by the curve $Z(\cdot,t)$, each multiplied by the corresponding winding number. Thus $\mathscr{A}(t)$ provides an upper bound for the area of the convex hull of $Z(\cdot,t)$.

By virtue of (15.6.3),

$$\partial_t Z(x,t) + \sigma(x,t)\partial_x Z(x,t) - \partial_x^2 Z(x,t) = \Phi(x,t), \tag{15.6.8}$$

where

$$\Phi(x,t) = \Big(\int_{-\infty}^{x} \varphi(y,t)dy \,, \quad \int_{-\infty}^{x} \psi(y,t)dy \Big). \tag{15.6.9}$$

The plan is to show that the rate of growth of $\mathscr{L}(t)$ and $\mathscr{A}(t)$ is controlled by $\|\varphi(\cdot,t)\|_{L^1(-\infty,\infty)}$ and $\|\psi(\cdot,t)\|_{L^1(-\infty,\infty)}$, and, in particular, that these functionals are nonincreasing when φ and ψ vanish identically.

15.6.1 Lemma.

$$\frac{d}{dt}\mathscr{A}(t) \le -\int_{-\infty}^{\infty} \big|\omega(x,t)\partial_x w(x,t) - w(x,t)\partial_x \omega(x,t)\big|\, dx \tag{15.6.10}$$

$$+\|w(\cdot,t)\|_{L^1(-\infty,\infty)}\|\psi(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\omega(\cdot,t)\|_{L^1(-\infty,\infty)}\|\varphi(\cdot,t)\|_{L^1(-\infty,\infty)}.$$

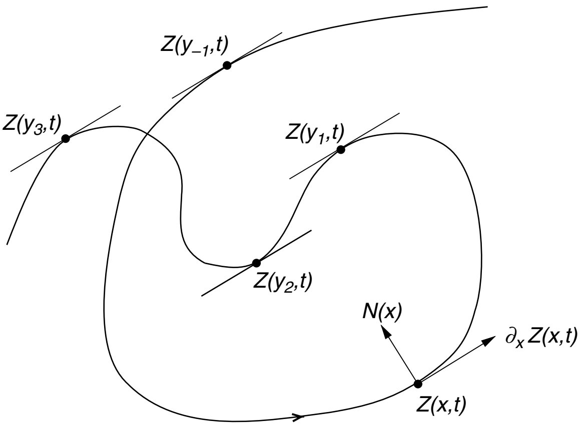

Fig. 15.6.1

Proof. Let us fix $t \in [\bar{t}, T)$ and consider the curve $Z(\cdot,t)$ in $\mathbb{R}^2$; see Fig. 15.6.1. With any $x \in (-\infty,\infty)$ we associate the unit vector $N(x)$ in $\mathbb{R}^2$ that is perpendicular to the tangent vector $Z'(x,t)$ and is oriented by

$$Z'(x,t) \wedge N(x) = |Z'(x,t)|. \tag{15.6.11}$$

In particular, for any $W \in \mathbb{R}^2$,

(15.6.12)
$$Z'(x,t) \wedge W = |Z'(x,t)|[N(x) \cdot W].$$

We now compute

(15.6.13)
$$\frac{d}{dt}\mathscr{A}(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{x<y} \operatorname{sgn}[Z'(x,t) \wedge Z'(y,t)]\left[\partial_t Z'(x,t) \wedge Z'(y,t) + Z'(x,t) \wedge \partial_t Z'(y,t)\right] dxdy$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sgn}\left[Z'(x,t) \wedge Z'(y,t)\right] \left[Z'(x,t) \wedge \partial_t Z'(y,t)\right] dydx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Z'(x,t)| \operatorname{sgn} \partial_y z(y,x,t)\, \partial_t \partial_y z(y,x,t) dydx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} |Z'(x,t)|\, \partial_t TV_{(-\infty,\infty)} z(\cdot,x,t)\, dx,$$

where we are using the notation

(15.6.14)
$$z(y,x,t) = N(x) \cdot Z(y,t).$$

Since $N(x) \cdot Z'(x,t) = 0$, x is a critical point of $z(\cdot,x,t)$. Assume, for simplicity, there is

a finite number of critical points $y_{-p} < \cdots < y_{-1} < y_0 = x < y_1 < \cdots < y_q$, and none of them is degenerate. As minima and maxima alternate,

(15.6.15)
$$\operatorname{sgn} \partial_y^2 z(y_r,x,t) = (-1)^r \operatorname{sgn} \partial_y^2 z(x,x,t).$$

A simple calculation yields

(15.6.16)
$$\partial_t\, TV_{(-\infty,\infty)} z(\cdot,x,t) = -2 \sum_{-p \le r \le q} \operatorname{sgn} \partial_y^2 z(y_r,x,t)\, \partial_t z(y_r,x,t).$$

We substitute into (15.6.16) $\partial_t z = N \cdot \partial_t Z$, with $\partial_t Z$ taken from (15.6.8). Since $\partial_y z(y_r,x,t) = 0$, and by virtue of (15.6.15),

(15.6.17)
$$\partial_t\, TV_{(-\infty,\infty)} z(\cdot,x,t)$$
$$= -2 \sum_{-p \le r \le q} |\partial_y^2 z(y_r,x,t)| - 2 \operatorname{sgn} \partial_y^2 z(x,x,t) \sum_{-p \le r \le q} (-1)^r [N(x) \cdot \Phi(y_r,t)].$$

Furthermore,

$$\sum_{-p\le r\le q} |\partial_y^2 z(y_r,x,t)| \ge |\partial_y^2 z(y_0,x,t)| = |N(x)\cdot Z''(x,t)|, \tag{15.6.18}$$

$$\left|\sum_{-p\le r\le q} (-1)^r [N(x)\cdot \Phi(y_r,t)]\right| \le \int_{-\infty}^{\infty} |N(x)\cdot \Phi'(y,t)|\,dy. \tag{15.6.19}$$

By combining (15.6.13) with (15.6.17), (15.6.18), (15.6.19), (15.6.14) and (15.6.12) we conclude that

$$\frac{d}{dt}\mathscr{A}(t) \le -\int_{-\infty}^{\infty} Z'(x,t)\wedge Z''(x,t)|\,dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Z'(x,t)\wedge \Phi'(y,t)|\,dydx. \tag{15.6.20}$$

Since $Z' = (w,\omega)$ and $\Phi' = (\varphi,\psi)$, (15.6.20) yields (15.6.10). The proof is complete.

15.6.2 Lemma. *Under the assumption* $w^2+\omega^2 \ne 0$,

$$\frac{d}{dt}\mathscr{L}(t) \le -\frac{1}{(1+9\delta_1^2)^{3/2}} \int_{-\infty}^{\infty} |w(x,t)| \left|\partial_x\left(\frac{\omega(x,t)}{w(x,t)}\right)\right|^2 \chi_{\{|\omega/w|<3\delta_1\}}dx + \|\varphi(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\psi(\cdot,t)\|_{L^1(-\infty,\infty)}\,. \tag{15.6.21}$$

Proof. Since $w^2+\omega^2 \ne 0$,

$$\frac{d}{dt}\mathscr{L}(t) = \int_{-\infty}^{\infty} \frac{w\partial_t w + \omega\partial_t w}{(w^2+\omega^2)^{1/2}}\,dx. \tag{15.6.22}$$

We substitute $\partial_t w$ and $\partial_t\omega$ from (15.6.3) into (15.6.22). Upon using the elementary identities

$$\frac{w\partial_x(\sigma w)+\omega\partial_x(\sigma\omega)}{(w^2+\omega^2)^{1/2}} = \partial_x[\sigma(w^2+\omega^2)^{1/2}], \tag{15.6.23}$$

$$\frac{w\partial_x^2 w+\omega\partial_x^2\omega}{(w^2+\omega^2)^{1/2}} = \partial_x^2(w^2+\omega^2)^{\frac{1}{2}} - \frac{|w|\left|\partial_x\left(\dfrac{\omega}{w}\right)\right|^2}{\left[1+\left(\dfrac{\omega}{w}\right)^2\right]^{3/2}}, \tag{15.6.24}$$

we deduce

$$\frac{d}{dt}\mathscr{L}(t) = -\int_{-\infty}^{\infty} \frac{|w|\left|\partial_x\left(\dfrac{\omega}{w}\right)\right|^2}{\left[1+\left(\dfrac{\omega}{w}\right)^2\right]^{3/2}}\,dx + \int_{-\infty}^{\infty} \frac{w\varphi+\omega\psi}{(w^2+\omega^2)^{1/2}}\,dx, \tag{15.6.25}$$

which easily yields (15.6.21). This completes the proof.

In order to show (15.6.1), we integrate (15.6.10) over $[\bar{t},T)$ to get the estimate

$$
\begin{aligned}
&\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} |\omega(x,t)\partial_x w(x,t) - w(x,t)\partial_x\omega(x,t)|dxdt \leq \mathscr{A}(\bar{t}) \\
&+\sup_{[\bar{t},T)}\Big(\|w(\cdot,t)\|_{L^1(-\infty,\infty)} + \|\omega(\cdot,t)\|_{L^1(-\infty,\infty)}\Big)\int_{\bar{t}}^{T}\int_{-\infty}^{\infty}(|\varphi(x,t)|+|\psi(x,t)|)\,dxdt.
\end{aligned}
\tag{15.6.26}
$$

Recalling (15.2.6), we apply (15.6.26) for $w = w_i$, $\omega = \omega_i$, $\varphi = \varphi_i$, $\psi = \psi_i$ and $\sigma = \sigma_i$. In that case, the right-hand side of (15.6.26) is $O(\delta_0^2)$, by virtue of (15.2.7) and (15.4.23).

To verify (15.6.2), we integrate (15.6.21) over $[\bar{t},T)$, choosing $\delta_1 \leq \frac{1}{3}$. We thus obtain

$$
\begin{aligned}
&\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} |w(x,t)|\left|\partial_x\left(\frac{\omega(x,t)}{w(x,t)}\right)\right|^2 \chi_{\{|\omega/w|<3\delta_1\}}dxdt \\
&\qquad\qquad \leq 4\mathscr{L}(\bar{t}) + 4\int_{\bar{t}}^{T}\int_{-\infty}^{\infty}(|\varphi(x,t)|+|\psi(x,t)|)dxdt.
\end{aligned}
\tag{15.6.27}
$$

We now apply this inequality for $w = w_i$, $\omega = \omega_i$, $\varphi = \varphi_i$, $\psi = \psi_i$ and $\sigma = \sigma_i$, after performing the change of variable $x \mapsto x - \lambda_i^* t$, which renders $\lambda_i^* = 0$. In that case the right-hand side of (15.6.27) is $O(\delta_0)$, on account of (15.2.7) and (15.4.23). Hence

$$
\int_{\bar{t}}^{T}\int_{-\infty}^{\infty} |w_i|\left|\partial_x\left(\frac{\omega_i}{w_i}\right)\right|^2 \chi_{\{|\lambda_i^*+\omega_i/w_i|<3\delta_1\}}dxdt = O(\delta_0).
\tag{15.6.28}
$$

Since $\|w_i\|_{L^\infty} = O(\delta_0^2)$, (15.6.2) follows directly from (15.6.28).

15.7 Energy Estimates

Here we estimate the last term on the right-hand side of (15.4.21), which stems from our fixing the speed s_i according to (15.4.18). The aim is to show that

$$
\int_{\bar{t}}^{T}\int_{-\infty}^{\infty}(|\partial_x w_i| + |\partial_x\omega_i|)|\omega_i + s_i w_i|\,dxdt = O(\delta_0^2).
\tag{15.7.1}
$$

The proof, which relies on energy estimates, is technical and does not provide as much insight as the discussion in the previous two sections. Consequently, only an outline will be given here. The reader may find the details in the references cited in Section 15.9.

Since one may perform the change of variable $x \mapsto x - \lambda_i^* t$, we may assume, without loss of generality, that $\lambda_i^* = 0$ and hence $\sigma_i = O(\delta_0)$.

In addition to θ, defined by (15.4.19), we will employ the "cutoff" functions

$$\eta(r) = \begin{cases} 0 \text{ if } |r| \leq \frac{3}{5}\delta_1 \\ \\ 1 \text{ if } |r| \geq \frac{4}{5}\delta_1 \end{cases} \qquad |\eta'(r)| \leq \frac{20}{\delta_1}, \quad |\eta''(r)| \leq \frac{100}{\delta_1^2}, \tag{15.7.2}$$

and $\bar{\eta}(r) = \eta(|r| - \frac{1}{5}\delta_1)$. We write $\eta_i = \eta(\omega_i/w_i)$, and $\bar{\eta}_i = \bar{\eta}(\omega_i/w_i)$. We also choose $\delta_0 << \delta_1 << 1$.

15.7.1 Lemma. *When* $|\omega_i/w_i| \geq \frac{3}{5}\delta_1$,

$$\begin{cases} |w_i| \leq \dfrac{5}{2\delta_1}|\partial_x w_i| + O(\delta_0)\displaystyle\sum_{j\neq i}|w_j| \\ \\ |\omega_i| \leq 2|\partial_x w_i| + O(\delta_0)\displaystyle\sum_{j\neq i}|w_j|, \end{cases} \tag{15.7.3}$$

while when $|\omega_i/w_i| \leq \delta_1$,

$$|\partial_x w_i| \leq 2\delta_1|w_i| + O(\delta_0)\sum_{j\neq i}|w_j|. \tag{15.7.4}$$

Sketch of Proof. We substitute $\partial_x U$ and $\partial_t U$ from (15.4.18) into (15.2.1) to get

$$\sum_{j=1}^n \omega_j S_j + \sum_{j=1}^n w_j A S_j = \sum_{j=1}^n \partial_x w_j S_j + \sum_{j=1}^n w_j \partial_x S_j. \tag{15.7.5}$$

Multiplying, from the left, by $S_i^\top$, recalling that $S_i^\top S_i = 1$, $S_i^\top \partial_x S_i = 0$, and using (15.4.9) yields

$$\begin{aligned} \omega_i - \sigma_i w_i - \partial_x w_i &= \sum_{j\neq i}\{[\partial_x w_j - \omega_j]S_i^\top S_j + w_j S_i^\top \partial_x S_j - w_j S_i^\top A S_j\} \\ &= O(\delta_0)\sum_{j\neq i}(|w_j| + |\partial_x w_j - \omega_j|). \end{aligned} \tag{15.7.6}$$

Assertions (15.7.3) and (15.7.4) follow from careful analysis of the above equation. This completes the proof.

Since $|\omega_i + s_i w_i|$ vanishes when $|\omega_i/w_i| \leq \delta_1$ and is otherwise bounded by $|\omega_i|$, we have

$$|\omega_i + s_i w_i| \leq |\bar{\eta}_i \omega_i| \leq \bar{\eta}_i \left[2|\partial_x w_i| + O(\delta_0)\sum_{j\neq i}|w_j|\right]. \tag{15.7.7}$$

Therefore,

$$
\begin{aligned}
(15.7.8)\qquad (|\partial_x w_i| + |\partial_x \omega_i|)|\omega_i + s_i w_i| &\le 2\bar{\eta}_i|\partial_x w_i|^2 + 2\bar{\eta}_i|\partial_x w_i||\partial_x \omega_i| + \sum_{j\neq i}(|w_j\partial_x w_i| + |w_j\partial_x\omega_i|)\\
&\le 3\eta_i|\partial_x w_i|^2 + \bar{\eta}_i|\partial_x\omega_i|^2 + \sum_{j\neq i}(|w_j\partial_x w_i| + |w_j\partial_x\omega_i|).
\end{aligned}
$$

As shown in Section 15.5, the integral over $(-\infty,\infty)\times(\bar{t},T)$ of the third term on the right-hand side of (15.7.8) is $O(\delta_0^2)$. Thus, in order to verify (15.7.1) it will suffice to show

$$
(15.7.9)\qquad \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \eta_i|\partial_x w_i|^2 dxdt = O(\delta_0^2),
$$

$$
(15.7.10)\qquad \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \bar{\eta}_i|\partial_x \omega_i|^2 dxdt = O(\delta_0^2).
$$

The first step towards establishing (15.7.9) is to multiply $(15.2.6)_1$ by $2\eta_i w_i$, integrate the resulting equation over $(-\infty,\infty)$, and integrate by parts. This yields

$$
\begin{aligned}
(15.7.11)\qquad \int_{-\infty}^{\infty}\{\partial_t(\eta_i w_i^2) + \partial_x(\eta_i\sigma_i)w_i^2 - (\partial_t\eta_i + 2\sigma_i\partial_x\eta_i - \partial_x^2\eta_i)w_i^2&\\
+\, 2\eta_i(\partial_x w_i)^2 + 4(\partial_x\eta_i)w_i\partial_x w_i\}dx &= \int_{-\infty}^{\infty} 2\eta_i w_i\varphi_i dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(15.7.12)\qquad 2\int_{-\infty}^{\infty}\eta_i|\partial_x w_i|^2 dx = -\frac{d}{dt}\int_{-\infty}^{\infty}\eta_i w_i^2 dx + \int_{-\infty}^{\infty}(\partial_t\eta_i + 2\sigma_i\partial_x\eta_i - \partial_x^2\eta_i)w_i^2 dx&\\
+2\int_{-\infty}^{\infty}\eta_i\sigma_i w_i\partial_x w_i dx - 4\int_{-\infty}^{\infty}(\partial_x\eta_i)w_i\partial_x w_i dx + 2\int_{-\infty}^{\infty}\eta_i w_i\varphi_i dx.&
\end{aligned}
$$

We proceed to estimate the right-hand side of the above equation.

Recalling the definition of η_i and using (15.2.6), we obtain, after a short calculation,

$$
\begin{aligned}
(15.7.13)\qquad (\partial_t\eta_i + 2\sigma_i\partial_x\eta_i - \partial_x^2\eta_i)w_i^2&\\
= \eta_i'(\psi_i w_i - \varphi_i\omega_i) + 2\eta_i' w_i(\partial_x w_i)\partial_x(\omega_i/w_i) - \eta_i'' w_i^2[\partial_x(\omega_i/w_i)]^2.&
\end{aligned}
$$

Furthermore, using $(15.7.3)_1$ and since $\sigma_i = O(\delta_0) << \delta_1$,

$$\left| 2\int_{-\infty}^{\infty} \eta_i \sigma_i w_i \partial_x w_i dx \right| \leq \int_{-\infty}^{\infty} \eta_i |\partial_x w_i|^2 dx + O(\delta_0) \int_{-\infty}^{\infty} \sum_{j\neq i} |w_j \partial_x w_i| dx. \tag{15.7.14}$$

On the range where $\eta_i' \neq 0$, we have $|\omega_i/w_i| < \delta_1$ and hence (15.7.4) applies. One then obtains

$$\begin{aligned} |(\partial_x \eta_i) w_i \partial_x w_i| &= |\eta_i' w_i (\partial_x w_i) \partial_x(\omega_i/w_i)| \\ &\leq O(1)|w_i \partial_x \omega_i - \omega_i \partial_x w_i| + O(\delta_0) \sum_{j\neq i} (|w_j \partial_x w_i| + |w_j \partial_x \omega_i|). \end{aligned} \tag{15.7.15}$$

We now combine (15.7.12) with (15.7.13), (15.7.14), (15.7.15) and integrate the resulting inequality over $(\bar{t}, T)$. This yields an estimate of the form

$$\begin{aligned} \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \eta_i |\partial_x w_i|^2 dx dt \leq& \int_{-\infty}^{\infty} (\eta_i w_i^2)(x,\bar{t})\, dx \\ &+ O(1) \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} (|w_i \psi_i| + |w_i \varphi_i| + |\omega_i \varphi_i|) dx dt \\ &+ O(1) \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} |w_i \partial_x \omega_i - \omega_i \partial_x w_i|\, dx dt \\ &+ O(\delta_0) \int_{\bar{t}}^{T}\int_{-\infty}^{\infty} \sum_{j\neq i} (|w_j \partial_x w_i| + |w_j \partial_x \omega_i|) dx dt \\ &+ O(1) \int_{\bar{t}}^{T} \int_{|\omega_i/w_i|<\delta_1} |w_i \partial_x(\omega_i/w_i)|^2 dx dt. \end{aligned} \tag{15.7.16}$$

By virtue of (15.4.20), (15.2.7), (15.5.1), (15.6.1) and (15.6.2), we conclude that the right-hand side of (15.7.16) is $O(\delta_0^2)$, which verifies (15.7.9).

The estimate (15.7.10) is established by a similar procedure. For the details the reader should consult the references in Section 15.9.

15.8 Stability Estimates

This section provides a sketch of the proof of the stability estimates (15.1.5), (15.1.6) and (15.1.7).

On account of the rescaling $U(x,t) = U_\mu(\mu x, \mu t)$, the estimates (15.1.6) and (15.1.7), for solutions of (15.1.3), (15.1.2), are respectively equivalent to

$$\|U(\cdot,t) - U(\cdot,\tau)\|_{L^1(-\infty,\infty)} \leq b(|t-\tau| + |\sqrt{t} - \sqrt{\tau}|), \quad \tau, t \in [0,\infty), \tag{15.8.1}$$

$$\|U(\cdot,t) - \bar{U}(\cdot,t)\|_{L^1(-\infty,\infty)} \leq a\|U_{0\mu}(\cdot) - \bar{U}_{0\mu}(\cdot)\|_{L^1(-\infty,\infty)}, \; t \in (0,\infty), \tag{15.8.2}$$

for solutions of (15.2.1), (15.2.2).

The estimate (15.8.1) is obtained by integrating over (τ,t) the inequality

$$\|\partial_t U(\cdot,t)\|_{L^1(-\infty,\infty)} \leq b\left(1 + \frac{1}{2\sqrt{t}}\right), \qquad t \in (0,\infty), \tag{15.8.3}$$

which follows from (15.2.1), by virtue of (15.2.3), (15.3.1) and (15.2.4).

The estimate (15.8.2) is established by means of the following homotopy argument. We have

$$U(x,t) - \bar{U}(x,t) = \int_0^1 \frac{d}{d\xi} U_\xi(x,t) d\xi, \tag{15.8.4}$$

where U_ξ denotes the solution of (15.2.1) with initial data $\xi \bar{U}_{0\mu} + (1-\xi)U_{0\mu}$. The "tangent" vector

$$W_\xi(x,t) = \frac{d}{d\xi} U_\xi(x,t) \tag{15.8.5}$$

is the solution of the linearized equation

$$\partial_t W_\xi(x,t) + \partial_x \left[A(U_\xi(x,t))W_\xi(x,t)\right] = \partial_x^2 W_\xi(x,t), \tag{15.8.6}$$

with initial value

$$W_\xi(\cdot,0) = \bar{U}_{0\mu}(\cdot) - U_{0\mu}(\cdot). \tag{15.8.7}$$

Equation (15.8.6) bears a close resemblance to the equation satisfied by the derivative $\partial_x U$ of solutions to (15.2.1), and may thus be treated by the methods employed in earlier sections. The analysis, which is found in the references cited in Section 15.9, shows that, as $\|\partial_x U_\xi(\cdot,t)\|_{L^2(-\infty,\infty)} < \delta_0$ on $(0,\infty)$, there exists a constant $a > 1$ such that, for any $\delta > 0$, $\|W_\xi(\cdot,0)\|_{L^1(-\infty,\infty)} < \delta/a$ implies $\|W_\xi(\cdot,t)\|_{L^1(-\infty,\infty)} < \delta$, for all $t \in (0,\infty)$. Since (15.8.6) is linear, the above assertion is equivalent to

$$\|W_\xi(\cdot,t)\|_{L^1(-\infty,\infty)} \leq a\|W_\xi(\cdot,0)\|_{L^1(-\infty,\infty)}, \qquad 0 < t < \infty. \tag{15.8.8}$$

Upon combining (15.8.8) with (15.8.4), (15.8.5) and (15.8.7), we arrive at (15.8.2), thus establishing (15.1.7).

The remaining estimate (15.1.5) is an immediate corollary of (15.1.7). Indeed, we apply (15.1.7) for the two solutions $U_\mu(x,t)$ and $\bar{U}_\mu(x,t) = U_\mu(x+h,t)$, with corresponding initial values $U_0(x)$ and $\bar{U}_0(x) = U_0(x+h)$, we multiply the resulting inequality by h^{-1} and then let $h \to 0$, which yields (15.1.5).

Solutions of (15.1.1) constructed by the vanishing viscosity method have the finite speed of propagation property. Indeed, by use of the properties of the Green function it can be shown that when U_0 and $\bar{U}_0$ coincide inside an interval (y,z), in which case $U_{0\mu}(x) = \bar{U}_{0\mu}(x)$ for $x \in (y/\mu, z/\mu)$, then the corresponding solutions U and $\bar{U}$ of (15.2.1), (15.2.2) satisfy

(15.8.9)
$$|U(x,t) - \bar{U}(x,t)| \leq c\|U_0(\cdot) - \bar{U}_0(\cdot)\|_{L^\infty(-\infty,\infty)} \left\{ \exp\left(\nu t - x + \frac{y}{\mu}\right) + \exp\left(\nu t + x - \frac{z}{\mu}\right) \right\}$$

for some positive constants c, ν and all (x,t) in $(-\infty,\infty) \times (0,\infty)$. Upon rescaling, $(x,t) \mapsto (x/\mu, t/\mu)$, so as to return to U_μ, $\bar{U}_\mu$, we conclude that the two solutions $U = \lim_{k\to\infty} U_{\mu_k}$ and $\bar{U} = \lim_{k\to\infty} \bar{U}_{\mu_k}$ of (15.1.1), with initial values U_0 and $\bar{U}_0$, must coincide for all (x,t) with $x \in (y + \nu t, z - \nu t)$.

It follows from the above that in the place of (15.1.10) and (15.1.8) we have the more precise estimates

$$\int_y^z |U(x,t) - \bar{U}(x,t)|dx \leq a \int_{y-\nu t}^{z+\nu t} |U_0(x) - \bar{U}_0(x)|dx, \tag{15.8.10}$$

$$TV_{(y,z)} U(\cdot,t) \leq a TV_{(y-\nu t, z+\nu t)} U_0(\cdot), \tag{15.8.11}$$

for any $-\infty \leq y < z \leq \infty$.

We next demonstrate that the finite speed of propagation property in conjunction with the stability estimate (15.1.8) imply that any solution U of (15.1.1), (15.1.2) constructed by the vanishing viscosity method satisfies the Tame Oscillation Condition 14.10.1. In turn, by virtue of Theorem 14.10.2, this will imply that, when all characteristic families are either genuinely nonlinear or linearly degenerate, then U must coincide with the unique solution constructed by the random choice method.

Because solutions of (15.1.1) are preserved under spatial and temporal translations, it will suffice to verify (14.10.3) at the origin, $x = 0$, $t = 0$. We fix $\lambda > \nu$ and consider the solution $\bar{U}$ of (15.1.1) with initial data

$$\bar{U}_0(x) = \begin{cases} U_0(-\lambda h+) & -\infty < x \leq -\lambda h \\ U_0(x) & -\lambda h < x < \lambda h \\ U_0(\lambda h-) & \lambda h < x < \infty. \end{cases} \tag{15.8.12}$$

Then we have $TV_{(-\infty,\infty)} \bar{U}_0(\cdot) = TV_{(-\lambda h, \lambda h)} U_0(\cdot)$, $\bar{U}(0\pm, h) = U(0\pm, h)$ and $\bar{U}(\infty, h) = U_0(\lambda h-)$. Therefore, on account of (15.1.8),

$$\begin{aligned} |U(0\pm,h) - U_0(0\pm)| &\leq |\bar{U}(0\pm,h) - \bar{U}(\infty,h)| + |U_0(\lambda h-) - U_0(0\pm)| \\ &\leq (a+1) TV_{(-\lambda h, \lambda h)} U_0(\cdot), \end{aligned} \tag{15.8.13}$$

which establishes (14.10.3).

15.9 Notes

The construction of *BV* solutions by the vanishing viscosity method had been a central open problem of long standing in the theory of hyperbolic systems of conservation laws. It has finally been solved, in a spectacular way, by Bianchini and Bressan [5]. The presentation in this section abridges that fundamental paper. An informative survey of the theory is found in Bressan [14]. The ground had been prepared by the preliminary papers, Bianchini and Bressan [1,2,3,4]. For the rate of convergence, see Bressan and Yang [1]. The method has now been extended to cover initial-boundary value problems for hyperbolic conservation laws (Bianchini and Spinolo [1], Spinolo [1]). The principal underlying ideas of this approach have been fruitfully employed for constructing solutions to the Riemann Problem (Bianchini [6]), and for establishing convergence of semidiscrete upwind schemes for hyperbolic conservation laws (Bianchini [7,8], Bressan, Baiti and Jenssen [1]), of Godunov's method, for special systems (Bressan and Jenssen [1], Bressan, Jenssen and Baiti [1]), and of the linear Jin-Xin relaxation scheme (Bianchini [9]). See also Bianchini [4] and Bressan and Shen [1]. At the same time, resonance phenomena have also been detected in Godunov's scheme and the Lax-Friedrichs scheme that may drive the total variation of approximate solutions to infinity; see Bressan and Jenssen [1], Baiti, Bressan and Jenssen [1], and Bressan, Jenssen and Baiti [1]. A multitude of additional applications are to be expected in the near future. It should also be emphasized that these techniques apply to general quasilinear strictly hyperbolic systems, regardless of whether they are in conservation form. Of course, in the nonconservative case the constructed "solutions" do not necessarily satisfy the equations in the sense of distributions but should be interpreted in the context of the theory of nonconservative shocks by LeFloch *et al.,* outlined in Section 8.7.

The first steps towards understanding how viscous shocks form and interact, and how the viscous approximation converges, have been taken in Bressan and Donadello [1,2], Shen and Park [1], and Shen and Xu [1].

The rate of convergence, in L^1, of solutions of (15.1.3) to solutions of (15.1.1), as the viscosity coefficient μ tends to zero, is discussed in Bressan, Huang, Wang and Yang [1].

There is extensive literature on alternative aspects of the vanishing viscosity approach. We have already seen, in Chapter VI, how this method applies to scalar conservation laws, in the L^∞ or *BV* setting. In Chapter XVI we shall encounter applications to certain systems of conservation laws, in the L^p setting. Yet another direction is to investigate how solutions of the system with viscosity approximate given, piecewise smooth solutions of the hyperbolic system; see, for instance Goodman and Xin [1], Lin and Yang [1], Hoff and Liu [1], Serre [14], Rousset [4], Shih-Hsien Yu [1], Tang, Teng and Xin [1], and Jiang, Ni and Sun [1]. One may pursue the same objective in the context of relaxation schemes; see Lattanzio and Serre [1], and Li and Pan [1].

A major open problem is whether the method still applies when the identity matrix, on the right-hand side of (15.1.3), is replaced by a more general viscosity matrix. In particular, it is not known whether one may realize *BV* solutions to the (one-dimensional) Euler equations for isentropic gas flow as solutions to the Navier-Stokes equations with vanishingly small coefficient of (physical) viscosity.

In the vanishing viscosity approach, the approximate solutions U_μ carry information on the viscous shock profiles, which is especially valuable, when one employs genuine physical viscosity, but it is lost in the limit $\mu \to 0$. This loss of information also occurs when solutions are constructed by a vanishing capillarity or relaxation method, or even by the approach outlined in Section 8.7, in which the shock profile itself determines the notion of weak solution. As a remedy, LeFloch [6] suggests attaching the information on internal shock structure to the solution U of the hyperbolic system, by means of the following interesting device. Instead of tracking $U(x,t)$ as an evolving discontinuous function of x, one should realize it as a moving continuous curve $(\xi(s,t), V(s,t))$, where $\xi(\cdot,t)$ is a smooth nondecreasing function of the parameter s, having the following properties:

(a) $\xi(\pm\infty,t) = \pm\infty$;

(b) $\xi(\cdot,t)$ is invertible on the set of points x of continuity of $U(\cdot,t)$, and the inverse satisfies $V(\xi^{-1}(x,t),t) = U(x,t)$;

(c) If x is a point of discontinuity of $U(\cdot,t)$, then $\xi(s,t) = x$ for s on some closed interval, say $[s_-, s_+]$, with $V(s_\pm,t) = U(x\pm,t)$ and $V(\cdot,t)$ on (s_-,s_+) tracing the profile of the discontinuity that joins $U(x-,t)$ to $U(x+,t)$.

The discontinuity profile will be a shock profile, when (x,t) is a point of approximate jump discontinuity of U, or a full wave fan profile, when (x,t) belongs to the set of irregular points. The above idea is conceptually pleasing and will likely find technical applications as well. In that direction, see Glass and LeFloch [1].

XVI

BV Solutions for Systems of Balance Laws

The aim here is to discuss the existence and long time behavior of *BV* solutions to the Cauchy problem for (possibly inhomogeneous) strictly hyperbolic systems of balance laws. Thus, this chapter may be viewed as the counterpart of Section 5.5, where the same issues are addressed in the context of classical solutions. For the reasons presented in the preceding chapters, the investigation shall be confined to systems in a single spatial dimension and initial data of small total variation; however, modulo these limitations, the analogy to the results of Section 5.5 goes quite far. Thus, the existence of local solutions will be established under moderate restrictions on the flux and on the source, while global existence will hinge on the presence of damping. As in Section 5.5, damping shall be induced by a dissipative source incurring nonnegative entropy production.

A strongly dissipative source that would yield global solutions, without any assistance, is rarely encountered in the applications. In the common situation, where the source is merely partially dissipative, the existence of global solutions results from the synergy between flux and source, encoded in the Kawashima condition.

The solution may be constructed through any one of the three methods for solving systems of conservation laws, presented in the preceding chapters, namely, random choice, front tracking or vanishing viscosity, in conjunction with operator splitting, so as to account for the effects of the source and/or inhomogeneity. Random choice will be the method of choice here.

The construction of classical solutions, in Section 5.5, is in the realm of L^2-type Sobolev spaces, and the estimates induced by a convex entropy are well suited for that purpose. By contrast, the *BV* theory, here, requires L^1-type bounds. For systems of two balance laws, which are endowed with a rich family of entropies, L^1 estimates may be derived by constructing convex entropies with a conical singularity at the equilibrium state. For general systems, in which all available entropy functions are smooth at the equilibrium state, L^1 estimates will be derived from L^2 bounds, by exploiting the finite domain of dependence property of hyperbolic systems.

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16.1 The Cauchy Problem

We consider a generally inhomogeneous, strictly hyperbolic system of balance laws

$$\partial_t U(x,t) + \partial_x F(U(x,t),x,t) + P(U(x,t),x,t) = 0. \tag{16.1.1}$$

The flux F and source P are Lipschitz functions defined on $\mathscr{O} \times (-\infty,\infty) \times [0,\infty)$ and taking values in $\mathbb{R}^n$. Furthermore, the Jacobian matrix $\mathrm{D}F(U,x,t)$ is Lipschitz and possesses real distinct eigenvalues $\lambda_1(U,x,t) < \cdots < \lambda_n(U,x,t)$ that are strictly separated, uniformly in (x,t).

We assign initial conditions

$$U(x,0) = U_0(x), \qquad -\infty < x < \infty, \tag{16.1.2}$$

and seek the solution to the Cauchy problem.

The effects of inhomogeneity and the source term will be accounted for via an *operator splitting* algorithm, which marches in time steps. At each step, the approximate solution to the inhomogeneous balance law is obtained by concatenating approximate solutions of ordinary differential equations in the form $\partial_x F(\bar{U},x,\bar{t}) = 0$, $\partial_t U + P(\bar{U},\bar{x},t) = 0$ and homogeneous conservation laws $\partial_t U + \partial_x F(U,\bar{x},\bar{t}) = 0$. This general scheme allows for a number of variations. To begin with, one may handle the homogeneous conservation laws part by any one of the methods developed in the previous three chapters, namely, random choice, front tracking or vanishing viscosity. Here we opt for the random choice method. Moreover, as we shall see, the specifics of the algorithm may have to be adapted to suit particular features of the system. In its simplest, generic form, the operator splitting algorithm proceeds as follows.

As in Section 13.2, we start out with a random sequence $\wp = \{a_0, a_1, \dots\}$, with $a_s \in (-1,1)$. We fix the spatial mesh-length h, with associated time mesh-length $\tau = \lambda^{-1}h$, and build the staggered grids of mesh-points (x_r,t_s), for $r+s$ even, and sampling points (y_s^r,t_s), $y_s^r = x_r + a_s h$, for $r+s$ odd.

Assuming U_h is already known on $\{(x,t) : -\infty < x < \infty,\, 0 \le t < t_s\}$, we define U_s^r, for $r+s$ odd, by means of (13.2.5), and then set

$$\hat{U}_s^r = U_s^r - \tau P(U_s^r, x_r, t_s). \tag{16.1.3}$$

Next we determine V_s^r and W_s^r, for $r+s$ odd, as solutions to the equation

$$F(W_s^r, x_{r+1}, t_s) = F(\hat{U}_s^r, x_r, t_s) = F(V_s^r, x_{r-1}, t_s). \tag{16.1.4}$$

To make (16.1.4) solvable, one may have to change coordinates $(x,t) \mapsto (y,t)$, with $y = y(x,t)$, so as to eliminate any zero characteristic speeds. Finally, we define U_h on $\{(x,t) : x_{r-1} \le x < x_{r+1}\,,\ t_s \le t < t_{s+1}\}$, for $r+s$ even, as the restriction to this rectangle of the solution to the Riemann problem

$$\partial_t U_h(x,t) + \partial_x F(U_h(x,t),x_r,t_s) = 0, \qquad t \ge t_s\,, \tag{16.1.5}$$

$$U_h(x,t_s) = \begin{cases} W_s^{r-1}, & x < x_r \\ V_s^{r+1}, & x > x_r\,. \end{cases} \tag{16.1.6}$$

The algorithm is initiated, at $s = 0$, by (13.1.10).

Inhomogeneity and the source term may amplify the total variation of approximate solutions, driving it beyond the range of currently available analytical tools. In order to keep the effect of inhomogeneity under control, we impose the following restrictions on the functions F and P: for any $U \in \mathscr{O}$, $x \in (-\infty,\infty)$ and $t \in [0,\infty)$[1],

$$|\mathrm{D}F_x(U,x,t)| \leq \omega, \qquad |\mathrm{D}F_t(U,x,t)| \leq \omega, \tag{16.1.7}$$

$$|\mathrm{D}F_x(U,x,t)| \leq f(x), \qquad |P_x(U,x,t)| \leq f(x), \tag{16.1.8}$$

where $f(x)$ is a $W^{1,1}(-\infty,\infty)$ function such that

$$\int_{-\infty}^{\infty} f(x)dx \leq \omega, \tag{16.1.9}$$

and ω is a nonnegative number. Under these conditions, the Cauchy problem admits at least local BV solutions:

16.1.1 Theorem. *For sufficiently small positive numbers ω and δ, there exists time $T = T(\omega,\delta)$, with $T(\omega,\delta) \to \infty$ as $(\omega,\delta) \to 0$, such that when* (16.1.7), (16.1.8), (16.1.9) *hold and*

$$TV_{(-\infty,\infty)}U_0(\cdot) = \delta, \tag{16.1.10}$$

then there exists an admissible BV solution U of (16.1.1), (16.1.2) *on the time interval $[0,T)$. For each fixed $t \in [0,T)$, $U(\cdot,t)$ is a function of bounded variation on $(-\infty,\infty)$ and*

$$TV_{(-\infty,\infty)}U(\cdot,t) \leq c_1(\delta+\omega)\,e^{\rho t}, \tag{16.1.11}$$

for some positive constants c_1 and ρ.

The proof of the above proposition, which rests on a fairly straightforward, though tedious, adaptation of the analysis in Chapter XIII that culminated in the proof of Theorem 13.1.1, can be found in the references cited in Section 16.7. Actually, in Section 16.4, the reader will find a detailed application of the operator splitting method, involving a variant of this algorithm, specially adapted to a particular form of systems (16.1.1). The exponential growth in the total variation is induced by both inhomogeneity and the source term, and the exponent ρ is $O(\omega+\gamma)$, where

[1] Throughout this chapter, n-vectors shall be regarded, and normed, as elements of ℓ_n^1, and $n \times n$ matrices shall be regarded, and normed, as linear operators on ℓ_n^1.

$\gamma = \sup|\mathrm{D}P|$. Of course, the solution cannot break down as long as $TV_{(-\infty,\infty)}U_h(\cdot,t)$ stays small.

Our next task is to identify classes of systems for which the Cauchy problem admits global *BV* solutions. The simplest mechanism that would keep the total variation small is rapid decay of the inhomogeneity and the source term as $t \to \infty$. Suppose that we replace the assumptions (16.1.7) and (16.1.8) by

$$|\mathrm{D}F_x(U,x,t)| \leq \omega g(t), \qquad |\mathrm{D}F_t(U,x,t)| \leq \omega g(t), \tag{16.1.12}$$

$$|P(U,x,t)| \leq \omega g(t), \qquad |\mathrm{D}P(U,x,t)| \leq \omega g(t), \tag{16.1.13}$$

$$|\mathrm{D}F_x(U,x,t)| \leq f(x)g(t), \quad |P_x(U,x,t)| \leq f(x)g(t), \tag{16.1.14}$$

for all $U \in \mathscr{O}$, $x \in (-\infty,\infty)$, $t \in [0,\infty)$, where $f(x)$ and ω are as above, while $g(t)$ is a bounded function in $L^1(0,\infty)$. Then a simple corollary of Theorem 16.1.1, and in particular of the estimate (16.1.11), is the following

16.1.2 Theorem. *For sufficiently small positive numbers ω and δ, when* (16.1.12), (16.1.13), (16.1.14), (16.1.9) *and* (16.1.10) *hold, then there exists a global admissible BV solution U of* (16.1.1), (16.1.2). *For each $t \in [0,\infty)$, $U(\cdot,t)$ is a function of bounded variation on $(-\infty,\infty)$ and*

$$TV_{(-\infty,\infty)}U(\cdot,t) \leq c_1(\delta+\omega). \tag{16.1.15}$$

A considerably subtler mechanism that induces global existence to the Cauchy problem is the rapid decay of the inhomogeneity and the source term as $|x| \to \infty$, in conjunction with nonzero characteristic speeds. Indeed, when all the characteristic speeds are bounded away from zero, one should expect that as t increases the bulk of the wave moves far away from the origin and eventually enters, and stays, in the region where inhomogeneity and the source term have negligible influence. To verify this conjecture requires delineating the global wave pattern and tracking the bulk of the wave. This may be effected only by the method of wave tracing, outlined in Section 13.8. A representative result in that direction is the following proposition, which is established in the references cited in Section 16.7.

16.1.3 Theorem. *Consider the strictly hyperbolic system of balance laws*

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + P(U(x,t),x) = 0, \tag{16.1.16}$$

with nonzero characteristic speeds, and characteristic families that are either genuinely nonlinear or linearly degenerate. Assume that for any U in $\mathscr{O}$ and x in $(-\infty,\infty)$,

$$|P(U,x)| \leq f(x), \qquad |\mathrm{D}P(U,x)| \leq f(x), \tag{16.1.17}$$

where $f(x)$ *satisfies* (16.1.9) *with* ω *sufficiently small. If the initial data* U_0 *have bounded variation,* (16.1.10) *with* δ *sufficiently small, then there exists a global admissible BV solution* U *of* (16.1.16), (16.1.2). *For each fixed* $t \in [0,\infty)$, $U(\cdot,t)$ *is a function of bounded variation on* $(-\infty,\infty)$ *and*

(16.1.18)
$$TV_{(-\infty,\infty)}U(\cdot,t) \leq c_1\,(\delta+\omega).$$

A typical application of the above proposition is to the system (7.1.24) that governs the isentropic flow of a gas through a duct of varying cross section $a(x)$. We rewrite (7.1.24) in the form (16.1.16):

(16.1.19)
$$\begin{cases} \partial_t v + \partial_x(\rho v) + a^{-1}(x)a'(x)\rho v = 0 \\ \\ \partial_t(\rho v) + \partial_x[\rho v^2 + p(\rho)] + a^{-1}(x)a'(x)\rho v^2 = 0. \end{cases}$$

Clearly, in order to meet the requirement (16.1.17) of Theorem 16.1.3, one needs to assume that $a(x)$ has sufficiently small total variation on $(-\infty,\infty)$.

Still another factor that may induce global existence to the Cauchy problem is the presence of a dissipative source. This will be the object of investigation in the remainder of this chapter.

16.2 Strong Dissipation

We here begin the investigation of systems of balance laws with dissipative source. It turns out that dissipation may secure global existence of BV solutions, with initial values of small total variation, even in the presence of inhomogeneity. However, in order to keep the analysis as simple as possible, we shall consider throughout only homogeneous hyperbolic systems of balance laws

(16.2.1)
$$\partial_t U(x,t) + \partial_x F(U(x,t)) + P(U(x,t)) = 0.$$

We assume the origin $U=0$ is contained in the domain $\mathscr{O}$ and $P(0)=0$, so that $U \equiv 0$ is an equilibrium solution.

Since the analysis is in BV space, we have to impose on P conditions that would render it dissipative in L^1. In order to identify the proper assumptions, we substitute $U = R(0)V$, where $R(U)$ is the $n\times n$ matrix with column vectors a set of linearly independent right eigenvectors $R_1(U),\cdots,R_n(U)$ of $\mathrm{D}F(U)$, and linearize (16.2.1) about 0. This yields the system

(16.2.2)
$$\partial_t V_i(x,t) + \lambda_i(0)\partial_x V_i(x,t) + \sum_{j=1}^{n} A_{ij}V_j(x,t) = 0, \qquad i=1,\cdots,n,$$

with

(16.2.3) $$A = L(0)\mathrm{D}P(0)R(0),$$

where $L(U)$ denotes the $n\times n$ matrix whose row vectors $L_1(U),\cdots,L_n(U)$ are the left eigenvectors of $\mathrm{D}F(U)$ normalized by $L(U)R(U)=I$. We multiply (16.2.2) by $\operatorname{sgn} V_i(x,t)$, integrate with respect to x over $(-\infty,\infty)$, and sum over $i=1,\cdots,n$, to deduce that when A is column-diagonally dominant, namely

(16.2.4) $$A_{ii} - \sum_{j\neq i} |A_{ji}| \geq \mu > 0, \qquad i = 1,\cdots,n,$$

then, as $t\to\infty$, solutions of (16.2.2) decay exponentially to zero in $L^1(-\infty,\infty)$.

It should be noted that whether the diagonal dominance property (16.2.4) holds may depend on the particular matrix $R(U)$ of right eigenvectors employed in the construction of A. Indeed, choosing the equivalent matrix $\hat{R}(U)=R(U)K$ of eigenvectors, where K is some positive diagonal matrix, would replace A with the matrix $\hat{A}=K^{-1}AK$; and diagonal dominance is not generally preserved under such similarity transformations. Given a matrix A, it is possible to find a positive diagonal matrix K that renders $K^{-1}AK$ column diagonally dominant if and only if all eigenvalues of the matrix $\tilde{A}$, with entries $\tilde{A}_{ii}=A_{ii}$, $i=1,\cdots,n$ and $\tilde{A}_{ij}=-|A_{ij}|$, for $i\neq j$, have positive real part (references in Section 16.7). In particular, this class of A encompasses positive triangular matrices as well as row-diagonally dominant matrices (by Geršgorin's theorem).

For any $\tau>0$, multiplying the linear system $(I+\tau A)X=Y$, from the left, by the row vector $\operatorname{sgn} X^\top$, yields

(16.2.5) $$|(I+\tau A)^{-1}| \leq (1+\mu\tau)^{-1}.$$

As we shall see, it is this property that induces existence of global solutions to the Cauchy problem for (16.2.1).

16.2.1 Theorem. *Consider the homogeneous, strictly hyperbolic system of balance laws* (16.2.1)*, with characteristic families that are either genuinely nonlinear or linearly degenerate. Let* 0 *be an equilibrium state,* $P(0)=0$. *Assume that for some selection of eigenvectors of* $\mathrm{D}F(0)$*, the matrix* A*, defined by* (16.2.3)*, is column-diagonally dominant* (16.2.4)*. Given initial data* U_0 *of bounded variation,* (16.1.10) *with* δ *sufficiently small, and* $U_0(x)\to 0$ *as* $x\to-\infty$ *, there exists a global admissible BV solution* U *of* (16.2.1), (16.1.2). *For each fixed* $t\in[0,\infty)$*,* $U(\cdot,t)$ *is a function of bounded variation on* $(-\infty,\infty)$ *and*

(16.2.6) $$TV_{(-\infty,\infty)}U(\cdot,t) \leq c_1\delta e^{-\nu t},$$

where c_1 *and* ν *are positive constants.*

Sketch of Proof. A detailed, technical proof of a stronger result, along the same lines, will be presented in the following sections of this chapter. Nevertheless, we provide here an outline of the proof of Theorem 16.2.1 so as to communicate to the reader, in relatively simple terms, how operator splitting works and the role played by the diagonal dominance assumption on the matrix A.

We construct the solution U by means of the scheme outlined in Section 16.1. The proof of consistency follows closely the argument used in the proof of Theorem 13.1.1 and need not be repeated here. It will suffice to establish a bound for the total variation $TV_{(-\infty,\infty)}U_h(\cdot,t)$ of the approximate solution U_h that will yield in the limit $h \downarrow 0$ the asserted estimate (16.2.6).

As in Section 13.3, for $r+s$ even we consider the diamond Δ_s^r with vertices (y_s^{r-1},t_s), (y_{s-1}^r,t_{s-1}), (y_s^{r+1},t_s) and (y_{s+1}^r,t_{s+1}), depicted in Fig. 13.3.1. The aim is to estimate the strength of the outgoing wave fan ε, emanating from (x_r,t_r), in terms of the strengths of the incoming wave fans α and β, which emanate from (x_{r-1},t_{s-1}) and (x_{r+1},t_{s-1}).

Since our system is homogeneous, (16.1.4) yields $V_s^r = \hat{U}_s^r = W_s^r$. According to the prescription of the algorithm,

(16.2.7) $$\Omega(\alpha;U_s^{r-1}) = \hat{U}_{s-1}^r,$$

(16.2.8) $$\Omega(\beta;\hat{U}_{s-1}^r) = U_s^{r+1},$$

(16.2.9) $$\Omega(\varepsilon;\hat{U}_s^{r-1}) = \hat{U}_s^{r+1},$$

where Ω is the wave fan function, defined by (9.3.4).

Let us consider the wave fan $\tilde{\varepsilon}$ that would have resulted from the interaction of α and β in the absence of a source term, i.e.,

(16.2.10) $$\Omega(\tilde{\varepsilon};U_s^{r-1}) = \Omega(\beta;\Omega(\alpha;U_s^{r-1})) = U_s^{r+1}.$$

By virtue of Theorem 9.9.1,

(16.2.11) $$\tilde{\varepsilon} = \alpha + \beta + O(1)\mathscr{D}(\Delta_s^r),$$

where the wave interaction term $\mathscr{D}(\Delta_s^r)$ is defined by (13.3.2).

We proceed to relate ε to $\tilde{\varepsilon}$. Since $\Omega(0;U) = U$, for any $U \in \mathscr{O}$, (16.2.9) together with (16.1.3) and $P(0) = 0$ yield

(16.2.12) $$\Omega(\varepsilon;U_s^{r-1}) = U_s^{r+1} - \tau[P(U_s^{r+1}) - P(U_s^{r-1})] + o(1)h|\varepsilon|,$$

where $o(1)$ denotes a quantity that becomes arbitrarily small when $\sup|U_h|$ is sufficiently small. By virtue of (9.3.8),

(16.2.13) $$\Omega(\varepsilon;U_s^{r-1}) - \Omega(\tilde{\varepsilon};U_s^{r-1}) = \tilde{R}(\varepsilon - \tilde{\varepsilon}),$$

where $\tilde{R}$ is some matrix close to the matrix $R(0)$ of right eigenvectors of $\mathrm{D}F(0)$. Furthermore, on account of (16.1.3),

(16.2.14) $$P(U_s^{r+1}) - P(U_s^{r-1}) = H[\hat{U}_s^{r+1} - \hat{U}_s^{r-1}] + \tau H[P(U_s^{r+1}) - P(U_s^{r-1})],$$

where H is some matrix close to $\mathrm{D}P(0)$. Finally, by (9.3.8) and (16.2.9),

$$\hat{U}_s^{r+1} - \hat{U}_s^{r-1} = \Omega(\varepsilon; \hat{U}_s^{r-1}) - \Omega(0; \hat{U}_s^{r-1}) = \hat{R}\varepsilon, \tag{16.2.15}$$

where $\hat{R}$ is some matrix close to $R(0)$. We now combine (16.2.10), (16.2.12), (16.2.13), (16.2.14) and (16.2.15) to get

$$\tilde{\varepsilon} = [I + \tau\tilde{A}]\varepsilon + o(1)h|\varepsilon|, \tag{16.2.16}$$

where

$$\tilde{A} = \tilde{R}^{-1}[I - \tau H]^{-1} H\hat{R} \tag{16.2.17}$$

is close to the matrix A, defined by (16.2.3).

On account of (16.2.11), (16.2.16) and (16.2.5), we conclude that, for as long as $\sup|U_h|$ stays sufficiently small,

$$|\varepsilon| \leq (1 - 3\nu\tau)(|\alpha| + |\beta|) + c\mathscr{D}(\Delta_s^r), \tag{16.2.18}$$

with $\nu = \mu/4 > 0$.

From (16.2.11), (16.2.16) and (16.2.18), we also deduce

$$|\varepsilon - (\alpha + \beta)| \leq ch(|\alpha| + |\beta|) + c\mathscr{D}(\Delta_s^r). \tag{16.2.19}$$

As in Section 13.4, we consider mesh curves I and associate with them the functionals $\mathscr{L}(I)$, $\mathscr{Q}(I)$ and $\mathscr{G}(I)$, defined by (13.4.2), (13.4.5) and (13.4.8). Assuming J is the immediate successor to I, depicted in Fig. 13.4.1, we may retrace the analysis in Section 13.4, using (16.2.18) to get

$$\mathscr{L}(J) \leq \mathscr{L}(I) - 3\nu\tau(|\alpha| + |\beta|) + c\mathscr{D}(\Delta_s^r), \tag{16.2.20}$$

in the place of (13.4.4), and then using (16.2.19) to get

$$\mathscr{Q}(J) - \mathscr{Q}(I) \leq ch\mathscr{L}(I)(|\alpha| + |\beta|) + [c\mathscr{L}(I) - 1]\mathscr{D}(\Delta_s^r), \tag{16.2.21}$$

in the place of (13.4.7). Thus, for κ sufficiently large and $\mathscr{L}(I)$ sufficiently small,

$$\mathscr{G}(J) \leq \mathscr{G}(I) - 2\nu\tau(|\alpha| + |\beta|). \tag{16.2.22}$$

Next, for fixed $s = 0, 1, 2, \dots$, we consider the mesh curve J_s with vertices all the sampling points (y_s^{r-1}, t_s) and (y_{s+1}^r, t_{s+1}) with $r + s$ even. Then assuming that $\sup|U_h|$ is so small that $\mathscr{G}(J_{s-1}) \leq 2\mathscr{L}(J_{s-1})$, (16.2.22) yields

$$\mathscr{G}(J_s) \leq (1 - \nu\tau)\mathscr{G}(J_{s-1}). \tag{16.2.23}$$

Thus, for any $t_s < t < t_{s+1}$,

$$TV_{(-\infty,\infty)} U_h(\cdot, t) \leq (1 - \nu\tau)^s \mathscr{G}(J_0), \tag{16.2.24}$$

where the total variation is measured by $\mathscr{L}(J_s)$.

Since $U_h(x,t) \to 0$ as $x \to -\infty$, the right-hand side of (16.2.24) also bounds $\sup_{(-\infty,\infty)}|U_h(\cdot,t)|$.

On the right-hand side of (16.2.24), $\mathscr{G}(J_0)$ is bounded by $cTV_{(-\infty,\infty)}U_0(\cdot)$. Therefore, letting $h \downarrow 0$, (16.2.24) yields (16.2.6). The proof is complete.

Sources inducing diagonally dominant matrices A have the same standing as the dissipative definite sources encountered in Section 5.5: they readily yield global existence and exponential decay of solutions to the Cauchy problem, but they are rarely encountered in the systems arising in physical applications. In the following sections we shall see how one may establish global existence for a broader class of sources, akin to dissipative semidefinite sources in Section 5.5.

16.3 Redistribution of Damping

Since diagonal dominance (16.2.4) of the matrix A is too restrictive for the intended applications, we seek here alternative assumptions that would still induce existence of BV solutions, in the large, to the Cauchy problem (16.2.1), (16.1.2).

For $\omega > 0$, we let $\mathscr{C}_\omega$ denote the class of initial data U_0 with the property that any admissible BV solution U of (16.2.1), (16.1.2) must satisfy

$$\int_{-\infty}^{\infty} |U(x,t)|dx \leq \omega, \tag{16.3.1}$$

on any time interval it may exist. The aim is to show that admissible BV solutions to (16.2.1), with initial values in $\mathscr{C}_\omega$, exist in the large, provided that the entries along the principal diagonal of the matrix A are positive:

$$A_{ii} > 0, \qquad i = 1, \dots, n. \tag{16.3.2}$$

As we shall see in Sections 16.5 and 16.6, (16.3.2) is a typical property of systems modeling relaxation phenomena. Furthermore, in such systems, the class $\mathscr{C}_\omega$ is sufficiently rich to encompass initial data of interest, near the equilibrium state.

16.3.1 Theorem. *Consider the homogeneous, strictly hyperbolic system of balance laws* (16.2.1), *with characteristic families that are either genuinely nonlinear or linearly degenerate. Let* 0 *be an equilibrium state,* $P(0) = 0$, *and assume that the entries* A_{ii} *of the matrix* A, *defined by* (16.2.3), *are positive* (16.3.2). *Then there are positive numbers* ω_0 *and* δ_0 *such that for any initial data* U_0 *that belong to* $\mathscr{C}_\omega$, *with* $\omega < \omega_0$, *and satisfy* (16.1.10), *with* $\delta < \delta_0$, *the Cauchy problem* (16.2.1), (16.1.2) *possesses an admissible* BV *solution* U *on* $(-\infty,\infty) \times [0,\infty)$, *with the property*

$$TV_{(-\infty,\infty)}U(\cdot,t) \leq c_0\omega + c_1\delta e^{-\nu t}, \qquad 0 \leq t < \infty, \tag{16.3.3}$$

for positive constants c_0, c_1 *and* ν, *independent of* U_0. *Furthermore, if*

$$(16.3.4)\qquad \int_{-\infty}^{\infty} |U(x,t)|dx \to 0, \qquad \text{as } t \to \infty,$$

then

$$(16.3.5)\qquad TV_{(-\infty,\infty)}U(\cdot,t) \to 0, \qquad \text{as } t \to \infty.$$

We lay down here the road map for the proof of the above theorem, while the technical details are deferred to Section 16.4.

In the system governing relaxation phenomena, the diagonal dominance condition (16.2.4) on A fails to hold, even though the source is dissipative, because the damping action is unevenly distributed among the equations of the system. This deficiency will be remedied here by redistributing the damping, on a more equitable basis, through a change of state vector, as follows.

Assuming U is an admissible BV solution of (16.2.1), (16.1.2), defined on a strip $(-\infty,\infty)\times[0,T)$ and $U(\cdot,t)$ is integrable on $(-\infty,\infty)$, for all $t \in [0,T)$, we introduce the functions

$$(16.3.6)\qquad \Phi(x,t) = \int_{-\infty}^{x} NU(y,t)dy,$$

$$(16.3.7)\qquad Z(x,t) = \int_{-\infty}^{x} NP(U(y,t))dy,$$

where N is a $n \times n$ matrix to be specified below. We note that Φ is Lipschitz with

$$(16.3.8)\qquad \partial_x\Phi(x,t) = NU(x,t), \quad \partial_t\Phi(x,t) = -NF(U(x,t)) - Z(x,t).$$

We now replace U by the new state vector

$$(16.3.9)\qquad \hat{U} = U - \Phi$$

and rewrite (16.2.1) as a system for $\hat{U}$, namely

$$(16.3.10)\qquad \partial_t\hat{U}(x,t) + \partial_x\hat{F}(\hat{U}(x,t),\Phi(x,t)) + \hat{P}(\hat{U}(x,t),\Phi(x,t),Z(x,t)) = 0,$$

where

$$(16.3.11)\qquad \hat{F}(\hat{U},\Phi) = F(\hat{U}+\Phi) - F(\Phi),$$

$$(16.3.12)\qquad \hat{P}(\hat{U},\Phi,Z) = P(\hat{U}+\Phi) - NF(\hat{U}+\Phi) + \mathrm{D}F(\Phi)N[\hat{U}+\Phi] - Z.$$

The motivation for switching from the relatively simple (16.2.1) to the cumbersome (16.3.10), which is not even a closed system, is that if one presumes that Φ and Z are somehow known, and regards (16.3.10) as an inhomogeneous system of balance laws for $\hat{U}$, then, in the place of (16.2.3), one gets the matrix $\hat{A}$ with entries

$$\hat{A}_{ij} = L_j(0)\mathrm{D}\hat{P}(0,0,0)R_j(0) = A_{ij} + [\lambda_i(0) - \lambda_j(0)]\Delta_{ij}\,, \tag{16.3.13}$$

where $\Delta = L(0)NR(0)$. This presents the opportunity of making $\hat{A}$ diagonally dominant by properly selecting the matrix N. In particular, the choice $N = R(0)\Delta L(0)$, with $\Delta_{ii} = 0$, for $i = 1, \dots, n$ and

$$\Delta_{ij} = \frac{A_{ij}}{\lambda_i(0) - \lambda_j(0)}\,, \qquad \text{for } i \neq j, \tag{16.3.14}$$

renders a diagonal $\hat{A}$, with $\hat{A}_{ii} = A_{ii}$, for $i = 1, \dots, n$, and $\hat{A}_{ij} = 0$, for $i \neq j$. In that case, when (16.3.2) holds, $\hat{A}$ is diagonally dominant.

The above considerations suggest the following procedure for proving Theorem 16.3.1. Starting out with U_0 that lies in $\mathscr{C}_\omega$ and satisfies (16.1.10), for ω and δ sufficiently small, Theorem 16.1.1 guarantees the existence of a local admissible *BV* solution U to the Cauchy problem (16.2.1), (16.1.2), on some time interval $[0,T)$. In fact it is known (references in Section 16.7) that, similar to the cases of systems of conservation laws discussed in Section 14.9, homogeneous systems of balance laws in the form (16.2.1) generate standard Riemann semigroups, inducing uniqueness of the solution U.

From the solution U one may determine Φ and Z on $(-\infty,\infty) \times [0,T)$, through (16.3.6) and (16.3.7), and then construct $\hat{U}$ by solving the Cauchy problem for (16.3.10) under initial conditions $\hat{U}(x,0) = \hat{U}_0(x)$,

$$\hat{U}_0(x) = U_0(x) - \int_{-\infty}^{x} NU_0(y)dy, \qquad -\infty < x < \infty. \tag{16.3.15}$$

Finally, one may recover U as $\hat{U} + \Phi$. The gain is that the dissipative action on the source $\hat{P}$, manifested in the diagonal dominance of $\hat{A}$, will emerge through the reconstruction of U from $\hat{U}$, inducing the estimate (16.3.3) on the variation, with constants c_0, c_1 and μ independent of T. The details of the derivation of (16.3.3), via the above process, will be presented in the next section 16.4.

Armed with the estimate (16.3.3), we may immediately extend the solution U to the entire upper half-plane $(-\infty,\infty) \times [0,\infty)$. Furthermore, in the presence of (16.3.3), (16.3.4) implies (16.3.5). To see this, assuming (16.3.4) holds, we fix any $\varepsilon > 0$, set $\bar{\omega} = \varepsilon/2c_0$ and identify $\bar{t} > 0$ such that $\|U(\cdot,t)\|_{L^1} < \bar{\omega}$ for $t \in [\bar{t},\infty)$. Then we have $U(\cdot,\bar{t}) \in \mathscr{C}_{\bar{\omega}}$. Moreover, by virtue of (16.3.3), $TV_{(-\infty,\infty)}U(\cdot,\bar{t}) < c_0\omega + c_1\delta$. Next we apply (16.3.3), after shifting the origin of time from 0 to $\bar{t}$, which yields

$$TV_{(-\infty,\infty)}U(\cdot,t) \leq c_0\bar{\omega} + c_1(c_0\omega + c_1\delta)e^{-\nu(t-\bar{t})}, \quad \bar{t} \leq t < \infty. \tag{16.3.16}$$

It is now clear that there is $t_\varepsilon > \bar{t}$ such that $TV_{(-\infty,\infty)}U(\cdot,t) < \varepsilon$, for all $t \geq t_\varepsilon$.

16.4 Bounds on the Variation

Following the procedure outlined in Section 16.3, we complete here the proof of Theorem 16.3.1 by establishing the estimate (16.3.3).

For fixed initial data U_0 that lie in $\mathscr{C}_\omega$ and satisfy (16.1.10), with ω and δ small, we let U denote the (local) admissible *BV* solution to (16.2.1), (16.1.2), defined on some strip $(-\infty,\infty)\times[0,T)$. The aim is to derive bounds for the function

$$X(t) = TV_{(-\infty,\infty)}U(\cdot,t), \qquad 0 \le t < T. \tag{16.4.1}$$

The solution U induces, through (16.3.6) and (16.3.7), Φ and Z as functions of (x,t) on $(-\infty,\infty)\times[0,T)$, so that one may regard $\hat{F}$ and $\hat{P}$, defined by (16.3.11) and (16.3.12), as given functions of the variables $(\hat{U},x,t)$. The plan is to (re)construct $\hat{U}$ on $(-\infty,\infty)\times[0,T)$ as solution to the inhomogeneous system of balance laws (16.3.10), with initial data $\hat{U}_0$, defined by (16.3.15).

The construction of $\hat{U}$ will be effected by means of the random choice method, in conjunction with operator splitting. Our system is in the form (16.1.1) but it has special structure as inhomogeneity enters implicitly through the functions Φ and Z. Furthermore, for fixed $\hat{U}$, the functions $\hat{F}$ and $\hat{P}$ are merely Lipschitz. These special features will force us to employ a cumbersome variation of the generic algorithm outlined in Section 16.1.

As in Section 16.1, we assume, without loss of generality, that none of the characteristic speeds may vanish, we fix a random sequence $\mathscr{P}=\{a_0,a_1,\dots\}$, with $a_s\in(-1,1)$, set the spatial mesh-length h and the temporal mesh-length $\tau=\lambda^{-1}h$, and identify the staggered grids of mesh-points (x_r,t_s), with $x_r=rh$, $t_s=s\tau$, for $r+s$ even, and sampling points (y_s^r,t_s), with $y_s^r=x_r+a_sh$, for $r+s$ odd.

The approximate solution $\hat{U}_h$ will be determined sequentially on the strips

$$\mathscr{S}_s=\{(x,t)\,:\,-\infty<x<\infty,\ t_s\le t<t_{s+1}\}, \qquad s=0,1,\dots,s^*, \tag{16.4.2}$$

where s^* is the largest integer with $(s^*+1)\tau\le T$.

The algorithm is initiated by setting $\hat{U}_h(x,0)=\hat{U}_0(x)$, for $-\infty<x<\infty$.

Assuming now that $\hat{U}_h$ has been determined on $\bigcup_{k=0}^{s-1}\mathscr{S}_k$, we extend its domain to $\mathscr{S}_s$ by the following procedure.

For $r=0,\pm1,\pm2,\dots$ and $s=0,1,\dots,s^*$, we define

$$\Phi_s^r=\Phi(x_r,t_s), \quad Z_s^r=Z(x_r,t_s) \tag{16.4.3}$$

and then set, for $r+s$ odd,

$$\hat{U}_s^r=\hat{U}_h(y_s^r,t_s-), \quad U_s^r=\hat{U}_s^r+\Phi_s^r, \tag{16.4.4}$$

$$Q_s^r=\mathrm{D}F(U_s^r)^{-1}\mathrm{D}F(\Phi_s^r), \tag{16.4.5}$$

$$\hat{P}_s^r=\hat{P}(\hat{U}_s^r,\Phi_s^r,Z_s^r), \tag{16.4.6}$$

$$\begin{cases} V_s^r=U_s^r+Q_s^r[\Phi_s^{r-1}-\Phi_s^r]-\tau\hat{P}_s^r \\ W_s^r=U_s^r+Q_s^r[\Phi_s^{r+1}-\Phi_s^r]-\tau\hat{P}_s^r, \end{cases} \tag{16.4.7}$$

$$(16.4.8)\qquad \begin{cases} \hat{V}_s^r = V_s^r - \Phi_s^{r-1} \\ \\ \hat{W}_s^r = W_s^r - \Phi_s^{r+1}. \end{cases}$$

Next, for $r = 0, \pm 1, \pm 2, \dots$ and $s = 0, 1, \dots, s^*$, with $r + s$ even, we define $\hat{U}_h$ on the rectangle

$$(16.4.9)\qquad \mathscr{R}_s^r = \{(x,t) \,:\, x_{r-1} < x < x_{r+1},\ t_s \leq t < t_{s+1}\}$$

by

$$(16.4.10)\qquad \hat{U}_h(x,t) = \tilde{U}(x - x_r, t - t_s) - \Phi_s^r,$$

where $\tilde{U}$ is the solution to the Riemann problem

$$(16.4.11)\qquad \partial_t \tilde{U} + \partial_x F(\tilde{U}) = 0,$$

$$(16.4.12)\qquad \tilde{U}(x,0) = \begin{cases} W_s^{r-1}, & -\infty < x < 0 \\ \\ V_s^{r+1}, & 0 < x < \infty. \end{cases}$$

To see the motivation for the above construction, first notice that by virtue of (16.4.8), (16.4.10), (16.4.11) and (16.4.12), $\hat{U}_h$ satisfies the equation

$$(16.4.13)\qquad \partial_t \hat{U}_h + \partial_x \hat{F}(\hat{U}_h, \Phi_s^r) = 0,$$

on the rectangle $\mathscr{R}_s^r$, together with the initial condition

$$(16.4.14)\qquad \hat{U}_h(x,t_s) = \begin{cases} \hat{W}_s^{r-1}, & x_{r-1} < x < x_r \\ \\ \hat{V}_s^{r+1}, & x_r < x < x_{r+1}, \end{cases}$$

along the base of $\mathscr{R}_s^r$. Thus, (16.4.13) is the homogeneous system of conservation laws resulting from (16.3.10) by dropping the source term $\hat{P}$ and freezing Φ in $\hat{F}$ at its value Φ_s^r at the mesh-point (x_r, t_s). On the other hand, by the construction (16.4.7), (16.4.8) of $\hat{V}_s^{r+1}$ and $\hat{W}_s^{r-1}$, (16.4.14) accounts for the effect of the source and also of the inhomogeneity induced by Φ and Z (operator splitting).

As in Chapter XIII, our task here is to establish compactness and consistency of the algorithm. Compactness will be demonstrated by bounding the total variation of the approximate solutions $\hat{U}_h(\cdot,t)$, uniformly in t and h. We will be operating under the precondition that ω and the variation of $\hat{U}_h(\cdot,t)$ are bounded by a small positive constant ρ. This shall be verified a posteriori, upon proving (16.3.3), with ω and δ sufficiently small. In order to avoid proliferation of symbols, we shall employ throughout c as a generic positive constant, independent of ω, δ, ρ, T and h.

It is clear that the total variation of $\hat{U}_h(\cdot,t)$ takes the same value for any t in the interval $(t_s\,,\, t_{s+1})$, $s = 0, 1, \dots, s^*$:

$$TV_{(-\infty,\infty)}\hat{U}_h(\cdot,t) = \hat{X}_s\,, \qquad t_s < t < t_{s+1}\,. \tag{16.4.15}$$

Furthermore,

$$\hat{X}_s = J_s + K_s\,, \qquad s = 0,1,\dots,s^*, \tag{16.4.16}$$

where J_s is the part of the variation generated by the jumps of $\hat{U}_h$ across x_r, for $r+s$ odd, and K_s is the part of the variation generated by the wave fans emanating from the mesh-points (x_r,t_s), for $r+s$ even.

Across $x = x_r$, with $r+s$ odd, $\hat{U}_h$ jumps from $\hat{V}_s^r$ to $\hat{W}_s^r$, and the jump is determined by (16.4.7) and (16.4.8):

$$\hat{W}_s^r - \hat{V}_s^r = [Q_s^r - I][\Phi_s^{r+1} - \Phi_s^{r-1}]. \tag{16.4.17}$$

Since $U_0 \in \mathscr{C}_\omega$, U satisfies (16.3.1), for $t \in [0,T)$, whence

$$TV_{(-\infty,\infty)}\Phi(\cdot,t) \le c\omega, \quad TV_{(-\infty,\infty)}Z(\cdot,t) \le c\omega, \qquad 0 \le t < T. \tag{16.4.18}$$

Therefore, recalling also (16.4.5),

$$J_s = \sum_{r+s \text{ odd}} |\hat{W}_s^r - \hat{V}_s^r| \le c\rho\,\omega, \qquad s = 0,1,\dots,s^*. \tag{16.4.19}$$

The next step is to estimate K_s, and this will require certain preparation. When a state W, on the left, and a state V, on the right, are joined by an admissible wave fan $\gamma = (\gamma_1,\dots,\gamma_n)$, then

$$V = \Omega(\gamma;W), \qquad W = \Theta(\gamma;V), \qquad \gamma = G(W,V), \tag{16.4.20}$$

where Ω is the wave fan function, defined by (9.3.4), $\Theta(\gamma;\cdot)$ is the inverse function of $\Omega(\gamma;\cdot)$, for fixed γ, and $G(W,\cdot)$ is the inverse function of $\Omega(\cdot\,;W)$, for fixed W. (Equivalently, $G(\cdot\,,V)$ is the inverse function of $\Theta(\cdot\,;V)$, for fixed V). These functions are defined on some neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}^n$ and satisfy

$$\Omega(0;W) = W, \qquad \Theta(0;V) = V, \qquad G(W,W) = 0, \tag{16.4.21}$$

$$\Omega_W(0;W) = I, \qquad \Theta_V(0;V) = I, \tag{16.4.22}$$

$$\Omega_\gamma(0;W) = R(W), \qquad \Theta_\gamma(0;V) = -R(V), \tag{16.4.23}$$

$$G_W(W,W) = -L(W), \qquad G_V(V,V) = L(V), \tag{16.4.24}$$

for all W and V in the domain. In the place of G it is more convenient to work with the function

$$H(W,Y) = G(W,W+Y), \tag{16.4.25}$$

for which we have

$$H(W,0)=0, \tag{16.4.26}$$

$$H_W(W,0)=0, \qquad H_Y(W,0)=L(W). \tag{16.4.27}$$

As in Chapter XIII, K_s will be measured by the sum of the strengths of the elementary waves that are crossing the t-time line, for any fix $t \in (t_s, t_{s+1})$. These waves emanate from the mesh-points (x_r, t_s), with $r+s$ even. We will estimate K_s inductively, by relating it to K_{s-1}.

We fix a mesh-point (x_r, t_s) and consider the familiar diamond-shaped region Δ_s^r, with vertices at the four surrounding sampling points $(y_s^{r-1}, t_s), (y_{s-1}^r, t_{s-1}), (y_s^{r+1}, t_s)$ and (y_{s+1}^r, t_{s+1}), referring to Fig. 13.3.1, in Chapter XIII. The wave fan ε emanating from (x_r, t_s) contributes $|\varepsilon|$ to K_s. On the other hand, a part α (possibly all or none) of the wave fan emanating from (x_{r-1}, t_{s-1}) enters Δ_s^r through its "southwestern" edge and a part β (possibly all or none) of the wave fan emanating for (x_{r+1}, t_{s-1}) enters Δ_s^r through its "southeastern" edge, contributing $|\alpha|+|\beta|$ to K_{s-1}.

By virtue of the construction of $\hat{U}_h$ together with (16.4.20) and (16.4.25),

$$U_s^{r-1} = \Theta(\alpha; V_{r-1}^s) + \Phi_s^{r-1} - \Phi_{s-1}^{r-1}, \tag{16.4.28}$$

$$U_s^{r+1} = \Omega(\beta; W_{s-1}^r) + \Phi_s^{r+1} - \Phi_{s-1}^{r+1}, \tag{16.4.29}$$

$$\varepsilon = G(W_s^{r-1}, V_s^{r+1}) = H(W_s^{r-1}, V_s^{r+1} - W_s^{r-1}). \tag{16.4.30}$$

We now define

$$\tilde{U}_s^{r-1} = \Theta(\alpha; U_{s-1}^r), \tag{16.4.31}$$

$$\tilde{U}_s^{r+1} = \Omega(\beta; U_{s-1}^r), \tag{16.4.32}$$

$$\tilde{\varepsilon} = G(\tilde{U}_s^{r-1}, \tilde{U}_s^{r+1}) = H(\tilde{U}_s^{r-1}, \tilde{U}_s^{r+1} - \tilde{U}_s^{r-1}). \tag{16.4.33}$$

By virtue of (13.3.1),

$$|\tilde{\varepsilon} - \alpha + \beta| \leq c\mathscr{D}(\Delta_s^r), \tag{16.4.34}$$

where $\mathscr{D}(\Delta_s^r)$ is given by (13.3.2).

The next step is to estimate $\varepsilon - \tilde{\varepsilon}$. From (16.4.30) and (16.4.33),

$$\varepsilon - \tilde{\varepsilon} = \bar{H}_W[W_s^{r-1} - \tilde{U}_s^{r-1}] + \bar{H}_Y[U_s^{r+1} - W_s^{r-1} - \tilde{U}_s^{r+1} + \tilde{U}_s^{r-1}] \tag{16.4.35}$$

where

$$\bar{H}_W = \int_0^1 H_W(\xi W_s^{r-1} + (1-\xi)\tilde{U}_m^{r-1}, \tilde{U}_s^{r+1} - \tilde{U}_s^{r-1})d\xi, \tag{16.4.36}$$

$$\bar{H}_Y = \int_0^1 H_Y(W_s^{r-1}, \xi(V_s^{r+1} - W_s^{r-1}) + (1-\xi)(\tilde{U}_s^{r+1} - \tilde{U}_s^{r-1}))d\xi. \tag{16.4.37}$$

Combining (16.4.7), (16.4.28), (16.4.29), (16.4.31) and (16.4.32),

$$\begin{aligned} W_s^{r-1} - \tilde{U}_s^{r-1} = {} & \bar{\Theta}_V\{Q_{s-1}^r[\Phi_{s-1}^{r-1} - \Phi_{s-1}^r] - \tau P_{s-1}^r\} \\ & + \Phi_s^{r-1} - \Phi_{s-1}^{r-1} + Q_s^{r-1}[\Phi_s^r - \Phi_s^{r-1}] - \tau P_s^{r-1}, \end{aligned} \tag{16.4.38}$$

$$\begin{aligned} V_s^{r+1} - \tilde{U}_s^{r+1} = {} & \bar{\Omega}_W\{Q_{s-1}^r[\Phi_{s-1}^{r+1} - \Phi_{s-1}^r] - \tau P_{s-1}^r\} \\ & + \Phi_s^{r+1} - \Phi_{s-1}^{r+1} + Q_s^{r+1}[\Phi_s^r - \Phi_s^{r+1}] - \tau P_s^{r+1}, \end{aligned} \tag{16.4.39}$$

where

$$\bar{\Theta}_V = \int_0^1 \Theta_V(\alpha; \xi V_{s-1}^r + (1-\xi)U_{s-1}^r)d\xi, \tag{16.4.40}$$

$$\bar{\Omega}_W = \int_0^1 \Omega_W(\beta; \xi W_{s-1}^r + (1-\xi)U_{s-1}^r)d\xi. \tag{16.4.41}$$

By virtue of (14.6.36), (16.4.27), (16.4.31), (16.4.21) and (16.4.32), we infer that $|\bar{H}_W| \leq c(|\alpha| + |\beta|)$. Furthermore, (16.4.38), (16.4.3), (16.3.8), (16.4.6) and (16.3.12) yield $|W_s^{r-1} - \tilde{U}_s^{r-1}| \leq c\rho h$. Therefore, we can bound the first term on the right-hand side of (16.4.35) as follows:

$$|\bar{H}_W[W_s^{r-1} - \tilde{U}_s^{r-1}]| \leq c\rho(|\alpha| + |\beta|)h. \tag{16.4.42}$$

The second term on the right-hand side of (16.4.35) requires more delicate treatment. From (16.4.38) and (16.4.39), we deduce

$$\begin{aligned} & W_s^{r-1} - \tilde{U}_s^{r-1} - V_s^{r+1} + \tilde{U}_s^{r+1} \\ & \quad = \tau[P_s^{r+1} - P_s^{r-1}] \\ & \qquad + [Q_{s-1}^r - I][\Phi_s^{r+1} - \Phi_{s-1}^{r+1} - \Phi_s^{r-1} + \Phi_{s-1}^{r-1}] \\ & \qquad + [\bar{\Theta}_V - I]\{Q_{m-1}^r[\Phi_{s-1}^{r-1} - \Phi_{s-1}^r] - \tau P_{s-1}^r\} \\ & \qquad - [\bar{\Omega}_W - I]\{Q_{m-1}^r[\Phi_{s-1}^{r+1} - \Phi_{s-1}^r] - \tau P_{s-1}^r\} \\ & \qquad + [Q_s^{r-1} - Q_{s-1}^r][\Phi_s^r - \Phi_s^{r-1}] \\ & \qquad - [Q_s^{r+1} - Q_{s-1}^r][\Phi_s^r - \Phi_s^{r+1}]. \end{aligned} \tag{16.4.43}$$

Recalling (16.4.6) and (16.3.12),

$$P_s^{r+1} - P_s^{r-1} = \bar{\hat{P}}_{\hat{U}}[\hat{U}_s^{r+1} - \hat{U}_s^{r-1}] + \bar{\hat{P}}_{\Phi}[\Phi_s^{r+1} - \Phi_s^{r-1}] - [Z_s^{r+1} - Z_s^{r-1}], \tag{16.4.44}$$

where

$$\bar{\hat{P}}_{\hat{U}} = \int_0^1 \hat{P}_{\hat{U}}(\xi \hat{U}_s^{r+1} + (1-\xi)\hat{U}_s^{r-1}, \Phi_s^{r+1}, 0) d\xi, \tag{16.4.45}$$

$$\bar{\hat{P}}_{\Phi} = \int_0^1 \hat{P}_{\Phi}(\hat{U}_s^{r-1}, \xi \Phi_s^{r+1} + (1-\xi)\Phi_s^{r-1}, 0) d\xi. \tag{16.4.46}$$

Furthermore, combining (16.4.4), (16.4.28), (16.4.29), (16.4.21), (16.4.7) and after a short calculation, we obtain

$$\hat{U}_s^{r+1} - \hat{U}_s^{r-1} = \bar{\Omega}_\gamma \beta - \bar{\Theta}_\gamma \alpha + [Q_{s-1}^r - I][\Phi_{s-1}^{r+1} - \Phi_{s-1}^{r-1}], \tag{16.4.47}$$

where

$$\bar{\Omega}_\gamma = \int_0^1 \Omega_\gamma(\xi \beta; W_{s-1}^r) d\xi, \qquad \bar{\Theta}_\gamma = \int_0^1 \Theta_\gamma(\xi \alpha; V_{s-1}^r) d\xi. \tag{16.4.48}$$

Continuing with the estimation of the remaining terms on the right-hand side of (16.4.43), we employ (16.4.22), (16.4.40) and (16.4.41) to get

$$|\bar{\Theta}_V - I| \le c|\alpha|, \qquad |\bar{\Omega}_W - I| \le c|\beta|. \tag{16.4.49}$$

Moreover, using (16.4.5), (16.4.3), (16.4.28), (16.4.7), (16.4.6), (16.3.12) and (16.3.8),

$$|Q_s^{r-1} - Q_{s-1}^r| \le c|\alpha| + c\rho h, \qquad |Q_s^{r+1} - Q_{s-1}^r| \le c|\beta| + c\rho h. \tag{16.4.50}$$

We now combine (16.4.34), (16.4.35), (16.4.42), (16.4.43), (16.4.44), (16.4.47), (16.4.49) and (16.4.50) to conclude

$$\begin{aligned} |\varepsilon - \alpha - \beta| &\le c\rho(|\alpha| + |\beta|)h + c\mathscr{D}(\Delta_s^r) \\ &\quad + c\{|\Phi_s^{r+1} - \Phi_s^r| + |\Phi_s^r - \Phi_s^{r-1}| + |Z_s^{r+1} - Z_s^{r-1}|\}h \\ &\quad + c\rho|\Phi_s^{r+1} - \Phi_{s-1}^{r+1} - \Phi_s^{r-1} + \Phi_{s-1}^{r-1}|, \end{aligned} \tag{16.4.51}$$

$$\begin{aligned} |\varepsilon| &\le |I + \tau \bar{H}_Y \bar{\hat{P}}_{\hat{U}} \bar{\Theta}_\gamma||\alpha| + |I - \tau \bar{H}_Y \bar{\hat{P}}_{\hat{U}} \bar{\Omega}_\gamma||\beta| \\ &\quad + c\rho(|\alpha| + |\beta|)h + c\mathscr{D}(\Delta_s^r) \\ &\quad + c\{|\Phi_s^{r+1} - \Phi_s^r| + |\Phi_s^r - \Phi_s^{r-1}| + |Z_s^{r+1} - Z_s^{r-1}|\}h \\ &\quad + c\rho|\Phi_s^{r+1} - \Phi_{s-1}^{r+1} - \Phi_s^{r-1} + \Phi_{s-1}^{r-1}|. \end{aligned} \tag{16.4.52}$$

For ρ sufficiently small, by virtue of (16.4.37), (16.4.27), (16.4.48), (16.4.23) and (16.4.45), $\bar{H}_Y$ is close to $L(0)$, $\bar{\Omega}_\gamma$ is close to $R(0)$, $\bar{\Theta}_\gamma$ is close to $-R(0)$ and $\bar{\hat{P}}_{\hat{U}}$ is close to $\hat{P}_{\hat{U}}(0,0,0)$. Therefore, both $-\bar{H}_Y \bar{\hat{P}}_{\hat{U}} \bar{\Theta}_\gamma$ and $\bar{H}_Y \bar{\hat{P}}_{\hat{U}} \bar{\Omega}_\gamma$ are close to the matrix $\hat{A}$ defined by (16.3.13). In particular, these matrices are column diagonally dominant, so that

$$|I + \tau \bar{H}_\gamma \bar{\hat{P}}_{\hat{U}} \bar{\Theta}_\gamma| \leq 1 - 4\nu\tau, \qquad |I - \tau \bar{H}_Y \bar{\hat{P}}_{\hat{U}} \bar{\Omega}_\gamma| \leq 1 - 4\nu\tau, \tag{16.4.53}$$

for some $\nu > 0$. Thus, when ρ is sufficiently small, (16.4.52) yields the estimate

$$\begin{aligned} |\varepsilon| \leq (1 - 3\nu\tau)(|\alpha| + |\beta|) + c\mathscr{D}(\Delta_s^r) \\ + c\{|\Phi_s^{r+1} - \Phi_s^r| + |\Phi_s^r - \Phi_s^{r-1}| + |Z_s^{r+1} - Z_s^{r-1}|\}h \\ + c\rho|\Phi_s^{r+1} - \Phi_{s-1}^{r+1} - \Phi_s^{r-1} + \Phi_{s-1}^{r-1}|. \end{aligned} \tag{16.4.54}$$

Holding s fixed, we sum the inequality (16.4.54) over all r with $r + s$ even. Referring back to (16.4.16), we notice that the sum of the $|\varepsilon|$ terms yields K_s, while the sum of the $|\alpha| + |\beta|$ terms gives K_{s-1}. The sums of $|\Phi_s^{r+1} - \Phi_s^r|$, $|\Phi_s^r - \Phi_s^{r-1}|$ and $|Z_s^{r+1} - Z_s^{r-1}|$ are all bounded by $c\omega$, on account of (16.4.3) and (16.4.18). Finally, combining (16.4.3) with (16.4.8),

$$\begin{aligned} &\Phi_s^{r+1} - \Phi_{s-1}^{r+1} - \Phi_s^{r-1} + \Phi_{s-1}^{r-1} \\ &= -\int_{t_{s-1}}^{t_s} N[F(U(x_{r+1},t)) - F(U(x_{r-1},t))]dt - \int_{t_{s-1}}^{t_s} [Z(x_{r+1},t) - Z(x_{r-1},t)]dt, \end{aligned} \tag{16.4.55}$$

whence, by virtue of (16.4.1) and (16.4.18),

$$\sum_{r+s \text{ even}} |\Phi_s^{r+1} - \Phi_{s-1}^{r+1} - \Phi_s^{r-1} + \Phi_{s-1}^{r-1}| \leq c(\omega + X_s)\tau, \tag{16.4.56}$$

with

$$X_s = \frac{1}{\tau}\int_{t_{s-1}}^{t_s} X(t)dt. \tag{16.4.57}$$

We thus have

$$K_s \leq (1 - 3\nu\tau)K_{s-1} + c\omega\tau + c\rho X_s\tau + c\Xi_s\,, \tag{16.4.58}$$

where

$$\Xi_s = \sum_{r+s \text{ even}} \mathscr{D}(\Delta_s^r). \tag{16.4.59}$$

In order to estimate the term Ξ_s, we employ the procedure used in Section 13.4. We introduce the *Glimm functional*

$$\Gamma_s = K_s + \kappa M_s\,, \tag{16.4.60}$$

where κ is a positive number to be fixed below and

$$M_s = \sum |\gamma_i||\theta_j|, \tag{16.4.61}$$

with summation running over all pairs (γ_i, θ_j) of approaching elementary waves emanating from mesh-points (x_r, t_s), with $r+s$ even.

Clearly, $K_s \le \Gamma_s \le (1+c\rho)K_s$. Furthermore,

$$M_s \le M_{s-1} + (K_s + K_{s-1})E_s - \Xi_s\,, \tag{16.4.62}$$

where E_s denotes the sum of $|\varepsilon - \alpha - \beta|$ associated with the Δ_s^r for $r+s$ even. Upon summing the inequalities (16.4.51) over all r with $r+s$ even, we deduce

$$E_s \le c\rho K_{s-1}\tau + c\omega\tau + c\rho X_s\,. \tag{16.4.63}$$

Hence, in view of (16.4.60), (16.4.58), (16.4.62) and (16.4.63), we conclude that for κ large and ρ small,

$$\Gamma_s \le (1-2\nu\tau)\Gamma_{s-1} + c\omega\tau + c\rho X_s\tau + c\Xi_s\,. \tag{16.4.64}$$

By iterating (16.4.64), for $s = 1, 2, \dots$, we obtain

$$\Gamma_s \le (1-2\nu\tau)^s\Gamma_0 + c\omega + c\rho\tau\sum_{k=1}^{s}(1-2\nu\tau)^{s-k}X_k\,. \tag{16.4.65}$$

Recalling (16.4.15), (16.4.16), (16.4.19) and (16.3.15), we conclude

$$\hat{X}_s \le c(1-2\nu\tau)^s\delta + c\omega + c\rho\tau\sum_{k=1}^{s}(1-2\nu\tau)^{s-k}X_k\,. \tag{16.4.66}$$

Thus, for any $t \in [0,T)$, the total variation of $\hat{U}_h(\cdot,t)$ is bounded, uniformly in h.

As in Chapter XIII, in order to establish the compactness of the family $\{\hat{U}_h\}$ of approximate solutions, we also need equicontinuity in the t-direction. To that end, let us fix $\ell > 1$. For any ζ and ξ in $[t_s\,, t_{s+1})$, with $\zeta < \xi$:

$$\int_{-\ell}^{\ell}|\hat{U}_h(x,\xi) - \hat{U}_h(x,\zeta)|dx \le c\hat{X}_s(\xi-\zeta) \le c\rho\tau. \tag{16.4.67}$$

Recalling (16.4.10), (16.4.9) and (16.4.12),

$$\begin{aligned}\int_{-\ell}^{\ell}&|\hat{U}_h(x,t_s) - \hat{U}_h(x,t_s-)|dx \\ &\le \sum\int_{x_{r-1}}^{x_{r+1}}|\hat{U}_h(y_s^r,t_s-) - \hat{U}_h(x,t_s-)|dx \\ &\quad + \sum h|Q_s^r - I||\Phi_s^{r+1} + \Phi_s^{r-1} - 2\Phi_s^r| + \sum 2h\tau|P_s^r| + O(h),\end{aligned} \tag{16.4.68}$$

where the summations run over all r with $(|r|+1)h < \ell$ and $r+s$ odd. On the right-hand side of (16.4.68), the first sum is bounded by $2h\hat{X}_{s-1}$, the second sum is majorised by $c\rho h\omega$ and the last sum is bounded by $c\rho\ell\tau$. Therefore, upon combining (16.4.67) with (16.4.68), we infer that, for any $0 \le \zeta < \xi < T$,

$$\int_{-\ell}^{\ell} |\hat{U}_h(x,\xi) - \hat{U}_h(x,\zeta)|dx \leq c\ell(|\xi - \zeta| + h). \tag{16.4.69}$$

We have thus established the required compactness of the family $\{\hat{U}_h\}$ for extracting a sequence $\{\hat{U}_{h_m}\}$, with $h_m \to 0$ as $m \to \infty$, that converges to a *BV* function $\hat{U}$, for each $t \in [0,\infty)$ and almost all x in $(-\infty,\infty)$. We now demonstrate that $\hat{U}$ is a solution to (16.3.10), with initial values $\hat{U}_0$ given by (16.3.15).

We fix any smooth test function ψ, with compact support on $(-\infty,\infty) \times [0,T)$. We multiply (16.4.13) by ψ, integrate over the rectangles $\mathscr{R}_s^r$, identified by (16.4.9), integrate by parts and sum over r and s with $r+s$ even. This yields

$$\begin{aligned}
&\sum_{s=0}^{s^*} \sum_{r+s \text{ even}} \int_{t_s}^{t_{s+1}} \int_{x_{r-1}}^{x_{r+1}} [\partial_t \psi \hat{U}_h + \partial_x \psi \hat{F}(\hat{U}_h, \Phi_s^r)] dxdt \\
&+ \sum_{s=0}^{s^*} \sum_{r+s \text{ odd}} [\int_{t_s}^{t_{s+1}} \psi(x_r,t)dt][\hat{F}(\hat{W}_s^r, \Phi_s^{r+1}) - \hat{F}(\hat{V}_s^r, \Phi_s^{r-1})] \\
&+ \sum_{s=0}^{s^*} \int_{-\infty}^{\infty} \psi(x,t_s)[\hat{U}_h(x,t_s) - \hat{U}_h(x,t_s-)]dx + \int_{-\infty}^{\infty} \psi(x,0)\hat{U}_0(x)dx = 0.
\end{aligned} \tag{16.4.70}$$

We examine the behavior of each term on the left-hand side of the above equation as $h \to 0$ along the sequence $\{h_m\}$. The first term converges to

$$\int_0^T \int_{-\infty}^{\infty} [\partial_t \psi \hat{U} + \partial_x \psi \hat{F}(\hat{U}, \Phi)] dxdt. \tag{16.4.71}$$

The second term is $O(h)$, because, on account of (16.4.7) and (16.4.8),

$$\hat{F}(\hat{W}_s^r, \Phi_s^{r+1}) - \hat{F}(\hat{V}_s^r, \Phi_s^{r-1}) = O(h^2). \tag{16.4.72}$$

For the third term, we use (16.4.14), (16.4.8), (16.4.7) and (16.4.4) to get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi(x,t_s)[\hat{U}_h(x,t_s) - \hat{U}_h(x,t_s-)]dx \\
&= \sum_{r+s \text{ odd}} \int_{x_{r-1}}^{x_{r+1}} \psi(x,t_s)[\hat{U}_h(y_s^r,t_s-) - \hat{U}_h(x,t_s-)]dx \\
&\quad + \sum_{r+s \text{ odd}} h\psi(x_r,t_s)[Q_s^r - I][\Phi_s^{r+1} + \Phi_s^{r-1} - 2\Phi_s^r] \\
&\quad + \sum_{r+s \text{ odd}} \{\int_{x_{r-1}}^{x_r} [\psi(x,t_s) - \psi(x_r,t_s)]dx\}[Q_s^r - I][\Phi_s^{r-1} - \Phi_s^r] \\
&\quad + \sum_{r+s \text{ odd}} \{\int_{x_r}^{x_{r+1}} [\psi(x,t_s) - \psi(x_r,t_s)]dx\}[Q_s^r - I][\Phi_s^{r+1} - \Phi_s^r] \\
&\quad - \sum_{r+s \text{ odd}} \tau\{\int_{x_{r-1}}^{x_{r+1}} \psi(x,t_s)dx\}\hat{P}_s^r.
\end{aligned} \tag{16.4.73}$$

We next estimate each term on the right-hand side of the above equation, with an eye to determining their contribution to Equation (16.4.70). The second term is bounded by $c\rho h^2 X(s)$, while both the third and the fourth terms are bounded by $c\rho h^2\omega$. Thus the contribution of the above three terms to (16.4.70) is $O(h)$. By contrast, the first term on the right-hand side of (16.4.73) is bounded by $ch\hat{X}_{s-1}$, so it may seemingly contribute $O(1)$ to (16.4.70). However, almost surely, the contribution of this term is $o(1)$, the reason being that, as discussed in Chapter XIII, the sampling points (y_s^r, t_s) were picked at random. As regards the last term in (16.4.73), we have

(16.4.74)
$$\left| \tau \int_{x_{r-1}}^{x_{r+1}} \psi(x,t_s)\hat{P}_s^r dx - \int_{t_{s-1}}^{t_s} \int_{x_{r-1}}^{x_{r+1}} \psi(x,t)\hat{P}(\hat{U}_h(x,t),\Phi(x,t),Z(x,t))dxdt \right|$$
$$\leq ch^2(u_{s-1}^r + h),$$

where u_{s-1}^r denotes the oscillation of $\hat{U}_h$ over the rectangle $\mathscr{R}_{s-1}^r$. Notice that by the construction of $\hat{U}_h$, its oscillation over $\mathscr{R}_{s-1}^r$ is bounded by the variation of $\hat{U}_h(\cdot, t_s-)$ over the interval (x_{r-1}, x_{r+1}) and hence the sum of u_{s-1}^r over all r with $r+s$ odd is bounded by $\hat{X}_{s-1}$. Thus the contribution to (16.4.70) of the last term in (16.4.73) is

$$-\int_0^T \int_{-\infty}^{\infty} \psi\hat{P}(\hat{U}_h,\Phi,Z)dxdt + O(h). \tag{16.4.75}$$

We now combine all of the above and pass to the $h \to 0$ limit in (16.4.70) to get

$$\int_0^T \int_{-\infty}^{\infty} [\partial_t\psi\hat{U} + \partial_x\psi\hat{F}(\hat{U},\Phi) - \psi\hat{P}(\hat{U},\Phi,Z)]dxdt \tag{16.4.76}$$
$$+ \int_{-\infty}^{\infty} \psi(x,0)\hat{U}_0(x)dx = 0,$$

which verifies that $\hat{U}$ is a solution to (16.3.10) with initial value $\hat{U}_0$.

Upon setting

$$\hat{X}(t) = TV_{(-\infty,\infty)}\hat{U}(\cdot,t), \qquad 0 \leq t < T, \tag{16.4.77}$$

(16.4.66) yields the estimate

$$\hat{X}(t) \leq c\delta e^{-2\nu t} + c\omega + c\rho \int_0^t e^{-2\nu(t-\xi)} X(\xi)d\xi. \tag{16.4.78}$$

In what follows, we shall be taking for granted that the Cauchy problem for (16.3.10) admits a unique admissible BV solution. Proving this would require the extension of the analysis presented in Section 14.10 from homogeneous systems of conservation laws to non homogeneous systems of balance laws – an arduous task that should not involve major new insights. On the basis of the presumption of uniqueness, since $U - \Phi$ is a solution of (16.3.10) with initial value $\hat{U}_0$ defined by (16.3.15), we conclude that $\hat{U} = U - \Phi$ and in particular $X(t) \leq \hat{X}(t) + c\omega$. Therefore, (16.4.78) gives

(16.4.79)
$$X(t) \le c\delta e^{-2\nu t} + c\omega + c\rho \int_0^t e^{-2\nu(t-\xi)} X(\xi) d\xi .$$

We set

(16.4.80)
$$Y(t) = \int_0^t e^{2\nu\xi} X(\xi) d\xi ,$$

in which case (16.4.79) yields the differential inequality

(16.4.81)
$$\dot{Y}(t) \le c\rho Y(t) + c\delta + c\omega e^{2\nu t}.$$

We fix ρ so small that the coefficient $c\rho$ multiplying $Y(t)$ is smaller than ν. Then, integrating (16.4.81) and substituting back into (16.4.79) yields

(16.4.82)
$$X(t) \le c\omega + c\delta e^{-\nu t}, \qquad 0 \le t \le T,$$

which establishes the estimate (16.3.3) and completes the proof of Theorem 16.3.1.

16.5 L^1 Stability Via Entropy with Conical Singularity at the Origin

One may apply Theorem 16.3.1 only for initial data in $\mathscr{C}_\omega$, having the property of generating solutions that satisfy (16.3.1). The aim of the present and the next section of this chapter is to demonstrate that, when the source is dissipative, the class $\mathscr{C}_\omega$ is quite rich, encompassing the initial data of interest, in the vicinity of the equilibrium state.

We assume that the system (16.2.1) is endowed with an entropy-entropy flux pair (η, q), where $\eta(U)$ is convex, so that admissible solutions U must satisfy the entropy inequality

(16.5.1)
$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + \mathrm{D}\eta(U(x,t))P(U(x,t)) \le 0.$$

As in Section 5.5, we call the source P *dissipative*, relative to η, if the entropy production is nonnegative:

(16.5.2)
$$\mathrm{D}\eta(U)P(U) \ge 0, \qquad U \in \mathscr{O}.$$

The most direct way for achieving our goal is by seeking a convex entropy η that exhibits a conical singularity, $\beta^{-1/2}|U| \le \eta(U) \le \beta^{1/2}|U|$, for U near the origin, and renders the source dissipative (16.5.2). Indeed, in that case, by virtue of (16.5.1), $\mathscr{C}_\omega$ must contain all initial data U_0 with $\|U_0\|_{L^1} \le \omega/\beta$. We now show how an entropy function with the above specifications may be constructed for the system (5.5.52), which has been playing in the literature the role of the paradigm for systems modeling relaxation phenomena, under the subcharacteristic condition (5.5.54). We assume here that (5.5.54) holds in strict inequality form.

We assume $f(0) = 0$ so that the origin is an equilibrium state. The characteristic speeds are $\lambda = -a(u)$ and $\mu = a(u)$, with associated eigenvectors

$$R = \frac{1}{2a}\begin{pmatrix} 1 \\ -a \end{pmatrix}, \qquad S = \frac{1}{2a}\begin{pmatrix} 1 \\ a \end{pmatrix}. \tag{16.5.3}$$

Then, from (16.2.3),

$$A = \frac{1}{2a}\begin{pmatrix} a+f' & -a+f' \\ -a-f' & a-f' \end{pmatrix}, \tag{16.5.4}$$

evaluated at $u = 0$. Because of the subcharacteristic condition, the main diagonal entries of A are positive, so that (16.3.2) is satisfied. However, since $\det A = 0$, the strict diagonal dominance condition (16.2.4) fails to hold for any choice of eigenvectors and hence Theorem 16.2.1 does not apply. With an eye towards using Theorem 16.3.1 in the place of Theorem 16.2.1, we seek a convex entropy $\eta(u,v)$, defined on some neighborhood of the origin, that exhibits a conical singularity,

$$\beta^{-1/2}(|u|+|v|) \le \eta(u,v) \le \beta^{1/2}(|u|+|v|), \tag{16.5.5}$$

and incurs nonnegative production:

$$\eta_v(u,v)[v - f(u)] \ge 0. \tag{16.5.6}$$

Following the discussion in Section 12.2, we will determine the entropy as a function $\eta(z,w)$ of the Riemann invariants

$$z = \int_0^u a(s)ds - v, \qquad w = \int_0^u a(s)ds + v. \tag{16.5.7}$$

Equation (12.2.2) here reduces to

$$\eta_{zw}(z,w) + \kappa(z,w)[\eta_z(z,w) + \eta_w(z,w)] = 0, \tag{16.5.8}$$

where κ stands for $(2a)^{-2}a'$ expressed as a function of (z,w). Furthermore, (12.2.3), stating that $\eta(u,v)$ is convex, here take the form

$$\eta_{zz} - \eta_{zw} \ge 0, \qquad \eta_{ww} - \eta_{zw} \ge 0. \tag{16.5.9}$$

The local equilibrium curve $v = f(u)$ turns into $w = g(z)$ with slope

$$g_z = \frac{a+f'}{a-f'}, \tag{16.5.10}$$

which is positive by virtue of the subcharacteristic condition. The entropy $\eta(z,w)$ with the requisite properties will be constructed on a rectangle

$$\mathscr{R} = \{(z,w):\ -r < z < r,\ g(-r) < w < g(r)\}, \tag{16.5.11}$$

for some $r > 0$ to be specified below.

Let us fix a smooth function $\phi(z)$ on $(-\infty,\infty)$, with $\phi(0) = 0, \phi_z(0) = 0$, and $|\phi_z(z)| \leq 3$, $\phi_{zz}(z) \geq 0$, for all $z \in (-\infty,\infty)$. In what follows, the order of magnitude symbol O shall be understood to hold uniformly for all ϕ with the above specifications, so in particular will not depend on the size of ϕ_{zz}. We then construct $\eta(z,w)$ on $\mathscr{R}$ as the solution of (16.5.8) with Cauchy data

$$\eta(z,g(z)) = \phi(z), \qquad -r < z < r, \tag{16.5.12}$$

$$\eta_w(z,g(z)) - \eta_z(z,g(z)) = 0, \qquad -r < z < r, \tag{16.5.13}$$

along the space-like equilibrium curve $w = g(z)$.

After lengthy but elementary estimations on (16.5.8) and (12.2.4), which are found in the references cited in Section 16.7, one deduces

$$\eta(z,w) = \frac{1}{1+g_z(z)}\phi(z) + \frac{g_z(g^{-1}(w))}{1+g_z(g^{-1}(w))}\phi(g^{-1}(w)) + O(z^2+w^2), \tag{16.5.14}$$

$$\eta_z(z,g(z)) = \eta_w(z,g(z)) = \frac{\phi_z(z)}{1+g_z(z)} = O(1), \tag{16.5.15}$$

$$\chi(z,w)\eta_{zz}(z,w) = \frac{\phi_{zz}(z)}{1+g_z(z)} + O(1), \tag{16.5.16}$$

$$\chi^*(z,w)\eta_{ww}(z,w) = \frac{\phi_{zz}(g^{-1}(w))}{g_z(g^{-1}(w))[1+g_z(g^{-1}(w))]} + O(1), \tag{16.5.17}$$

where $\chi(z,w)$ and $\chi^*(z,w)$ are integrating factors:

$$\chi(z,w) = \exp\int_{g(z)}^{w} \kappa(z,\xi)d\xi, \qquad \chi^*(z,w) = \exp\int_{g^{-1}(w)}^{z} \kappa(\zeta,w)d\zeta. \tag{16.5.18}$$

We take $\phi(z) = \psi(z) + \gamma z^2$, where $\psi(0) = 0, \psi_z(0) = 0$, and $|\psi_z(z)| \leq 1$, $\psi_{zz}(z) \geq 0$ for all $z \in (-\infty,\infty)$. The positive constant γ is fixed sufficiently large, and the constant r is fixed proportionally small, so that $\gamma r < 1$, in order to secure that $|\phi_z(z)| \leq 3$ and (16.5.7) all hold for $(z,w) \in \mathscr{R}$. Then $\eta(u,v)$ will be convex. Moreover, by the chain rule, $\eta_v = \eta_w - \eta_z$ and $\eta_{vv} = \eta + \eta_{ww} - 2\eta_{zw}$ so that (16.5.13) and (16.5.9) together imply (16.5.6).

At this point, holding r and γ fixed as above, we consider a sequence of $\psi(z)$ that converges to the Lipschitz function $|z|$. The sequence of associated entropies will then converge to some Lipschitz function η on $\mathscr{R}$, which is an entropy for (5.5.52), that is convex as a function of (u,v) and satisfies (16.5.6). Furthermore, by virtue of (16.5.14),

(16.5.19)
$$\eta(z,w) = \frac{1}{1+g_z(z)}|z| + \frac{g_z(g^{-1}(w))}{1+g_z(g^{-1}(w))}|g^{-1}(w)| + O(z^2+w^2) + \gamma O(z^2+w^2).$$

As a final step, holding γ fixed as above, we impose, if necessary, a further reduction on the size of r so that the terms with linear growth in (16.5.19) dominate the terms of quadratic order, in which case η satisfies (16.5.5) as well.

Since the constructed η satisfies (16.5.5) and (16.5.6), the entropy inequality (16.5.1) implies that $\mathscr{C}_\omega$ for the system (5.5.52) contains all initial data

$$\int_{-\infty}^{\infty} \{|u_0(x)| + |v_0(x)|\}dx = \sigma, \tag{16.5.20}$$

with $\sigma \leq \beta^{-1}\omega$. Therefore, when the initial data satisfy (16.5.20) as well as

$$TV_{(-\infty,\infty)}u_0(\cdot) + TV_{(-\infty,\infty)}v_0(\cdot) = \delta, \tag{16.5.21}$$

with σ and δ sufficiently small, Theorem 16.3.1 establishes the existence of a *BV* solution, in the large.

A construction of an entropy with the above specifications – and in particular with a conical singularity at the origin – seems feasible only for systems endowed with a rich family of entropies, most notably for systems of just two balance laws. For systems of larger size, with a limited collection of entropies, one must use an alternative approach, which will be developed in the next section.

16.6 L^1 Stability when the Source is Partially Dissipative

The aim here is to demonstrate that the assumptions, and thereby also the conclusions, of Theorem 16.3.1 apply to systems modeling relaxation phenomena. Thus this section is the counterpart of Section 5.5, with *BV* weak solutions in the role of classical solutions.

As in Section 5.5, we assume that our system (16.2.1) is endowed with a smooth entropy-entropy flux pair (η,q), where η is convex and has been normalized by $\eta(0)=0$, $\mathrm{D}\eta(0)=0$. Thus any admissible *BV* solution will satisfy the entropy inequality (16.5.1). We assume, further, that the source P is dissipative semidefinite relative to η, in the terminology introduced in Section 5.5, namely

$$\mathrm{D}\eta(U)P(U) \geq a|P(U)|^2, \qquad U \in \mathscr{O}, \tag{16.6.1}$$

with $a>0$. Thus the entropy inequality readily yields bounds on the spatial integral of $|U|^2$ and the space-time integral of $|P(U)|^2$.

Since we shall need bounds on the space-time integral of $|U|^2$, we must invoke the synergy between flux and source manifested in the Kawashima condition, introduced in Section 5.5,

$$\mathrm{D}P(0)R_i(0) \neq 0, \qquad i=1,\dots,n, \tag{16.6.2}$$

which guarantees that the system resulting by linearizing (16.2.1) about the equilibrium state 0 does not admit solutions in the form $u(x-\lambda_i(0)t)R_i(0)$, manifesting undamped propagating fronts.

We demonstrate that in the presence of (16.6.1) the Kawashima condition is equivalent to the assumption (16.3.2) of Theorem 16.3.1. On account of (16.6.1), the function

$$\theta(U) = \mathrm{D}\eta(U)P(U) - a|P(U)|^2 \tag{16.6.3}$$

is minimized at $U=0$, and hence the Hessian matrix

$$\mathrm{D}^2\theta(0) = \mathrm{D}^2\eta(0)\mathrm{D}P(0) + \mathrm{D}P(0)^\top \mathrm{D}^2\eta(0) - 2a\mathrm{D}P(0)^\top \mathrm{D}P(0) \tag{16.6.4}$$

is positive semidefinite. Multiplying, from the left, (7.4.2) by $R_i^\top$ yields

$$R_i^\top \mathrm{D}^2\eta \mathrm{D}F = \lambda_i R_i^\top \mathrm{D}^2\eta, \tag{16.6.5}$$

which shows that $R_i^\top \mathrm{D}^2\eta$ is collinear to L_i, and in particular

$$R_i^\top \mathrm{D}^2\eta = (R_i^\top \mathrm{D}^2\eta R_i)L_i\,. \tag{16.6.6}$$

We now multiply (16.6.4), from the left by $R_i^\top(0)$ and from the right by $R_i(0)$. Using that $\mathrm{D}^2\theta(0)$ is positive semidefinite, together with (16.6.5) at $U=0$, we conclude

$$[R_i^\top(0)\mathrm{D}^2\eta(0)R_i(0)]A_{ii} \geq a|\mathrm{D}P(0)R_i(0)|^2, \tag{16.6.7}$$

which verifies that (16.6.2) implies (16.3.2). On the other hand, it is easily seen from (16.2.2) that $A_{ii} \neq 0$, for $i = 1,\dots,n$, implies the Kawashima condition.

The next task is to identify and expose the component of the state vector that is directly affected by damping. Let us assume that the kernel of $\mathrm{D}P(0)$ has dimension $k, 1 \leq k < n$. Thus, $\mathrm{D}P(0) = S^{-1}\Gamma S$, where S is a nonsingular $n \times n$ matrix and Γ is an $n \times n$ matrix in the form

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}, \tag{16.6.8}$$

with C a nonsingular $\ell \times \ell$ matrix, $\ell = n-k$. Passing from U to a new state vector $\hat{U} = SU$, with projections V and W on $\mathbb{R}^k$ and $\mathbb{R}^\ell$, reduces (16.2.1) to a system

$$\begin{cases} \partial_t V + \partial_x G(V,W) + X(V,W) = 0 \\[2ex] \partial_t W + \partial_x H(V,W) + CW + Y(V,W) = 0, \end{cases} \tag{16.6.9}$$

where X,Y and their first derivatives vanish at the origin. We may thus assume, without loss of generality, that our system (16.2.1) has the form (16.6.9). In what follows, we shall be using either form, (16.2.1) or (16.6.9), as is convenient.

As already noted in Section 5.5, the systems encountered in the applications, modeling relaxation phenomena, generally result from combining conservation laws with balance laws and thus assume the form (16.6.9) with $X \equiv 0$. In that case, it is easy to see that the entropy inequality (16.6.1) requires $Y(V,0) = 0$ and so (16.6.9) may be written in the more compact form

$$\text{(16.6.10)} \qquad \begin{cases} \partial_t V + \partial_x G(V,W) = 0 \\ \\ \partial_t W + \partial_x H(V,W) + C(V,W)W = 0. \end{cases}$$

The spatial mean of the component V of solutions to (16.6.10) is time-invariant, and this simplifies the analysis considerably. Accordingly, we shall discuss below, in detail, the special case of systems (16.2.1) in the form (16.6.10), returning briefly to the general case (16.6.9) at the conclusion of this section.

With the entropy η as function of (V,W), positive semidefiniteness of the Hessian matrix (16.6.4) holds if and only if the $k \times \ell$ matrix $\eta_{VW}(0,0)$ vanishes and the $\ell \times \ell$ matrix $\eta_{WW}(0,0)C(0,0)$ is positive definite. Furthermore, the Kawashima condition (16.6.2) is satisfied when, for $i = 1, \dots, n$, the projection of $R_i(0)$ on $\mathbb{R}^\ell$ does not vanish.

We have seen already that the above conditions imply (16.3.2). With an eye to the remaining assumptions of Theorem 16.3.1, we proceed to identify the class $\mathscr{C}_\omega$ of initial data that generate solutions with the property (16.3.1). This will be achieved through the following

16.6.1 Theorem. *Assume the system* (16.2.1) *is in the form* (16.6.10), *the source is dissipative semidefinite, relative to the entropy* η, *and the Kawashima condition holds. Let* $U = (V,W)$ *be an admissible BV solution, with initial values* $U_0 = (V_0, W_0)$, *defined on a strip* $(-\infty,\infty) \times [0,T)$ *and taking values in a ball* $\mathscr{B}_\rho$ *of small radius* ρ, *centered at the origin. Suppose that* U_0 *decays, as* $|x| \to \infty$, *sufficiently fast to render the integral*

$$\text{(16.6.11)} \qquad \int_{-\infty}^{\infty} (1+x^2)|U_0(x)|^2 dx = \sigma^2$$

finite. Furthermore, let

$$\text{(16.6.12)} \qquad \int_{-\infty}^{\infty} V_0(x) dx = 0.$$

Then there is $\sigma_0 > 0$, *independent of* T, *such that, for* $\sigma < \sigma_0$,

$$\text{(16.6.13)} \qquad \int_{-\infty}^{\infty} |U(x,t)| dx \le b\sigma, \qquad 0 \le t < T,$$

with b *independent of* T. *Furthermore, if* $T = \infty$,

$$\text{(16.6.14)} \qquad \int_{-\infty}^{\infty} |U(x,t)| dx \to 0, \quad \text{as } t \to \infty.$$

It is this theorem that guarantees that $\mathscr{C}_\omega$ contains all initial data U_0 satisfying (16.6.2) and (16.6.11) for $\sigma \leq \omega/b$. The proof will be the culmination of a priori estimates to be established below. As in Section 16.4, in order to avoid the proliferation of symbols, we will employ throughout c as a generic positive constant that may depend on bounds of F, P and their derivatives on some fixed neighborhood of the origin, containing $\mathscr{B}_\rho$, but is independent of ρ, σ and T.

A useful tool will be the "potential" function

$$\Psi(x,t) = \int_{-\infty}^{x} V(y,t)dy. \tag{16.6.15}$$

Clearly, Ψ is Lipschitz, with derivatives

$$\partial_x \Psi = V, \quad \partial_t \Psi = -G(V,W). \tag{16.6.16}$$

The role of the assumption (16.6.12) is to secure that the initial value $\Psi_0(x)$ of Ψ satisfies $\Psi_0(\pm\infty) = 0$. Then, integrating by parts,

$$\begin{aligned} \int_{-\infty}^{\infty} |\Psi_0(x)|^2 dx &= \int_{-\infty}^{0} \left| \int_{-\infty}^{x} V_0(y)dy \right|^2 dx + \int_{0}^{\infty} \left| \int_{\infty}^{x} V_0(y)dy \right|^2 dx \\ &= -2\int_{-\infty}^{0} xV_0^\top(x)\Psi_0(x)dx - 2\int_{0}^{\infty} xV_0^\top(x)\Psi_0(x)dx \\ &\leq 2\int_{-\infty}^{\infty} x^2 |V_0(y)|^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} |\Psi_0(x)|^2 dx, \end{aligned} \tag{16.6.17}$$

so that, by virtue of (16.6.11),

$$\int_{-\infty}^{\infty} |\Psi_0(x)|^2 dx \leq 4\sigma^2. \tag{16.6.18}$$

The following proposition provides the first hint that the spatial L^2 norm of the solution decays as time tends to infinity.

16.6.2 Lemma. *Under the assumptions of Theorem* 16.6.1, *there is a constant* $\bar{\omega}$ *independent of* T, *such that if*

$$|\Psi(x,t)| < \bar{\omega}, \qquad -\infty < x < \infty, \ 0 \leq t < T, \tag{16.6.19}$$

then

$$\int_0^T \int_{-\infty}^{\infty} |U(x,t)|^2 dxdt \leq c\sigma^2. \tag{16.6.20}$$

Proof. By virtue of (16.6.1), integration of the entropy inequality (16.5.1) over $(-\infty,\infty)\times[0,T)$ readily yields that $\int\int |W|^2 dxdt$ is bounded by $c\sigma^2$. However, showing that this will also be the case for the complementary component V will require considerable effort, as it rests on the synergy between the partially dissipative source and the flux, encoded in the Kawashima condition.

We introduce the following notations:

$$B = G_V(0,0), \quad J = G_W(0,0), \quad E = H_V(0,0), \quad D = H_W(0,0), \tag{16.6.21}$$

$$K = \eta_{VV}(0,0), \quad M = \eta_{WW}(0,0), \tag{16.6.22}$$

$$Q = [\eta_{WW}(0,0)C(0,0)]^{-1}. \tag{16.6.23}$$

The $k \times k$ matrix K and the $\ell \times \ell$ matrix M are symmetric and positive definite. Furthermore, since $\mathrm{D}^2\eta(U)\mathrm{D}F(U)$ is symmetric and $\eta_{VW}(0,0) = 0$,

$$(KB)^\top = KB, \quad (MD)^\top = MD, \quad (ME)^\top = KJ. \tag{16.6.24}$$

Finally, the $\ell \times \ell$ matrix Q is positive definite.

If N is any eigenvector of the matrix B, then $EN \neq 0$, since $EN = 0$ would imply that $R = \begin{pmatrix} N \\ 0 \end{pmatrix}$ is an eigenvector of $\mathrm{D}F(0)$ with $\mathrm{D}P(0)R = 0$, in contradiction to the Kawashima condition. It may then be shown (reference in Section 16.7) that there exists a $k \times k$ matrix Ω such that ΩK is skew-symmetric and ΩKB is positive on the kernel of ME.

We now define the following functions:

$$\Theta(V,W,\Psi) = \Psi^\top K\Psi - 2\Psi^\top KJQMW - \kappa\Psi^\top \Omega KV + \gamma\eta(V,W), \tag{16.6.25}$$

$$\Xi(V,W,\Psi) = \Psi^\top KB\Psi - 2\Psi^\top KJQMH(V,W) - \kappa\Psi^\top \Omega KG(V,W) + \gamma q(V,W), \tag{16.6.26}$$

$$\begin{aligned} \Pi(V,W,\Psi) = {} & 2\Psi^\top KG(V,W) - 2\Psi^\top KBV - 2G^\top(V,W)KJQMW \\ & +2V^\top KJQMH(V,W) - 2\Psi^\top KJQMC(V,W)W - \kappa G^\top(V,W)\Omega KV \\ & +\kappa V^\top \Omega KG(V,W) + \gamma\eta_W(V,W)C(V,W)W, \end{aligned} \tag{16.6.27}$$

where κ and γ are positive constants to be fixed below.

A lengthy but straightforward calculation, using (16.6.10), (16.6.16) and (16.5.1) yields

$$\partial_t\Theta(V,W,\Psi) + \partial_x\Xi(V,W,\Psi) + \Pi(V,W,\Psi) \leq 0. \tag{16.6.28}$$

We perform a (finite) Taylor expansion of $\Pi(V,W,\Psi)$ about the origin. Using the symmetry relations (16.6.24) and recalling that $|U| < \rho$ and $|\Psi| < \bar{\omega}$, we obtain

$$\Pi = V^\top \Lambda V + V^\top \Gamma W + W^\top \Delta W + O(\rho + \bar{\omega})(|V|^2 + |W|^2), \tag{16.6.29}$$

where

$$\Lambda = 2(ME)^\top QME + 2\kappa\Omega KB, \tag{16.6.30}$$

$$\Delta = -2J^\top KJQM + \gamma Q^{-1}, \tag{16.6.31}$$

$$\Gamma = -2KBJQM + 2KJQMD + 2\kappa\Omega KJ. \tag{16.6.32}$$

The crucial observation is that the second term on the right-hand side of (16.6.30) is positive on the kernel of ME and the first term is positive on the complementary space, whence, for κ sufficiently small, Λ is positive definite. We may thus fix ρ and $\bar{\omega}$ sufficiently small and γ sufficiently large so both Θ and Π become positive definite at the origin. Then, integrating (16.6.28) over $(-\infty,\infty)\times[0,T)$ and using (16.6.11) and (16.6.18), we arrive at (16.6.20). This completes the proof.

16.6.3 Lemma. *Under the assumptions of Theorem* 16.6.1 *and so long as* (16.6.19) *holds,*

$$t\int_{-\infty}^{\infty} |U(x,t)|^2 dx \leq c\sigma^2, \qquad 0 \leq t < T. \tag{16.6.33}$$

Furthermore, if $T=\infty$,

$$t\int_{-\infty}^{\infty} |U(x,t)|^2 dx \to 0, \qquad \textit{as } t\to\infty. \tag{16.6.34}$$

Proof. On account of (16.5.1), (16.5.2),

$$\partial_t[t\eta(U)] + \partial_x[tq(U)] \leq \eta(U). \tag{16.6.35}$$

Integrating the above inequality over $(-\infty,\infty)\times[0,t]$, $t\in[0,T)$, and using (16.6.20), we arrive at (16.6.33).

Suppose now $T=\infty$ and fix any $\varepsilon>0$. By virtue of (16.6.20), there exists $\tau>0$ such that

$$\tau\int_{-\infty}^{\infty} \eta(U(x,\tau))dx < \frac{\varepsilon}{2}, \tag{16.6.36}$$

$$\int_{\tau}^{\infty}\int_{-\infty}^{\infty} \eta(U(x,t))dxdt < \frac{\varepsilon}{2}. \tag{16.6.37}$$

For any $t\in(\tau,\infty)$, integrating (16.6.35) over $(-\infty,\infty)\times[\tau,t]$ yields

$$t\int_{-\infty}^{\infty} \eta(U(x,t))dx < \varepsilon, \tag{16.6.38}$$

whence (16.6.34) follows. This completes the proof.

Proof of Theorem 16.6.1. Suppose that (16.6.19) holds. We identify $\lambda > 0$ such that

$$|q(U)| \leq \lambda \eta(U) \tag{16.6.39}$$

holds for all U in a neighborhood of the origin containing $\mathscr{B}_\rho$.

On account of (16.6.33) and Schwarz's inequality,

$$\int_{-2\lambda t}^{2\lambda t} |U(x,t)|dx \leq \left\{ 4\lambda t \int_{-2\lambda t}^{2\lambda t} |U(x,t)|^2 dx \right\}^{1/2} \leq c\sigma, \tag{16.6.40}$$

for any $t \in [0,T)$. Furthermore, if $T = \infty$, (16.6.40) and (16.6.34) imply

$$\int_{-2\lambda t}^{2\lambda t} |U(x,t)|dx \to 0, \qquad \text{as } t \to \infty. \tag{16.6.41}$$

We now proceed to estimate the integral of $|U(x,t)|$ over the interval $(2\lambda t, \infty)$. For $k = 1,2,\dots$, we integrate the entropy inequality (16.5.1) over the trapezoid with vertices $(2^k\lambda t, t)$, $(2^{k+1}\lambda t, t)$, $((2^k - 1)\lambda t, 0)$ and $((2^{k+1}+1)\lambda t, 0)$. On account of (16.6.39) this yields

$$\int_{2^k\lambda t}^{2^{k+1}\lambda t} |U(x,t)|^2 dx \leq \beta \int_{(2^k-1)\lambda t}^{(2^{k+1}+1)\lambda t} |U_0(x)|^2 dx. \tag{16.6.42}$$

Therefore,

$$\begin{aligned} \int_{2^k\lambda t}^{2^{k+1}\lambda t} (x^2+1)|U(x,t)|^2 dx &\leq \beta(4^{k+1}\lambda^2 t^2 + 1) \int_{(2^k-1)\lambda t}^{(2^{k+1}+1)\lambda t} |U_0(x)|^2 dx \\ &\leq 16\beta \int_{(2^k-1)\lambda t}^{(2^{k+1}+1)\lambda t} (x^2+1)|U_0(x)|^2 dx. \end{aligned} \tag{16.6.43}$$

Summing the above inequalities over $k = 1,2,\dots$, we deduce

$$\int_{2\lambda t}^{\infty} (x^2+1)|U(x,t)|^2 dx \leq 32\beta \int_{\lambda t}^{\infty} (x^2+1)|U_0(x)|^2 dx \leq c\sigma^2. \tag{16.6.44}$$

Finally, by Schwarz's inequality,

(16.6.45)

$$\int_{2\lambda t}^{\infty} |U(x,t)|dx \leq \left[\int_{2\lambda t}^{\infty} (x^2+1)^{-1} dx \right]^{\frac{1}{2}} \left[\int_{2\lambda t}^{\infty} (x^2+1)|U_0(x)|^2 dx \right]^{\frac{1}{2}} \leq c\sigma(\lambda t + 1)^{-\frac{1}{2}}.$$

A similar bound is obtained for the integral of $|U(x,t)|$ over $(-\infty, -2\lambda t)$. These bounds together with (16.6.40) and (16.6.41) establish (16.6.13), for some b, and (16.6.14), albeit subject to (16.6.19). In order to show that this restriction is superfluous, we first note that, upon increasin g, if necessary, the size of b, (16.6.13) holds at $t = 0$, independently of (16.6.19). We thus set $\sigma_0 = \bar{\omega}/b$ and fix $\sigma < \sigma_0$. Since $|\Psi(x,t)| \le \|V(\cdot,t)\|_{L^1}$, we deduce that when (16.6.13) is satisfied for some $t \in [0,T)$, (16.6.19) must hold on $(-\infty,\infty) \times [t+\varepsilon)$, by virtue of (16.6.16). A simple continuation argument then establishes that, for any $\sigma < \sigma_0$, (16.6.19) holds on $(-\infty,\infty) \times [0,T)$. This completes the proof.

We may now combine Theorems 16.3.1 and 16.6.1, arriving at the following existence theorem:

16.6.4 Theorem. *Consider a system* (16.2.1) *of balance laws in the form* (16.6.10), *with genuinely nonlinear or linearly degenerate characteristic fields, endowed with a convex entropy* η. *Assume that the source is dissipative semidefinite* (16.6.1), *relative to* η, *and the Kawashima condition* (16.6.2) *holds. Then there are positive constants* $\delta_0, \sigma_0, c_0, c_1, \nu$ *and* b *so that the Cauchy problem under initial data* $U_0 = (V_0, W_0)$, *with*

$$\int_{-\infty}^{\infty} (1+x^2)|U_0(x)|^2 dx = \sigma^2 < \sigma_0^2\,, \tag{16.6.46}$$

$$TV_{(-\infty,\infty)} U_0(\cdot) = \delta < \delta_0\,, \tag{16.6.47}$$

$$\int_{-\infty}^{\infty} V_0(x)dx = 0, \tag{16.6.48}$$

possesses an admissible BV solution U *on* $(-\infty,\infty) \times [0,\infty)$ *and*

$$\int_{-\infty}^{\infty} |U(x,t)|dx \le b\sigma, \qquad 0 \le t < \infty, \tag{16.6.49}$$

$$TV_{(-\infty,\infty)} U(\cdot,t) \le c_0\sigma + c_1\delta e^{-\nu t}, \qquad 0 \le t < \infty, \tag{16.6.50}$$

$$\int_{-\infty}^{\infty} |U(x,t)|dx \to 0, \qquad \textit{as } t \to \infty, \tag{16.6.51}$$

$$TV_{(-\infty,\infty)} U(\cdot,t) \to 0, \qquad \textit{as } t \to \infty. \tag{16.6.52}$$

For illustration, we consider again the system (5.5.52), discussed in Section 16.5. In order to treat it in the context of the present section, we need a convex entropy η that renders the source dissipative semidefinite:

$$\eta_v(u,v)[v - f(u)] \ge \frac{a}{\mu}|v - f(u)|^2, \tag{16.6.53}$$

with $a > 0$.

Following the procedure in Section 16.5, we will construct an entropy with the above specifications, as a function $\eta(z,w)$ of the Riemann invariants (16.5.7) defined on the rectangle (16.5.11), by solving the Cauchy problem for the equation (16.5.8) under Cauchy data (16.5.12), (16.5.13). However, here we employ any convex function φ, with $\varphi(0)=0, \varphi_z(0)=0$ and $\varphi_{zz}(0)=1$. We perform similar estimations as in Section 16.5, getting

$$\eta(z,w) = O(r^2), \tag{16.6.54}$$

$$\eta_z(z,w) = O(r), \qquad \eta_w(z,w) = O(r), \qquad \eta_{zw}(z,w) = O(r), \tag{16.6.55}$$

$$\eta_{zz}(z,w) = \frac{1}{1+g_z(z)} + O(r), \tag{16.6.56}$$

$$\eta_{ww}(z,w) = \frac{1}{g_z(g^{-1}(w))[1+g_z(g^{-1}(w))]} + O(r), \tag{16.6.57}$$

where now the order symbol O is tied to the assumption $\varphi_{zz}(0)=1$.

Thus, for r small, η satisfies the convexity condition (16.5.9). Furthermore, since $\eta_v = \eta_w - \eta_z$ and $\eta_{vv} = \eta_{zz} + \eta_{ww} - 2\eta_{zw}$, we deduce that $\eta_v(u, f(u)) = 0$ and $\eta_{vv}(u,v) \geq a/\mu$, so that (16.6.53) is indeed satisfied.

Given functions u_0 and v_0 on $(-\infty,\infty)$, such that

$$\int_{-\infty}^{\infty} (1+x^2)[u_0^2(x) + v_0^2(x)]dx = \sigma^2, \tag{16.6.58}$$

$$TV_{(-\infty,\infty)} u_0(\cdot) + TV_{(-\infty,\infty)} v_0(\cdot) = \delta, \tag{16.6.59}$$

$$\int_{-\infty}^{\infty} u_0(x)dx = 0, \tag{16.6.60}$$

with σ and δ sufficiently small, Theorem 16.6.4 implies that the Cauchy problem for the system (5.5.52), with initial data (u_0, v_0), possesses an admissible BV solution (u,v), in the large, and

$$\int_{-\infty}^{\infty} [|u(x,t)| + |v(x,t)|]dx \to 0, \qquad \text{as } t \to \infty, \tag{16.6.61}$$

$$TV_{(-\infty,\infty)} u(\cdot,t) + TV_{(-\infty,\infty)} v(\cdot,t) \to 0, \qquad \text{as } t \to \infty. \tag{16.6.62}$$

Comparing the above with the treatment of the system (5.5.52) in Section 16.5, one notes that the assumption (16.6.58) is more restrictive than (16.5.20) and that the condition (16.6.60) imposed here was not needed there. On the other hand, in contrast to the approach of Section 16.5, the present treatment also delivers the long time behavior of solutions.

As a matter of fact, the question of existence and long time behavior of *BV* solutions to the more general class of systems (16.6.9), with nonzero X, or even to the restricted class (16.6.10) without the condition (16.6.12), is still open. It is clear that U, or at least its V-component, will not generally decay to zero in $L^1(-\infty,\infty)$, as $t \to \infty$. The following, particular, result provides a first indication of what should be expected under such circumstances.

We consider the Cauchy problem for the simple system

$$\begin{cases} \partial_t u - \partial_x v + \alpha v^2 = 0 \\ \\ \partial_t v + \partial_x p(u) + v = 0, \end{cases} \tag{16.6.63}$$

as a model for systems in the class (16.6.9), where $p'(u) < 0$, and α is a constant, positive, negative or zero. The following analog to Theorem 16.6.4 holds:

16.6.5 Theorem. *Assume* (u_0, v_0) *are given functions on* $(-\infty,\infty)$ *satisfying* (16.6.58) *and* (16.6.59), *with* σ and δ *sufficiently small. Then there exists an admissible BV solution* (u,v) *of the system* (16.6.63) *on* $(-\infty,\infty)\times[0,\infty)$, *with initial data* (u_0,v_0). *Furthermore,*

$$\int_{-\infty}^{\infty} [|u(x,t) - \theta(x,t)| + |v(x,t)|]dx \leq b\sigma(t+1)^{-\frac{1}{4}}, \qquad 0 \leq t < \infty, \tag{16.6.64}$$

(16.6.65)
$$TV_{(-\infty,\infty)}u(\cdot,t) + TV_{(-\infty,\infty)}v(\cdot,t) \leq c_0\sigma(t+1)^{-\frac{1}{4}} + c_1\delta e^{-\nu t}, \quad 0 \leq t < \infty,$$

where c_0, c_1, ν *and* b *are positive constants, independent of the initial data, and* θ *is the solution*

$$\theta(x,t) = M(4\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right) \tag{16.6.66}$$

to the heat equation, with M *some constant depending on* (u_0,v_0).

Sketch of proof. For simplicity, we discuss only the special case $\alpha = 0$, so that (16.6.63) is still in the form (16.6.10). However, we no longer impose (16.6.60), assuming instead

$$\int_{-\infty}^{\infty} u_0(x)dx = M. \tag{16.6.67}$$

We merely sketch the proof. The details, together with the treatment of the general case $\alpha \neq 0$, are found in the literature cited in Section 16.7.

The objective is to demonstrate that, for any $0 < t < \infty$,

$$\int_{-\infty}^{\infty} [|u(x,t) - \hat{u}(x,t)| + |v(x,t) - \hat{v}(x,t)|]dx \leq c\sigma(t+1)^{-\frac{1}{4}}, \tag{16.6.68}$$

where $(\hat{u}, \hat{v})$ is the solution to the system

$$\begin{cases} \partial_t \hat{u} - \partial_x \hat{v} = 0 \\ \hat{v} = -\partial_x p(\hat{u}), \end{cases} \tag{16.6.69}$$

with the same initial values (u_0, v_0) as (u, v). Thus $\hat{u}$ satisfies the porous media equation

$$\partial_t \hat{u} + \partial_x^2 p(\hat{u}) = 0. \tag{16.6.70}$$

By means of lengthy but straightforward analysis, involving elementary "energy" estimates, it is possible to establish bounds for the solution $\hat{u}$ of (16.6.70) and thereby for $\hat{v}$, including the following:

$$\int_0^\infty \int_{-\infty}^\infty \hat{v}^2(x,t)dxdt \leq c\sigma^2, \tag{16.6.71}$$

$$\int_0^\infty \int_{-\infty}^\infty (x^2 + t + 1)[\partial_t \hat{u}(x,t)]^2 dxdt \leq c\sigma^2, \tag{16.6.72}$$

$$\int_0^\infty \int_{-\infty}^\infty (x^2 + t + 1)^2 [\partial_t \hat{v}(x,t)]^2 dxdt \leq c\sigma^2, \tag{16.6.73}$$

$$\int_{-\infty}^\infty |\hat{u}(x,t) - \theta(x,t)| dx \leq c\sigma(t+1)^{-\frac{1}{4}}, \tag{16.6.74}$$

$$\int_{-\infty}^\infty |\hat{v}(x,t)| dx \leq c\sigma(t+1)^{-\frac{1}{2}}, \tag{16.6.75}$$

where θ is defined by (16.6.66). Thus proving (16.6.68) will establish the assertion (16.6.64). The asserted decay (16.6.65) in the variation will then follow from Theorem 16.3.1.

With an eye to verifying (16.6.68), we set $w = u - \hat{u}$ and $z = v - \hat{v}$, noting that (w, z) satisfies the system

$$\begin{cases} \partial_t w - \partial_x z = 0 \\ \partial_t z + \partial_x \hat{p}(w, \hat{u}) + z + \partial_t \hat{v} = 0, \end{cases} \tag{16.6.76}$$

with zero initial data. In (16.6.76), $\hat{p}$ stands for the "relative pressure" defined by

$$\hat{p}(w, \hat{u}) = p(w + \hat{u}) - p(\hat{u}). \tag{16.6.77}$$

The admissibility of solutions to (16.6.76) is encoded in the "relative entropy" inequality

$$\partial_t\left[\hat{\psi}(w,\hat{u})+\frac{1}{2}z^2\right]+\partial_x[\hat{p}(w,\hat{u})z]+z^2\le-[\hat{p}(w,\hat{u})-p'(\hat{u})w]\partial_t\hat{u}-z\partial_t\hat{v}, \tag{16.6.78}$$

where $\hat{\psi}$ denotes the "relative internal energy" defined by

$$\hat{\psi}(w,\hat{u})=-\int_{\hat{u}}^{w+\hat{u}}p(\xi)d\xi+p(\hat{u})w. \tag{16.6.79}$$

A priori bounds on (w,z) will be derived by combining (16.6.78) with the balance law

$$\partial_t\left[\frac{1}{2}\Phi^2+(z+\hat{v})\Phi\right]+\partial_x[\hat{p}(w,\hat{u})\Phi]-\hat{p}(w,\hat{u})w=z^2+z\hat{v}, \tag{16.6.80}$$

where Φ is the "potential function"

$$\Phi(x,t)=\int_{-\infty}^{x}w(y,t)dy. \tag{16.6.81}$$

Terms with the "good sign" appearing in (16.6.78) and (16.6.80) include $\hat{\psi}(w,\hat{u})$, z^2, Φ^2 and $-\hat{p}(w,\hat{u})w$. It is easy to see that, when σ is sufficiently small, the terms of indefinite sign may be balanced, with the help of (16.6.71), (16.6.72) and (16.6.73), against the terms with the good sign, yielding bounds

$$\int_{-\infty}^{\infty}[w^2(x,t)+z^2(x,t)]dx\le c\sigma^2, \tag{16.6.82}$$

$$\int_0^{\infty}\int_{-\infty}^{\infty}[w^2(x,t)+z^2(x,t)]dxdt\le c\sigma^2. \tag{16.6.83}$$

To get the next round of estimates, we multiply (16.6.78), first by t and then by x^2, which yields

$$\begin{aligned}\partial_t\left[t[\hat{\psi}(w,\hat{u})+\frac{1}{2}z^2]\right]&+\partial_x[t\hat{p}(w,\hat{u})z]+tz^2\\ &\le\hat{\psi}(w,\hat{u})+\frac{1}{2}z^2-t[\hat{p}(w,\hat{u})-p'(\hat{u})w]\partial_t\hat{u}-tz\partial_t\hat{v},\end{aligned} \tag{16.6.84}$$

$$\begin{aligned}\partial_t\left[x^2[\hat{\psi}(w,\hat{u})+\frac{1}{2}z^2]\right]&+\partial_x[x^2\hat{p}(w,\hat{u})z]+x^2z^2\\ &\le 2x\hat{p}(w,\hat{u})z-x^2[\hat{p}(w,\hat{u})-p'(\hat{u})w]\partial_t\hat{u}-x^2z\partial_t\hat{v}.\end{aligned} \tag{16.6.85}$$

With the help of (16.6.82) and (16.6.83), together with (16.6.71), (16.6.72) and (16.6.73), one may balance the terms of indefinite sign in (16.6.84) and (16.6.85) against the terms $t\hat{\psi}$, $x^2\hat{\psi}$, tz^2 and x^2z^2, with the good sign, to obtain the estimate

$$\int_{-\infty}^{\infty}(x^2+t+1)[w^2(x,t)+z^2(x,t)]dx\le c\sigma^2. \tag{16.6.86}$$

Finally, we apply Schwarz's inequality to (16.6.86) to get

$$\left[\int_{-\infty}^{\infty}[|w(x,t)|+|z(x,t)|]dx\right]^2 \tag{16.6.87}$$

$$\leq 2\int_{-\infty}^{\infty}(x^2+t+1)^{-1}dx\int_{-\infty}^{\infty}(x^2+t+1)[w^2(x,t)+z^2(x,t)]dx$$

$$\leq c\sigma^2(t+1)^{-\frac{1}{2}},$$

whence we obtain (16.6.68). This completes the sketch of the proof.

In the long time behavior of solutions to the Cauchy problem for the class of systems of balance laws considered in this chapter, with initial data decaying as $|x| \to \infty$, the damping action of the source is moderated by wave dispersion. By contrast, the dissipative effect of the source becomes absolutely dominant when the waves are confined as in the case of spatially periodic solutions. An illustration is provided by the following proposition, whose proof is found in the literature cited in Section 16.7.

16.6.6 Theorem. *Consider a system* (16.2.1) *of balance laws in the form* (16.6.10) *with genuinely nonlinear or linearly degenerate characteristic fields, endowed with a convex entropy* η. *Assume that the source is dissipative semidefinite* (16.6.1), *relative to* η, *and the Kawashima condition* (16.6.2) *holds. Then there are positive constants* δ_0, c_0, c_1 *and* ν *so that the Cauchy problem with initial data* U_0, *where*

$$U_0(x+2) = U_0(x), \qquad -\infty < x < \infty, \tag{16.6.88}$$

$$|U_0(x)| < \delta < \delta_0, \qquad -\infty < x < \infty, \tag{16.6.89}$$

$$TV_{[-1,1]}U_0(\cdot) = \delta < \delta_0, \tag{16.6.90}$$

possesses an admissible BV solution U on $(-\infty,\infty)\times[0,\infty)$, *which is* 2*-periodic and satisfies*

$$|U(x,t)| \leq c_0\delta e^{-\nu t}, \qquad -1 \leq x < 1,\ 0 \leq t < \infty, \tag{16.6.91}$$

$$TV_{[-1,1]}U(\cdot,t) \leq c_1\delta e^{-\nu t}, \qquad 0 \leq t < \infty. \tag{16.6.92}$$

In particular, under the subcharacteristic condition (5.5.54) in strict inequality form, the system (5.5.52) meets the conditions of Theorem 16.6.6, and thereby possesses spatially periodic BV solutions that decay exponentially as time tends to infinity.

The existence of spatially periodic BV solutions to genuinely nonlinear systems of two conservation laws was established in Chapter XII; however it is doubtful that this extends to systems of conservation laws of larger size. We see here that in the presence of a dissipative source these difficulties disappear.

16.7 Notes

Local and global solutions to the Cauchy problem for systems of balance laws were constructed, under the assumptions of Theorems 16.1.1 and 16.2.1, by the random choice method, in Dafermos and Hsiao [1], then through the front tracking algorithm, in Amadori and Guerra [1,3], and finally via the vanishing viscosity approach, in Christoforou [1]. In particular, Amadori and Guerra [1,3] establish the uniqueness of solutions and recast the diagonal dominance condition (16.2.4) in a form that does not depend on a particular choice of eigenvectors of $\mathrm{D}F$. Uniqueness via vanishing viscosity is discussed in Christoforou [2].

For the rate of convergence of the vanishing viscosity approximation, see Christoforou and Trivisa [2]. The rate of decay of positive waves is established in Goatin and Gosse [1], and Christoforou and Trivisa [1,3]. The vanishing viscosity approach to systems with memory is pursued in Christoforou [3] and Chen and Christoforou [1]. For initial-boundary value problems, see Colombo and Guerra [3].

For a detailed proof of a proposition akin to Theorem 16.1.2, via operator splitting, see Christoforou [5]. Alternatively, instead of employing operator splitting, one may adapt the Glimm scheme to inhomogeneous systems of balance laws by solving at each step a generalized Riemann problem (see Section 9.11). In that connection, see Hong and LeFloch [1], Chou, Hong and Su [1], and Su, Hong and Chou [1].

Theorem 16.1.3 is taken from Tai-Ping Liu [14]. Amadori, Gosse and Guerra [1], and Seung-Yeal Ha [1] improve this result by establishing L^1 stability. The effects of resonance between the waves and the source term may be seen in Tai-Ping Liu [18], Cai Zhong Li and Tai-Ping Liu [1], Pego [4], Isaacson and Temple [4], Klingenberg and Risebro [2], Ha and Yang [1], Lien [1], Lan and Lin [1], Asakura [3], Hong and Temple [1] and Hong [1].

The radially symmetric form (7.1.29) of the Euler equations, possibly with damping, combustion, self generated gravitational force, or other manifestations of the Euler-Poisson equations, provide interesting examples of inhomogeneous systems of balance laws. A variety of related problems have been studied by Gui-Qiang Chen [6], Chen and Glimm [1,2], Gui-Qiang Chen and Tiang-Hong Li [1], Hsiao, Luo and Yang [1], Tong Yang [1], Wang and Wang [2,3], Chen and Wagner [1], Tsuge [1,2,3,4,5,6], and Ha, Huang, and Lien [1]. In the majority of the above works, one has to exclude the origin from the domain of solutions in order to avoid the singular behavior of the system at $r = 0$. See Section 18.9.

Redistribution of damping was introduced in Dafermos [23,25,34]. In particular, the details of the derivation of the L^1 estimate for the system (5.5.52), expounded in Section 16.5, are found in Dafermos [25]. The derivation of L^1 bounds in Section 16.6, which is taken from Dafermos [36, 43], makes essential use of a Liapunov functional invented by Ruggeri and Serre [1]. The detailed proof of Theorem 16.6.5 is in Dafermos [41]. Theorem 16.6.6, on periodic solutions, is from Dafermos [42,44]. Applications of redistribution of damping to systems of balance laws with memory, arising in viscoelasticity and heat conduction, are found in Dafermos [38,39,40]. For a survey see Dafermos [45].

XVII

Compensated Compactness

Approximate solutions to hyperbolic systems of conservation laws may be generated in a variety of ways: by the method of vanishing viscosity, through difference approximations, by relaxation schemes, etc. The topic for discussion in this chapter is whether solutions may be constructed as limits of sequences of approximate solutions that are bounded only in some L^p space. Since the systems are nonlinear, the difficulty lies in that the construction schemes are generally consistent only when the sequence of approximating solutions converges strongly, whereas the assumed L^p bounds guarantee only weak convergence: Approximate solutions may develop high-frequency oscillations of finite amplitude which play havoc with consistency. The aim is to demonstrate that entropy inequalities may save the day by quenching rapid oscillations, thus enforcing strong convergence of the approximating solutions. Some indication of this effect was alluded to in Section 1.9.

The principal tools in the investigation will be the notion of Young measure and the functional analytic method of compensated compactness. The former naturally induces the very general class of measure-valued solutions and the latter is employed to verify that nonlinearity reduces measure-valued solutions to traditional ones. As it relies heavily on entropy dissipation, the approach appears to be applicable mainly to systems endowed with a rich family of entropy-entropy flux pairs, most notably the scalar conservation law and systems of just two conservation laws. Despite this limitation, the approach is quite fruitful, not only because of the abundance of important systems with such structure, but also because it provides valuable insight into the stabilizing role of entropy dissipation as well as into the "conflicted" stabilizing-destabilizing behavior of nonlinearity. Different manifestations of these factors were already encountered in earlier chapters.

Out of a host of known applications of the method, only the simplest shall be presented here, pertaining to the scalar conservation law, genuinely nonlinear systems of two conservation laws, and the system of isentropic elasticity and gas dynamics.

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17.1 The Young Measure

The stumbling block for establishing consistency of construction schemes that generate weakly convergent sequences of approximate solutions lies in that it is not generally permissible to pass weak limits under nonlinear functions. Suppose Ω is an open subset of $\mathbb{R}^m$ and $\{U_k\}$ is a sequence in $L^\infty(\Omega;\mathbb{R}^n)$ which converges in L^∞ weak* to some limit $\bar{U}$. If g is any continuous real-valued function on $\mathbb{R}^n$, the sequence $\{g(U_k)\}$ will contain subsequences that converge in L^∞ weak*, say to $\bar{g}$, but in general $\bar{g} \neq g(\bar{U})$. It turns out that the limit behavior of such sequences, for all continuous g, is encoded in a family $\{\nu_X : X \in \Omega\}$ of probability measures on $\mathbb{R}^n$, which is constructed by the following procedure.

Let $M(\mathbb{R}^n)$ denote the space of bounded Radon measures on $\mathbb{R}^n$, which is isometrically isomorphic to the dual of the space $C(\mathbb{R}^n)$ of bounded continuous functions. With $k = 1, 2, \cdots$ and any $X \in \Omega$, we associate the Dirac mass $\delta_{U_k(X)}$ in $M(\mathbb{R}^n)$, centered at the point $U_k(X)$, and realize the family $\{\delta_{U_k(X)} : X \in \Omega\}$ as an element ν_k of the space $L^\infty_w(\Omega;M(\mathbb{R}^n))$, which is isometrically isomorphic to the dual of $L^1(\Omega;C(\mathbb{R}^n))$. By virtue of standard weak compactness and separability theorems, there is a subsequence $\{\nu_j\}$ of $\{\nu_k\}$ which converges weakly* to some $\nu \in L^\infty_w(\Omega;M(\mathbb{R}^n))$. Thus, $\nu = \{\nu_X : X \in \Omega\}$ and, as $j \to \infty$,

(17.1.1)

$$\int_\Omega \psi(X, U_j(X))dX = \int_\Omega < \delta_{U_j(X)}, \psi(X,\cdot) > dX \to \int_\Omega < \nu_X, \psi(X,\cdot) > dX,$$

for any $\psi \in C(\Omega \times \mathbb{R}^n)$. The supports of the $\delta_{U_j(X)}$ are uniformly bounded and hence the ν_X must have compact support. Furthermore, since the $\delta_{U_j(X)}$ are probability measures, so are the ν_X. In particular, applying (17.1.1) for $\psi(X,U) = \phi(X)g(U)$, where $\phi \in C(\Omega)$ and $g \in C(\mathbb{R}^n)$, we arrive at the following

17.1.1 Theorem. *Let Ω be an open subset of $\mathbb{R}^m$. Then any bounded sequence $\{U_k\}$ in $L^\infty(\Omega;\mathbb{R}^n)$ contains a subsequence $\{U_j\}$, and a measurable family $\{\nu_X : X \in \Omega\}$ of probability measures with compact support, such that, for any $g \in C(\mathbb{R}^n)$,*

(17.1.2) $$g(U_j) \rightharpoonup \bar{g}, \qquad \text{as } j \to \infty,$$

in L^∞ weak*, where

(17.1.3) $$\bar{g}(X) = < \nu_X, g > = \int_{\mathbb{R}^n} g(U)d\nu_X(U).$$

The collection $\{\nu_X : X \in \Omega\}$ constitutes the family of *Young measures* associated with the subsequence $\{U_j\}$. To gain some insight, let us consider the ball $\mathscr{B}_r(X)$ in Ω, with center at some $X \in \Omega$, radius r and measure $|\mathscr{B}_r|$. On account of our construction of ν_X, it is easy to see that

(17.1.4) $$\nu_X = \lim_{r\downarrow 0} \lim_{j\uparrow\infty} \frac{1}{|\mathscr{B}_r|} \int_{\mathscr{B}_r(X)} \delta_{U_j(Y)} dY, \qquad \text{a.e. on } \Omega,$$

where the limits are to be understood in the weak* sense. Notice that the averaged integral on the right-hand side of (17.1.4) may be interpreted as the probability distribution of the values of $U_j(Y)$ as Y is selected uniformly at random from $\mathscr{B}_r(X)$. Thus, according to (16.1.4), ν_X represents the limiting probability distribution of the values of U_j near X.

By virtue of (17.1.2) and (17.1.3), the subsequence $\{U_j\}$ converges, in L^∞ weak*, to the mean $\bar{U} =< \nu_X, U >$ of the Young measures. The limit $\bar{g}$ of $\{g(U_j)\}$ will satisfy $\bar{g} = g(\bar{U})$, for all $g \in C(\mathbb{R}^n)$, if and only if ν_X reduces to the Dirac mass $\delta_{\bar{U}(X)}$ centered at $\bar{U}(X)$. In that case, $\{|U_j|\}$ will converge to $|\bar{U}|$, which implies that $\{U_j\}$ will converge to $\bar{U}$ strongly in $L^p_{\mathrm{loc}}(\Omega)$, for any $1 \le p < \infty$, and some subsequence of $\{U_j\}$ will converge to $\bar{U}$ a.e. on Ω. Hence, to establish strong convergence of $\{U_j\}$, one needs to verify that the support of the Young measure is confined to a point.

Certain applications require more general versions of Theorem 17.1.1. Young measures ν_X are defined even when the sequence $\{U_k\}$ is merely bounded in some $L^p(\Omega;\mathbb{R}^n)$, with $1 < p < \infty$. If Ω is bounded, the ν_X are still probability measures and (17.1.2), (17.1.3) hold for all continuous functions g which satisfy a growth condition $|g(U)| \le c(1 + |U|^q)$, for some $0 < q < p$. In that case, convergence in (17.1.2) is weakly in $L^r(\Omega)$, for $1 < r < p/q$. By contrast, when Ω is unbounded, the ν_X may have mass less than one, because in the process of constructing them, as one passes to the $j \to \infty$ limit, part of the mass may leak out at infinity.

17.2 Compensated Compactness and the div-curl Lemma

The theory of compensated compactness strives to classify bounded (weakly compact) sets in L^p space endowed with additional structure that falls short of (strong) compactness but still manages to render certain nonlinear functions weakly continuous. This is nicely illustrated by means of the following proposition, the celebrated *div-curl lemma*, which commands a surprisingly broad gamut of applications.

17.2.1 Theorem. *Given an open subset Ω of $\mathbb{R}^m$, let $\{G_j\}$ and $\{H_j\}$ be sequences of vector fields in $L^2(\Omega;\mathbb{R}^m)$ converging weakly to respective limits $\bar{G}$ and $\bar{H}$, as $j \to \infty$. Assume both $\{\operatorname{div} G_j\}$ and $\{\operatorname{curl} H_j\}$ lie in compact subsets of $W^{-1,2}(\Omega)$. Then*

$$G_j \cdot H_j \to \bar{G} \cdot \bar{H}, \qquad \text{as } j \to \infty, \tag{17.2.1}$$

in the sense of distributions.

Proof. It will suffice to establish (17.2.1) for Ω bounded. Moreover, on account of $G_j \cdot \bar{H} \to \bar{G} \cdot \bar{H}$, we may assume, without loss of generality, that $\bar{H} = 0$.

Let $\Phi_j \in W_0^{1,2}(\Omega;\mathbb{R}^m) \cap W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ denote the solution of the boundary value problem $\Delta\Phi_j = H_j$ in Ω, $\Phi_j = 0$ on $\partial\Omega$. Then $\{\Phi_j\}$ converges to zero weakly in $W^{2,2}_{\mathrm{loc}}$, and hence $\{\operatorname{div}\Phi_j\}$ converges to zero weakly in $W^{1,2}_{\mathrm{loc}}$. On the other hand, since $\Delta(\operatorname{curl}\Phi_j) = \operatorname{curl} H_j$, $\{\operatorname{curl}\Phi_j\}$ converges to zero strongly in $W^{1,2}_{\mathrm{loc}}$.

We now set

$$V_j = H_j - \operatorname{grad} \operatorname{div} \Phi_j \tag{17.2.2}$$

and observe that, for $\alpha = 1, \dots, m$,

$$V_{j\alpha} = \sum_{\beta=1}^{m} \partial_\beta (\partial_\beta \Phi_{j\alpha} - \partial_\alpha \Phi_{j\beta}), \tag{17.2.3}$$

so that $\{V_j\}$ converges to zero strongly in L^2_{loc}.

With the help of (17.2.2), we obtain

$$G_j \cdot H_j = G_j \cdot V_j + \operatorname{div}[(\operatorname{div} \Phi_j) G_j] - (\operatorname{div} \Phi_j)(\operatorname{div} G_j). \tag{17.2.4}$$

Each term on the right-hand side of (17.2.4) tends to zero, in the sense of distributions, as $j \to \infty$, and this establishes (17.2.1). The proof is complete.

In the applications, the following technical result is often helpful for verifying the hypotheses of Theorem 17.2.1.

17.2.2 Lemma. *Let Ω be an open subset of $\mathbb{R}^m$ and $\{\phi_j\}$ a bounded sequence in $W^{-1,p}(\Omega)$, for some $p > 2$. Furthermore, let $\phi_j = \chi_j + \psi_j$, where $\{\chi_j\}$ lies in a compact set of $W^{-1,2}(\Omega)$, while $\{\psi_j\}$ lies in a bounded set of the space of measures $M(\Omega)$. Then $\{\phi_j\}$ lies in a compact set of $W^{-1,2}(\Omega)$.*

Proof. Consider the (unique) functions g_j and h_j in $W_0^{1,2}(\Omega)$ which solve the equations

$$\Delta g_j = \chi_j, \qquad \Delta h_j = \psi_j. \tag{17.2.5}$$

By standard elliptic theory, $\{g_j\}$ lies in a compact set of $W_0^{1,2}(\Omega)$ while $\{h_j\}$ lies in a compact set of $W_0^{1,q}(\Omega)$, for $1 < q < \frac{m}{m-1}$. Since $\phi_j = \Delta(g_j + h_j)$, $\{\phi_j\}$ is contained in a compact set of $W^{-1,q}(\Omega)$. But $\{\phi_j\}$ is bounded in $W^{-1,p}(\Omega)$, with $p > 2$, hence, by interpolation between $W^{-1,q}$ and $W^{-1,p}$, it follows that $\{\phi_j\}$ lies in a compact set of $W^{-1,2}(\Omega)$. The proof is complete.

17.3 Measure-Valued Solutions for Systems of Conservation Laws and Compensated Compactness

Consider a system of conservation laws,

$$\partial_t U + \partial_x F(U) = 0, \tag{17.3.1}$$

and suppose $\{U_k\}$ is a sequence of approximate solutions in an open subset Ω of $\mathbb{R}^2$, namely

$$\partial_t U_k + \partial_x F(U_k) \to 0, \qquad \text{as } k \to \infty, \tag{17.3.2}$$

in the sense of distributions on Ω. For example, $\{U_k\}$ may have been derived via the vanishing viscosity approach, that is $U_k = U_{\mu_k}$, with $\mu_k \downarrow 0$ as $k \to \infty$, where U_μ is the solution of the parabolic system

$$\partial_t U + \partial_x F(U) = \mu \partial_x^2 U. \tag{17.3.3}$$

When $\{U_k\}$ lies in a bounded set of $L^\infty(\Omega;\mathbb{R}^n)$, following the discussion in Section 16.1, one may extract a subsequence $\{U_j\}$, associated with a family of Young probability measures $\{\nu_{x,t} : (x,t) \in \Omega\}$ such that $h(U_j) \rightharpoonup < \nu, h >$, as $j \to \infty$, in L^∞ weak*, for any continuous h. In particular, on account of (17.3.2),

$$\partial_t < \nu_{x,t}\,, U > + \partial_x < \nu_{x,t}\,, F(U) >= 0\,. \tag{17.3.4}$$

One may thus interpret $\nu_{x,t}$ as a new type of weak solution for (17.3.1):

17.3.1 Definition. A *measure-valued solution for the system of conservation laws* (17.3.1), *in an open subset Ω of $\mathbb{R}^2$, is a measurable family $\{\nu_{x,t} : (x,t) \in \Omega\}$ of probability measures that satisfies* (17.3.4) *in the sense of distributions on Ω.*

Clearly, any traditional weak solution $U \in L^\infty(\Omega;\mathbb{R}^n)$ of (17.3.1) may be identified with the measure-valued solution $\nu_{x,t} = \delta_{U(x,t)}$. However, the class of measure-valued solutions is definitely broader than the class of traditional solutions. For instance, if U and V are any two traditional solutions of (17.3.1) in $L^\infty(\Omega;\mathbb{R}^n)$, then for any fixed $\alpha \in (0,1)$,

$$\nu_{x,t} = \alpha \delta_{U(x,t)} + (1-\alpha)\delta_{V(x,t)} \tag{17.3.5}$$

defines a nontraditional, measure-valued solution.

At first glance, the notion of measure-valued solution may appear too broad to be relevant. However, abandoning the premise that solutions should assign at each point (x,t) a specific value to the state vector provides the means for describing effectively a class of physical phenomena, such as phase transitions, where at the macroscopic level a mixture of phases may occupy the same point in space-time. We shall not develop these ideas here, but rather regard measure-valued solutions as stepping stones towards constructing traditional solutions.

The notion of admissibility naturally extends from traditional to measure-valued solutions. The measure-valued solution $\nu_{x,t}$ on Ω is said to satisfy the *entropy admissibility condition*, relative to the entropy-entropy flux pair (η, q) of (17.3.1), if

$$\partial_t < \nu_{x,t}\,, \eta(U) > + \partial_x < \nu_{x,t}\,, q(U) > \leq 0, \tag{17.3.6}$$

in the sense of distributions on Ω.

Returning to our earlier example, suppose $\nu_{x,t}$ is generated through a sequence $\{U_{\mu_j}\}$ of solutions to the parabolic system (17.3.3). If (η, q) is any entropy-entropy flux pair for (17.3.1), multiplying (17.3.3) by $\mathrm{D}\eta(U_\mu)$ and using (7.4.1) yields the identity

$$\partial_t \eta(U_\mu) + \partial_x q(U_\mu) = \mu \partial_x^2 \eta(U_\mu) - \mu \partial_x U_\mu^\top \mathrm{D}^2 \eta(U_\mu) \partial_x U_\mu \,. \tag{17.3.7}$$

In particular, when η is convex the last term on the right-hand side of (17.3.7) is nonpositive. We thus conclude that any measure-valued solution $\nu_{x,t}$ of (17.3.1), constructed by the vanishing viscosity approach relative to (17.3.3), satisfies the entropy admissibility condition (17.3.6), for any entropy-entropy flux pair (η, q) with η convex.

Lest it be thought that admissibility suffices to reduce measure-valued solutions to traditional ones, it should be noted that when two traditional solutions U and V satisfy the entropy admissibility condition for an entropy-entropy flux pair (η, q), then so does also the nontraditional measure-valued solution $\nu_{x,t}$ defined by (17.3.5). On the other hand, admissibility may be an agent for uniqueness and stability in the framework of measure-valued solutions as well. In that direction, it has been shown (references in Section 17.9) that any measure-valued solution $\nu_{x,t}$ of a scalar conservation law, on the upper half-plane, that satisfies the entropy admissibility condition for all convex entropy-entropy flux pairs, and whose initial values are Dirac masses, $\nu_{x,0} = \delta_{u_0(x)}$ for some $u_0 \in L^\infty(-\infty, \infty)$, necessarily reduces to a traditional solution, i.e., $\nu_{x,t} = \delta_{u(x,t)}$, where u is the unique admissible solution of the conservation law with initial data $u(x,0) = u_0(x)$. In particular, this implies that for scalar conservation laws any measure-valued solution constructed by the vanishing viscosity approach, with traditional initial data, reduces to a traditional solution.

Returning to the system (17.3.1), a program will be outlined for verifying that the measure-valued solution that is induced by the family of Young measures $\{\nu_{x,t} : (x,t) \in \Omega\}$ associated with a sequence $\{U_j\}$ of approximate solutions reduces to a traditional solution. This program will then be implemented for special systems. As already noted in Section 1.9, when (17.3.1) is hyperbolic, approximate solutions may develop sustained rapid oscillations, which prevent strong convergence of the sequence $\{U_j\}$. Thus, our enterprise is destined to fail, unless the approximate solutions somehow embody a mechanism that quenches oscillations. From the standpoint of the theory of compensated compactness, such a mechanism is manifested in the condition

$$\partial_t \eta(U_j) + \partial_x q(U_j) \subset \text{ compact set in } W^{-1,2}_{\mathrm{loc}}(\Omega), \tag{17.3.8}$$

for any entropy-entropy flux pair (η, q) of (17.3.1).

To see the implications of (16.3.8), consider any two entropy-entropy flux pairs (η_1, q_1) and (η_2, q_2). As $j \to \infty$, the sequences $\{\eta_1(U_j)\}$, $\{\eta_2(U_j)\}$, $\{q_1(U_j)\}$ and $\{q_2(U_j)\}$ converge, respectively, to $\bar{\eta}_1 = <\nu, \eta_1>$, $\bar{\eta}_2 = <\nu, \eta_2>$, $\bar{q}_1 = <\nu, q_1>$ and $\bar{q}_2 = <\nu, q_2>$, where for brevity we set $\nu_{x,t} = \nu$. By (17.3.8), both $\mathrm{div}(q_2(U_j)\,,\, \eta_2(U_j))$ and $\mathrm{curl}(\eta_1(U_j), -q_1(U_j))$ lie in compact sets of $W^{-1,2}_{\mathrm{loc}}(\Omega)$. Hence, on account of Theorem 17.2.1,

$$\eta_1(U_j) q_2(U_j) - \eta_2(U_j) q_1(U_j) \rightharpoonup \bar{\eta}_1 \bar{q}_2 - \bar{\eta}_2 \bar{q}_1 \,, \qquad \text{as } j \to \infty, \tag{17.3.9}$$

in $L^\infty(\Omega)$ weak*, or equivalently

$$< \nu, \eta_1 >< \nu, q_2 > - < \nu, \eta_2 >< \nu, q_1 > = < \nu, \eta_1 q_2 - \eta_2 q_1 > \,. \tag{17.3.10}$$

The plan is to use (17.3.10), for strategically selected entropy-entropy flux pairs, in order to demonstrate that the support of the Young measure ν is confined to a single point. Clearly, such a program may have a fair chance for success only when there is flexibility to construct entropy-entropy flux pairs with prescribed specifications. For all practical purposes, this requirement limits the applicability of the method to scalar conservation laws, systems of two conservation laws, and the special class of systems of more than two conservation laws that are endowed with a rich family of entropies (see Section 7.4). On the other hand, the method offers considerable flexibility in regard to construction scheme, as it requires only that the approximate solutions satisfy (17.3.8).

For illustration, let us verify (17.3.8) under the assumption that the system (17.3.1) is endowed with a uniformly convex entropy, Ω is the upper half-plane, and the sequence $\{U_j\}$ of approximate solutions is generated by the vanishing viscosity approach, $U_j = U_{\mu_j}$, where U_μ is the solution of (17.3.3) on the upper half-plane, with initial data

$$U(x,0) = U_{0\mu}(x), \qquad -\infty < x < \infty, \tag{17.3.11}$$

lying in a bounded set of $L^\infty(-\infty,\infty) \cap L^2(-\infty,\infty)$.

Let η be a uniformly convex entropy, so that $\mathrm{D}^2\eta(U)$ is positive definite. We can assume $0 \le \eta(U) \le c|U|^2$, since otherwise we simply substitute η by the entropy $\eta^*(U) = \eta(U) - \eta(0) - \mathrm{D}\eta(0)U$. Upon integrating (17.3.7) over the upper half-plane, we obtain the estimate

$$\mu \int_0^\infty \int_{-\infty}^\infty |\partial_x U_\mu(x,t)|^2 dxdt \le a, \tag{17.3.12}$$

where a is independent of μ.

Consider now any, not necessarily convex, entropy-entropy flux pair (η, q), and fix some open bounded subset Ω of the upper half-plane. Let us examine (17.3.7). The left-hand side is bounded in $W^{-1,p}(\Omega)$, for any $1 \le p < \infty$. The right-hand side is the sum of two terms: By virtue of (17.3.12), the first term tends to zero, as $\mu \downarrow 0$, in $W^{-1,2}(\Omega)$, and thus lies in a compact set of $W^{-1,2}(\Omega)$. The second term lies in a bounded set of $M(\Omega)$, again on account of (17.3.12). Therefore, (17.3.8) follows from Lemma 17.2.2.

17.4 Scalar Conservation Laws

Here we shall see how the program outlined in the previous section may be realized in the case of the scalar conversation law

$$\partial_t u + \partial_x f(u) = 0. \tag{17.4.1}$$

17.4.1 Theorem. *Let Ω be an open subset of $\mathbb{R}^2$ and $\{u_k(x,t)\}$ a bounded sequence in $L^\infty(\Omega)$ with*

(17.4.2) $$\partial_t \eta(u_k) + \partial_x q(u_k) \subset \textit{ compact set in } W^{-1,2}_{\mathrm{loc}}(\Omega),$$

for any entropy-entropy flux pair of (17.4.1). *Then there is a subsequence* $\{u_j\}$ *such that*

(17.4.3) $$u_j \rightharpoonup \bar{u}, \qquad f(u_j) \rightharpoonup f(\bar{u}), \qquad \text{as } j \to \infty,$$

in L^∞ *weak**. *Furthermore, if the set of u with* $f''(u) \neq 0$ *is dense in* $\mathbb{R}$, *then* $\{u_j\}$ *converges almost everywhere to* $\bar{u}$ *on* Ω.

Proof. By applying Theorem 17.1.1, we extract the subsequence $\{u_j\}$ and the associated family of Young measures $\nu = \nu_{x,t}$ so that $h(u_j) \rightharpoonup < \nu, h >$, for any continuous function h. Thus, $u_j \rightharpoonup \bar{u} =< \nu, u >$ and $f(u_j) \rightharpoonup < \nu, f >$. We thus have to show that $< \nu, f >= f(\bar{u})$, and that ν reduces to the Dirac mass when there is no interval on which $f'(u)$ is constant.

We employ (17.3.10) for the particular entropy-entropy flux pairs $(u, f(u))$ and $(f(u), g(u))$, where

(17.4.4) $$g(u) = \int_0^u [f'(v)]^2 dv,$$

to get

(17.4.5) $$< \nu, u >< \nu, g > - < \nu, f >< \nu, f >=< \nu, ug - f^2 > .$$

From Schwarz's inequality,

(17.4.6) $$[f(u) - f(\bar{u})]^2 \leq (u - \bar{u})[g(u) - g(\bar{u})],$$

we deduce

(17.4.7) $$< \nu\,, [f(u) - f(\bar{u})]^2 - (u - \bar{u})[g(u) - g(\bar{u})] > \leq 0.$$

Upon using (16.4.5), (16.4.7) reduces to

(17.4.8) $$[< \nu, f > - f(\bar{u})]^2 \leq 0,$$

whence $< \nu, f >= f(\bar{u})$. In particular, the left-hand side of (17.4.7) will vanish. Hence, (17.4.6) must hold as an equality for u in the support of ν. However, Schwarz's inequality (17.4.6) may hold as equality only if f' is constant on the interval with endpoints $\bar{u}$ and u. When no such interval exists, the support of ν collapses to a single point and ν reduces to the Dirac mass $\delta_{\bar{u}}$. The proof is complete.

As indicated in the previous section, one may generate a sequence $\{u_k\}$ that satisfies the assumptions of Theorem 17.4.1 by the method of vanishing viscosity, setting $u_k = u_{\mu_k}$, $\mu_k \to 0$ as $k \to \infty$, where u_μ is the solution of

(17.4.9) $$\partial_t u + \partial_x f(u) = \mu \partial_x^2 u,$$

on the upper half-plane, with initial data

$$u(x,0) = u_{0\mu}(x), \qquad -\infty < x < \infty, \tag{17.4.10}$$

that are uniformly bounded in $L^\infty(-\infty,\infty) \cap L^2(-\infty,\infty)$. Indeed, the resulting $\{u_k\}$ will be bounded in L^∞, since $\|u_\mu\|_{L^\infty} \leq \|u_{0\mu}\|_{L^\infty}$ by the maximum principle. Moreover, (17.4.2) will hold for all entropy-entropy flux pairs (η, q), by the general argument of Section 17.3, which applies here, in particular, because (17.4.1) possesses the uniformly convex entropy u^2. Finally, $\mu \partial_x^2 u_\mu \to 0$, as $\mu \downarrow 0$, in the sense of distributions. We thus arrive at the following

17.4.2 Theorem. *Suppose $u_{0\mu} \rightharpoonup u_0$, as $\mu \downarrow 0$, in $L^\infty(-\infty,\infty)$ weak*. Then there is a sequence $\{\mu_j\}$, $\mu_j \to 0$ as $j \to \infty$, such that the sequence $\{u_{\mu_j}\}$ of solutions of* (17.4.9), (17.4.10) *converges in L^∞ weak* to some function $\bar{u}$, which is a solution of* (16.4.1), *on the upper half-plane, with initial value $\bar{u}(x,0) = u_0(x)$ on $(-\infty,\infty)$. Furthermore, if the set of u with $f''(u) \neq 0$ is dense in $\mathbb{R}$, then $\{u_{\mu_j}\}$, or a subsequence thereof, converges almost everywhere to $\bar{u}$ on the upper half-plane.*

17.5 A Relaxation Scheme for Scalar Conservation Laws

The aim here is to pass to the limit, as $\mu \downarrow 0$, in the system (5.2.18), with the help of the theory of compensated compactness. Such an exercise may serve a dual purpose: for the case one is interested in (5.2.18) itself, as a model for some physical process, it will demonstrate relaxation to local equilibrium governed by the scalar conservation law (17.4.1); as a byproduct, it will establish that solutions to the Cauchy problem for (17.4.1) exist, and will suggest a method for computing them. For the latter purpose, it shall be advantageous to make the non-relaxed system (5.2.18) as simple as possible, namely semilinear,

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \partial_t v(x,t) + a^2 \partial_x u(x,t) + \frac{1}{\mu}[v(x,t) - f(u(x,t))] = 0, \end{cases} \tag{17.5.1}$$

where a is some positive constant. In order to simplify the analysis, we shall deal here with this semilinear system. At this point it may be helpful for the reader to review the introduction to relaxation theory presented in Section 5.2.

We assume f' is bounded and select a sufficiently large so that the strict subcharacteristic condition (recall (5.2.29))

$$-a + \delta < f'(u) < a - \delta, \qquad u \in (-\infty,\infty), \tag{17.5.2}$$

holds, for some $\delta > 0$. We normalize v by postulating $f(0) = 0$.

Entropy-entropy flux pairs $(\eta(u,v), q(u,v))$ for (17.5.1) satisfy the linear hyperbolic system

$$\text{(17.5.3)} \qquad \begin{cases} q_u(u,v) - a^2\eta_v(u,v) = 0 \\ \\ q_v(u,v) - \eta_u(u,v) = 0, \end{cases}$$

with general solution

$$\text{(17.5.4)} \qquad \begin{cases} \eta(u,v) = r(au+v) + s(au-v) \\ \\ q(u,v) = ar(au+v) - as(au-v). \end{cases}$$

The subcharacteristic condition (17.5.2) implies that the curve $v = f(u)$ is nowhere characteristic for the system (17.5.3), and hence, given any entropy-entropy flux pair $(\hat{\eta}(u), \hat{q}(u))$ for the scalar conservation law (17.4.1), one may construct an entropy-entropy flux pair $(\eta(u,v), q(u,v))$ for (17.5.1) with Cauchy data

$$\text{(17.5.5)} \qquad \eta(u, f(u)) = \hat{\eta}(u), \qquad q(u, f(u)) = \hat{q}(u), \qquad u \in (-\infty, \infty).$$

Differentiating (17.5.5) with respect to u and using that $\hat{q}'(u) = \hat{\eta}'(u) f'(u)$, together with (17.5.3) and (17.5.2), we deduce that $\eta_v(u, f(u)) = 0$. This, in turn, combined with (17.5.5) and (17.5.4), yields

$$\text{(17.5.6)} \qquad r'(au + f(u)) = s'(au - f(u)) = \frac{1}{2a}\hat{\eta}'(u),$$

whence one determines r and s on $\mathbb{R}$, and thereby η and q on $\mathbb{R}^2$. In particular, $\hat{\eta}'' \geq 0$ on $\mathbb{R}$ implies $r'' \geq 0$, $s'' \geq 0$ on $\mathbb{R}$, and hence $\eta_{vv} \geq 0$ on $\mathbb{R}^2$. Since $\eta_v(u, f(u)) = 0$, we then conclude that the dissipativeness condition (5.2.4) holds:

$$\text{(17.5.7)} \qquad \eta_v(u,v)[v - f(u)] \geq 0, \qquad (u,v) \in \mathbb{R}^2.$$

Under the stronger hypothesis $\hat{\eta}''(u) \geq \beta > 0$, $u \in \mathbb{R}$, (17.5.7) becomes stricter:

$$\text{(17.5.8)} \qquad \eta_v(u,v)[v - f(u)] \geq \gamma |v - f(u)|^2, \qquad (u,v) \in \mathbb{R}^2,$$

with $\gamma > 0$.

We have now laid the groundwork for establishing the existence of solutions to the Cauchy problem for (17.5.1) and for passing to the limit, as $\mu \downarrow 0$.

17.5.1 Theorem. *Under the subcharacteristic condition* (17.5.2)*, the Cauchy problem for the system* (17.5.1)*, with initial data*

$$\text{(17.5.9)} \qquad (u(x,0), v(x,0)) = (u_{0\mu}(x), v_{0\mu}(x)), \qquad -\infty < x < \infty,$$

in $L^\infty(-\infty,\infty) \cap L^2(-\infty,\infty)$*, possesses a bounded (weak) solution* (u_μ, v_μ) *on the upper half-plane. Furthermore,*

$$\text{(17.5.10)} \qquad \frac{1}{\mu}\int_0^\infty \int_{-\infty}^\infty [v_\mu - f(u_\mu)]^2 dxdt \leq b \int_{-\infty}^\infty [u_{0\mu}^2(x) + v_{0\mu}^2(x)]dx,$$

where b is independent of μ.

Proof. Since (17.5.1) is semilinear hyperbolic, a local solution (u_μ, v_μ) exists and may be continued for as long as it remains bounded in L^∞. Furthermore, if (η, q) is any entropy-entropy flux pair,

$$\partial_t \eta(u_\mu, v_\mu) + \partial_x q(u_\mu, v_\mu) + \frac{1}{\mu}\eta_v(u_\mu, v_\mu)[v_\mu - f(u_\mu)] = 0. \tag{17.5.11}$$

We construct the entropy-entropy flux pair (η_m, q_m), induced by (17.5.5), with $\hat{\eta}(u) = |u|^m$, $m = 2, 3, \dots$, and normalized by $\eta_m(0,0) = 0$, $q_m(0,0) = 0$. Notice that, necessarily, the first derivatives of η_m also vanish at the origin. We integrate (17.5.11) over $(-\infty,\infty)\times[0,t]$ and use (17.5.7) to get

$$\int_{-\infty}^{\infty} \eta_m(u_\mu(x,t), v_\mu(x,t))dx \le \int_{-\infty}^{\infty} \eta_m(u_{0\mu}(x), v_{0\mu}(x))dx. \tag{17.5.12}$$

By (17.5.6) and (17.5.2), it follows easily that $(\hat{c}|w|)^m \le r_m(w) \le (\hat{C}|w|)^m$ and $(\hat{c}|w|)^m \le s_m(w) \le (\hat{C}|w|)^m$, whence

$$c^m(|u|^m + |v|^m) \le \eta_m(u,v) \le C^m(|u|^m + |v|^m), \qquad (u,v)\in\mathbb{R}^2. \tag{17.5.13}$$

Therefore, raising (17.5.12) to the power $\frac{1}{m}$ and letting $m\to\infty$ we conclude that $\|u_\mu(\cdot,t)\|_{L^\infty(-\infty,\infty)}$ and $\|v_\mu(\cdot,t)\|_{L^\infty(-\infty,\infty)}$ are bounded in terms of $\|u_{0\mu}(\cdot)\|_{L^\infty(-\infty,\infty)}$ and $\|v_{0\mu}(\cdot)\|_{L^\infty(-\infty,\infty)}$, uniformly in t and μ. Thus the solution (u_μ, v_μ) exists on the entire upper half-plane.

Next we write (17.5.11) for the entropy-entropy flux pair (η_2, q_2), and integrate it over $(-\infty,\infty)\times[0,\infty)$. For this case, the stronger dissipativeness inequality (17.5.8) applies and thus we deduce (17.5.10). The proof is complete.

17.5.2 Theorem. *Consider the family $\{(u_\mu, v_\mu)\}$ of solutions of the Cauchy problem* (17.5.1), (17.5.9)*, where $\{(u_{0\mu}, v_{0\mu})\}$ is bounded in $L^\infty(-\infty,\infty)\cap L^2(-\infty,\infty)$ and $u_{0\mu}\to u_0$, as $\mu\downarrow 0$, in L^∞ weak*. Then there is a sequence $\{\mu_j\}$, with $\mu_j\downarrow 0$ as $j\to\infty$, such that $\{(u_{\mu_j}, v_{\mu_j})\}$ converges, in L^∞ weak*, to $(\bar{u}, f(\bar{u}))$, where $\bar{u}$ is a solution of* (17.4.1)*, on the upper half-plane, with initial value $\bar{u}(x,0) = u_0(x)$ on $(-\infty,\infty)$. Furthermore, if the set of u with $f''(u)\neq 0$ is dense in $\mathbb{R}$, then $\{(u_{\mu_j}, v_{\mu_j})\}$ converges to $(\bar{u}, f(\bar{u}))$, almost everywhere on the upper half-plane.*

Proof. By Theorem 17.5.1, $\{(u_\mu, v_\mu)\}$ is contained in a bounded set of the space $L^\infty((-\infty,\infty)\times[0,\infty))$.

We fix any entropy-entropy flux pair $(\hat{\eta}, \hat{q})$ for (17.4.1), consider the entropy-entropy flux pair (η, q) for (17.5.1) generated by solving the Cauchy problem (17.5.3), (17.5.5), and use (17.5.11) to write

$$\begin{aligned} &\partial_t \hat{\eta}(u_\mu) + \partial_x \hat{q}(u_\mu) \\ &\quad = \partial_t[\eta(u_\mu, f(u_\mu)) - \eta(u_\mu, v_\mu)] + \partial_x[q(u_\mu, f(u_\mu)) - q(u_\mu, v_\mu)] \\ &\qquad - \frac{1}{\mu}\eta_v(u_\mu, v_\mu)[v_\mu - f(u_\mu)]. \end{aligned} \tag{17.5.14}$$

By virtue of (17.5.10), both $\eta(u_\mu, f(u_\mu)) - \eta(u_\mu, v_\mu)$ and $q(u_\mu, f(u_\mu)) - q(u_\mu, v_\mu)$ tend to zero in L^2, as $\mu \downarrow 0$. Therefore, the first two terms on the right-hand side of (17.5.14) tend to zero in $W^{-1,2}$, as $\mu \downarrow 0$. On the other hand, the third term lies in a bounded set of L^1, again on account of (17.5.10), recalling that $\eta_v(u, f(u)) = 0$.

We now fix any sequence $\{\mu_k\}$, with $\mu_k \downarrow 0$ as $k \to \infty$, and set $(u_k, v_k) = (u_{\mu_k}, v_{\mu_k})$. In virtue of the above, Lemma 17.2.2 implies that (17.4.2) holds for any entropy-entropy flux pair $(\hat{\eta}, \hat{q})$ of (17.4.1), where Ω is the upper half-plane. Theorem 17.4.1 then yields (17.4.3), for some subsequence $\{u_j\}$. In turn, (17.4.3) together with (17.5.10) imply $v_j \to f(\bar{u})$, in L^∞ weak*. In particular, $\bar{u}$ is a solution of (17.4.1), with initial values u_0, because of $(17.5.1)_1$.

When the set of u with $f''(u) \neq 0$ is dense in $\mathbb{R}$, $\{u_j\}$ converges to $\bar{u}$ almost everywhere, on account of Theorem 17.4.1. It then follows from (17.5.10) that, likewise, $\{v_j\}$ converges to $f(\bar{u})$ almost everywhere. The proof is complete.

By combining (17.5.11), (17.5.7), (17.5.10) and (17.5.5), we infer that, at least in the case where $\{u_j\}$ converges almost everywhere, the limit $\bar{u}$ will satisfy the entropy admissibility condition, for any entropy-entropy flux pair $(\hat{\eta}, \hat{q})$, with $\hat{\eta}$ convex.

Notice that Theorem 17.5.2 places no restriction on the initial values $v_{0\mu}$ of v_μ, save for the requirement that they be bounded. In particular, $v_{0\mu}$ may lie far apart from its local equilibrium value $f(u_{0\mu})$. In that situation v_k must develop a boundary layer across $t = 0$.

The reader should be warned that compensated compactness is not the most efficient method for handling the simple system (17.5.1). Indeed, it has been shown (references in Section 17.9) that if (u, v) and $(\bar{u}, \bar{v})$ are any pair of solutions of (17.5.1), with corresponding initial values (u_0, v_0) and $(\bar{u}_0, \bar{v}_0)$, then

$$
\begin{aligned}
(17.5.15) \qquad & \int_{-\ell}^{\ell} \{|u(x,t) - \bar{u}(x,t)| + |v(x,t) - \bar{v}(x,t)|\} dx \\
& \leq \frac{(1+a)^2}{a} \int_{-\ell-at}^{\ell+at} \{|u_0(x) - \bar{u}_0(x)| + |v_0(x) - \bar{v}_0(x)|\} dx
\end{aligned}
$$

holds, for any $\ell > 0$ and $t > 0$. Armed with this estimate, one may easily establish compactness in L^1 as well as in BV, and then pass to the $\mu \downarrow 0$ relaxation limit. Nevertheless, at the time of this writing, compensated compactness is the only approach that works for the nonlinear system (5.2.18), because no analog to the estimate (17.5.15) is currently known for that case.

17.6 Genuinely Nonlinear Systems of Two Conservation Laws

The program outlined in Section 17.3 will here be implemented for genuinely nonlinear systems (17.3.1) of two conservation laws. In particular, our system will be endowed with a coordinate system of Riemann invariants (z, w), normalized as in

(12.1.2), and the condition of genuine nonlinearity will be expressed by (12.1.3), namely $\lambda_z < 0$ and $\mu_w > 0$. Moreover, the system will be equipped with a rich family of entropy-entropy flux pairs, including the Lax pairs constructed in Section 12.2, which will play a pivotal role in the analysis.

We show that the entropy conditions, in conjunction with genuine nonlinearity, quench rapid oscillations:

17.6.1 Theorem. *Let Ω be an open subset of $\mathbb{R}^2$ and $\{U_k(x,t)\}$ a bounded sequence in $L^\infty(\Omega;\mathbb{R}^2)$ with*

$$\partial_t \eta(U_k) + \partial_x q(U_k) \subset \text{ compact set in } W^{-1,2}_{\mathrm{loc}}(\Omega), \tag{17.6.1}$$

for any entropy-entropy flux pair (η, q) of (16.3.1). *Then there is a subsequence $\{U_j\}$ which converges almost everywhere on Ω.*

Proof. By applying Theorem 17.1.1, we extract a subsequence $\{U_j\}$ and identify the associated family of Young measures $\nu_{x,t}$. We have to show that, for almost all (x,t), the support of $\nu_{x,t}$ is confined to a single point and so this measure reduces to the Dirac mass. It will be expedient to monitor the Young measure on the plane of the Riemann invariants (z,w), rather than in the original state space.

We thus let ν denote the Young measure at any fixed point $(x,t) \in \Omega$, relative to the (z,w) variables, and consider the smallest rectangle $\mathscr{R} = [z^-, z^+] \times [w^-, w^+]$ that contains the support of ν. We need to show $z^- = z^+$ and $w^- = w^+$. Arguing by contradiction, assume $z^- < z^+$.

We consider the Lax entropy-entropy flux pairs (12.2.5), which will be here labeled (η_k, q_k), so as to display explicitly the dependence on the parameter k. We shall use the η_k as weights for redistributing the mass of ν, reallocating it near the boundary of $\mathscr{R}$. To that end, with each large positive integer k we associate probability measures $\nu_k^\pm$ on $\mathscr{R}$, defined through their action on continuous functions $h(z,w)$:

$$< \nu_k^\pm, h > = \frac{< \nu, h\eta_{\pm k} >}{< \nu, \eta_{\pm k} >}. \tag{17.6.2}$$

Because of the factor e^{kz} in the definition of η_k, the measure ν_k^- (or ν_k^+) is concentrated near the left (or right) side of $\mathscr{R}$. As $k \to \infty$, the sequences $\{\nu_k^-\}$ and $\{\nu_k^+\}$, or subsequences thereof, will converge, weakly* in the space of measures, to probability measures ν^- and ν^+, which are respectively supported by the left edge $[z^-] \times [w^-, w^+]$ and the right edge $[z^+] \times [w^-, w^+]$ of $\mathscr{R}$.

We apply (17.3.10) for any fixed entropy-entropy flux pair (η, q) and the Lax pairs $(\eta_{\pm k}, q_{\pm k})$ to get

$$< \nu, q > - \frac{< \nu, q_{\pm k} >}{< \nu, \eta_{\pm k} >} < \nu, \eta > = \frac{< \nu, \eta_{\pm k} q - \eta q_{\pm k} >}{< \nu, \eta_{\pm k} >}. \tag{17.6.3}$$

From (12.2.5) and (12.2.7) we infer

$$q_{\pm k} = [\lambda + O(\frac{1}{k})]\eta_{\pm k}. \tag{17.6.4}$$

Therefore, letting $k \to \infty$ in (17.6.3) yields

$$< \nu, q > - < \nu^{\pm}, \lambda >< \nu, \eta > = < \nu^{\pm}, q - \lambda\eta > . \tag{17.6.5}$$

Next, we apply (16.3.10) for the Lax pairs (η_{-k}, q_{-k}) and (η_k, q_k), thus obtaining

$$\frac{< \nu, q_k >}{< \nu, \eta_k >} - \frac{< \nu, q_{-k} >}{< \nu, \eta_{-k} >} = \frac{< \nu, \eta_{-k} q_k - \eta_k q_{-k} >}{< \nu, \eta_{-k} >< \nu, \eta_k >} . \tag{17.6.6}$$

By (17.6.4), the left-hand side of (17.6.6) tends to $< \nu^+, \lambda > - < \nu^-, \lambda >$, as $k \to \infty$. On the other hand, the right-hand side tends to zero, because the numerator is $O(k^{-1})$ while, for k large,

$$< \nu, \eta_{\pm k} > \geq c \exp[\pm \tfrac{k}{2}(z^- + z^+)]. \tag{17.6.7}$$

Hence,

$$< \nu^-, \lambda > = < \nu^+, \lambda > . \tag{17.6.8}$$

Combining (17.6.5) with (17.6.8),

$$< \nu^-, q - \lambda\eta > = < \nu^+, q - \lambda\eta > . \tag{17.6.9}$$

We apply (17.6.9) for $(\eta, q) = (\eta_k, q_k)$. On account of (12.2.12), for k large,

$$\begin{cases} < \nu^-, q_k - \lambda\eta_k > \leq C\dfrac{1}{k}\exp(kz^-) \\[2ex] < \nu^+, q_k - \lambda\eta_k > \geq c\dfrac{1}{k}\exp(kz^+), \end{cases} \tag{17.6.10}$$

which yields the desired contradiction to $z^- < z^+$. Similarly one shows $w^- = w^+$, so that $\mathscr{R}$ collapses to a single point. The proof is complete.

The stumbling block in employing the above theorem for constructing solutions to our system (17.3.1) is that, at the time of this writing, it has not been established that sequences of approximate solutions produced by any of the available schemes are bounded in L^∞. Thus, boundedness has to be imposed as an extraneous (and annoying) assumption. On the other hand, once boundedness is taken for granted, it is not difficult to verify the other requirement of Theorem 17.6.1, namely (17.6.1). In particular, when the sequence of U_k is generated via the vanishing viscosity approach, as solutions of the parabolic system (17.3.3), condition (17.6.1) follows directly from the discussion in Section 17.3, because genuinely nonlinear systems of two conservation laws are always endowed with uniformly convex entropies. For example, as shown in Section 12.2, under the normalization condition (12.1.4), the Lax entropy η_k is convex, for k sufficiently large. We thus have

17.6.2 Theorem. *For $\mu > 0$, let U_μ denote the solution on the upper half-plane of the genuinely nonlinear parabolic system of two conservation laws* (17.3.3) *with initial* *data* (17.3.11), *where* $U_{0\mu} \rightharpoonup U_0$ *in* $L^\infty(-\infty,\infty)$ *weak**, *as* $\mu \downarrow 0$. *Suppose the family* $\{U_\mu\}$ *lies in a bounded subset of* L^∞. *Then, there is a sequence* $\{\mu_j\}$, $\mu_j \to 0$ *as* $j \to \infty$, *such that* $\{U_{\mu_j}\}$ *converges, almost everywhere on the upper half-plane, to a solution* $\bar{U}$ *of* (17.3.1) *with initial value* $\bar{U}(x,0) = U_0(x)$, $-\infty < x < \infty$.

One obtains entirely analogous results for sequences of approximate solutions generated by a class of one-step difference schemes with a three-point domain of dependence:

(17.6.11)
$$U(x,t+\Delta t) - U(x,t) = \frac{\alpha}{2}G(U(x,t),U(x+\Delta x,t)) - \frac{\alpha}{2}G(U(x-\Delta x,t),U(x,t)),$$

where $\alpha = \Delta t/\Delta x$ is the ratio of mesh-lengths and G, possibly depending on α, is a function that satisfies the consistency condition $G(U,U) = F(U)$. The class includes the *Lax-Friedrichs scheme*, with

$$G(V,W) = \frac{1}{2}[F(V)+F(W)] + \frac{1}{\alpha}(V-W), \tag{17.6.12}$$

and also the *Godunov scheme*, where $G(V,W)$ denotes the state in the wake of the solution to the Riemann problem for (17.3.1), with left state V and right state W. The condition of uniform boundedness on L^∞ of the approximate solutions has to be extraneously imposed in these cases as well.

17.7 The System of Isentropic Elasticity

The assertion of Theorem 17.6.1 is obviously false when the system (17.3.1) is linear. On the other hand, genuine nonlinearity is far too strong a restriction: it may be allowed to fail along a finite collection of curves in state space, so long as these curves intersect transversely the level curves of the Riemann invariants. This will be demonstrated here in the context of the system (7.1.11) of conservation laws of one-dimensional, isentropic thermoelasticity,

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \\ \partial_t v - \partial_x \sigma(u) = 0, \end{cases} \tag{17.7.1}$$

under the assumption $\sigma''(u) \neq 0$ for $u \neq 0$, but $\sigma''(0) = 0$, so that genuine nonlinearity fails along the line $u = 0$ in state space. Nevertheless, the analog of Theorem 17.6.1 still holds:

17.7.1 Theorem. *Let* Ω *be an open subset of* $\mathbb{R}^2$ *and* $\{(u_k,v_k)\}$ *a bounded sequence in* $L^\infty(\Omega;\mathbb{R}^2)$ *with*

$$\partial_t \eta(u_k,v_k) + \partial_x q(u_k,v_k) \subset \text{ compact set in } W^{-1,2}_{\mathrm{loc}}(\Omega), \tag{17.7.2}$$

for any entropy-entropy flux pair (η, q) *of* (17.7.1). *Then there is a subsequence* $\{(u_j, v_j)\}$ *which converges almost everywhere on* Ω.

Proof. As in the proof of Theorem 17.6.1, we extract a subsequence $\{(u_j, v_j)\}$ and identify the associated family of Young measures $\nu_{x,t}$. We fix (x,t) in Ω and monitor the Young measure ν at (x,t) relative to the Riemann invariants

$$z = \int_0^u [\sigma'(\omega)]^{\frac{1}{2}} d\omega + v, \qquad w = -\int_0^u [\sigma'(\omega)]^{\frac{1}{2}} d\omega + v. \tag{17.7.3}$$

We need to show that the smallest rectangle $\mathscr{R} = [z^-, z^+] \times [w^-, w^+]$ that contains the support of ν collapses to a single point.

By retracing the steps in the proof of Theorem 17.6.1 that do not depend on the genuine nonlinearity of the system, we rederive (17.6.9). The remainder of the argument will depend on the relative positions of $\mathscr{R}$ and the straight line $z = w$ along which genuine nonlinearity fails.

Suppose first the line $z = w$ does not intersect the right edge of $\mathscr{R}$, that is, $z^+ \notin [w^-, w^+]$. In that case, (17.6.10) are still in force, yielding $z^- = z^+$. Hence $\mathscr{R}$ collapses to $[z^+] \times [w^-, w^+]$, which, according to our assumption, lies entirely in the genuinely nonlinear region, and so by the familiar argument $w^- = w^+$, verifying the assertion of the theorem. Similar arguments apply when the line $z = w$ misses any one of the other three edges of $\mathscr{R}$.

It thus remains to examine the case where the line $z = w$ intersects all four edges of $\mathscr{R}$, i.e., $z^- = w^-$ and $z^+ = w^+$. Even in that situation, by virtue of (12.2.12), $q_k - \lambda\eta_k$ does not change sign along $[z^-] \times [w^- + \varepsilon, w^+]$ and $[z^+] \times [w^-, w^+ - \varepsilon]$, so the familiar argument still goes through, showing $z^- = z^+$, unless the measures ν^- and ν^+ are respectively concentrated in the vertices (z^-, w^-) and (z^+, w^+). When that happens, (17.6.9) reduces to

$$q(z^-, w^-) - \lambda(z^-, w^-)\eta(z^-, w^-) = q(z^+, w^+) - \lambda(z^+, w^+)\eta(z^+, w^+). \tag{17.7.4}$$

In particular, let us apply (17.7.4) for the trivial entropy-entropy flux pair $(u, -v)$. At the "southwestern" vertex, $u^- = 0$ and $v^- = z^- = w^-$, while at the "northeastern" vertex, $u^+ = 0$ and $v^+ = z^+ = w^+$. Hence, (17.7.4) yields $z^- = z^+ = w^- = w^+$. The proof is complete.

Smoothness of $\sigma(u)$ cannot be generally relaxed as examples indicate that the assertion of the above proposition may break down when $\sigma''(u)$ is discontinuous at $u = 0$.

In particular, Theorem 17.7.1 applies when the elastic medium responds like a "hard spring," that is, σ is concave at $u < 0$ and convex at $u > 0$:

$$u\sigma''(u) > 0, \qquad u \neq 0. \tag{17.7.5}$$

For that case, it is possible to establish L^∞ bounds on the approximate solutions constructed by the vanishing viscosity method, namely, as solutions to a Cauchy problem

$$\begin{cases} \partial_t u - \partial_x v = \mu \partial_x^2 u \\ \partial_t v - \partial_x \sigma(u) = \mu \partial_x^2 v, \end{cases} \tag{17.7.6}$$

$$(u(x,0), v(x,0)) = (u_{0\mu}(x), v_{0\mu}(x)), \qquad -\infty < x < \infty. \tag{17.7.7}$$

17.7.2 Theorem. *Under the assumption* (17.7.5), *for any* $M > 0$, *the set* $\mathscr{U}_M$, *defined by*

$$\mathscr{U}_M = \{(u,v) : -M \le z(u,v) \le M,\ -M \le w(u,v) \le M\}, \tag{17.7.8}$$

where z *and* w *are the Riemann invariants* (17.7.3) *of* (17.7.1), *is a (positively) invariant region for solutions of* (17.7.6), (17.7.7).

Proof. The standard proof is based on the maximum principle. An alternative proof will be presented here, which relies on entropies and thus is closer to the spirit of the hyperbolic theory. It has the advantage of requiring less regularity for solutions of (17.7.6). Moreover, it readily extends to any other approximation scheme, which, like (17.7.6), is dissipative under convex entropies of (17.7.1).

For the system (17.7.1), the equations (7.4.1) that determine entropy-entropy flux pairs (η, q) reduce to

$$\begin{cases} q_u(u,v) = -\sigma'(u)\eta_v(u,v) \\ q_v(u,v) = -\eta_u(u,v). \end{cases} \tag{17.7.9}$$

Notice that (17.7.9) admits the family of solutions

$$\eta_m(u,v) = Y_m(u)\cosh(mv) - 1, \tag{17.7.10}$$

$$q_m(u,v) = -\frac{1}{m} Y_m'(u)\sinh(mv), \tag{17.7.11}$$

where $m = 1, 2, \cdots$ and Y_m is the solution of the ordinary differential equation

$$Y_m''(u) = m^2 \sigma'(u) Y_m(u), \qquad -\infty < u < \infty, \tag{17.7.12}$$

with initial conditions

$$Y_m(0) = 1, \qquad Y_m'(0) = 0. \tag{17.7.13}$$

A simple calculation gives

$$\eta_{muu}\eta_{mvv} - \eta_{muv}^2 \ge m^2[m^2\sigma' Y_m^2 - Y_m'^2]. \tag{17.7.14}$$

Moreover, by virtue of (17.7.12),

$$[m^2\sigma' Y_m^2 - Y_m'^2]' = m^2\sigma'' Y_m^2. \tag{17.7.15}$$

Consequently, (17.7.5) implies that the right-hand side of (17.7.14) is positive and hence $\eta_m(u,v)$ is a convex function on $\mathbb{R}^2$. Furthermore, $\eta_m(0,0) = 0$ and $\eta_{mu}(0,0) = \eta_{mv}(0,0) = 0$, so that $\eta_m(u,v)$ is positive definite.

Next we examine the asymptotics of $\eta_m(u,v)$ as $m \to \infty$. The change of variables $(u, Y_m) \mapsto (\xi, X_m)$:

$$\xi = \int_0^u [\sigma'(\omega)]^{\frac{1}{2}} d\omega, \tag{17.7.16}$$

$$X_m = (\sigma')^{\frac{1}{4}} Y_m\,, \tag{17.7.17}$$

transforms (17.7.12) into

$$\ddot{X}_m = m^2 X_m + [\tfrac{1}{4}(\sigma')^{-2}\sigma''' - \tfrac{5}{16}(\sigma')^{-3}(\sigma'')^2] X_m\,, \tag{17.7.18}$$

with asymptotics, derived by the variation of parameters formula,

$$X_m(\xi) = \left[\sigma'(0)^{\frac{1}{4}} + O\left(\frac{1}{m}\right)\right] \cosh(m\xi), \tag{17.7.19}$$

as $m \to \infty$, and for ξ confined in any fixed bounded interval.

Upon combining (17.7.10) with (17.7.17), (17.7.19), (17.7.16) and (17.7.3), we deduce

$$\lim_{m\to\infty} \eta_m(u,v)^{\frac{1}{m}} = \begin{cases} \exp[z(u,v)], & \text{if } u > 0\,, v > 0, \\ \exp[w(u,v)], & \text{if } u < 0\,, v > 0, \\ \exp[-w(u,v)], & \text{if } u > 0\,, v < 0, \\ \exp[-z(u,v)], & \text{if } u < 0\,, v < 0. \end{cases} \tag{17.7.20}$$

We now consider the solution (u_μ, v_μ) of (17.7.6), (17.7.7), where $(u_{0\mu}, v_{0\mu})$ lie in $L^2(-\infty,\infty)$ and take values in the region $\mathscr{U}_M$, defined by (17.7.8). We write (17.3.7), with $U_\mu = (u_\mu, v_\mu)$, $\eta = \eta_m$, $q = q_m$, and integrate it over the strip $(-\infty,\infty) \times [0,t]$, to get

$$\int_{-\infty}^{\infty} \eta_m(u_\mu(x,t), v_\mu(x,t))dx \le \int_{-\infty}^{\infty} \eta_m(u_{0\mu}(x), v_{0\mu}(x))dx. \tag{17.7.21}$$

Raising (17.7.21) to the power $1/m$, letting $m \to \infty$ and using (17.7.20), we conclude that $(u_\mu(\cdot,t), v_\mu(\cdot,t))$ takes values in the region $\mathscr{U}_M$. The proof is complete.

The above proposition, in conjunction with Theorem 17.7.1, yields an existence theorem for the system (17.7.1), which is free from extraneous assumptions:

17.7.3 Theorem. *Let (u_μ, v_μ) be the solution of the initial value problem* (17.7.6), (17.7.7), *on the upper half-plane, where $(u_{0\mu}, v_{0\mu}) \rightharpoonup (u_0, v_0)$ in $L^\infty(-\infty,\infty)$ weak*.*

Under the condition (17.7.5), *there is a sequence* $\{\mu_j\}$, $\mu_j \to 0$ *as* $j \to \infty$, *such that* $\{(u_{\mu_j}, v_{\mu_j})\}$ *converges almost everywhere on the upper half-plane to a solution* $(\bar{u}, \bar{v})$ *of* (17.7.1) *with initial values* $(\bar{u}(x,0), \bar{v}(x,0)) = (u_0(x), v_0(x))$, *for* $-\infty < x < \infty$.

The assumption (17.7.5) and the use of the special, artifical viscosity (17.7.6) are essential in the proof of Theorem 17.7.3, because they appear to be indispensable for establishing uniform L^∞ bounds on approximate solutions. At the same time, it is interesting to know whether one may construct solutions to (17.7.1) by passing to the zero viscosity limit in the system (8.6.3) of viscoelasticity, or at least in the model system

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u) = \mu \partial_x^2 v, \end{cases} \tag{17.7.22}$$

which is close to it.

Even though we do not have uniform L^∞ estimates for solutions of (17.7.22), as this system is not dissipative with respect to all convex entropies of (17.7.1), we still have a number of estimates of L^p type, the most prominent among them being the "energy inequality" induced by the physical entropy-entropy flux pair (7.4.10). It is thus natural to inquire whether the method of compensated compactness is applicable in conjunction with such estimates. Of course, this would force us to abandon L^∞ and consider Young measures in the framework of L^p, a possibility already raised in Section 17.1. It turns out that this approach is effective for the problem at hand, albeit at the expense of elaborate analysis, so only the conclusion shall be recorded here. The proof is found in the references cited in Section 17.9.

17.7.4 Theorem. *Consider the system* (17.7.22), *where* (a) $\sigma'(u) \geq \sigma_0 > 0$, *for all* $-\infty < u < \infty$; (b) σ'' *may vanish at most at one point on* $(-\infty, \infty)$; (c) $\sigma'(u)$ *grows like* $|u|^\alpha$, as $|u| \to \infty$, *for some* $\alpha \geq 0$; *and* (d) $\sigma''(u)$ *and* $\sigma'''(u)$ *grow no faster than* $|u|^{\alpha-1}$, *as* $|u| \to \infty$. *Let* (u_μ, v_μ) *be the solution of the Cauchy problem* (17.7.22), (17.7.7), *where* $\{(u_{0\mu}, v_{0\mu})\}$ *are functions in* $W^{1,2}(-\infty,\infty)$, *which have uniformly bounded total energy,*

$$\int_{-\infty}^{\infty} [\tfrac{1}{2} v_{0\mu}^2(x) + e(u_{0\mu})] dx \leq a, \tag{17.7.23}$$

have relatively tame oscillations,

$$\mu \int_{-\infty}^{\infty} [\partial v_{0\mu}(x)]^2 dx \to 0, \qquad \text{as } \mu \to 0, \tag{17.7.24}$$

and converge, $u_{0\mu} \to u_0$, $v_{0\mu} \to v_0$, *as* $\mu \to 0$, *in the sense of distributions. Then there is a sequence* $\{\mu_j\}$, $\mu_j \to 0$ *as* $j \to \infty$, *such that* $\{(u_{\mu_j}, v_{\mu_j})\}$ *converges in* L^p_{loc}, *for any* $1 < p < 2$, *to a solution* (u,v) *of* (17.7.1) *with initial values* (u_0, v_0) *on* $(-\infty, \infty)$.

17.8 The System of Isentropic Gas Dynamics

The system (7.1.13) of isentropic gas dynamics, for an ideal gas, in Eulerian coordinates, the first hyperbolic system of conservation laws ever to be derived, has served over the past two centuries as proving ground for testing the theory. It is thus fitting to conclude this chapter with the application of the method of compensated compactness to that system.

It is instructive to monitor the system simultaneously in its original form (7.1.13), with state variables density ρ and velocity v, as well as in its canonical form

$$\begin{cases} \partial_t \rho + \partial_x m = 0 \\ \partial_t m + \partial_x \left[\dfrac{m^2}{\rho} + \kappa \rho^\gamma \right] = 0, \end{cases} \tag{17.8.1}$$

with state variables density ρ and momentum $m = \rho v$. The physical range for density is $0 \le \rho < \infty$, while v and m may take any values in $(-\infty, \infty)$.

For convenience, we scale the state variables so that $\kappa = (\gamma - 1)^2/4\gamma$, and use the notation $\theta = \frac{1}{2}(\gamma - 1)$, in which case the characteristic speeds (7.2.10) and the Riemann invariants (7.3.3) assume the form

$$\lambda = -\theta \rho^\theta + v = -\theta \rho^\theta + \frac{m}{\rho}, \qquad \mu = \theta \rho^\theta + v = \theta \rho^\theta + \frac{m}{\rho}, \tag{17.8.2}$$

$$z = -\rho^\theta + v = -\rho^\theta + \frac{m}{\rho}, \qquad w = \rho^\theta + v = \rho^\theta + \frac{m}{\rho}. \tag{17.8.3}$$

It is not difficult to construct sequences of approximate solutions taking values in compact sets of the state space $[0, \infty) \times (-\infty, \infty)$. For example, one may follow the vanishing viscosity approach relative to the system

$$\begin{cases} \partial_t \rho + \partial_x m = \mu \partial_x^2 \rho \\ \partial_t m + \partial_x \left[\dfrac{m^2}{\rho} + \kappa \rho^\gamma \right] = \mu \partial_x^2 m, \end{cases} \tag{17.8.4}$$

which admits the family of (positively) invariant regions

$$\mathscr{U}_M = \{(\rho, m) : \rho \ge 0,\ -M \le z(\rho, m) \le w(\rho, m) \le M\}. \tag{17.8.5}$$

Furthermore, solutions (ρ_μ, m_μ) of (17.8.4) on the upper half-plane, with initial data that are bounded in $L^\infty(-\infty, \infty) \cap L^2(-\infty, \infty)$, satisfy

$$\partial_t \eta(\rho_\mu, m_\mu) + \partial_x q(\rho_\mu, m_\mu) \subset \text{ compact set in } W^{-1,2}_{\text{loc}}, \tag{17.8.6}$$

for any entropy-entropy flux pair (η, q) of (17.8.1). Approximate solutions with analogous properties are also constructed by finite difference schemes, such as the Lax-Friedrichs scheme and the Godunov scheme. They all lead to the following existence theorem:

17.8.1 Theorem. *For any $\gamma > 1$, there exists a bounded solution (ρ, v) of the system* (7.1.13) *on the upper half-plane, with assigned initial value*

$$(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x)), \qquad -\infty < x < \infty, \tag{17.8.7}$$

where (ρ_0, v_0) are in $L^\infty(-\infty,\infty)$ and $\rho_0(x) \geq 0$, for $-\infty < x < \infty$. Furthermore, the solution satisfies the entropy admissibility condition

$$\partial_t \eta(\rho,m) + \partial_x q(\rho,m) \leq 0, \tag{17.8.8}$$

for any entropy-entropy flux pair (η, q) of (17.8.1), *with $\eta(\rho,m)$ convex.*

The proof employs (17.3.10) to establish that the support of the Young measure, associated with a sequence of approximate solutions, either reduces to a single point in state space or is confined to the axis $\rho = 0$ (vacuum state).

As function of (ρ, v), any entropy η of (7.1.13) satisfies the integrability condition

$$\eta_{\rho\rho} = \theta^2 \rho^{\gamma-3} \eta_{vv} \,. \tag{17.8.9}$$

The above equation is singular along the axis $\rho = 0$, and the nature of the singularity changes as one crosses the threshold $\gamma = 3$. Accordingly, different arguments have to be used for treating the cases $\gamma < 3$ and $\gamma > 3$.

Of relevance here are the so-called *weak entropy-entropy flux pairs*, which vanish at $\rho = 0$. They admit the representation

$$\begin{cases} \eta(\rho,v) = \displaystyle\int_{-\infty}^{\infty} \chi(\rho, \omega - v) g(\omega) d\omega \\ q(\rho,v) = \displaystyle\int_{-\infty}^{\infty} \chi(\rho, \omega - v)(\theta\omega + (1-\theta)v) g(\omega) d\omega, \end{cases} \tag{17.8.10}$$

where

$$\chi(\rho,v) = \begin{cases} (\rho^{2\theta} - v^2)^s, & \text{if } \rho^{2\theta} > v^2 \\ 0, & \text{if } \rho^{2\theta} \leq v^2, \end{cases} \tag{17.8.11}$$

with $s = \frac{1}{2}\frac{3-\gamma}{\gamma-1}$. Thus χ is the fundamental solution of (17.8.9) under initial conditions $\eta(0,v) = 0$, $\eta_\rho(0,v) = \delta_0(v)$.

As already noted in Section 2.5, the classical kinetic theory predicts the value $\gamma = 1 + \frac{2}{n}$ for the adiabatic exponent of a gas with n degrees of freedom. When the number of degrees of freedom is odd, $n = 2\ell + 1$, the exponent s in (17.8.11) is the integer ℓ. In this special situation the analysis of weak entropies – and thereby the reduction of the Young measure – is substantially simplified. However, even in that simpler case the proof is quite technical and shall be relegated to the references cited in Section 17.9. Only the degenerate case $\gamma = 3$ will be presented here.

For $\gamma = 3$, i.e., $\theta = 1$, (17.8.2) and (17.8.3) yield $\lambda = z$ and $\mu = w$, in which case the two characteristic families totally decouple. In particular, (12.2.1) reduce to $q_z = z\eta_z$, $q_w = w\eta_w$, so that there are entropy-entropy flux pairs (η, q) which depend solely on z, for example $(2z, z^2)$ and $(3z^2, 2z^3)$.

Suppose now a sequence $\{(\rho_{\mu_k}, m_{\mu_k})\}$ of solutions of (17.8.4), with $\mu_k \to 0$ as $k \to \infty$, induces a weakly convergent subsequence $\{(z_j, w_j)\}$ of Riemann invariants with associated family $\nu_{x,t}$ of Young measures. We fix (x,t), set $\nu_{x,t} = \nu$ and apply (17.3.10) for the two entropy-entropy flux pairs $(2z, z^2)$ and $(3z^2, 2z^3)$ to get

$$4 < \nu, z >< \nu, z^3 > -3 < \nu, z^2 >< \nu, z^2 > = < \nu, z^4 > . \tag{17.8.12}$$

Next we consider the inequality

$$z^4 - 4z^3\bar{z} + 6z^2\bar{z}^2 - 4z\bar{z}^3 + \bar{z}^4 = (z - \bar{z})^4 \geq 0, \tag{17.8.13}$$

where $\bar{z} =< \nu, z >$, and apply the measure ν to it, thus obtaining

$$< \nu, z^4 > -4 < \nu, z^3 >< \nu, z > +6 < \nu, z^2 >< \nu, z >^2 -3 < \nu, z >^4 \geq 0. \tag{17.8.14}$$

Combining (17.8.14) with (17.8.12) yields

$$-3[< \nu, z^2 > - < \nu, z >^2]^2 \geq 0, \tag{17.8.15}$$

whence $< \nu, z^2 > = < \nu, z >^2$. Therefore, $\{z_j\}$ converges strongly to $\bar{z} =< \nu, z >$.

Similarly one shows that $\{w_j\}$ converges strongly to $\bar{w} =< \nu, w >$. In particular, $(\bar{z}, \bar{w})$ induces a solution $(\bar{\rho}, \bar{v})$ of (7.1.13) by $\bar{\rho} = \frac{1}{2}(\bar{w} - \bar{z})$ and $\bar{v} = \frac{1}{2}(\bar{w} + \bar{z})$.

From the standpoint of continuum physics, it is more natural to employ physical, rather than artificial, vanishing viscosity and construct approximate solutions (ρ_μ, m_μ) to (17.8.1) through the system

$$\begin{cases} \partial_t \rho + \partial_x m = 0 \\ \\ \partial_t m + \partial_x \left[\dfrac{m^2}{\rho} + \kappa \rho^\gamma \right] = \mu \partial_x^2 v, \end{cases} \tag{17.8.16}$$

in the place of (17.8.4). However, this stumbles on the need for L^∞ bounds on solutions of (17.8.16), uniformly valid for $\mu > 0$, which are at present unavailable. One may attempt to use, instead, L^p bounds, but then the task of reducing the Young measure to a point mass becomes quite laborious. What follows is a simplified version of a more general result established in that direction. It pertains to initial data (ρ_0, m_0) that tend to a state $(\bar{\rho}, 0)$, as $|x| \to \infty$, with $\bar{\rho} > 0$.

17.8.2 Theorem. *Consider the family $\{(\rho_{0\mu}, m_{0\mu}),\ \mu > 0\}$ of smooth initial data with the following properties: The density is positive,*

$$\rho_{0\mu}(x) \geq a_\mu > 0, \qquad -\infty < x < \infty, \tag{17.8.17}$$

and its oscillation is controlled by

$$\mu^2 \int_{-\infty}^{\infty} \frac{[\partial_x \rho_{0\mu}(x)]^2}{\rho_{0\mu}^3(x)} dx \leq N. \tag{17.8.18}$$

The momentum is uniformly bounded,

$$\int_{-\infty}^{\infty} |m_{0\mu}(x)| dx \leq P, \tag{17.8.19}$$

and the mechanical energy relative to the state $(\bar{\rho}, 0)$ *is also uniformly bounded:*

$$\int_{-\infty}^{\infty} \left(\frac{\kappa}{\gamma - 1} [\rho_{0\mu}^{\gamma}(x) - \bar{\rho}^{\gamma} - \gamma \bar{\rho}^{\gamma-1}(\rho_{0\mu}(x) - \bar{\rho})] + \frac{1}{2} \frac{m_{0\mu}^2(x)}{\rho_{0\mu}(x)} \right) dx \leq E. \tag{17.8.20}$$

Finally,

$$(\rho_{0\mu}, m_{0\mu}) \to (\rho_0, m_0), \qquad as\ \ \mu \to 0, \tag{17.8.21}$$

in the sense of distributions. Then for any $\mu > 0$ *there exists a classical solution* (ρ_μ, m_μ) *of the Cauchy problem for* (17.8.16) *with initial values* $(\rho_{0\mu}, m_{0\mu})$. *Furthermore, there are sequences* $\{\mu_j\}$, *with* $\mu_j \to 0$ *as* $j \to \infty$, *such that* $\{(\rho_{\mu_j}, m_{\mu_j})\}$ *converges a.e. to a solution* (ρ, m) *of the Cauchy problem for* (17.8.1), *with initial data* (ρ_0, m_0). *This solution satisfies the entropy admissibility condition.*

The lengthy, technical proof of the above proposition, which is found in the references cited in Section 17.9, proceeds by first establishing L^p bounds on solutions (ρ_μ, m_μ) of (17.8.16) that hold uniformly in $\mu > 0$, and then showing that the induced Young measure reduces to a point mass, so that the $\mu \to 0$ weak limit (ρ, m) of (ρ_μ, m_μ) is a solution of (17.8.1). In order to convey a taste of the methodology of the proof, we sketch below the derivation of the central a priori estimates. However, so as to simplify the presentation, we will establish the estimates under the extraneous assumptions that $\bar{\rho} = 0$ and the total mass of the initial data is finite and uniformly bounded,

$$\int_{-\infty}^{\infty} \rho_{0\mu} dx \leq M. \tag{17.8.22}$$

Even though these requirements violate the conditions of the theorem, they do not affect the essence of the argument, as far as the derivation of the estimates goes.

From the balance laws of mass and mechanical energy, in conjunction with (17.8.22) and (17.8.20), follows

$$\int_{-\infty}^{\infty} \rho_\mu(x,t) dx \leq M, \tag{17.8.23}$$

(17.8.24)
$$\int_{-\infty}^{\infty} \left(\frac{\kappa}{\gamma - 1} \rho_\mu^{\gamma}(x,t) + \frac{1}{2} \frac{m_\mu^2(x,t)}{\rho_\mu(x,t)} \right) dx + \mu \int_0^t \int_{-\infty}^{\infty} [\partial_x v_\mu(x,\tau)]^2 dx d\tau \leq E.$$

Next we introduce the function

$$\Phi_\mu(x,t) = \int_{-\infty}^{x} m_\mu(y,t)dy, \tag{17.8.25}$$

which serves as a potential, noting that, by virtue of (17.8.23), (17.8.24) and Schwarz's inequality, we have

$$|\Phi_\mu(x,t)|^2 \le \int_{-\infty}^{\infty} \frac{m_\mu^2(x,t)}{\rho_\mu(x,t)}dx \int_{-\infty}^{\infty} \rho_\mu(x,t)dx \le 2EM. \tag{17.8.26}$$

On account of (17.8.16) and (17.8.25),

$$\partial_t(\rho_\mu \Phi_\mu) + \partial_x(m_\mu \Phi_\mu) = -\kappa\rho_\mu^{\gamma+1} + \mu\rho_\mu\partial_x v_\mu. \tag{17.8.27}$$

Integrating (17.8.27) over $(-\infty,\infty)\times(0,t)$, using (17.8.23), (17.8.24), (17.8.26), and assuming $\mu \le \kappa$, we conclude

$$\int_0^t \int_{-\infty}^{\infty} \rho_\mu^{\gamma+1}(x,\tau)dxd\tau \le \frac{2}{\kappa}E + \frac{8}{\kappa}M^3 + \frac{\mu t}{\kappa}M, \tag{17.8.28}$$

which provides the first basic new estimate.

To get the next estimate, we switch to Lagrangian coordinates (ξ,t). To avoid confusion, partial derivatives with respect to ξ and t in Lagrangian coordinates will be denoted by subscripts ξ and t, while the symbols ∂_x and ∂_t are retained for partial derivatives with respect to x and t in Eulerian coordinates. In particular, $x_\xi = \rho^{-1}$, $x_t = v$, $\partial_x\xi = \rho$.

In Lagrangian coordinates, the system (17.8.16) reads

$$\begin{cases} (\rho^{-1})_t - v_\xi = 0 \\ v_t + (\kappa\rho^\gamma)_\xi = \mu(\rho v_\xi)_\xi\,. \end{cases} \tag{17.8.29}$$

After a straightforward calculation, using $(17.8.29)_1$,

$$[(\partial_x\rho_\mu^{-1})^2]_t = [\rho_\mu^{-2}(\rho_{\mu\xi})^2]_t = -2\rho_\mu^{-1}\rho_{\mu\xi}(\rho_\mu v_{\mu\xi})_\xi\,. \tag{17.8.30}$$

Multiplying (17.8.30) by μ and then substituting the term $\mu(\rho_\mu v_{\mu\xi})_\xi$ from $(17.8.29)_2$ into the resulting equation, we arrive at

$$\begin{aligned} \mu[(\partial_x\rho_\mu^{-1})^2]_t &= -2\rho_\mu^{-1}\rho_{\mu\xi}v_{\mu t} - 2\kappa\gamma\rho_\mu^{\gamma-2}(\rho_{\mu\xi})^2 \\ &= -2(\rho_\mu^{-1}\rho_{\mu\xi}v_\mu)_t - 2(\rho_\mu v_\mu v_{\mu\xi})_\xi + 2\rho_\mu(v_{\mu\xi})^2 - 2\kappa\gamma\rho_\mu^{\gamma-2}(\rho_{\mu\xi})^2. \end{aligned} \tag{17.8.31}$$

We now integrate (17.8.31), first with respect to ξ over $(-\infty,\infty)$ and then with respect to time over $(0,t)$. Finally, we switch back to Eulerian coordinates (x,t). The end result is the following identity:

$$\begin{aligned}(17.8.32)\quad &\mu\int_{-\infty}^{\infty}\rho_\mu^{-3}(\partial_x\rho_\mu)^2dx+2\kappa\gamma\int_0^t\int_{-\infty}^{\infty}\rho_\mu^{\gamma-3}(\partial_x\rho_\mu)^2dxd\tau\\ &=-2\int_{-\infty}^{\infty}\rho_\mu^{-1}(\partial_x\rho_\mu)v_\mu dx+2\int_0^t\int_{-\infty}^{\infty}(\partial_x v_\mu)^2dxd\tau\\ &+\mu\int_{-\infty}^{\infty}\rho_{0\mu}^{-3}(\partial_x\rho_{0\mu})^2dx+2\int_{-\infty}^{\infty}\rho_{0\mu}^{-1}(\partial_x\rho_{0\mu})v_{0\mu}dx.\end{aligned}$$

We multiply (17.8.32) by μ and estimate the first and the last term on the right-hand side as follows:

$$(17.8.33)\quad -2\mu\int_{-\infty}^{\infty}\rho_\mu^{-1}(\partial_x\rho_\mu)v_\mu dx\le\frac{\mu^2}{2}\int_{-\infty}^{\infty}\rho_\mu^{-3}(\partial_x\rho_\mu)^2dx+2\int_{-\infty}^{\infty}\rho_\mu^{-1}m_\mu^2dx,$$

$$(17.8.34)\quad 2\mu\int_{-\infty}^{\infty}\rho_{0\mu}^{-1}(\partial_x\rho_{0\mu})v_{0\mu}dx\le\frac{\mu^2}{2}\int_{-\infty}^{\infty}\rho_{0\mu}^{-3}(\partial_x\rho_{0\mu})^2dx+2\int_{-\infty}^{\infty}\rho_{0\mu}^{-1}m_{0m}^2dx.$$

The third term on the right-hand side of (18.7.32) is estimated by (17.8.18), while the second terms on the right-hand sides of (17.8.32), (17.8.33) and (17.8.34) are all estimated with the help of (17.8.24). We thus arrive at the desired estimate:

(17.8.35)
$$\mu^2\int_{-\infty}^{\infty}\rho_\mu^{-3}(x,t)(\partial_x\rho_\mu(x,t))^2dx+4\mu\kappa\gamma\int_0^t\int_{-\infty}^{\infty}\rho_\mu^{\gamma-3}(\partial_x\rho_\mu)^2dxd\tau\le 20E+3N,$$

which controls the oscillation of the density.

Similar, albeit merely local, estimates apply even when $\bar\rho>0$. In order to build a satisfactory compactness framework, even higher local L^p bounds are needed on ρ and on v, which can be obtained with the assistance of the weak entropy-entropy flux pair (17.8.10), for $g(\omega)=\frac{1}{2}|\omega|\omega$. However, the derivation of these estimates hinges on keeping the density at a distance from zero on a substantial portion of the upper half-plane, and thus requires $\bar\rho>0$.

Armed with the above estimates, it is possible to show that the Young measure reduces to a point mass. However, the task is technically challenging since the measure takes values on the phase space $\{(\rho,m):\,0<\rho<\infty,\,-\infty<m<\infty\}$, which is not compact. The difficulty is resolved by an insightful compactification of the phase space. The reader may consult the literature cited in Section 17.9.

The use of L^p, rather than L^∞, bounds becomes indispensable for a host of one-dimensional systems in gas dynamics that contain inhomogeneities and/or source terms with singularities, manifesting geometric effects. An illustrative example is provided by the system (7.1.29) governing three-dimensional radial isentropic flow of an ideal gas, which we write here in the form

$$(17.8.36)\qquad \begin{cases}\partial_t(r^2\rho)+\partial_r(r^2m)=0\\[2ex] \partial_t(r^2m)+\partial_r\left[r^2\left(\dfrac{m^2}{\rho}+\kappa\rho^\gamma\right)\right]=2\kappa r\rho^\gamma.\end{cases}$$

So long as the flow is obstructed from approaching the origin, for example by a solid spherical obstacle, the system may be treated via L^∞ estimates. However, for flows defined for all $r \geq 0$, density and velocity may become infinite at the origin, so a treatment through L^p estimates provides the only hope for success. Accordingly, the following existence theorem has been established for the Cauchy problem on $r \geq 0$, $t \geq 0$, in the class of functions that conserve mass,

$$\int_0^\infty r^2 \rho(r,t) dr = M\,, \qquad 0 \leq t < \infty, \tag{17.8.37}$$

and have finite energy,

$$\int_0^\infty r^2 \left[\frac{\kappa}{\gamma - 1} \rho^\gamma(r,t) + \frac{m^2(r,t)}{2\rho(r,t)} \right] dr \leq E\,, \qquad 0 \leq t < \infty. \tag{17.8.38}$$

17.8.3 Theorem. *The Cauchy problem for the system* (17.8.36), *with* $3/2 < \gamma \leq 3$, *under initial data that have finite mass* M *and finite energy* E, *possesses a weak solution* $(\rho(r,t), m(r,t))$ *on* $[0,\infty) \times [0,\infty)$, *which satisfies* (17.8.37) *and* (17.8.38).

The proof of the above proposition, which is found in the references cited in Section 17.9, bears close resemblance to the proof of Theorem 17.8.2. In particular, a central role is played by the estimate

$$\int_0^t \int_0^\infty r^4 \rho^{\gamma+1}(r,\tau) dr d\tau \leq \frac{1}{\kappa} \sqrt{2M^3 E}, \tag{17.8.39}$$

namely, the analog of (17.8.28). For a formal derivation of (17.8.39), notice that $(17.8.36)_1$ implies the existence of a Lipschitz continuous potential Ψ:

$$\partial_r \Psi = r^2 \rho\,, \qquad \partial_t \Psi = -r^2 m. \tag{17.8.40}$$

By combining $(17.8.36)_2$ with (17.8.40), one easily verifies the identity

$$\partial_t (r^2 m \Psi) + \partial_r \left[r^2 \left(\frac{m^2}{\rho} + \kappa \rho^\gamma \right) \Psi \right] = \kappa r^4 \rho^{\gamma+1} + 2\kappa r \rho^\gamma \Psi. \tag{17.8.41}$$

We normalize Ψ by $\Psi(0,t) = 0$, in which case (17.8.40) and (17.8.37) imply that $0 \leq \Psi(r,t) \leq M$. Then Schwarz's inequality, together with (17.8.37) and (17.8.38), yield

$$\left[\int_0^\infty r^2 m \Psi dr \right]^2 \leq M^2 \int_0^\infty r^2 \frac{m^2}{\rho} dr \int_0^\infty r^2 \rho dr \leq 2M^3 E. \tag{17.8.42}$$

Thus, upon integrating (17.8.41) over $[0,\infty) \times [0,t)$, one arrives at (17.8.39).

17.9 Notes

The method of compensated compactness was introduced by Murat [1] and Tartar [1,2]. The program of employing the method for constructing solutions to hyperbolic conservation laws was designed by Tartar [2,3], who laid down the fundamental condition (17.3.10) and demonstrated its use in the context of the scalar case. The first application to systems, due to DiPerna [6], provided the impetus for intensive development of these ideas, which has produced a substantial body of research. The presentation here only scratches the surface. A clear introduction is also found in the lecture notes of Evans [1], the text by Hörmander [2], the monograph by Malek, Neças, and Růžička [1], as well as the treatise by M.E. Taylor [2]. For more detailed and deeper development of the subject the reader is referred to the book by Serre [11] and the monograph by Yun-guang Lu [1]. An informative presentation of research trends in the area is provided by the survey article by Gui-Qiang Chen [8].

The Young measure was introduced in L.C. Young [1]. The presentation here follows Ball [3], where the reader may find generalizations beyond the L^∞ framework, as well as commentary and references to alternative constructions.

For an introduction to the theory of compensated compactness, see the lecture notes of Tartar [1,2,3]. The div-curl lemma is due to Murat and Tartar. The proof presented here is taken from Evans [1]. Lemma 16.2.2 is generally known as *Murat's lemma* (Murat [2]).

The notion of a measure-valued solution was introduced by DiPerna [11]. For further developments of the theory and applications to the construction of solutions to systems of conservation laws, including those of mixed type modeling phase transitions, see Chen and Frid [2], Coquel and LeFloch [1], Demengel and Serre [1], Frid [3], Poupaud and Rascle [1], Roytburd and Slemrod [1], Schochet [2], and Szepessy [1].

The scalar conservation law was first treated via the method of compensated compactness by Tartar [2]. The clever argument employed in the proof of Theorem 17.4.1 was discovered, independently, by Tartar (private communication to the author in May 1986) and by Chen and Lu [1]. See also Vecchi [1]. This approach has been extended, by Tadmor, Rascle and Bagnerini [1], to scalar conservation laws in two space dimensions, for which purpose one needs to employ two entropy-entropy flux pairs. The scalar conservation law is treated in the L^p framework by Yang, Zhu and Zhao [2]. For the case where the flux is merely continuous, see Panov [7].

For scalar balance laws with singular source see Schonbek [2], Amadori and Coclite [1], and Kwon [2].

The Cauchy problem for scalar conservation laws in several spatial dimensions can also be solved in L^∞ by way of measure-valued solutions (DiPerna [11], Szepessy [2], Panov [2]). An alternative approach, combining a kinetic formulation with ideas from the theory of compensated compactness, is carried out in Hwang and Tzavaras [1]. The initial-boundary value problem has been considered by Szepessy [1], and Ben Moussa and Szepessy [1].

The competition between viscosity and dispersion, in scalar conservation laws, is investigated by Schonbek [1] in one space dimension, and by Kondo and LeFloch [2], LeFloch and Natalini [1], and Hwang and Tzavaras [1] in several space dimensions. The corresponding question for the singular limit of solutions to the Camassa-Holm shallow water equation is discussed by Coclite and Karlsen [1].

Other cases of singular perturbations of the scalar conservation law by differential operators of order four, which have been treated by the method of compensated compactness, include the article by Tadmor [3] on vanishing hyperviscosity, and the paper by Otto and Westdickenberg [1] on the thin film approximation.

The active investigation of relaxation for hyperbolic conservation laws in recent years has produced voluminous literature, so it would be impossible to include here an exhaustive list of references. The survey paper by Natalini [3] contains an extensive bibliography. A number of relevant references have already been recorded in Sections 5.6 and 6.11. A seminal role in the development of the theory was played by the work of Tai-Ping Liu [21], motivated by Whitham [2]. The method of compensated compactness was first employed in this context by Gui-Qiang Chen and Tai-Ping Liu [1] and by Chen, Levermore and Liu [1], for systems of two conservation laws whose relaxed form is the scalar conservation law. The particular efficacy (for theoretical and computational purposes) of the semilinear system (17.5.1) was first recognized by Jin and Xin [1]. The treatment of that system in Section 17.5 is an adaptation of the analysis in Chen, Levermore and Liu [1], Lattanzio and Marcati [1,2], and Coquel and Perthame [1]. See also Yun-guang Lu [4]. For applications to the model system (7.1.26) for combustion, see Hanouzet, Natalini and Tesei [1]; for applications to the chromatography equations, see Collet and Rascle [1], and Klingenberg and Lu [1]. Furthermore, Lu and Klingenberg [1], Tzavaras [4,5], Gosse and Tzavaras [1], Serre [15], and Lattanzio and Serre [2] apply the method of compensated compactness to systems of three or four conservation laws whose relaxed form is a system of two conservation laws.

Interesting contributions to relaxation theory also include Coquel and Perthame [1], Marcati and Natalini [2], Marcati and Rubino [3], Rubino [3], Luo and Xin [1], and Luo and Yang [3].

The L^1-Lipschitz estimate (17.5.15) for the semilinear system (17.5.1), which leads to a treatment of the relaxation problem in the framework of the space BV, is due to Natalini [1]. Existence of BV solutions on the upper half-plane for the nonlinear system (5.2.18) has been established by Dafermos [25], but BV estimates independent of μ that would allow passing to the relaxation limit, as $\mu \downarrow 0$, are currently known only for the special case $p(u) = -u^{-1}$ (Luo, Natalini and Yang [1], Amadori and Guerra [2]). For other special systems that have been treated in BV, see Tveito and Winther [2], Gosse [1], and Luo and Natalini [1].

The treatment of the genuinely nonlinear system of two conservation laws, in Section 17.6, and the system of isentropic elasticity with a single inflection point, in Section 17.7, follows the pioneering paper of DiPerna [8]. See also Gripenberg [1] and Chen, Li and Li [1]. Counterexamples to Theorem 17.7.1, when $\sigma''(u)$ is discontinuous at $u = 0$, are exhibited in Greenberg [3], and Greenberg and Rascle [1].

The system of isentropic elasticity was treated in the L^p framework by J.W. Shearer [1], Peixiong Lin [1] and Serre and Shearer [1]. An alternative, original construction of solutions in L^∞ (Demoulini, Stuart and Tzavaras [1], Miroshnikov and Tzavaras [1,2]) is based on the observation that the system resulting from discretizing the time variable can be solved through a variational principle. The initial-boundary value problem in L^∞ is solved by Heidrich [1].

The theory of invariant regions via the maximum principle is due to Chueh, Conley and Smoller [1] (see also Hoff [2]). A systematic discussion, with several examples, is found in Serre [11]. The connection between stability of relaxation schemes and existence of invariant regions is discussed in Serre [15]. The proof of Theorem 17.7.2 is taken from Dafermos [16]. See also Serre [3], and Venttsel' [1].

The system of isentropic gas dynamics was first treated by the method of compensated compactness in DiPerna [9], for the special values $\gamma = 1 + \frac{2}{n}$, $n = 2\ell + 1$, of the adiabatic exponent. Subsequently, Gui-Qiang Chen [1] and Ding, Chen and Luo [1] extended the analysis to any γ within the range $(1, \frac{5}{3}]$. For a survey, see Gui-Qiang Chen [2]. The case $\gamma \geq 3$ was solved by Lions, Perthame and Tadmor [1], and the full range $1 < \gamma < \infty$, as stated in Theorem 17.8.1, is covered in Lions, Perthame and Souganidis [1]. The isothermal case, $\gamma = 1$, is singular and was treated by Huang and Wang [2], and LeFloch and Shelukhin [1]. The argument presented here, for the special case $\gamma = 3$, was communicated to the author by Gui-Qiang Chen. Extra regularity for this special value of γ is shown by Vasseur [1]. The more general, genuinely nonlinear system (7.1.12), for a nonpolytropic gas, was treated by Chen and LeFloch [2,3] under the assumption that the pressure function $p(\rho)$ and the function $\kappa\rho^\gamma$, together with their derivatives, up to fourth order, coincide asymptotically, as $\rho \to 0$.

The use of physical, in contrast to artificial, viscosity, culminating in Theorem 17.8.2, is more recent and is due to Chen and Perepelitsa [1]. See also Chen and Perepelitsa [2] on the shallow water equations. These works make essential use of techniques introduced by LeFloch and Westdickenberg [1] for treating the radially symmetric case, leading to Theorem 17.8.3. On the same problem, Chen and Perepelitsa [3] address the issue of focusing of waves and show that no mass concentration forms at the origin, despite the singular behavior of the system at $r = 0$. (Compare with Section 16.7).

The approach of Serre [2,11] has rendered the method of compensated compactness sufficiently flexible to treat systems of two conservation laws even when characteristic families are linearly degenerate, strict hyperbolicity fails, etc. Solutions to many interesting systems are constructed by Chen and Wang [1,2], Dehua Wang [1,2], Chen and Kan [1], Pui-Tak Kan [1], Kan, Santos and Xin [1], Heibig [2], Yun-guang Lu [1], Marcati and Natalini [2,3], Marcati and Rubino [1,2], Rubino [1,2], Huijiang Zhao [1], Frid and Santos [1,2], and Yun-guang Lu [3]. Since the analysis relies heavily on the availability of a rich family of entropies, the application of the method to systems of more than two conservation laws is presently limited to special systems in which the shock and rarefaction wave curves coincide for all but at most two characteristic families (Benzoni-Gavage and Serre [1]) and to the system of nonisentropic gas dynamics for very special equations of state (Chen and Dafermos [1], Chen, Li and Li [1], Bereux, Bonnetier and LeFloch [1], and Frid, Holden and Karlsen [1]).

The compensated compactness method has also been applied successfully to the system of Euler equations for steady irrotational sonic-subsonic or transonic gas flow.

In that case the system is degenerate or it changes type; see Morawetz [2,4], Chen, Dafermos, Slemrod and Wang [1], Chen, Slemrod and Wang [1], Huang and Wang [1], and Chen, Huang and Wang [1]. As we shall see in Section 18.7, the method of compensated compactness has also found interesting applications in differential geometry.

For a variety of systems, the large-time behavior of solutions with initial value s that are either periodic or L^1 perturbations of Riemann data is established in Chen and Frid [1,3,4,6], by combining scale invariance with compactness. See also Frid [6], and Frid and Rendón [1]. The method of compensated compactness has been employed to demonstrate that the large-time behavior of solutions to the Euler equations with frictional damping is governed by the porous media equation; see Serre and Xiao [1], Huang and Pan [1,2], Pan and Zhao [1], Luo and Yang [1], and Lattanzio and Rubino [1]. For the large-time behavior of solutions to systems with relaxation, see Serre [19].

The kinetic formulation, which was applied effectively in Chapter VI to scalar conservation laws in several spatial dimensions, has been successfully extended to certain systems of conservation laws in one space dimension, including the Euler equations of isentropic gas flow (Berthelin and Bouchut [1,2,3]), as well as to the system of isentropic elastodynamics (Perthame and Tzavaras [1], Tzavaras [6]). A detailed discussion and a comprehensive list of references are found in the monograph by Perthame [2]. Refined properties of solutions are derived by combining the kinetic formulation with techniques from the theory of compensated compactness. In particular, for strictly hyperbolic systems of two conservation laws, Tzavaras [6] obtains an explicit formula for the coupling of oscillations between the two characteristic fields.

Valuable insight on the effects of nonlinearity in hyperbolic conservation laws is gained from the investigation of how the solution operator interacts with highly oscillatory initial data, say $U_{0\varepsilon}(x) = V(x, x/\varepsilon)$, where $V(x,\cdot)$ is periodic and ε is a small positive parameter. When the system is linear, the rapid oscillations are transported along characteristics and their amplitude is not attenuated. On the opposite extreme, when the system is strictly hyperbolic and genuinely nonlinear, the analysis in Sections 17.4 and 17.6 indicates that, as $\varepsilon \to 0$, the resulting family of solutions $U_\varepsilon(x,t)$ contains sequences which converge strongly to solutions with initial value the weak limit of $\{U_{0\varepsilon}\}$, that is for $t > 0$ the solution operator quenches high frequency oscillations of the initial data. It is interesting to investigate intermediate situations, where some characteristic families may be linearly degenerate, strict hyperbolicity fails, etc. Following the study of many particular examples (cf. Bonnefille [1], Gui-Qiang Chen [3,4,5], E [1], Heibig [1], Rascle [1], Cheverry, Gues and Métivier [1,2], Bourdarias, Gisclon and Junca [3], and Serre [5,8]), a coherent theory of *propagation of oscillations* seems to be emerging (Serre [11]).

There is a well-developed theory of propagation of oscillations based on the method of *weakly nonlinear geometric optics* which derives asymptotic expansions for solutions of hyperbolic systems under initial data oscillating with high frequency and small amplitude. Following the pioneering work of Landau [1], Lighthill [1], and Whitham [1], extensive literature has emerged, of purely formal, semirigorous or rigorous nature, dealing with the cases of a single phase, or possibly resonating multiphases, etc. See, for example, Choquet-Bruhat [1], Hunter and Keller [1,2], Majda and Rosales [1], Majda, Rosales and Schonbek [1], Pego [4], Hunter [1], Joly, Métivier and Rauch [1,3], and Cheverry [1]. It is remarkable that the asymptotic expansions remain valid even after shocks develop in the solution; see DiPerna and Majda [1], Schochet [5] and Cheverry [2]. A survey is found in Majda [5] and a systematic presentation is given in Serre [11]. For the scalar conservation law in several space dimensions the validity of nonlinear geometric optics has been established by Chen, Junca and Rascle [1]. For a recent treatment of geometric optics for systems of conservation laws, based on the Standard Riemann Semigroup, discussed in Chapter XIV, see Chen, Xiang and Zhang [1].

XVIII

Steady and Self-similar Solutions in Multi-Space Dimensions

As noted earlier in this book, the general theory of nonlinear hyperbolic systems of conservation laws in several space dimensions is terra incognita. Nevertheless, a number of important problems in two space dimensions are currently tractable, as they admit stationary or self-similar solutions, in which case the number of independent variables is reduced to two.

The chapter begins with an introduction to the Riemann problem for scalar conservation laws in two space dimensions. The resulting equation in the (two) self-similar variables retains hyperbolicity. The emerging wave pattern is quite intricate and, depending on the data, may assume any one of 32 distinct configurations, of which two representative cases will be recorded here.

The next task is to consider stationary or self-similar solutions of the Euler equations for planar isentropic gas flow. The number of independent variables is again reduced to two; however, the price to pay is that the resulting system is no longer hyperbolic but of mixed elliptic-hyperbolic type. In a transonic flow, the border between the elliptic and the hyperbolic region is a free boundary that has to be determined as part of the solution. The recently solved, classical problem of regular shock reflection by a wedge, for irrotational flow, will be discussed in some detail, as an illustrative example of this type.

Section 18.7 presents an unexpected connection between the classical problem in differential geometry of embedding isometrically a two-dimensional Riemannian manifold into three-dimensional Euclidean space and steady irrotational flow of a Chaplygin gas.

The chapter closes with a discussion of self-similar solutions in nonlinear elastodynamics, modeling the phenomenon of cavitation.

18.1 Self-Similar Solutions for Multidimensional Scalar Conservation Laws

We consider a scalar conservation law in two space dimensions,

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Grundlehren der mathematischen Wissenschaften 325,
DOI 10.1007/978-3-662-49451-6_18

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0, \tag{18.1.1}$$

and seek self-similar solutions $u(x,y,t) = v(x/t, y/t)$. Letting $\xi = x/t$ and $\zeta = y/t$, $v(\xi,\zeta)$ satisfies the equation

$$-\xi v_\xi + f(v)_\xi - \zeta v_\zeta + g(v)_\zeta = 0. \tag{18.1.2}$$

The characteristics of (18.1.2), along which v is constant, are determined by the ordinary differential equation

$$[f'(v(\xi,\zeta)) - \xi]d\zeta - [g'(v(\xi,\zeta)) - \zeta]d\xi = 0. \tag{18.1.3}$$

Notice the set of singular points

$$\mathscr{B} = \{(\xi,\zeta) \colon \, \xi = f'(v), \, \zeta = g'(v)\}, \tag{18.1.4}$$

parametrized by v. For simplicity, we make the assumption

$$f''(u) > 0, \qquad g''(u) > 0, \qquad [f''(u)/g''(u)]' > 0, \tag{18.1.5}$$

in which case $\mathscr{B}$ is a strictly increasing, concave curve $\zeta = \zeta(\xi)$.

The Rankine-Hugoniot jump condition across a shock curve reads

$$[\lambda(u_-,u_+) - \xi]d\zeta - [\mu(u_-,u_+) - \zeta]d\xi = 0, \tag{18.1.6}$$

where

$$\lambda(u_-,u_+) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \qquad \mu(u_-,u_+) = \frac{g(u_+) - g(u_-)}{u_+ - u_-}. \tag{18.1.7}$$

Notice that any shock curve joining two fixed states u_- and u_+ lies on some straight line emanating from the nodal point $\xi = \lambda(u_-,u_+)$, $\zeta = \mu(u_-,u_+)$. Under the convention that the normal vector $(d\zeta, -d\xi)$ is pointing towards the $(+)$ side of shock curves, admissible shocks should satisfy Oleinik's E-condition, namely

$$[\lambda(u_-,u_0) - \lambda(u_-,u_+)]d\zeta - [\mu(u_-,u_0) - \mu(u_-,u_+)]d\xi \geq 0, \tag{18.1.8}$$

for any u_0 between u_- and u_+.

The objective is to construct BV solutions of (18.1.2) on $\mathbb{R}^2$ that satisfy assigned boundary conditions at infinity: $v(r\cos\theta, r\sin\theta) \to h(\theta)$, as $r \to \infty$. In particular, a natural extension of the classical Riemann problem to two space dimensions is to determine a self-similar solution of (18.1.1) with initial values

$$u(x,y,0) = \begin{cases} u_{NE} & 0 < x < \infty, \quad 0 < y < \infty \\ u_{SE} & 0 < x < \infty, \quad -\infty < y < 0 \\ u_{NW} & -\infty < x < 0, \quad 0 < y < \infty \\ u_{SW} & -\infty < x < 0, \quad -\infty < y < 0, \end{cases} \tag{18.1.9}$$

where u_{NE}, u_{SE}, u_{NW} and u_{SW} are given constants.

If $u = v(x/t, y/t)$ is the solution of (18.1.1), (18.1.9), then for large ξ (or $-\xi$), v depends solely on ζ and depicts an admissible shock or rarefaction wave that joins the states u_{NE} and u_{SE} (or u_{SW} and u_{NW}) and propagates in the y-direction. Similarly, for large ζ (or $-\zeta$), v depends solely on ξ and depicts an admissible shock or rarefaction wave that joins the states u_{NW} and u_{NE} (or u_{SW} and u_{SE}) and propagates in the x-direction. An interesting wave pattern emerges in the region of the ξ-ζ plane where the above four waves interact. In fact, depending on the relative positions of u_{NE} , u_{SE} , u_{NW} and u_{SW} on the real axis, there are 32 distinct wave configurations, which are described and classified in the literature cited in Section 18.7. For illustration, the two simplest cases will be recorded below.

Assume first $u_{SW} < u_{NW} < u_{SE} < u_{NE}$. In that case the solution is Lipschitz continuous on $\mathbb{R}^2$, with level curves depicted in Figure 18.1.1. Indeed, the pairs of states (u_{NW}, u_{NE}), (u_{SW}, u_{SE}), (u_{NE}, u_{SE}) and (u_{NW}, u_{SW}) are all connected by rarefaction waves. The line $\mathscr{B}$ of singular points, defined by (18.1.4), marks the border between these rarefaction waves, and serves as a "roof valley" allowing for Lipschitz continuous transition of the solution across it.

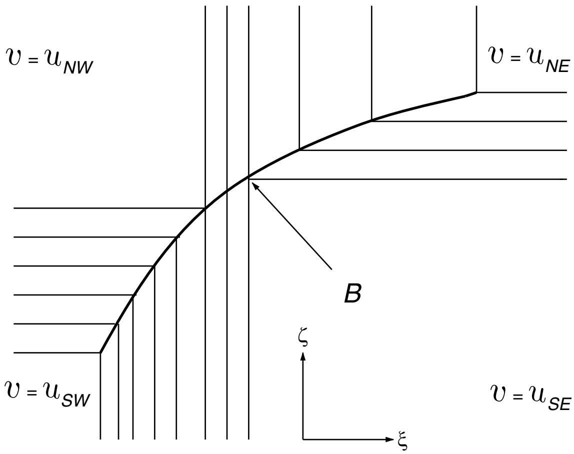

Fig. 18.1.1

Assume next $u_{SW} > u_{SE} > u_{NW} > u_{NE}$. In that case the solution comprises constant states joined by admissible shocks, as depicted in Figure 18.1.2. Indeed, the pairs of states (u_{NW}, u_{NE}) and (u_{NW}, u_{SW}) are connected by two shocks which collide at the point $A = (\lambda(u_{NE}, u_{NW}), \mu(u_{NW}, u_{SW}))$; and the pairs of states (u_{SW}, u_{SE}) and (u_{SE}, u_{NE}) are similarly connected by two shocks which collide at the point $B = (\lambda(u_{SW}, u_{SE}), \mu(u_{NE}, u_{SE}))$. The wave pattern is completed by two shocks joining u_{NE} with u_{SW}. Both emanate from the node $O = (\lambda(u_{NE}, u_{SW}), \mu(u_{NE}, u_{SW}))$; one terminates at the point A and the other at the point B.

The remaining cases involve combinations of shocks and rarefaction waves, which may interact to generate more complex wave patterns. In the absence of condi-

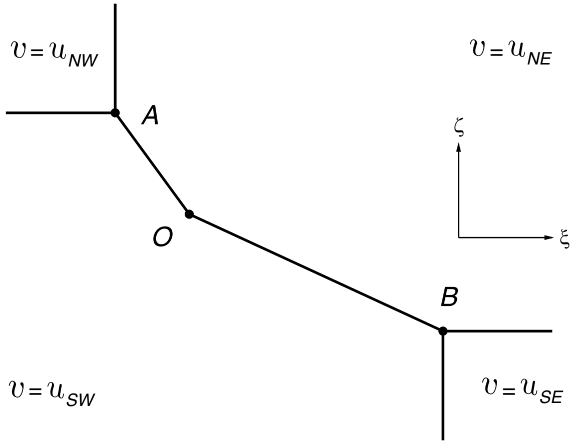

Fig. 18.1.2

tions (18.1.5), the wave configuration is even more intricate. See the references cited in Section 18.8.

It is not to be expected that multi-dimensional Riemann problems will play as pivotal a role as their one-dimensional counterparts. Nevertheless, they are valuable, as they provide a graphic illustration of the geometric complexity of solutions of systems of conservation laws in several space dimensions.

18.2 Steady Planar Isentropic Gas Flow

Turning to systems of conservation laws in two space dimensions, we consider the steady planar isentropic flow of a gas with equation of state $p(\rho)$, where $p'(\rho) > 0$ and $p''(\rho) > 0$. In Eulerian coordinates, the density and the velocity do not depend on time. Thus, letting (x,y) denote the spatial variables and (u,v) stand for the velocity components, the two-dimensional version of the Euler equations (3.3.36), with zero body force, reduces to

$$\left\{\begin{array}{l} \partial_x(\rho u) + \partial_y(\rho v) = 0 \\ \\ \partial_x(\rho u^2 + p(\rho)) + \partial_y(\rho u v) = 0 \\ \\ \partial_x(\rho u v) + \partial_y(\rho v^2 + p(\rho)) = 0. \end{array}\right. \tag{18.2.1}$$

The sound speed is

$$c = \sqrt{p'(\rho)}\,. \tag{18.2.2}$$

Testing hyperbolicity, albeit in the $x-$ rather than in the $t-$ direction, one finds eigenvalues

$$\lambda_{1,3} = \frac{uv \pm c\sqrt{u^2+v^2-c^2}}{u^2-c^2}, \qquad \lambda_2 = \frac{v}{u}, \tag{18.2.3}$$

so that the system (18.2.1) is hyperbolic in the supersonic regime, $u^2+v^2 > c^2$, and of composite type in the subsonic regime, $u^2+v^2 < c^2$.

The Rankine-Hugoniot jump conditions for a steady shock with slope $s = dy/dx$ are obtained from (3.3.40) for $\sigma = 0$ and ν collinear to $(-s,1)$:

$$\begin{cases} s[\![\rho u]\!] = [\![\rho v]\!] \\ s[\![\rho u^2+p]\!] = [\![\rho uv]\!] \\ s[\![\rho uv]\!] = [\![\rho v^2+p]\!]. \end{cases} \tag{18.2.4}$$

The family of jump discontinuities associated with the eigenvalue λ_2 comprises the vortex sheets. The velocity is tangential on both sides of a vortex sheet, so that $s = v/u = \lambda_2$, and the density does not jump across. By contrast, gas molecules are crossing shocks associated with the eigenvalues λ_1 and λ_3, entering through what will be termed as the *front side* and exiting by the *back side*. Following standard practice, we normalize the symbol $[\![\;]\!]$ to denote the jump, front minus back value, of the enclosed quantity. Under this convention, the entropy admissibility condition (3.3.41) takes the form

$$s\left[\!\!\left[u(\rho\varepsilon + \frac{1}{2}\rho(u^2+v^2))\right]\!\!\right] \geq \left[\!\!\left[v(\rho\varepsilon + \frac{1}{2}\rho(u^2+v^2))\right]\!\!\right]. \tag{18.2.5}$$

Under the standard assumption of genuine nonlinearity, (18.2.5) is equivalent to the requirement that the shock be compressible, i.e., $[\![\rho]\!] < 0$, and also equivalent to the Lax E-condition, namely that the back value of λ_1 (or λ_3) is smaller than the slope s of a 1- (or a 3-) shock, and in turn s is smaller than the front value of λ_1 (or λ_3).

We now investigate the structure of the admissible branch of the 1- or 3-shock curve emanating from any specified front state (ρ_f, u_f, v_f). For convenience, we select coordinate axes in state space so that $u_f > 0$ and $v_f = 0$. Admissible back states (ρ_b, u_b, v_b) must satisfy the Rankine-Hugoniot conditions (18.2.4), namely

$$\begin{cases} s(\rho_f u_f - \rho_b u_b) = -\rho_b v_b \\ s(\rho_f u_f^2 + p(\rho_f) - \rho_b u_b^2 - p(\rho_b)) = -\rho_b u_b v_b \\ -s\rho_b u_b v_b = p(\rho_f) - p(\rho_b) - \rho_b v_b^2, \end{cases} \tag{18.2.6}$$

together with the admissibility condition $\rho_f < \rho_b$. We will parametrize the shock curves by ρ_b. Upon setting

$$c_0^2 = \frac{\rho_b}{\rho_f}\,\frac{p(\rho_b) - p(\rho_f)}{\rho_b - \rho_f}, \tag{18.2.7}$$

and after a long but straightforward calculation, we derive from (18.2.6):

(18.2.8)
$$s = \pm c_0 (u_f^2 - c_0^2)^{-\frac{1}{2}},$$

(18.2.9)
$$u_b = u_f - \frac{\rho_b - \rho_f}{\rho_b} \frac{c_0^2}{u_f},$$

(18.2.10)
$$v_b^2 = \left(\frac{\rho_b - \rho_f}{\rho_b} \right)^2 (u_f^2 - c_0^2) \frac{c_0^2}{u_f^2}.$$

Note that $c_0^2 > c_f^2$ and $c_0^2 \downarrow c_f^2$ as $\rho_b \downarrow \rho_f$. Hence 1- or 3-shocks may exist only when $u_f^2 > c_f^2$, i.e., the front state is supersonic. However, the back state may be supersonic, sonic or subsonic, and accordingly the shock is termed *supersonic, sonic,* or *transonic*.

The above equations describe the configuration in which the flow upstream is horizontal, moving from left to right, but upon crossing a 1-shock (or a 3-shock) with slope $s < 0$ (or $s > 0$) it bends downwards, $v_b < 0$, (or upwards, $v_b > 0$) and moves away from the shock, $u_b > 0$ and $v_b/u_b > s$ (or $v_b/u_b < s$).

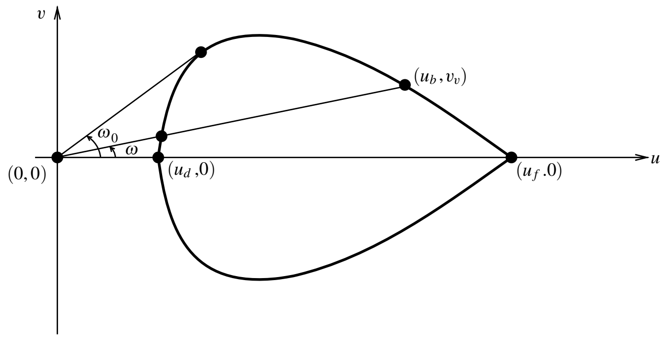

Fig. 18.2.1

Equations (18.2.9) and (18.2.10) define the so-called *shock polar* of the state $(\rho_f, u_f, 0)$, which is depicted in Figure 18.2.1. Notice that the shock polar is symmetric relative to the u-axis, and its lower half is the projection of the 1-shock curve while its upper half is the projection of the 3-shock curve, on the u-v plane. The angle $\omega = \arctan(v/u)$ by which the flow direction is deflected upon crossing the shock is easily read from the shock polar. In addition to $(\rho_f, u_f, 0)$, the 1- and the 3-shock curves meet, and terminate, at the state $(\rho_d, u_d, 0)$, identified through the conditions $c_0 = u_f$ and $\rho_d u_d = \rho_f u_f$. At that point $s = \infty$, so the flow, before and after impinging on the shock, is directed along the x-axis, perpendicular to the shock.

The squared speed of the shock, determined from (18.2.9) and (18.2.10), (18.2.11) $$u_b^2 + v_b^2 = u_f^2 - \frac{\rho_b^2 - \rho_f^2}{\rho_b^2} c_0^2,$$

turns out to be a decreasing function of ρ_b. We already know that $(\rho_f, u_f, 0)$ is supersonic, and it is easily seen that $(\rho_d, u_d, 0)$ is subsonic. Also, $c^2 = p'(\rho_b)$ is an increasing function of ρ_b. It follows that there is $\rho_0 \in (\rho_f, \rho_d)$ with the following property: the back state (ρ_b, u_b, v_b) of any 1- or 3-shock with front state $(\rho_f, u_f, 0)$ is subsonic when $\rho_b \in (\rho_0, \rho_d]$, sonic if $\rho_b = \rho_0$, and supersonic when $\rho_b \in (\rho_f, \rho_0)$.

Next we discuss the same issues in the context of steady irrotational isentropic flow,

(18.2.12) $$\partial_x v - \partial_y u = 0, \qquad u = \partial_x \phi, \quad v = \partial_y \phi,$$

where ϕ is the velocity potential. In that case, as explained in Section 3.3.6, it is common practice to retain the balance of mass equation $(18.2.1)_1$, namely,

(18.2.13) $$\partial_x(\rho u) + \partial_y(\rho v) = 0,$$

but abandon the balance of momentum equations $(18.2.1)_2$ and $(18.2.1)_3$, replacing them by the Bernoulli law (see (3.3.49)):

(18.2.14) $$\frac{1}{2} q^2 + h(\rho) = \text{constant}, \qquad q^2 = u^2 + v^2 = |\nabla \phi|^2,$$

where $h = \varepsilon + p/\rho$ is the enthalpy, with derivative $h'(\rho) = p'(\rho)/\rho$. Note that $(18.2.1)_{2,3}$ still hold when density and velocity are Lipschitz and that even when they fail, in the presence of shocks, $(\rho u^2 + p\,, \rho uv)$ and $(\rho uv\,, \rho v^2 + p)$ still constitute entropy-entropy flux pairs that are useful when treating the system by the method of compensated compactness. This observation also implies that when $(18.2.1)_{2,3}$ are abandoned for (18.2.14), momentum is balanced to third order in shock strength and hence the error may be deemed negligible in the context of flows with weak shocks.

In the presence of a shock with slope $dy/dx = s$, the jump conditions take the form

(18.2.15) $$[\![\phi]\!] = 0,$$

(18.2.16) $$s[\![\rho u]\!] = [\![\rho v]\!],$$

(18.2.17) $$\left[\!\!\left[\frac{1}{2} q^2 + h(\rho)\right]\!\!\right] = 0.$$

Notice that (18.2.12) implies the jump condition

(18.2.18) $$s[\![v]\!] = -[\![u]\!],$$

so that the tangential component of the velocity is continuous across the shock. As already noted in Section 3.3.6, (18.2.18) implies that vortex sheets are incompatible with irrotational flow. However, we still have 1- and 3-shocks, subject to the admissibility condition $[\![\rho]\!] < 0$.

We now consider, in the setting of irrotational flow, the configuration discussed earlier, namely admissible 1- and 3-shocks, with back states (ρ_b, u_b, v_b) and prescribed front state $(\rho_f, u_f, 0)$, where $u_f > 0$. The jump conditions (18.2.18), (18.2.16) and (18.2.17) now take the form

$$\begin{cases} -sv_b = u_b - u_f \\ s(\rho_f u_f - \rho_b u_b) = -\rho_b v_b \\ \frac{1}{2}u_f^2 + h(\rho_f) = \frac{1}{2}(u_b^2 + v_b^2) + h(\rho_b). \end{cases} \tag{18.2.19}$$

As above, we parametrize the (admissible branch of the) 1- and 3-shock curves by ρ_b. Upon setting

$$c_1^2 = 2[h(\rho_b) - h(\rho_f)], \tag{18.2.20}$$

we obtain from (18.2.19)

$$s = \pm c_1 \left[\frac{\rho_b^2 - \rho_f^2}{\rho_b^2} u_f^2 - c_1^2 \right]^{-\frac{1}{2}}, \tag{18.2.21}$$

$$u_b = u_f - \frac{\rho_b}{\rho_b + \rho_f} \frac{c_1^2}{u_f}, \tag{18.2.22}$$

$$v_b^2 = \left(\frac{\rho_b}{\rho_b + \rho_f} \right)^2 \left[\frac{\rho_b^2 - \rho_f^2}{\rho_b^2} u_f^2 - c_1^2 \right] \frac{c_1^2}{u_f^2}. \tag{18.2.23}$$

Equations (18.2.22), (18.2.23) determine the shock polar of $(\rho_f, u_f, 0)$, for irrotational flow. It has similar form to the shock polar for non-irrotational flow, discussed above, and so it will be depicted here by the same Figure 18.2.1. The state $(\rho_d, u_d, 0)$ where the 1- and 3-shock curves meet and terminate is now determined by $c_1^2 = (1 - \rho_f^2/\rho_d^2)u_f^2$ and $\rho_d u_d = \rho_f u_f$. At that point $s = \infty$, the shock being perpendicular to the flow direction. In the place of (17.2.11), we now have

$$u_b^2 + v_b^2 = u_f^2 - c_1^2 \tag{18.2.24}$$

so again there is $\rho_0 \in (\rho_f, \rho_d)$ with the property that (ρ_b, u_b, v_b) is subsonic when $\rho_b \in (\rho_0, \rho_d]$, sonic if $\rho_b = \rho_0$, and supersonic when $\rho_b \in (\rho_f, \rho_0)$.

Applications of the above will be presented in Section 18.4.

18.3 Self-Similar Planar Irrotational Isentropic Gas Flow

In planar irrotational isentropic flow, the velocity v derives from the velocity potential ϕ, by (3.3.48), and the governing equations are the two-dimensional versions of the continuity equation $(3.3.36)_1$ and the Bernoulli equation (3.3.49). Because of their scaling properties, these equations admit self-similar solutions in the form

$$\rho = \rho(\xi), \quad v = v(\xi), \quad \phi = t\psi(\xi), \quad \xi = \frac{1}{t}x. \tag{18.3.1}$$

Indeed, under this premise, upon letting ∇ denote the gradient operator with respect to the ξ-variable, (3.3.48), $(3.3.36)_1$, and (3.3.49), with $g = 0$, become

$$v = \nabla\psi, \tag{18.3.2}$$

$$-\xi\cdot\nabla\rho + \nabla\cdot(\rho\nabla\psi) = 0, \tag{18.3.3}$$

$$\psi - \xi\cdot\nabla\psi + \frac{1}{2}|\nabla\psi|^2 + h(\rho) = h(\rho_0). \tag{18.3.4}$$

On any open subset Ω of $\mathbb{R}^2$ on which ρ takes a constant value, (18.3.3) reduces to $\Delta\psi = 0$. In that case, applying the Laplace operator to (18.3.4) yields $|\nabla v|^2 = 0$, i.e., the velocity is also constant. In accordance with earlier usage of the term, such an Ω will be called a *constant state* of the flow.

We can write (18.3.3) and (18.3.4) in the more elegant form

$$\nabla\cdot(\rho\nabla\chi) + 2\rho = 0, \tag{18.3.5}$$

$$\chi + \frac{1}{2}|\nabla\chi|^2 + h(\rho) = h(\rho_0), \tag{18.3.6}$$

in terms of

$$\chi(\xi) = \psi(\xi) - \frac{1}{2}|\xi|^2, \tag{18.3.7}$$

which is called the *pseudopotential*. Accordingly,

$$\nabla\chi = v - \xi \tag{18.3.8}$$

is called the *pseudovelocity*, and the "flow" generated by it is called *pseudoflow*.

In particular, on a constant state Ω on which the velocity is constant, say $\bar{v}$, the pseudovelocity is $\bar{v} - \xi$ and the pseudopotential is

$$\chi(\xi) = \bar{v}\cdot\xi - \frac{1}{2}|\xi|^2 + \text{constant}. \tag{18.3.9}$$

From (18.3.4), by slightly abusing the notation,

(18.3.10)
$$\rho = h^{-1}(h(\rho_0) - \chi - \frac{1}{2}|\nabla\chi|^2) = \rho(|\nabla\chi|^2, \chi, \rho_0).$$

We may thus express the sound speed in terms of χ:

(18.3.11)
$$c^2 = p'(\rho(|\nabla\chi|^2, \chi, \rho_0)) = c^2(|\nabla\chi|^2, \chi, \rho_0).$$

Substituting ρ from (18.3.10) into (18.3.5), we derive a nonlinear, second-order equation for χ:

(18.3.12)
$$\nabla\cdot(\rho(|\nabla\chi|^2, \chi, \rho_0)\nabla\chi) + 2\rho(|\nabla\chi|^2, \chi, \rho_0) = 0.$$

Recalling that $h' = c^2/\rho$, (18.3.12) may be written as

(18.3.13)
$$c^2\Delta\chi - \sum_{i,j=1}^{2} \chi_{\xi_i}\chi_{\xi_j}\chi_{\xi_i\xi_j} = |\nabla\chi|^2 - 2c^2.$$

In terms of the (pseudo)-Mach number

(18.3.14)
$$L = \frac{|\nabla\chi|}{c},$$

the equations (18.3.12), (18.3.13) are elliptic, at points where $L < 1$, or hyperbolic, at points where $L > 1$.

An instructive, geometric method of testing the type of the equation (18.3.12) is as follows. With any fixed state (ρ, v), we associate on $\mathbb{R}^2$ the circle $|\xi - v| = c(\rho)$, with center v and radius $c(\rho)$, which will be called the *sonic circle* of (ρ, v). It is now clear that, given a flow $(\rho(\xi), v(\xi))$, (18.3.12) will be elliptic (or hyperbolic) at a point $\bar{\xi}$ if and only if $\bar{\xi}$ lies inside (or outside) the sonic circle of the state $(\rho(\bar{\xi}), v(\bar{\xi}))$.

Weak solutions of (18.3.12) are Lipschitz functions χ that satisfy this equation in the sense of distributions. Shocks are associated with curves across which partial derivatives of χ experience jump discontinuities. Since χ itself is continuous, its tangential derivatives must also be continuous across shocks and thus the tangential components of the pseudovelocity and the velocity cannot jump. In particular, the (pseudo) flow cannot support (pseudo) vortex sheets. By contrast, the normal derivatives of χ, and thereby the normal component of the pseudovelocity and the velocity, must jump across shocks, in accordance with the Rankine-Hugoniot condition

(18.3.15)
$$\left[\!\!\left[\rho\left(|\nabla\chi|^2, \chi, \rho_0\right)\frac{\partial\chi}{\partial\nu}\right]\!\!\right] = [\![\rho(v-\xi)\cdot\nu]\!] = 0,$$

where ν is the unit normal to the shock. The Bernoulli equation (18.3.6) dictates the additional jump condition

(18.3.16)
$$\left[\!\!\left[\frac{1}{2}|\nabla\chi|^2 + h(\rho)\right]\!\!\right] = \left[\!\!\left[\frac{1}{2}[(v-\xi)\cdot\nu]^2 + h(\rho)\right]\!\!\right] = 0.$$

Thus the jump conditions for self-similar and for steady irrotational flow are identical, with pseudovelocity $u = v - \xi$, in the former, playing the role of velocity, in the latter. In particular, the shock polar is here relevant as well.

It should be noted that the shock speed at the point ξ is $\xi \cdot \nu$. Thus straight-line shocks propagate with constant speed. In particular, straight-line shocks passing through the origin are stationary. In that case $\xi \cdot \nu = 0$ and (18.3.15), (18.3.16) reduce to the jump conditions (18.2.16), (18.2.17) for steady flow, recorded in Section 17.2.

As in the steady case, the front state of any admissible shock must be pseudosupersonic, while the back state may be pseudosupersonic, pseudosonic or pseudosubsonic, and accordingly the shock is termed *supersonic*, *sonic*, or *transonic*.

The hope is that the modicum of stability induced by the ellipticity of (18.3.13) in the subsonic regime will prevent the formation of very complex flow patterns exhibiting finely intermingled subsonic, sonic and supersonic regions. Credence to this expectation is provided by the following version of the maximum principle for solutions of (18.3.13), which is proved in the references cited in Section 18.9:

18.3.1 Theorem. *Let χ be a smooth solution of* (18.3.13), *on an open bounded set Ω, with $L \leq 1$ and $c \leq \bar{c}$. Then there is $\delta > 0$, depending on Ω, such that either $L^2 \leq 1 - \delta$ or else, for any function $f \in C^2(\Omega)$, with $|\nabla f| \leq \delta / \bar{c}$ and $|\nabla^2 f| \leq \delta / \bar{c}^2$, the function $L^2 + f$ does not attain its maximum in Ω.*

It follows that small perturbations of Ω that retain $L \leq 1$ on the boundary also preserve ellipticity in the interior. Another corollary of Theorem 18.3.1 is that there are no smooth flows that are sonic on an open set.

Since sonic regions must be "thin", it is reasonable to conjecture that, in generic flows, the interface between a subsonic and a supersonic region is a curve. This curve may be a transonic shock across which density and velocity jump, in accordance with (18.3.15), (18.3.16). Alternatively, the transition from a subsonic to a supersonic region may occur continuously, across a sonic curve. In order to gain insight into the latter situation, we investigate below the simplest setting in which a subsonic region is interfaced with a supersonic region that is a constant state $(\bar{\rho}, \bar{v})$. Then, as noted above, the interface will be an arc of the sonic circle with center at the point $\bar{v}$ and radius $\bar{c} = c(\bar{\rho})$. Without loss of generality, we assume $\bar{v} = 0$, and express the pseudopotential χ in polar coordinates (r, θ), with $\xi_1 = r\cos\theta$, $\xi_2 = r\sin\theta$. In the supersonic region, $r > \bar{c}$, we have $\chi = \frac{1}{2}r^2$. We are interested in the behavior of χ in the vicinity of the sonic circle, on the subsonic side, $r < \bar{c}$. The continuity of ρ and v across the sonic circle implies that both χ and its gradient are continuous on the sonic circle. The pertinent question is whether the gradient of χ is Lipschitz, or even smoother, across the interface. For definiteness, we assume that the gas is ideal, with adiabatic exponent $\gamma > 1$ and equations of state normalized by

$$p(\rho) = \frac{\gamma - 1}{\gamma}\rho^\gamma, \qquad c^2(\rho) = (\gamma - 1)\rho^{\gamma-1}. \qquad h(\rho) = \rho^{\gamma-1}. \tag{18.3.17}$$

In polar coordinates, Equation (18.3.13) takes the form

(18.3.18) $$[c^2-(\chi_r-r)^2]\chi_{rr}-\frac{2}{r^2}(\chi_r-r)\chi_\theta\chi_{r\theta}+\frac{1}{r^2}(c^2-\frac{1}{r^2}\chi_\theta^2)\chi_{\theta\theta}$$
$$+\frac{c^2}{r}\chi_r+\frac{1}{r^3}(\chi_r-2r)\chi_\theta^2=0.$$

So as to zero in into the transition of χ and its derivatives across the interface, we introduce in (18.3.18) new variables $s=\bar{c}-r$ and $w=\chi-\frac{1}{2}r^2$, in the place of r and χ. In view of the vanishing of w, w_θ, $w_{\theta\theta}$, w_s, $w_{s\theta}$ and $w_{s\theta\theta}$ at $s=0$, comparison of the anticipated orders of magnitude of the terms in (18.3.18) leads to the ansatz that, for s positive small, this equation must reduce to

(18.3.19) $$[2s-(\gamma+1)w_s]w_{ss}-w_s=R,$$

with $R=o(s)$. The reader may find the proof of this ansatz in the references cited in Section 18.9, together with precise estimates on solutions. Here we shall take $R=o(s)$ for granted and just sketch the derivation of

(18.3.20) $$w_s=\frac{s}{\gamma+1}+o(s),$$

which establishes that the transition of (ρ,v) across the sonic circle is Lipschitz, but not C^1, and also secures that the flow is subsonic, for s positive small.

Under the change of variable

(18.3.21) $$w=\frac{z}{\gamma+1}+\frac{s^2}{\gamma+1},$$

(18.3.19) yields

(18.3.22) $$z_sz_{ss}+3z_s+2s=o(s),$$

which may be integrated to give

(18.3.23) $$z_s^2+6z+2s^2=o(s^2).$$

Notice that, for s positive small, $z<0$, by (18.3.23), and $z_s\neq 0$, by (18.3.22). Thus $z_s<0$. The transformation $z=-\frac{1}{6}(\varpi^2+2)s^2$ reduces (18.3.23) to a separable differential equation for ϖ, which readily yields $\varpi=1+o(1)$ and thereby $z=-\frac{1}{2}s^2+o(s^2)$, $z_s=-s+o(s)$, thus establishing (18.3.20).

Upon observing that $w=s^{3/2}$ is a solution to the linear equation $2sw_{ss}-w_s=0$, we realize that the nonlinear term w_sw_{ss} in (18.3.19) plays a pivotal role in securing the Lipschitz continuity of w_s at $s=0$.

We close this section with the remark that the Euler equations (3.3.36), with $b=0$, also admit self-similar solutions $\rho=\rho(\xi)$, $v=v(\xi)$, which satisfy the system

(18.3.24) $$\begin{cases} -\xi\cdot\nabla\rho+\nabla\cdot(\rho v)=0 \\ -\xi\cdot\nabla(\rho v)+\nabla\cdot(\rho v\otimes v)+\nabla p(\rho)=0, \end{cases}$$

or, in terms of the *pseudovelocity* $u = v - \xi$,

$$\begin{cases} \nabla \cdot (\rho u) = -2\rho \\ \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = -3\rho u. \end{cases} \tag{18.3.25}$$

Again, the jump conditions for self-similar and for steady non-irrotational flow are identical, with pseudovelocity, in the former, playing the role of velocity, in the latter.

It is often instructive and helpful to view the configuration of (ρ, v), in the self-similar ξ-coordinates, as a snapshot of the actual flow, in the x-variables, taken at $t = 1$.

We shall encounter applications of the above in the following three sections.

18.4 Supersonic Isentropic Gas Flow Past a Ramp

On the upper half of the x-y plane, consider the isentropic flow of a gas with upstream state $(\rho_f, u_f, 0)$, $u_f > 0$, impinging on a solid ramp with foot at the origin and slope $\tan\omega$ (see Figure 18.4.1). Equivalently, one may regard this configuration as portraying the upper half of the (symmetric with respect to the x-axis) gas flow past a solid wedge with vertex angle 2ω.

We examine the feasibility of steady flow with the wave pattern depicted in Figure 18.4.1, namely two constant states $(\rho_f, u_f, 0)$ and (ρ_b, u_b, v_b) joined by a 3-shock with slope $s > \tan\omega$. For that purpose, $(\rho_f, u_f, 0)$ need be supersonic, and (u_b, v_b) should lie on the shock polar of $(\rho_f, u_f, 0)$ and must satisfy the boundary conditions imposed by the solid ramp, i.e., $v_b = u_b \tan\omega$, so that the normal velocity component vanishes on the ramp. Referring to Figure 18.2.1, we identify the angle ω_0 such that the number of wave configurations satisfying the above specifications is none when $\omega > \omega_0$, one if $\omega = \omega_0$, and two when $\omega < \omega_0$. In the last case, traditionally, the shock with the higher strength is referred to as the *strong shock* and the shock with the lower strength is called the *weak shock*. This should not be confused with the usage of the term "weak" earlier in the book. Indeed, the weak shock, in the present sense, can be quite strong.

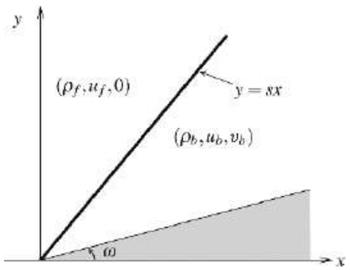

Fig. 18.4.1

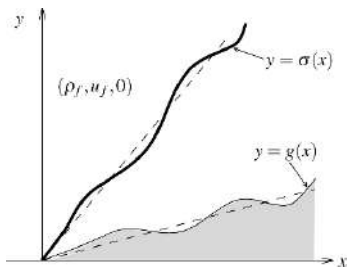

Fig. 18.4.2

The situation is exactly the same (albeit for different ω_0) in the setting of irrotational flow.

The question of whether one or both of the strong and the weak shocks are physically admissible has been vigorously debated over the past sixty years, but has not yet been settled in a definitive manner. On the basis of experimental and numerical evidence or stability considerations, there are strong indications that it is the weak shock solution that is physically admissible. Of course, there are different notions of stability, and even the strong shock solution meets certain stability criteria. The reader may find detailed information on these matters in the literature cited in Section 18.9. Here, we shall attempt to provide just a taste of the ongoing discussion by sketching two distinct approaches to the question of stability of the weak shock solution.

It follows from the presentation in Section 18.2 that the back state (ρ_b, u_b, v_b) of the weak shock solution will be supersonic when ω does not exceed a certain critical value $\omega_{cr} \in (0, \omega_0)$. It is this case that will be studied here.

The first test will be the stability of the supersonic weak shock solution under perturbation of the geometry of the ramp. Accordingly, we deform the straight ramp $y = x \tan \omega$, $x \geq 0$, so it becomes the curve $y = g(x)$, $x \geq 0$, where g is a C^1 function on $[0, \infty)$ such that $g(0) = 0$, $g'(0) = \tan \omega$, and g' has bounded variation on $[0, \infty)$; see Figure 18.4.2. Our task is to determine a steady isentropic flow with prescribed supersonic upstream state $(\rho_f, u_f, 0)$, $u_f > 0$, which impinges on the above obstacle. We assume $\omega \in (0, \omega_{cr})$, so that when $g(x) = x \tan \omega$, such a flow is determined by the supersonic weak shock solution depicted in Figure 18.4.1.

18.4.1 Theorem. *There are positive constants* ε *and* a *such that when*

$$TV_{[0,\infty)} g'(\cdot) < \varepsilon, \tag{18.4.1}$$

there exists a steady isentropic flow (ρ, u, v) *past the ramp, i.e., defined on the domain* $\{(x,y) : 0 \leq x < \infty,\ g(x) \leq y < \infty\}$, *and having the following structure:*

(a) *There is a Lipschitz curve* $y = \sigma(x)$ *on* $[0, \infty)$, *with* $\sigma(0) = 0$, $\sigma'(0+) = s$, *which is a supersonic weak shock. The slope* σ' *of* σ *is a function of bounded variation on* $[0, \infty)$ *and*

$$TV_{[0,\infty)} \sigma'(\cdot) \leq a\, TV_{[0,\infty)} g'(\cdot). \tag{18.4.2}$$

(b) *On the domain* $\{(x,y) : 0 \leq x < \infty,\ \sigma(x) < y < \infty\}$, $(\rho, u, v) = (\rho_f, u_f, 0)$.

(c) *On the domain* $\{(x,y) : 0 < x < \infty,\ g(x) \leq y < \sigma(x)\}$, (ρ, u, v) *is a BV solution of* (18.2.1), *whose trace on the curve* $y = g(x)$ *satisfies the boundary condition*

$$v(x, g(x)) = g'(x) u(x, g(x)), \tag{18.4.3}$$

expressing that the velocity component normal to the rigid obstacle vanishes.

The total variation is small, controlled by the total variation of g'*:*

(18.4.4) $$TV_{[g(x),\infty)}(\rho(x,\cdot),u(x,\cdot),v(x,\cdot)) \le aTV_{[0,\infty)}g'(\cdot)\,,$$

for every $x \in (0,\infty)$*. Finally,*

(18.4.5) $$\lim_{x\to\infty}\frac{v(x,y)}{u(x,y)} = \lim_{x\to\infty} g'(x)\,,$$

uniformly for $y \in [g(x),\sigma(x))$*.*

The lengthy and technical proof of the above proposition, which is found in the references cited in Section 18.9, is constructive, employing an adaptation of the random choice method to the present setting.

Our next task is to provide evidence that the steady supersonic weak shock solution is dynamically stable, in the sense that it describes the long-time behavior of an unsteady flow, on the domain $\{(x,y) : 0 \le x < \infty,\, x\tan\omega \le y < \infty\}$, with boundary conditions

(18.4.6) $$(\rho,u,v)(0,y,t) = (\rho_f,u_f,0), \qquad 0 \le y < \infty,\ 0 < t < \infty,$$

(18.4.7) $$v(x,x\tan\omega,t) = u(x,x\tan\omega,t)\tan\omega, \qquad 0 < x < \infty,\ 0 < t < \infty,$$

and initial condition

(18.4.8) $$(\rho,u,v)(x,y,0) = (\rho_f,u_f,0), \qquad 0 \le x < \infty,\ \ x\tan\omega < y < \infty.$$

We will carry out the construction of a flow with the above specifications in the realm of potential flow for an ideal gas with adiabatic exponent $\gamma > 1$ and equations of state (18.3.17). The assumptions on the data are that $(\rho_f,u_f,0)$, $u_f > 0$, is a supersonic state; and $\omega < \omega_{cr}$, in which case the back state (ρ_b,u_b,v_b), $v_b/u_b = \tan\omega$, of the weak steady shock with front state $(\rho_f,u_f,0)$ is also supersonic. The desired outcome will be that, as $t \to \infty$, this unsteady flow will converge to the steady flow depicted in Figure 18.4.1.

Because of the geometry of the domain and the invariance of the assigned initial and boundary data under coordinate stretching, the desired flow will be self-similar, as discussed in Section 18.3, namely with ρ,u,v and the pseudopotential χ functions of the two variables $\xi = x/t$ and $\zeta = y/t$. In particular, as explained in Section 18.3, the steady supersonic shock joining the states $(\rho_f,u_f,0)$ and (ρ_b,u_b,v_b) will appear in the ξ-ζ plane as a straight line passing through the origin.

The corner effect will not be felt immediately at points far from the origin, and thus in the short term the interaction between initial and boundary conditions on the ramp will generate a reflected supersonic shock, with slope $\tan\omega$, joining the front state $(\rho_f,u_f,0)$ with a state (ρ_d,u_d,v_d). The equation of this shock is in the form

(18.4.9) $$\zeta = (\xi+\ell)\tan\omega,$$

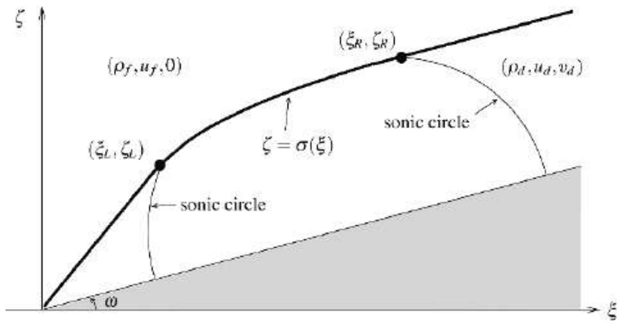

Fig. 18.4.3

where ℓ is a positive constant to be determined below. By the boundary condition (18.4.7), the vector (u_d, v_d) must be tangential to the shock, $v_d/u_d = \tan\omega$. Furthermore, this vector must coincide with the tangential component of $(u_f, 0)$, as the tangential component of velocity is continuous across shocks. Thus

(18.4.10) $$u_d = u_f \cos^2\omega, \quad v_d = u_f \sin\omega\cos\omega, \quad \sqrt{u_d^2 + v_d^2} = u_f \cos\omega.$$

To determine the remaining parameters ρ_d and ℓ, we appeal to the Rankine-Hugoniot condition (18.3.15), which expresses mass conservation, and here reduces to

(18.4.11) $$\ell(\rho_d - \rho_f) = \rho_f u_f\,,$$

and to (18.3.16), dictated by Bernoulli's equation, which yields

(18.4.12) $$\ell u_f \sin^2\omega = \rho_d^{\gamma-1} - \rho_f^{\gamma-1} - \frac{1}{2}u_f^2 \sin^2\omega.$$

Combining (18.4.11) with (18.4.12), we derive the equation

(18.4.13) $$\frac{\left(\rho_d^{\gamma-1} - \rho_f^{\gamma-1}\right)(\rho_d - \rho_f)}{\rho_d + \rho_f} = \frac{1}{2}u_f^2 \sin^2\omega,$$

which admits a unique solution ρ_d satisfying the admissibility condition $\rho_d > \rho_f$. Finally, ℓ is determined by (18.4.11).

The notion of the sonic circle was introduced in Section 18.3. The sonic circle of the state (ρ_b, u_b, v_b) is centered at the point (u_b, v_b), which lies on the ramp, and has radius $\sqrt{\gamma-1}\,\rho_b^{\frac{\gamma-1}{2}}$. Since (ρ_b, u_b, v_b) is supersonic, the origin $\xi = \zeta = 0$ lies outside this circle, but the steady shock $\zeta = s\xi$ is intersected by the circle at two points, of which the one closest to the origin will be denoted by (ξ_L, ζ_L). Similarly, the sonic circle of the state (ρ_d, u_d, v_d), which is centered at the point (u_d, v_d), also lying on the ramp, and has radius $\sqrt{\gamma-1}\,\rho_d^{\frac{\gamma-1}{2}}$, intersects the shock $\zeta = (\xi + \ell)\tan\omega$ at two points, of which the more distant from the origin will be denoted by (ξ_R, ζ_R).

18.4.2 Theorem. *Under the above assumptions, there exists a piecewise smooth, self-similar flow* (ρ, u, v), *with configuration depicted in Figure* 18.4.3, *having the following structure:*

(a) *It contains a single shock* $\zeta = \sigma(\xi)$, *with* $\sigma \in C^1[0,\infty)$. *For* $0 < \xi < \xi_L$, *the shock is stationary supersonic,* $\sigma(\xi) = s\xi$, *joining* $(\rho_f, u_f, 0)$ *with* (ρ_b, u_b, v_b); *for* $\xi_L < \xi < \xi_R$, *the shock is transonic; for* $\xi_R < \xi < \infty$, *the shock is supersonic,* $\sigma(\xi) = (\xi + \ell)\tan\omega$, *moving with speed* $\ell \sin\omega$ *and joining* $(\rho_f, u_f, 0)$ *with* (ρ_d, u_d, v_d). *On* (ξ_L, ξ_R), σ *is* C^3 *and convex.*
(b) *The domain ahead of the shock is the constant state* $(\rho_f, u_f, 0)$.
(c) *The domain behind the stationary supersonic part of the shock, lying outside the sonic circle of* (ρ_b, u_b, v_b), *is the constant state* (ρ_b, u_b, v_b).
(d) *The domain behind the moving supersonic part of the shock, lying outside the sonic circle of* (ρ_d, u_d, v_d), *is the constant state* (ρ_d, u_d, v_d).
(e) *In the domain behind the transonic part of the shock, bordered on the left by the sonic circle of* (ρ_b, u_b, v_b) *and on the right by the sonic circle of* (ρ_d, u_d, v_d), (ρ, u, v) *are smooth functions, with* (u, v) *derived from a smooth pseudopotential* χ:

$$u = \chi_\xi + \xi\,, \quad v = \chi_\zeta + \zeta\,, \tag{18.4.14}$$

satisfying the partial differential equation

$$\nabla\cdot\left[\left(\rho_0^{\gamma-1} - \chi - \frac{1}{2}|\nabla\chi|^2\right)^{\frac{1}{\gamma-1}}\nabla\chi\right] + 2\left(\rho_0^{\gamma-1} - \chi - \frac{1}{2}|\nabla\chi|^2\right)^{\frac{1}{\gamma-1}} = 0, \tag{18.4.15}$$

which is of elliptic type.

The construction of the density and velocity fields in the hyperbolic regime has been expounded above. It is then easy to determine the pseudopotential χ, with the help of (18.3.9). The constant is chosen so that χ is continuous across shocks. In particular, in the domain ahead of the shock,

$$\chi = u_f\xi - \frac{1}{2}(\xi^2 + \zeta^2). \tag{18.4.16}$$

Then, in the domain behind the stationary part of the shock, outside the sonic circle of (ρ_b, u_b, v_b),

$$\chi = u_b\xi + v_b\zeta - \frac{1}{2}(\xi^2 + \zeta^2). \tag{18.4.17}$$

Finally, in the domain behind the moving supersonic part of the shock, outside the sonic circle of (ρ_d, u_d, v_d), by virtue of (18.4.9) and (18.4.10),

$$\chi = u_d\xi + v_d\zeta - \frac{1}{2}(\xi^2 + \zeta^2) - \frac{1}{2}\ell u_f \sin 2\omega. \tag{18.4.18}$$

What remains to be done is the construction of χ in the elliptic regime, as solution of the equation (18.4.15) subject to the following boundary conditions. Along the ramp, $\partial\chi/\partial\nu = 0$, so that the normal component of the velocity vanishes. Along the arcs of the two sonic circles, the values of χ and its first derivatives are prescribed by the requirement of continuity of the velocity field across these curves. Finally, the jump conditions (18.3.15) and (18.3.16) must hold across the shock. Solving this problem is not an easy task. To begin with, the graph of the transonic part of the shock over the interval (ξ_L, ξ_R) is not known in advance, but it is a free boundary to be determined as part of the solution. Another major difficulty is that ellipticity degenerates along the arcs of the sonic circles, and at the same time both Dirichlet and Neuman boundary data are prescribed there. The hard and technical analysis required in order to overcome these obstacles lies beyond the scope of this book. The proof of Theorem 18.4.2 is found in the references cited in Section 18.9.

The supersonic region adjacent to the origin shrinks as ω increases, and vanishes when $\omega = \omega_{cr}$, in which case the transonic shock emanates from the foot of the ramp. This configuration persists even as ω increases beyond ω_{cr}, within the range $[\omega_{cr}, \omega_0)$. See the references in Section 18.9.

It is clear that the self-similar flow depicted in Figure 18.4.3, when transcribed to the original variables (x,y,t), tends, as $t \to \infty$, to the steady flow depicted in Figure 18.4.1.

18.5 Regular Shock Reflection on a Wall

In the realm of two-dimensional potential flow for an ideal gas with equation of state (18.3.17), we consider here a shock colliding with a wall and undergoing *regular reflection*, i.e., the reflected shock emanates from the point of contact of the incident shock with the wall.

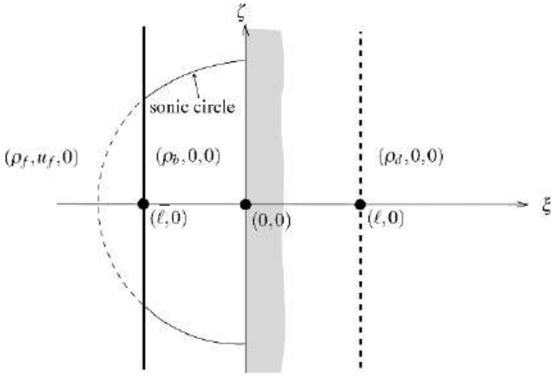

Fig. 18.5.1

We choose coordinates so that the incident shock stays parallel to the y-axis, and is moving along the x-axis in the direction of increasing x. The rigid wall is represented by the line $y = x\tan\omega$, for some angle $\omega \in (0, \frac{\pi}{2}]$. The incident shock joins states $(\rho_f, u_f, 0)$, on the left, and $(\rho_d, 0, 0)$, on the right, and at $t = 0$ is located along the y-axis. Because of the geometry and the scale invariance of the data, the flow will be self-similar, as explained in Section 18.3, with ρ, u, v and the pseudopotential χ functions of the two variables $\xi = x/t$, $\zeta = y/t$.

We begin with the simplest case of *normal reflection*, i.e., $\omega = \pi/2$. In the absence of the wall, the incident shock would have advanced in the x-direction, and at time $t = 1$ would occupy the coordinate line $x = \ell$, for some $\ell > 0$. Thus, in the ξ-ζ plane the incident shock would be represented by the stationary line $\xi = \ell$.

We can determine ℓ, together with the required relation between ρ_f, u_f and u_d, by appealing to the jump conditions (18.3.14) and (18.3.15), which here reduce to

$$\ell(\rho_f - \rho_d) = \rho_f u_f\,, \tag{18.5.1}$$

$$\ell u_f = \rho_f^{\gamma-1} - \rho_d^{\gamma-1} + \frac{1}{2}u_f^2\,. \tag{18.5.2}$$

Combining these equations yields

$$\frac{(\rho_f^{\gamma-1} - \rho_d^{\gamma-1})(\rho_f - \rho_d)}{\rho_f + \rho_d} = \frac{1}{2}u_f^2\,, \tag{18.5.3}$$

whence one determines uniquely $\rho_d < \rho_f$. Then ℓ is obtained from (18.5.1), in terms of u_f and ρ_f.

Because of the wall, at $t = 0$ the incident shock will be reflected as another shock, which will also stay parallel to the y-axis, but it will be moving along the x-axis in the direction of decreasing x. In the ξ-ζ plane, this shock will be represented by a coordinate line $\xi = \bar{\ell}$, for some $\bar{\ell} < 0$. The front state of the reflected shock will be $(\rho_f, u_f, 0)$ and the back state (ρ_b, u_b, v_b). The normal velocity component on the wall must vanish, and so $u_b = 0$. Furthermore, the tangential velocity component must be continuous across the shock, which implies $v_b = 0$. The remaining parameters ρ_b and $\bar{\ell}$ are determined with the help of the jump conditions (18.3.15) and (18.3.16), which now take the form

$$\bar{\ell}(\rho_f - \rho_b) = \rho_f u_f\,, \tag{18.5.4}$$

$$\bar{\ell} u_f = \rho_f^{\gamma-1} - \rho_b^{\gamma-1} + \frac{1}{2}u_f^2\,. \tag{18.5.5}$$

Combining (18.5.4) and (18.5.5) yields

$$\frac{(\rho_f^{\gamma-1} - \rho_b^{\gamma-1})(\rho_f - \rho_b)}{\rho_f + \rho_b} = \frac{1}{2}u_f^2\,, \tag{18.5.6}$$

which determines uniquely $\rho_b > \rho_f$. Then $\bar{\ell}$ is computed from (18.5.4).

An easy estimation verifies that, for any $\gamma > 1$, (18.5.7) $$|\bar{\ell}| < c(\rho_b) = \sqrt{\gamma - 1}\rho_b^{\frac{\gamma-1}{2}},$$

which implies that the shock intersects the sonic circle of the state $(\rho_b, 0, 0)$, as depicted in Figure 18.5.1. Thus, part of the reflected shock is supersonic and part is transonic.

We now turn to the case of *oblique reflection*, i.e., $\omega < \pi/2$. On the ξ-ζ plane, the incident shock will make contact with the wall at a point $(\ell, \ell \tan\omega)$ and will lie on the coordinate line $\xi = \ell$. One determines ℓ, together with the needed relation between ρ_f, u_f and ρ_d, exactly as above, by employing the jump conditions (18.5.1) and (18.5.2), then passing to (18.5.3), etc.

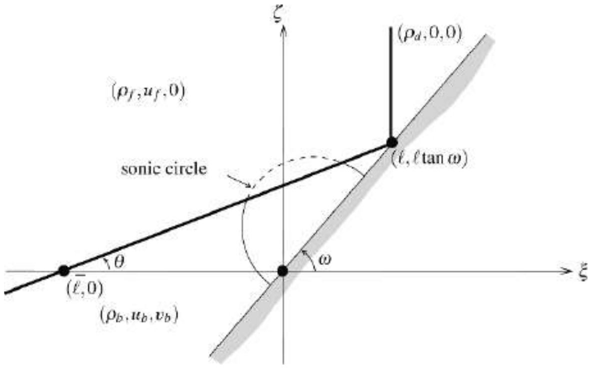

Fig. 18.5.2

For regular reflection to occur, it is necessary to fit a shock emanating from the point $(\ell, \ell\tan\omega)$, having slope say $\tan\theta$, $\theta < \omega$, and joining the front state $(\rho_f, u_f, 0)$ with some back state (ρ_b, u_b, v_b), with $\rho_b > \rho_f$; see Figure 18.5.2. The required conditions on the velocity field (u_b, v_b) are

(18.5.8) $$v_b - u_b \tan\omega = 0,$$

so that the normal velocity on the wall vanishes, and

(18.5.9) $$u_b + v_b \tan\theta = u_f,$$

in order to meet the requirement that the tangential component of the velocity be continuous across the shock. The above two equations yield

(18.5.10) $$u_b = \frac{1}{1 + \tan\omega\tan\theta} u_f, \qquad v_b = \frac{\tan\omega}{1 + \tan\omega\tan\theta} u_f.$$

To determine the remaining parameters ρ_b and θ, we appeal to the jump conditions (18.3.15) and (18.3.16), which here reduce to

(18.5.11) $$\ell(\rho_b - \rho_f) = \frac{\tan\theta}{\tan\omega - \tan\theta}\rho_f u_f + \frac{v_b - u_b\tan\theta}{\tan\omega - \tan\theta}\rho_b,$$

(18.5.12) $$\frac{1}{2}u_b^2+\frac{1}{2}v_b^2-\ell(u_b+v_b\tan\omega)=\rho_f^{\gamma-1}-\rho_b^{\gamma-1}+\frac{1}{2}u_f^2-\ell u_f\,.$$

Substituting u_b and v_b from (18.5.10) into (18.5.11), (18.5.12), and then eliminating ρ_b between the resulting two equations, yields a single equation for the unknown slope $\tan\theta$. As follows from the discussion in Section 18.2, this equation will have a solution, with $\tan\theta<\tan\omega$, so long as $\omega_d\le\omega<\pi/2$, for some $\omega_d\in(0,\frac{1}{2}\pi)$ termed the *detachment angle*. Two distinct states (ρ_b,u_b,v_b) may then be determined from $\tan\theta$ with the help of (18.5.10) and (18.5.11), one joined to $(\rho_f,u_f,0)$ by a strong shock, the other by a weak shock. As noted in Section 18.4, the strong shock is not observed in experiments, so here we shall retain only the weak shock. In fact, as $\omega\to\pi/2$, it is the wave pattern generated by the weak shock that converges to the wave pattern for normal reflection, described above.

For ω near $\pi/2$, the resulting wave pattern is depicted in Fig. 18.5.2. We observe that the sonic circle of the state (ρ_b,u_b,v_b), which is centered at the point (u_b,v_b) of the wall and has radius $c(\rho_b)$, intersects the reflected shock, so that this shock is partly supersonic and partly transonic. However, the contact point $(\ell,\ell\tan\omega)$ lies outside the sonic circle, so that the collision is supersonic. This pattern persists so long as $\omega_s<\omega<\pi/2$, for some $\omega_s\in(\omega_d,\frac{1}{2}\pi)$, which is termed the *sonic angle*. For $\omega\in(\omega_d,\omega_s)$, $(\ell,\ell\tan\omega)$ lies inside the sonic circle so that the reflection is transonic.

Regular reflection is impossible when $\omega<\omega_d$. Numerical and experimental evidence indicates that under such conditions the point of reflection gets detached from the wall, albeit it stays connected to it by a complex wave pattern called a *Mach stem*. Despite extensive studies of this phenomenon, called *Mach reflection*, a rigorous definitive analytic theory is still lacking. Vortex sheets play an important role in the structure of Mach stems and thus potential flow does not seem to provide the proper framework for such a theory.

18.6 Shock Collision with a Ramp

In the realm of two-dimensional potential gas flow for an ideal gas with equations of state (18.3.17), we will examine here the collision of a shock with a solid ramp. On the upper half of the x-y plane, the foot of the ramp is located at the origin and its slope is $\tan\omega$. The shock is parallel to the y-axis and is moving from left to right. At time zero, it lies on the y-axis. On its left, $x<0$, the gas has constant density ρ_f and constant velocity $(u_f,0)$, with $u_f>0$. On its right, $x>0$, over the ramp, the gas has constant density ρ_d, with $\rho_d<\rho_f$, and is at rest, i.e., $u_d=v_d=0$. Equivalently, one may regard this configuration as portraying the upper half of the collision of a shock with a wedge, the vertex angle being 2ω.

We thus seek some potential flow (ρ,u,v), that is defined on the domain $\{(x,y):x<0,\,0\le y<\infty\}\cup\{(x,y):x\ge 0,\,x\tan\omega\le y<\infty\}$, and admits boundary conditions

(18.6.1) $$v(x,0,t)=0,\qquad -\infty<x<0,\;\;0<t<\infty,$$

(18.6.2) $$v(x, x\tan\omega, t) = u(x, x\tan\omega, t)\tan\omega, \qquad 0 \le x < \infty,\ 0 < t < \infty,$$

and the initial condition

(18.6.3) $$(\rho, u, v)(x, y, 0) = \begin{cases} (\rho_f, u_f, 0) & -\infty < x < 0,\ 0 \le y < \infty \\ (\rho_d, 0, 0) & 0 \le x < \infty,\ x\tan\omega \le y < \infty, \end{cases}$$

where the states $(\rho_f, u_f, 0)$ and $(\rho_d, 0, 0)$, with $u_f > 0$ and $\rho_f > \rho_d$, are joined by an admissible shock.

Because of the geometry of the domain and the scale invariance of the data, the solution to the above problem will be self-similar, so that ρ, u, v, and the pseudopotential χ may be realized as functions of the two variables $\xi = x/t$ and $\zeta = y/t$ on $\{(\xi,\zeta) : \xi < 0, 0 \le \zeta < \infty\} \cup \{(\xi,\zeta) : \xi \ge 0, \xi\tan\omega \le \zeta < \infty\}$.

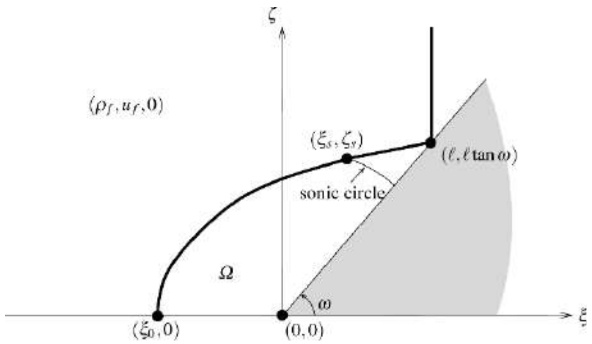

Fig. 18.6.1

On the ξ-ζ plane, the incoming shock will make contact with the ramp at a point $(\ell, \ell\tan\omega)$. The solution in the vicinity of that point will not be affected by the boundary data on the left of the foot of the ramp, i.e., for $\xi < 0$. Thus, under the assumption that $\omega > \omega_d$, where ω_d is the detachment angle identified in Section 18.5, the incoming shock will undergo regular reflection at $(\ell, \ell\tan\omega)$. In particular, ℓ and the necessary relation between ρ_f, u_f and ρ_d will again be determined through (18.5.1) and (18.5.3). The reflected shock will emanate from $(\ell, \ell\tan\omega)$, will have slope $\tan\theta$, and will join the front state $(\rho_f, u_f, 0)$ with a back state (ρ_b, u_b, v_b), determined through (18.5.10), (18.5.11) and (18.5.12).

Let us consider the case $\omega > \omega_s$, where ω_s is the sonic angle identified in Section 18.5. Then the point $(\ell, \ell\tan\omega)$ lies outside the sonic circle of the state (ρ_b, u_b, v_b), which is centered at the point (u_b, v_b) of the ramp and has radius $c(\rho_b) = \sqrt{\gamma - 1}\rho_b^{\frac{\gamma-1}{2}}$. Thus, initially the reflected shock will be supersonic and will retain its constant slope up until it crosses into the above sonic circle. Furthermore, the domain bordered from the left by the sonic circle, from the top by the reflected shock, and from the bottom by the ramp will be the constant state (ρ_b, u_b, v_b).

The pseudopotential χ on the constant states of the solution will be determined by (18.3.9), with the constant chosen so as to ensure continuity across shocks. Thus, on the constant state $(\rho_d, 0, 0)$,

$$\chi = -\frac{1}{2}(\xi^2 + \zeta^2), \tag{18.6.4}$$

on the constant state $(\rho_f, u_f, 0)$,

$$\chi = u_f(\xi - \ell) - \frac{1}{2}(\xi^2 + \zeta^2), \tag{18.6.5}$$

and on the constant state (ρ_b, u_b, v_b),

$$\chi = u_b\xi + v_b\zeta - \frac{1}{2}(\xi^2 + \zeta^2) - \frac{1 + \tan^2\omega}{1 + \tan\omega\tan\theta}\ell u_f. \tag{18.6.6}$$

After crossing the sonic circle of (ρ_b, u_b, v_b), at a point (ξ_s, ζ_s), the reflected shock is diffracted, as it is subjected to the influence of the boundary condition $v(\xi, 0) = 0$, $\xi < 0$, which induces it to become concave. It can be shown that if $u_f \leq c(\rho_f)$, as is the case when the incident shock is relatively weak, the diffracted shock eventually hits the ξ-axis at a right angle, at a point $(\xi_0, 0)$, with $\xi_0 < 0$, as depicted in Figure 18.6.1. To verify this shock configuration, and complete the construction of the flow, one has to determine the pseudopotential χ in the domain, marked as Ω in Figure 18.6.1, which is bordered from below by the semi-axis $\xi < 0$ and the ramp, and from above by the reflected shock and the sonic circle of (ρ_b, u_b, v_b). This is effected by solving the partial differential equation (18.4.15), which is elliptic on Ω, under the following boundary conditions. Along the bottom part of the boundary, $\partial\chi/\partial\nu = 0$, as required by (18.6.1) and (18.6.2). Along the arc of the sonic circle, $\chi_\xi = u_d - \xi$ and $\chi_\zeta = v_d - \zeta$, in order to ensure continuity of the velocity field across this curve. Finally, the jump conditions (18.3.15) and (18.3.16) must hold across the shock. This problem exhibits the difficulties already encountered in Section 18.4: first, that the graph of the transonic shock is not known in advance, since it is a free boundary to be determined as part of the solution, and second that both Dirichlet and Neuman data are prescribed on the sonic circle, on which ellipticity degenerates. Overcoming these obstacles requires hard and technical analysis, which lies beyond the scope of this book and is found in the references cited in Section 18.9. The conclusions are summarized in the following

18.6.1 Theorem. *Under the assumptions* $\omega > \omega_s$ *and* $u_f \leq c(\rho_f)$, *there exists a piecewise smooth, self-similar potential flow* (ρ, u, v) *with boundary conditions* (18.6.1), (18.6.2) *and initial condition* (18.6.3). *Its configuration, depicted in Figure* 18.6.1, *exhibits the following features*:

(a) *The incident shock, joining* $(\rho_f, u_f, 0)$ *with* $(\rho_d, 0, 0)$, *undergoes regular reflection at a point* $(\ell, \ell\tan\omega)$ *on the ramp.*
(b) *The reflected shock* $\zeta = \sigma(\xi)$ *emanates from* $(\ell, \ell\tan\omega)$ *and terminates at a point* $(\xi_0, 0)$ *of the* ξ*-axis. It is supersonic,* $\sigma(\xi) = \ell\tan\omega + (\xi - \ell)\tan\theta$, *for* $\xi_s < \xi < \ell$, *joining* $(\rho_f, u_f, 0)$ *with* (ρ_b, u_b, v_b); *and transonic for* $\xi_0 < \xi < \xi_s$. *Furthermore,* σ *is* C^2 *at* ξ_s *and* C^∞ *on* (ξ_0, ξ_s).
(c) *The domain ahead of the incident shock is the constant state* $(\rho_d, 0, 0)$.

(d) *The domain behind the incident shock but ahead of the reflected shock is the constant state* $(\rho_f, u_f, 0)$.
(e) *The domain behind the supersonic part of the reflected shock, lying outsie the sonic circle of* (ρ_b, u_b, v_b), *is the constant state* (ρ_b, u_b, v_b).
(f) *In the domain* Ω, *behind the transonic part of the reflected shock, bordered on the right by the sonic circle of* (ρ_b, u_b, v_b), (ρ, u, v) *is* C^∞, *with* (u, v) *derived through* (18.4.15) *from a* C^∞ *pseudopotential* χ, *which satisfies the elliptic equation* (18.4.16).
(g) (ρ, u, v) *is Lipschitz across the arc of the sonic circle of* (ρ_b, u_b, v_b) *that borders* Ω.

The situation where one or both of the assumptions $\omega > \omega_s$ and $u_f \leq c(\rho_f)$ are violated is discussed in the references cited in Section 18.9. It is shown that when $\omega \in [\omega_d, \omega_s]$, but still $u_f \leq c(\rho_f)$, the reflection at the point $(\ell, \ell \tan \omega)$ becomes transonic and one ends up with the same configuration depicted in Figure 18.6.1, except that now the points (ξ_s, ζ_s) and $(\ell, \ell \tan \omega)$ coalesce so that the reflected shock stays transonic all along its track, and the entire region Ω, now bordered from below by the semi-axis $\xi < 0$ and the ramp, and from above by the reflected shock, is subsonic.

The above description still applies when $u_f > c(\rho_f)$, but only as long as ω exceeds a certain critical angle $\omega_c \in [\omega_d, \frac{1}{2}\pi)$. As ω approaches the value ω_c, the point $(\xi_0, 0)$ in the Figure 18.6.1 moves closer to the origin $(0,0)$ and merges with it when $\omega = \omega_c$, in which case the reflected shock becomes attached to the foot of the ramp.

18.7 Isometric Immersions

Differential geometry is another rich source for systems of balance laws of mixed, elliptic-hyperbolic type. Here we consider the classical problem of isometric embedding of a two-dimensional Riemannian manifold in $\mathbb{R}^3$.

Consider a two-dimensional manifold $\Omega \subset \mathbb{R}^2$ equipped with a smooth Riemannian metric $[g_{ij}]$. The inverse matrix of $[g_{ij}]$ is denoted by $[g^{kl}]$ and $|g|$ stands for $\det[g_{ij}]$. We employ standard notation from differential geometry, including the summation convention.

The metric induces the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}), \tag{18.7.1}$$

and thence the curvature tensor

$$R_{ijkl} = g_{lm} (\partial_k \Gamma_{ij}^m - \partial_j \Gamma_{ik}^m + \Gamma_{ij}^n \Gamma_{nk}^m - \Gamma_{ik}^n \Gamma_{nj}^m) \tag{18.7.2}$$

and the Gauss curvature

$$\kappa = \frac{R_{1212}}{|g|}. \tag{18.7.3}$$

An immersion r of the manifold in $\mathbb{R}^3$ will preserve the first fundamental form $I = g_{ij}dx^i dx^j$, i.e.

$$\partial_i r \cdot \partial_j r = g_{ij}\,, \qquad i,j = 1,2, \tag{18.7.4}$$

if the second fundamental form $II = h_{ij}dx^i dx^j$ satisfies the classical Gauss-Codazzi equations. For convenience we relabel (x^1,x^2) as (x,y) and rescale the h_{ij} by

$$L = \frac{h_{11}}{\sqrt{|g|}}\,, \qquad M = \frac{h_{12}}{\sqrt{|g|}}\,, \qquad N = \frac{h_{22}}{\sqrt{|g|}}\,, \tag{18.7.5}$$

in which case the Gauss-Codazzi equations assume the form

$$\begin{cases} \partial_x N - \partial_y M = -\Gamma^1_{22} L + 2\Gamma^1_{12} M - \Gamma^1_{11} N \\ \\ \partial_x M - \partial_y L = \Gamma^2_{22} L - 2\Gamma^2_{12} M + \Gamma^2_{11} N, \end{cases} \tag{18.7.6}$$

$$LN - M^2 = \kappa. \tag{18.7.7}$$

The immersion is termed *regular* or *singular,* depending on whether the associated (L,M,N) is a Lipschitz continuous classical solution or a merely bounded measurable weak solution of the system (18.7.6), (18.7.7). Here we operate in the realm of singular immersions.

A crucial observation is that the L^∞ weak* limit of a sequence of weak solutions of (18.7.6), (18.7.7) is itself a weak solution, as follows from the div-curl lemma (Theorem 17.2.1) upon realizing the left-hand sides of $(18.7.6)_1$, $(18.7.6)_2$ and (18.7.7), respectively, as $\operatorname{div}(N,-M)$, $\operatorname{curl}(L,M)$ and $(N,-M)\cdot(L,M)$. This paves the way for treating the system by the method of compensated compactness.

From the standpoint of the theory of conservation laws, one may realize (18.7.6) as a system of balance laws that can be closed by prescribing "constitutive equations" for L, M and N, subject to the compatibility constraint (18.7.7). To that end, we select a state vector (ρ,u,v), together with a monotone function $p(\rho)$, and set

$$L = \rho v^2 + p(\rho), \qquad M = -\rho u v, \qquad N = \rho u^2 + p(\rho), \tag{18.7.8}$$

in which case (18.7.6) assumes the form

$$\begin{cases} \partial_x(\rho u^2 + p(\rho)) + \partial_y(\rho u v) = X \\ \\ \partial_x(\rho u v) + \partial_y(\rho v^2 + p(\rho)) = Y, \end{cases} \tag{18.7.9}$$

where

$$\begin{cases} X = -(\rho v^2 + p)\Gamma^1_{22} - 2\rho u v \Gamma^1_{12} - (\rho u^2 + p)\Gamma^1_{11} \\ \\ Y = -(\rho v^2 + p)\Gamma^2_{22} - 2\rho u v \Gamma^2_{12} - (\rho u^2 + p)\Gamma^2_{11}\,. \end{cases} \tag{18.7.10}$$

Thus (18.7.6) reduces to the equations of balance of momentum for the isentropic steady irrotational planar flow of a gas with equation of state $p(\rho)$. Indeed, (18.7.9) are akin to $(18.2.1)_{2,3}$, with (X,Y) playing the role of body force.

For equation of state we select

$$p(\rho) = -\frac{1}{\rho}, \tag{18.7.11}$$

which represents a gas of Chaplygin type (2.5.23). In order to satisfy (18.7.7), by virtue of (18.7.8) and (18.7.11),

$$-q^2 + \frac{1}{\rho^2} = \kappa, \qquad q^2 = u^2 + v^2. \tag{18.7.12}$$

The enthalpy for a gas with equation of state (18.7.11) is $-\frac{1}{2\rho^2}$, whence (18.7.12) may be regarded as the Bernoulli equation, akin to (18.2.14), albeit with a "body force" $\frac{1}{2}\operatorname{grad}\kappa$.

To complete the analogy between isometric immersion and gas flow, we derive the balance laws for mass and vorticity induced by (18.7.9) and (18.7.12). By combining these equations one gets

$$\begin{cases} u[\partial_x(\rho u) + \partial_y(\rho v)] - \rho v[\partial_x v - \partial_y u] = \frac{1}{2}\rho\partial_x\kappa + X \\ v[\partial_x(\rho u) + \partial_y(\rho v)] + \rho u[\partial_x v - \partial_y u] = \frac{1}{2}\rho\partial_y\kappa + Y, \end{cases} \tag{18.7.13}$$

whence

$$\partial_x(\rho u) + \partial_y(\rho v) = \frac{\rho u}{2q^2}\partial_x\kappa + \frac{\rho v}{2q^2}\partial_y\kappa + \frac{u}{q^2}X + \frac{v}{q^2}Y, \tag{18.7.14}$$

$$\partial_x v - \partial_y u = -\frac{v}{2q^2}\partial_x\kappa + \frac{u}{2q^2}\partial_y\kappa - \frac{v}{\rho q^2}X + \frac{u}{\rho q^2}Y, \tag{18.7.15}$$

to be compared with $(18.2.1)_2$ and (18.2.12).

Notice that the transformation $(\rho,u,v) \mapsto (L,M,N)$ is invertible:

$$u^2 = -p(N-p), \quad v^2 = -p(L-p), \quad M^2 = (N-p)(L-p). \tag{18.7.16}$$

Indeed, the third equality in (18.7.16) determines p, and thereby also $\rho = -1/p$, as functions of (L,M,N), and subsequently the first two equalities in (18.7.16) express u and v likewise as functions of (L,M,N).

The squared sonic speed in a gas with equation of state (18.7.11) is $c^2(\rho) = 1/\rho^2$, so that the flow is subsonic where $\kappa > 0$ and supersonic where $\kappa < 0$. By (18.7.11) and (18.7.12),

$$\rho = \frac{1}{\sqrt{q^2+\kappa}}, \qquad p = -\sqrt{q^2+\kappa}. \tag{18.7.17}$$

In the sequel, we shall be tacitly assuming that ρ and p, as functions of q, have been substituted into (18.7.9), (18.7.14) and (18.7.15), so these equations now involve just two unknowns, namely u and v.

In the realm of regular isometric immersions, (18.7.9) is equivalent to (18.7.14), (18.7.15), and either of these systems may be employed for determining the immersion. However, this is not the case for singular isometric immersions, in which (18.7.9) still hold, but (18.7.14) and (18.7.15) are not necessarily valid. The above should be compared and contrasted with the setting, in Section 18.2, for isentropic steady irrotational planar gas flow, where all of (18.2.1), (18.2.12) and (18.2.14) are satisfied in smooth flows, whereas, by common practice, $(18.2.1)_1$, (18.2.12) and (18.2.14) are retained but $(18.2.1)_{2,3}$ are abandoned, for flows containing shocks.

The system (18.7.9) – and thus its equivalent system (18.7.14), (18.7.15) – is elliptic when the flow is subsonic, i.e. $\kappa > 0$, hyperbolic when the flow is supersonic, i.e. $\kappa < 0$, and of mixed elliptic-hyperbolic type when the Gauss curvature changes sign in Ω.

Here we focus on the hyperbolic case, assuming henceforth that $\kappa < 0$, and in particular, without essential loss of generality, that $\kappa = -1$ (in the general case where κ is not constant, L, M and N must be renormalized as $L/\sqrt{-\kappa}$, $M/\sqrt{-\kappa}$ and $N/\sqrt{-\kappa}$).

What follows is a sketchy outline of a program aiming at constructing singular isometric immersions for the case Ω is $\{(x,y)\,:\, -\infty < x < \infty,\ 0 < y < \infty\}$. The details are found in the literature cited in Section 18.9.

Viewing y as the "time" variable, the task is to establish the existence of an $L^\infty(\Omega)$ weak solution (u,v) to the Cauchy problem for the hyperbolic system (18.7.9), with prescribed initial data on the x-axis. To that end, we employ the vanishing viscosity method, by considering the parabolic system

$$
(18.7.18)\qquad \begin{cases} \partial_x(\rho u^2+p)+\partial_y(\rho uv)=\varepsilon\partial_y^2(\rho u)+X \\[2ex] \partial_x(\rho uv)+\partial_y(\rho v^2+p)=\varepsilon\partial_y^2(\rho v)+Y, \end{cases}
$$

with ε a positive parameter that eventually goes to zero. The aim is to show that, under the prescribed initial data, the Cauchy problem for (18.7.18) possesses a classical solution $(u_\varepsilon, v_\varepsilon)$ on Ω, which is bounded in $L^\infty(\Omega)$, uniformly for $\varepsilon > 0$. Once this is achieved, $(u_\varepsilon, v_\varepsilon)$ generates, through (18.7.7) and (18.7.8), functions $(L_\varepsilon, M_\varepsilon, N_\varepsilon)$, likewise bounded in $L^\infty(\Omega)$, uniformly for $\varepsilon > 0$. One may thus extract a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0$, as $n \to \infty$, such that $\{(L_{\varepsilon_n}, M_{\varepsilon_n}, N_{\varepsilon_n})\}$ converges in $L^\infty(\Omega)$ weak* to functions (L,M,N) on Ω. By retracing the steps taken earlier for establishing, with the help of the div-curl lemma, the sequential compactness, in L^∞ weak*, of solutions to the system (18.7.6), (18.7.7), one readily shows that (L,M,N) satisfies (18.7.6), (18.7.7) and thereby induces the desired singular isometric immersion of the manifold into $\mathbb{R}^3$. Finally, in order to pursue the analogy between isometric immersion and gas flow to its completion, one derives from (L,M,N), via (18.7.16), $L^\infty(\Omega)$ functions (u,v) that solve the Cauchy problem for the system (18.7.9).

Clearly, the success of the program outlined above hinges on showing that solutions to the Cauchy problem for the parabolic system (18.7.18) are bounded uniformly in $\varepsilon > 0$. It turns out that it is more efficient to address this question in the context of the equivalent system

$$\partial_x(\rho u) + \partial_y(\rho v) = \varepsilon \frac{u}{q^2}\partial_y^2(\rho u) + \varepsilon \frac{v}{q^2}\partial_y^2(\rho v) + \frac{u}{q^2}X + \frac{v}{q^2}Y, \tag{18.7.19}$$

$$\partial_x v - \partial_y u = -\varepsilon \frac{v}{\rho q^2}\partial_y^2(\rho u) + \varepsilon \frac{u}{\rho q^2}\partial_y^2(\rho v) - \frac{v}{\rho q^2}X + \frac{u}{\rho q^2}Y, \tag{18.7.20}$$

which is obtained from (18.7.18) by retracing the steps taken above for deriving (18.7.14) and (18.7.15) from (18.7.9). The construction of solutions that are bounded uniformly in $\varepsilon > 0$ is achieved by locating a bounded invariant region for the system, independent of ε, and selecting initial data residing in that region.

The expectation is that invariant regions for the parabolic system (18.7.19), (18.7.20) will be bordered by level curves of Riemann invariants of the parent hyperbolic system (18.7.14), (18.7.15). After a lengthy calculation, in polar coordinates $u = q\cos\theta$, $v = q\sin\theta$, one shows that

$$z = \theta - \arccos\left(\frac{1}{q}\right), \qquad w = \theta + \arccos\left(\frac{1}{q}\right) \tag{18.7.21}$$

are Riemann invariants of (18.7.14), (18.7.15). In particular,

$$\begin{cases} u\cos z + v\sin z = 1 \\ u\cos w + v\sin w = 1, \end{cases} \tag{18.7.22}$$

which imply that the level curves of both z and w are straight lines on the u-v plane, so that prospective invariant regions for the system (18.7.19), (18.7.20) should be diamond-shaped. Indeed, a technical construction of invariant regions with the above specifications is carried out in the references cited in Section 18.9, under certain assumptions on the metric $[g_{ij}]$, which apply, for instance, to the *catenoid*.

18.8 Cavitation in Elastodynamics

This section serves a dual purpose: it demonstrates that unbounded weak solutions to hyperbolic systems of balance laws may exhibit a different kind of nonuniqueness than what was encountered in earlier chapters of this book; and also shows that the pathology attributable to the presence of vacuum is not peculiar to gas dynamics, as it extends to elastodynamics of solids.

We consider the Cauchy problem for the system (3.3.19) of isentropic thermoelasticity, in the absence of body force, $b = 0$, with initial data

$$F(x,0) = \lambda I, \qquad v(x,0) = 0, \qquad x \in \mathbb{R}^3, \tag{18.8.1}$$

where λ is a positive constant. It admits the steady solution $F(x,t) = \lambda I$, $v(x,t) = 0$, for $x \in \mathbb{R}^3$, $t \geq 0$, associated with the placement $\chi(x,t) = \lambda x$ of the elastic body. Furthermore, Theorem 5.3.3 asserts that when the Piola-Kirchhoff stress S derives, via (3.3.20), from a rank-one convex internal energy function ε, the above solution is unique within the class of weak solutions with small local oscillation that satisfy the entropy inequality (3.3.21). However, as we shall see here, if the restriction that weak solutions must be L^∞ functions is lifted, there exist additional solutions satisfying the entropy inequality.

We consider the case where the elastic medium is an isotropic solid. The new solutions to the Cauchy problem will be induced by self-similar motions in the form

$$\chi(x,t) = \frac{\phi(s)}{s}x, \qquad s = \frac{|x|}{t}, \tag{18.8.2}$$

where ϕ is a Lipschitz function on $[0,\infty)$, which is smooth on the intervals $[0,\sigma)$ and (σ,∞), for some $\sigma > 0$, and

$$\phi(s) > 0, \quad \dot{\phi}(s) > 0, \quad \ddot{\phi}(s) > 0, \qquad 0 < s < \sigma, \tag{18.8.3}$$

$$\dot{\phi}(\sigma-) = \mu < \lambda, \tag{18.8.4}$$

$$\phi(s) = \lambda s, \qquad \sigma \leq s < \infty. \tag{18.8.5}$$

Thus, in the Lagrangian realization of χ, the body remains in a rest state in the region $|x| > \sigma t$, which is bordered by a precursor spherical shock with radius σt growing linearly in time. Behind the shock, in the region $0 < |x| < \sigma t$, the material undergoes a smooth deformation. However, along the line $x = 0$ there is a singularity analogous to the delta shock encountered in Section 9.6. In the Eulerian realization of χ, the singular line represents a spherical cavity, which opens at the origin at $t = 0$, and expands as its radius $\phi(0)t$ grows linearly in time. We require that the cavity emerge spontaneously and be self-equilibrated, so that the free boundary of the body is stress-free: the radial component of the Cauchy stress vanishes on it.

The deformation gradient field and the velocity field generated by the motion (18.8.2) are

$$F = \frac{\phi(s)}{s}I - \left[\frac{\phi(s)}{s} - \dot{\phi}(s)\right]\frac{1}{|x|^2}\,xx^\top, \tag{18.8.6}$$

$$v = \left[\frac{\phi(s)}{s} - \dot{\phi}(s)\right]\frac{1}{t}x. \tag{18.8.7}$$

The aim is to construct ϕ such that (18.8.6), (18.8.7) provide a solution to (3.3.19), (18.8.1).

We first note that (F,v) satisfies $(3.3.19)_1$, in the sense of distributions. Indeed, $(3.3.19)_1$ holds pointwise for $0 < |x| < \sigma t$ and for $\sigma t < |x| < \infty$. Furthermore, the jump condition $(3.3.22)_1$ across the surface $|x| = \sigma t$ is satisfied as a direct consequence of the continuity of χ. Finally, the term $1/s$, which is singular at $x = 0$, does not cause any problem, since it is locally integrable.

We now turn to the equation $(3.3.19)_2$, which expresses the balance of momentum. As discussed in Section 2.5, the internal energy ε, at constant entropy, for any isotropic thermoelastic solid is a symmetric function

$$\varepsilon = \Phi(\lambda_1, \lambda_2, \lambda_3) \tag{18.8.8}$$

of the eigenvalues of the right stretch tensor U, defined by (2.1.7). Here F is positive symmetric, so that $U = F$ and

$$\lambda_1 = \dot{\phi}(s), \quad \lambda_2 = \frac{\phi(s)}{s}, \quad \lambda_3 = \frac{\phi(s)}{s}. \tag{18.8.9}$$

For convenience, we employ the notation Φ_i, Φ_{ij}, ... for the functions $\partial\Phi/\partial\lambda_i$, $\partial^2\Phi/\partial\lambda_i\partial\lambda_j$, etc. Furthermore, we denote by $\hat{\Phi}$, $\hat{\Phi}_i$, $\hat{\Phi}_{ij}$ the functions of s resulting from taking the composition of Φ, Φ_i, Φ_{ij} with (18.8.9).

From (3.3.20) and (18.8.8), after a lengthy calculation,

$$S = \hat{\Phi}_1(s)\left[\frac{1}{|x|^2}\, xx^\top\right] + \hat{\Phi}_2(s)\left[I - \frac{1}{|x|^2}\, xx^\top\right]. \tag{18.8.10}$$

Then $(3.3.19)_2$ will hold in the sense of distributions so long as it is satisfied pointwise for $0 < |x| < \sigma t$ and the jump condition $(3.3.22)_2$ holds across the surface $|x| = \sigma t$. By virtue of (18.8.7) and (18.8.10), these conditions here reduce to

$$\frac{1}{2}\left[\hat{\Phi}_{11}(s) - s^2\right] s\ddot{\phi}(s) = \hat{\Phi}_2(s) - \hat{\Phi}_1(s) + \left[\frac{\phi(s)}{s} - \dot{\phi}(s)\right]\hat{\Phi}_{12}(s), \tag{18.8.11}$$

for $0 < s < \sigma$, and

$$\sigma^2 = \frac{\Phi_1(\lambda,\lambda,\lambda) - \Phi_1(\mu,\lambda,\lambda)}{\lambda - \mu}. \tag{18.8.12}$$

We will construct ϕ with the above specifications under the assumption

$$\Phi(\lambda_1,\lambda_2,\lambda_3) = \sum_{i=1}^{3} g(\lambda_i) + h(\lambda_1\lambda_2\lambda_3), \tag{18.8.13}$$

with g and h smooth functions on $(0,\infty)$ satisfying

$$g''(p) > 0, \qquad g'''(p) < 0, \qquad 0 < p < \infty, \tag{18.8.14}$$

$$h''(b) > 0, \qquad h'''(b) < 0, \qquad 0 < b < \infty, \tag{18.8.15}$$

$$\lim_{p\to\infty} \frac{g'(p)}{p} = \gamma \geq 0, \tag{18.8.16}$$

$$h'(b) \to -\infty, \quad \text{as } b \to 0+, \qquad h'(b) \to \infty, \quad \text{as } b \to \infty, \tag{18.8.17}$$

whence it follows

$$\Phi_{11}(p,q,q) > 0, \quad \Phi_{111}(p,q,q) < 0, \qquad 0 < p < \infty, \quad 0 < q < \infty. \tag{18.8.18}$$

The above conditions are physically reasonable, modeling isotropic elastic solids with polyconvex strain energy functions exhibiting material softening.

The stretching parameter λ will not be prescribed in advance but shall be determined in the course of the construction.

We seek a solution to the ordinary differential equation (18.8.11), with initial conditions

$$\phi(0) = \phi_0, \qquad \lim_{s \to 0+} \dot{\phi}(s) \left[\frac{\phi(s)}{s} \right]^2 = a, \tag{18.8.19}$$

where ϕ_0 is a free positive parameter, while a is fixed in such a way that the radial component of the Cauchy stress vanishes on the free boundary. A calculation combining (2.3.6) with (18.8.10) yields that the above requirement is satisfied when $h'(a) = 0$.

Elementary, though hard, analysis, recorded in the references cited in Section 18.9, establishes that (18.8.11), (18.8.19) admit a unique solution ϕ, on a maximal interval $[0, \bar{s})$, such that

$$\phi(s) > 0, \quad \dot{\phi}(s) > 0, \quad \ddot{\phi}(s) > 0, \quad \frac{\phi(s)}{s} - \dot{\phi}(s) > 0, \qquad 0 < s < \bar{s}. \tag{18.8.20}$$

Furthermore, there is a unique σ in the interval $(0, \bar{s})$ such that (18.8.12) holds with $\lambda = \phi(\sigma)/\sigma$, $\mu = \dot{\phi}(\sigma)$. This completes the construction of the motion χ with the aforementioned specifications.

The analysis also shows that by varying the parameter ϕ_0 one may attain any value of the stretching parameter λ above a critical value λ_{cr}.

We now turn to the admissibility of the constructed solution. By virtue of (18.8.18), and since $\mu < \lambda$,

$$\Phi_{11}(\lambda,\lambda,\lambda) < \sigma^2 < \Phi_{11}(\mu,\lambda,\lambda), \tag{18.8.21}$$

which verifies that the shock satisfies the Lax E-condition.

Next, we check the entropy shock admissibility condition (3.3.23), in its equivalent form (3.3.24). A straightforward calculation, using (18.8.6), (18.8.8) and (18.8.10), yields

$$[\![\varepsilon]\!] - \operatorname{tr}\left(\frac{1}{2}(S_+ + S_-)^\top [\![F]\!] \right) \tag{18.8.22}$$

$$= \Phi(\lambda,\lambda,\lambda) - \Phi(\mu,\lambda,\lambda) - \frac{1}{2}(\lambda - \mu)[\Phi_1(\lambda,\lambda,\lambda) + \Phi_1(\mu,\lambda,\lambda)].$$

The right-hand side of the above equation is a function of μ that vanishes, together with its first derivative, at $\mu = \lambda$, while its second derivative $-\frac{1}{2}(\lambda-\mu)\Phi_{111}(\mu,\lambda,\lambda)$ is nonnegative, on account of (18.8.18). We thus conclude that the shock also satisfies the entropy admissibility condition (3.3.24).

Since both the steady solution and the solution with cavitation satisfy the Lax E-condition and the entropy condition, the issue of nonuniqueness raised by the existence of this new solution to the Cauchy problem (3.3.19), (18.8.1) cannot be resolved by appealing to the standard admissibility criteria. We thus resort to comparing the rates of entropy production by these two solutions, in the spirit of the discussion in Section 9.7.

The relative entropy of the new solution (F,v) with respect to the steady solution $(\lambda I, 0)$ reads

$$H = \varepsilon(F) + \frac{1}{2}|v|^2 - \varepsilon(\lambda I) - \operatorname{tr}\left(S(\lambda I)(F-\lambda I)^\top\right). \tag{18.8.23}$$

Therefore, the relative entropy rate is

$$\dot{\mathscr{H}}(t) = \int_{|x|<\sigma t} \partial_t H(x,t)dx + \sigma \int_{|x|=\sigma t} H(x,t)da. \tag{18.8.24}$$

After a straightforward calculation,

$$\begin{aligned} \dot{\mathscr{H}}(t) = -4\pi\sigma^3 t^2 \{ &\Phi(\lambda,\lambda,\lambda) - \Phi(\mu,\lambda,\lambda) \\ &-\frac{1}{2}(\lambda-\mu)[\Phi_1(\lambda,\lambda,\lambda)+\Phi_1(\mu,\lambda,\lambda)]\}, \end{aligned} \tag{18.8.25}$$

which is negative (compare with (18.8.22)). Thus the entropy rate criterion favors the solution with cavitation over the steady solution. Of course, this argument does not single out a unique solution, because one may generate infinitely many solutions in which cavities open simultaneously at different points of the body. Thus one may construct solutions with arbitrarily small entropy rate.

18.9 Notes

Early and more recent contributions to the solution of the Riemann problem for scalar conservation laws in two space dimensions are found in Guckenheimer [2], Wagner [1], Lindquist [1], Zhang and Zhang [1], Tong Zhang and Yuxi Zheng [1], Chen, Li and Tan [1], and Xiaozhou Yang [1]. A detailed treatment, providing a complete classification of solutions, is contained in the monographs by Chang and Hsiao [3], Li, Zhang and Yang [1], and Yuxi Zheng [2].

Gas flow past ducts, nozzles or obstacles of various shapes, and its applications to technology, have been studied for a long time by aerodynamicists, using analytical, numerical or experimental techniques. Background information, including the setting of the equations of steady or self-similar planar irrotational gas flow, recorded here in Sections 18.2, 18.3, and basic bibliography, can be found in any of the standard texts on theoretical aerodynamics, such as Courant and Friedrichs [1], von Mises [1], Hayes and Probstein [1], and Ferrari and Tricomi [1]. A very useful and reliable source is the survey article by Serre [24]. See also the commentary by Serre [28] on von Neumann [5].

Rigorous analysis in the 1950s originally focused on subsonic irrotational flow, where the equation satisfied by the pseudopotential is of elliptic type; see Shiffman [1] and Bers [1]. Morawetz [1,2,3] pioneered research on transonic flow, in which case the equation is of mixed type, elliptic-hyperbolic, degenerating on the sonic interface. After a hiatus of thirty years of relative inactivity, there has been a revival of interest in this area, buoyed up by advances in the theory of partial differential equations of elliptic or hyperbolic type. Tai-Ping Liu [29] provides an interesting historical account. For the state of the art in this ongoing research program, the reader should consult the forthcoming monograph by Chen and Feldman [9]. For an overview and perspectives, see Gui-Qiang Chen [12].

Because of the serious analytical difficulties encountered in treating the general Euler equations, a substantial part of research is conducted in the simpler setting of irrotational flow, or in the context of simplified systems that retain the salient features of the actual Euler equations. Examples are the *pressure gradient system*, derived from the Euler equations by deleting the nonlinear convective terms[1]; the *pressureless Euler equations*, in which the convective terms are retained but the pressure is set equal to zero; and the so-called *small disturbance equations*, obtained through heuristic asymptotic arguments. For details and a comprehensive bibliography, see the book by Yuxi Zheng [2] and the survey article by Gui-Qiang Chen [11]. Another way of facilitating the analysis is by considering the Chaplygin gas (2.5.23), which renders the system of the Euler equations linearly degenerate, thus simplifying considerably shock interactions and the wave pattern; see Serre [26,29].

Theorem 18.3.1, commonly referred to as the "ellipticity principle", is proved in Elling and Liu [1]. Another interesting maximum principle for the pressure in the subsonic (elliptic) regime is found in Serre [9].

For a different perspective on the transition from the hyperbolic to the elliptic regime, see Serre and Freistühler [1].

Theorem 18.4.1 is taken from Chen, Zhang and Zhu [1]. See also Chen, Xiao and Zhang [1]. Theorem 18.4.2, which provides the solution to the so-called *Prandtl's problem*, is due to Elling and Liu [2], with improvements by Bae, Chen and Feldman [2]. For related results see Gui-Qiang Chen and Beixiang Fang [1], Gui-Qiang Chen and Tian-Hong Li [2], Shuxing Chen [2,3,4,6], Shuxing Chen and Beixiang Fang [1], Shuxing Chen and Dening Li [1,2], Chen, Min and Zhang [1], Chen, Xin and Yin [1], Lien and Liu [1], Schaeffer [2], Xin and Yin [2], and Yongqian Zhang [1,2].

The definitive, theoretical solution to the classical problem of shock collision with a ramp, outlined here in Section 18.6, is given in Chen and Feldman [9]. In particular, this work has confirmed von Neumann's conjecture that regular reflection is possible so long as the slope of the ramp exceeds the detachment angle. For pre-

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[1] In that case the full system of adiabatic (nonisentropic) gas dynamics (3.3.29) must be employed, as the isentropic system (3.3.36) becomes trivial.

liminary results in that direction, see Chen and Feldman [4,6] and Bae, Chen and Feldman [1]. For further discussion in the same setting, see Elling [1,2,3,4], and Sever [13]. For shock collisions on curved ramps, see Chen, Chen and Feldman [1]. For regular reflection in the simpler setting of the pressure gradient system, the small disturbance equations or the Chaplygin gas, see Čanić, Keyfitz and Kim [1,2,3], Keyfitz and Lieberman [1], Jegdić [1], Jegdić, Keyfitz and Čanić [1], Keyfitz [3], Serre [26], and Yuxi Zheng [3,4].

There is ample numerical and experimental evidence that when the ramp is not steep enough to support regular reflection the shock will be detached, forming a pattern of *Mach reflection*; see the book by Ben-Dor [1]. However, a rigorous theoretical treatment of this phenomenon is still lacking. For issues related to Mach reflection, see Čanić, Keyfitz and Kim [4], Chen, Wang and Yang [1], Shuxing Chen [8,9], and Tesdall, Sanders and Keyfitz [1,2].

Transonic flow in nozzles or ducts is considered in Chen, Chen and Feldman [1], Chen, Chen and Song [1], Chen and Feldman [5,7,8], Gui-Qiang Chen and Hairong Yuan [1], Shuxing Chen [10], Shuxing Chen and Hairong Yuan [1], Xie and Xin [1,2,3], Xin and Yin [1,3,4], Bae and Feldman [1], Chen, Deng and Xiang [1], Tianyou Zhang and Yuxi Zheng [1], Du, Weng and Xin [1], Du, Xie and Xin [1], Chunpeng Wang and Zhouping Xin [1], Du, Xin and Yan [1], and Li, Xin and Yin [1,2,3,4,5]. Similarly, multidimensional piston problems are discussed by Chen, Chen, Wang and Wang [1], Shuxing Chen [5], and Chen, Wang and Zhang [1,2].

For diffraction of shocks by corners, see Chen, Deng and Xiang [2].

For global, shock-free solutions, involving rarefaction waves, see Heibig [3], Kim and Song [1], Jiequan Li [1,2], Kyungwoo Song [1], and Yuxi Zheng [5]. The interaction of rarefaction waves is studied in Bang [1], Glimm, Ji, Li, Li, Zhang, Zhang and Zheng [1], Lei and Zheng [1], Jiequan Li and Yuxi Zheng [1,2], Li, Zhang and Zheng [1], Li Yang and Zheng [1], Chen and Zheng [1], and Ji and Zheng [1].

The structure of self-similar small L^∞ perturbations of a steady supersonic two-dimensional gas flow is described in Elling and Roberts [1].

Solutions to Riemann problems for various systems in two space dimensions are constructed in Čanić [1,2], Čanić and Keyfitz [1,2], Chang, Chen and Yang [1], Schulz-Rinne [1], Shuxing Chen [1], Tan and Zhang [1], Yang and Huang [1], Zhang, Li and Zhang [1], Tong Zhang and Yuxi Zheng [3], and Yuxi Zheng [6].

Finally, for construction and stability of transonic shocks and vortex sheets, see Chen and Feldman [1,2,3], Shuxing Chen [7], Chen, Zhang and Zhu [2], Gui-Qiang Chen and Ya-Guang Wang [1], Chen, Kukreja and Yuan [1,2], Chen, Christoforou and Jegdić [1], Keyfitz, Tesdall, Payne and Popivanov [1], Mingjie Li and Yuxi Zheng [1], and Coulombel and Secchi [1,2,3,4].

The curious and interesting connection between the isometric immersion of surfaces in $\mathbb{R}^3$ and irrotational gas dynamics, discussed in Section 18.7, was conceived by M. Slemrod and developed in Chen, Slemrod and Wang [2,3,4]. See also Christoforou [4], and Christoforou and Slemrod [1].

Cavitation in nonlinear elasticity was originally treated by Ball [2], in the realm of statics. The extension of the theory to elastodynamics, outlined here in Section 18.8, was pioneered by Pericak-Spector and Spector [1,2] and further developed by

Miroshnikov and Tzavaras [3]. Interesting arguments on how to overrule the cavitation instability are presented in Giesselmann and Tzavaras [1].

There is extensive literature on self-similar, radially symmetric solutions for the Euler equations of gas dynamics; see Courant and Friedrichs [1], Yuxi Zheng [2], Tong Zhang and Yuxi Zheng [4,5], and Serre [10]. In particular, for solutions manifesting cavitation in fluids, see Wang and Li [1], and Baisheng Yan [1].

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