STRANGE FUNCTIONS IN REAL ANALYSIS

THIRD EDITION

Alexander Kharazishvili

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

CPD CERTIFIED
The CPD Certification Service

STRANGE FUNCTIONS IN REAL ANALYSIS

THIRD EDITION

STRANGE FUNCTIONS IN REAL ANALYSIS

THIRD EDITION

Alexander Kharazishvili

Razmadze Mathematical Institute
Tbilisi, Georgia

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed on acid-free paper
Version Date: 20170812

International Standard Book Number-13: 978-1-4987-7314-0 (Hardback)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Table of Contents

Preface

At the present time, many strange (or singular) objects in various fields of mathematics are known, and no working mathematician is greatly surprised if he meets some objects of this type during his investigations. In connection with strange (singular) objects, classical mathematical analysis must be noticed especially. It is sufficient to recall here the well-known examples of continuous nowhere differentiable real-valued functions, examples of Lebesgue measurable real-valued functions nonintegrable on any nonempty open subinterval of the real line, examples of Lebesgue integrable real-valued functions with everywhere divergent Fourier series, and others.

There is a very powerful technique in modern mathematics by means of which we can obtain various kinds of strange objects. This is the so-called category method based on the celebrated Baire theorem from general topology. Obviously, this theorem plays one of the most important roles in mathematical analysis and its applications. Let us recall that, according to the Baire theorem, in any complete metric space E (also, in any locally compact topological space E) the complement of a first category subset of E is everywhere dense in E, and it often turns out that this complement consists precisely of strange (in a certain sense) elements. Many interesting applications of the category method are presented in the excellent textbook by Oxtoby [202] in which the deep analogy between measure and category is thoroughly discussed as well. In this connection, the monograph by Morgan [192] must also be pointed out where an abstract concept generalizing the notions of measure and category is introduced and investigated in detail.

Unfortunately, the category method does not always work and we sometimes need an essentially different approach to questions concerning the existence of singular objects.

This book is devoted to some strange functions (and point sets) in real analysis and their applications. Those functions can be frequently met during various studies in analysis and play an essential role there, especially as counterexamples to numerous statements that at first sight seem to be very natural but, finally, fail to be true in certain extraordinary situations (see, e.g., [78]). Another important role of strange functions, with respect to given concepts of analysis, is to show that those concepts are, in some sense, not satisfactory and hence have to be revised, generalized, or extended in an appropriate direction. In this context, we may say that strange functions (point sets) stimulate and inspire the development of analysis.

The book deals with a number of important examples and constructions of strange functions. Primarily, we consider strange functions acting from the real line into itself. Notice that many such functions can be obtained by using the category method (for instance, a real-valued continuous function defined on the closed unit interval of the real line, which does not possess a finite derivative at each point of this interval). But, as mentioned above, there are some situations where the classical category method cannot be applied, and thus, in such a case, we have to appeal to an appropriate individual construction.

We begin with functions that can be constructed within the theory

ZF & **DC**,

where **ZF** denotes the Zermelo–Fraenkel set theory without the Axiom of Choice and **DC** denotes a certain weak form of this axiom: the so-called Axiom of Dependent Choices, which is enough for most domains of classical mathematics. Among strange functions whose existence can be established in **ZF** & **DC**, the following ones are of primary interest: Cantor and Peano type functions, semicontinuous functions that are not countably continuous, singular monotone functions, everywhere differentiable nowhere monotone functions, and Jarnik's continuous nowhere approximately differentiable functions.

Then we examine various functions whose constructions need essentially noneffective methods, i.e., they need an uncountable form of the Axiom of Choice: functions nonmeasurable in the Lebesgue sense, functions without the Baire property, functions associated with a Hamel basis of the real line, Sierpiński–Zygmund functions that are discontinuous on each subset of the real line having the cardinality continuum, etc.

Finally, we consider a number of examples of functions whose existence cannot be established without the aid of additional set-theoretical axioms. However, it is demonstrated in the book that the existence of such functions follows from (or is equivalent to) certain widely known set-theoretical hypotheses (e.g., the Continuum Hypothesis).

Among other topics presented in this book and closely connected with strange functions in real analysis, we wish to point out the following ones: the construction (under Martin's Axiom) of absolutely nonmeasurable solutions of the Cauchy functional equation and their application to the measure extension problem raised by Banach; Egorov type theorems on the uniform convergence of sequences of measurable functions; some relationships between the classical Sierpiński partition of the Euclidean plane and Fubini type theorems; the existence of a function on a second category subset of a topological space, which cannot be extended to a function defined on the whole space and possessing the Baire property; sup-measurable and weakly sup-measurable functions with their applications in the theory of ordinary differential equations.

In Chapter 22 of our book, we consider the family of all continuous nondifferentiable functions from the positions of category and measure. We present one general approach illuminating the basic reasons which necessarily imply that the above-mentioned family of functions has to be large in the sense of category or measure. Notice that, in connection with continuous nondifferentiable functions, a short scheme for constructing the classical Wiener measure is discussed in this chapter, too, and some simple but useful statements from the general theory of stochastic (random) processes are demonstrated.

This book is based on the course of lectures repeatedly given by the author at I. Vekua Institute of Applied Mathematics of Tbilisi State University and entitled

Some Pathological Functions in Real Analysis.

These lectures (their role is played by the corresponding chapters of the book) are, in fact, mutually independent from the logical point of view but are strictly related from the point of view of the topics discussed and the methods applied (such as purely set-theoretical arguments and constructions, measure-theoretical methods, the Baire category method, and so forth).

The material presented in the book is essentially self-contained and, consequently, is accessible to a wide audience of mathematicians, including graduate and postgraduate students. For the reader's convenience, the Introduction (i.e., Chapter 0) gives an overview of the subject. Here some preliminary notions and facts are presented that are useful in our further considerations. The reader can ignore this introductory chapter, returning to it if the need arises. In this connection, the standard graduate-level textbooks and well-known monographs (for instance, [26], [85], [86], [92], [97], [105], [146], [149], [153], [194], [196], [202], [212], [225]) should be pointed out containing all auxiliary notions and facts from set theory, general topology, classical descriptive set theory, integration theory, and real analysis.

In Chapter 0 we begin with basic set-theoretical concepts, such as binary relations of special type (namely, equivalence relations, orderings, and functional graphs), ordinal numbers and cardinal numbers, the Axiom of Choice and the Zorn Lemma, some weak forms of the Axiom of Choice (especially, the countable form of **AC** and the Axiom of Dependent Choices), the Continuum Hypothesis, the Generalized Continuum Hypothesis, and Martin's Axiom as a set-theoretical assertion which is essentially weaker than the Continuum Hypothesis but rather helpful in various constructions of set theory, topology, measure theory, and real analysis (cf. [18], [146]).

Then we briefly present some basic concepts of general topology and classical descriptive set theory, such as the notion of a first category set in a topological space, the Baire property (the Baire property in the restricted sense) of subsets of a topological space, the notion of a Polish space, Borel sets in a topological space, analytic (Suslin) subsets of a topological space, and the projective hierarchy of Luzin, which treats the Borel and analytic sets as the first two steps of this hierarchy. It is also stressed that Borel and analytic sets have a nice descriptive structure but this feature fails to be true for general projective sets (because, in certain models of set theory, there exist projective subsets of the real line that are not Lebesgue measurable and do not have the Baire property).

The final part of Chapter 0 is devoted to some classical facts and statements from real analysis. Namely, we recall here the notion of a real-valued lower (upper) semicontinuous function and demonstrate basic properties of such functions, formulate and prove the fundamental Vitali covering theorem, introduce the notion of a density point for a Lebesgue measurable set, and present the Lebesgue theorem on density points as a consequence of the above-mentioned Vitali theorem. In addition, we give a short proof of the existence of a real-valued continuous nowhere differentiable function, starting with the well-known Kuratowski lemma on closed projections. Let us emphasize once more that the problem of the existence of real-valued continuous nondifferentiable functions, with respect to various concepts of generalized derivative, is one of the central questions in this book. We develop this topic gradually and, as mentioned earlier, investigate the question from different points of view. However, we hope that the reader will be able to see that the main kernel is contained in purely logical and set-theoretical aspects of the question.

The third edition of this book differs from the first and second ones. The text of the manuscript is essentially revised and a lot of details are improved in order to reach a more clear presentation of the material. As before, many exercises are included in the text. Notice that relatively difficult exercises are marked by asterisks and, quite often, are provided with necessary hints or explanations. They contain valuable information about the topics discussed in the book and show deep relationships between them.

Moreover, five new chapters and two appendices are added. These new chapters highlight the following themes: a characterization of constant functions on the real line $\mathbf{R}$ via Dini's derived numbers, the cardinality of first Baire class within **ZF** set theory, connections between summation methods and Lebesgue nonmeasurable functions on $\mathbf{R}$, the distinction between the Riemann and Lebesgue iterated integrals, a construction (under Martin's Axiom) of an absolute null subset of the plane $\mathbf{R}^2$ whose orthogonal projections on all straight lines lying in $\mathbf{R}^2$ are absolutely nonmeasurable.

Appendix 1 is devoted to Luzin's theorem on the existence of a primitive for any real-valued Lebesgue measurable function on $\mathbf{R}$ that is finite almost everywhere.

In Appendix 2 a construction of a Banach limit on the family of all bounded real sequences is presented by starting with a nontrivial ultrafilter in the power set $\mathcal{P}(\mathbf{N})$ of the set $\mathbf{N}$ of all natural numbers.

The Bibliography is significantly expanded by adding some relatively recent works devoted to strange functions in real analysis and their applications.

A. B. Kharazishvili

Chapter 0
Introduction: Basic concepts

In this introductory chapter we would like to fix the notation and present some auxiliary facts from set theory, general topology, classical descriptive set theory, measure theory, and real analysis. We shall systematically utilize those facts in our further considerations.

The symbol **ZF** denotes the Zermelo–Fraenkel set theory, which is one of the most important formal systems of axioms for the whole of modern mathematics (in this connection, see [97], [146], and [153]; cf. also [26]). The basic notions of the Zermelo–Fraenkel system are sets and the membership relation $\in$ between them. Of course, the system **ZF** consists of several axioms, which formalize various properties of sets in terms of $\in$. We do not present here a list of these axioms and, actually, we shall work in the so-called naive set theory (cf. [86]).

The symbol **ZFC** denotes the Zermelo–Fraenkel theory with the Axiom of Choice. In other words, **ZFC** is the theory **ZF** & **AC**, where **AC** denotes, as usual, the Axiom of Choice (the precise formulation of **AC** will be given later with several of its equivalents).

At the present time, it is widely known that **ZFC** theory is a basis of modern mathematics, i.e., almost all fields of mathematics can be developed by starting with **ZFC**. The Axiom of Choice is a very powerful set-theoretical assertion which implies many extraordinary and interesting consequences. Sometimes, in order to get a required result, we do not need the whole power of the Axiom of Choice. In such cases, it is sufficient to apply various weak forms (versions, variants) of **AC**. Some of these forms will be discussed below.

If x and X are any two sets, then the relation $x \in X$ means that x belongs to X. In this situation, we also say that x is an element of X.

One of the axioms of set theory implies that any set y is an element of some set Y (certainly, depending on y). Thus, the notion of an element is equivalent to the notion of a set.

The relation $X \subset Y$ means that a set X is a subset of a set Y, i.e., each element of X is also an element of Y.

If $X \subset Y$ and $X \neq Y$, then we say that X is a proper subset of Y.

If $R(x)$ is a relation depending on an element x (or, in other words, $R(x)$ is a property of an element x), then the symbol $\{x : R(x)\}$ denotes the set (the family, the class) of all those elements x for which the relation $R(x)$ holds. In our further considerations we always suppose that $R(x)$ is such that the corresponding set $\{x : R(x)\}$ does exist. For example, a certain consequence of the axioms of **ZF** theory states that there always exists a set of the type

$$\{x \ : \ x \in X \ \& \ S(x)\},$$

where X is an arbitrarily given set and $S(x)$ is an arbitrary relation. In this case we write $\{x \in X : S(x)\}$ instead of $\{x : x \in X \ \& \ S(x)\}$. Also, if we have two relations $R_1(x)$ and $R_2(x)$, then we write $\{x : R_1(x), \ R_2(x)\}$ instead of $\{x : R_1(x) \ \& \ R_2(x)\}$.

The symbol $\emptyset$ denotes, as usual, the empty set, i.e., $\emptyset = \{x : x \neq x\}$.

If X is any set, then the symbol $\mathcal{P}(X)$ denotes the family of all subsets of X, i.e., we have $\mathcal{P}(X) = \{Y : Y \subset X\}$.

Notice that the existence of $\mathcal{P}(X)$ is stated by one of the axioms of **ZF** theory (see, e.g., [97], [146], [153]). The set $\mathcal{P}(X)$ is also called the power set of a given set X.

If x and y are any two elements, then the set $\{x, y\}$ is defined by the equality

$$\{x, y\} = \{z : z = x \ \vee \ z = y\}$$

and is called the unordered pair consisting of x and y. In addition, if $x = y$, then the short notation $\{x\}$ is used instead of $\{x, y\}$ and $\{x\}$ is called the singleton whose unique element coincides with x.

The set $(x, y) = \{\{x\}, \{x, y\}\}$ is called the ordered pair (or, simply, the pair) consisting of x and y. The reader can easily check that the implication

$$((x, y) = (x', y')) \Rightarrow (x = x' \ \& \ y = y')$$

is valid for all elements $x, \ y, \ x', \ y'$.

Let X and Y be any two sets. Then, as usual,

$X \cup Y$ denotes the union of X and Y;

$X \cap Y$ denotes the intersection of X and Y;

$X \setminus Y$ denotes the difference of X and Y;

$X \triangle Y$ denotes the symmetric difference of X and Y, i.e.,

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X).$$

We also put $X \times Y = \{(x, y) : x \in X,\ y \in Y\}$. The set $X \times Y$ is called the Cartesian product of two given sets X and Y. In a similar way, by using recursion, one can define the Cartesian product $X_1 \times X_2 \times\ \dots\ \times X_n$ of a finite family $\{X_1,\ X_2,\ \dots\ ,\ X_n\}$ of arbitrary sets.

If X is a set, then the symbol $\mathrm{card}(X)$ denotes the cardinality of X. Quite often, $\mathrm{card}(X)$ is also called the cardinal number of X.

ω $(= \omega_0)$ is the first infinite cardinal (ordinal) number. In fact, ω is the cardinality of the set $\mathbf{N} = \{0, 1, 2, ..., n, ...\}$ of all natural numbers. Sometimes, it is convenient to identify the sets ω and $\mathbf{N}$ and we always assume such an identification in our further considerations (cf. [97], [146], [153]).

A set X is finite if $\mathrm{card}(X) < \omega$; accordingly, X is infinite if $\mathrm{card}(X) \geq \omega$.

A set Y is (at most) countable if $\mathrm{card}(Y) \leq \omega$; accordingly, Y is uncountable if $\mathrm{card}(Y) > \omega$.

For an arbitrary set E, we put:

$[E]^{<\omega}$ = the family of all finite subsets of E;

$[E]^{\leq\omega}$ = the family of all countable subsets of E.

ω_1 is the first uncountable cardinal (ordinal) number. Notice that ω_1 is often identified with the set of all countable ordinal numbers (countable ordinals).

Various ordinal numbers (ordinals) are denoted by symbols α, β, γ, ξ, ζ, η, ...

Let α be an ordinal number. We say that α is a limit ordinal if

$$\alpha = \sup\{\beta\ :\ \beta < \alpha\}.$$

The cofinality of a limit ordinal α is the smallest ordinal ξ such that there exists a family $\{\alpha_\zeta : \zeta < \xi\}$ of ordinals satisfying the relations

$$(\forall \zeta < \xi)(\alpha_\zeta < \alpha), \qquad \alpha = \sup\{\alpha_\zeta\ :\ \zeta < \xi\}.$$

The cofinality of a limit ordinal α is denoted by the symbol $\mathrm{cf}(\alpha)$.

Clearly, we have the inequality $\mathrm{cf}(\alpha) \leq \alpha$ for all limit ordinals α.

A limit ordinal number α is called a regular ordinal if $\mathrm{cf}(\alpha) = \alpha$.

Accordingly, a limit ordinal number α is called a singular ordinal if $\mathrm{cf}(\alpha) < \alpha$.

Starting with the definitions of regular and singular limit ordinals, one can define, in the usual manner, regular infinite cardinals and singular infinite cardinals. For example, ω and ω_1 are regular ordinals (cardinals) and ω_ω is a singular ordinal (cardinal).

As mentioned above, in many considerations it is convenient to identify every ordinal α with the set of all those ordinals which are strictly less than α. Such an approach to the theory of ordinal numbers is due to von Neumann (see, e.g., [97], [146]). It is also convenient to identify every cardinal κ with the smallest ordinal number α such that $\mathrm{card}(\alpha) = \kappa$.

If κ is an arbitrary infinite cardinal number, then the symbol κ^+ denotes the smallest cardinal among all those cardinals which are strictly greater than κ. For example, we have $\omega^+ = \omega_1$, $\omega_2 = (\omega_1)^+$, ...

The symbol $\mathbf{Z}$ denotes the set of all integers.

The symbol $\mathbf{Q}$ denotes the set of all rational numbers.

The symbol $\mathbf{R}$ denotes the set of all real numbers.

If the set $\mathbf{R}$ is equipped with its standard structures (order structure, algebraic structure, topological structure), then $\mathbf{R}$ is usually called the real line. This object is basic for classical mathematical analysis. The so-called extended real line $\mathbf{R}^* = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$ is also frequently used in different topics of real analysis.

The symbol $\mathbf{c}$ denotes the cardinality of the continuum, i.e., we have

$$\mathbf{c} = 2^\omega = \mathrm{card}(\mathbf{R}).$$

The Continuum Hypothesis (denoted **CH**) is the assertion $\mathbf{c} = \omega_1$.

The Generalized Continuum Hypothesis (abbreviation **GCH**) asserts that $2^{\omega_\alpha} = \omega_{\alpha+1}$ for all ordinals α.

At the present time, it is well known that **ZFC** theory is consistent if and only if both theories

$$\mathbf{ZFC}\ \&\ (\text{the Continuum Hypothesis}),$$

$$\mathbf{ZFC}\ \&\ (\text{the negation of the Continuum Hypothesis})$$

are consistent (see, e.g., [97], [146]). Moreover, it is also well known that **ZFC** theory is consistent if and only if the theory

$$\mathbf{ZFC}\ \&\ (\mathbf{c} \text{ is a singular cardinal})$$

is consistent. More precisely, it was established that, for an infinite cardinal number ω_α satisfying the relation $\mathrm{cf}(\omega_\alpha) > \omega$, there exists a model of **ZFC** in which we have the equality $\mathbf{c} = \omega_\alpha$. Actually, if one starts with an arbitrary countable transitive model of **ZFC** (strictly speaking, of a relevant fragment of **ZFC**) satisfying the Generalized Continuum Hypothesis, then the above-mentioned equality is true in a certain Cohen model of **ZFC** extending the original model (for details, see [97], [146]).

The Generalized Continuum Hypothesis holds in a special model of set theory, first constructed by Gödel. This model is called the Constructible

Universe of Gödel and usually denoted by **L**. Actually, the Constructible Universe **L** is a subclass of the well-known von Neumann Universe, which is a natural model of set theory. Various facts and statements concerning **L** are discussed in [97] and [146] (see also [18]). It is reasonable to notice here that, in **L**, some effectively defined subsets of the real line are bad from the point of view of Lebesgue measure and Baire property (i.e., they are not measurable in the Lebesgue sense and do not possess the Baire property).

Let n be a fixed natural number. The symbol $\mathbf{R}^n$ denotes, as usual, the n-dimensional Euclidean space. If $n = 0$, then $\mathbf{R}^n$ is the one-element set consisting of zero only. If $n > 0$, then it is sometimes convenient to consider $\mathbf{R}^n$ as a vector space V over the field $\mathbf{Q}$ of all rational numbers. According to a fundamental statement of the theory of vector spaces (over arbitrary fields), there exists a basis in the space V (see, e.g., [52] where more general assertions are discussed for universal algebras). This basis is usually called a Hamel basis of V. Obviously, the cardinality of any Hamel basis of V is equal to the cardinality of the continuum. Notice also that the existence of a Hamel basis of V cannot be established without the aid of uncountable forms of the Axiom of Choice, because the existence of such a basis immediately implies the existence of a subset of $\mathbf{R}$ nonmeasurable with respect to the standard Lebesgue measure λ given on $\mathbf{R}$. Some nontrivial applications of Hamel bases will be discussed in Chapter 11 of this book.

Let X and Y be any two sets. A binary relation between X and Y is an arbitrary subset G of the Cartesian product of X and Y, i.e., $G \subset X \times Y$. In particular, if we have $X = Y$, then we say that G is a binary relation on the base (ground) set X. For any binary relation $G \subset X \times Y$, we put

$$\mathrm{pr}_1(G) = \{x \ : \ (\exists y)((x, y) \in G)\}, \qquad \mathrm{pr}_2(G) = \{y \ : \ (\exists x)((x, y) \in G)\}.$$

It is clear that $G \subset \mathrm{pr}_1(G) \times \mathrm{pr}_2(G)$.

The Axiom of Dependent Choices is the following set-theoretical statement:

If G is a binary relation on a nonempty set X and, for each element $x \in X$, there exists an element $y \in X$ such that $(x, y) \in G$, then there exists a sequence $(x_0,\ x_1,\ ...,\ x_n,\ ...)$ of elements of X, such that

$$(\forall n \in \mathbf{N})((x_n, x_{n+1}) \in G).$$

The Axiom of Dependent Choices is usually denoted by **DC**. Actually, the statement **DC** is a weak form of the Axiom of Choice. This form is completely sufficient for most fields of classical mathematics: geometry of a finite-dimensional Euclidean space, mathematical analysis on the real line, Lebesgue measure theory, etc. We shall deal with the axiom **DC** many times in our further considerations and we shall discuss some nontrivial applications of this axiom.

It was established by Blair that the following two assertions are equivalent in **ZF** theory:

(a) the Axiom of Dependent Choices (**DC**);

(b) no nonempty complete metric space is of first category on itself (the classical Baire theorem).

In connection with the equivalence of (a) and (b), see Exercise 1 for this chapter.

Let X be an arbitrary set. A binary relation $G \subset X \times X$ is called an equivalence relation on X if the following three conditions hold:

(1) $(x, x) \in G$ for all elements $x \in X$;

(2) $(x, y) \in G$ and $(y, z) \in G$ imply $(x, z) \in G$;

(3) $(x, y) \in G$ implies $(y, x) \in G$.

If G is an equivalence relation on X, then the pair (X, G) is called a set equipped with an equivalence relation. In this case, the set X is also called the base (ground) set for the given equivalence relation G.

Obviously, if G is an equivalence relation on X, then we have a partition of X canonically associated with G. This partition consists of the sets $G(x)$, where x ranges over X and $G(x)$ denotes the section of G corresponding to x, i.e., $G(x)$ is defined by the formula $G(x) = \{y : (x, y) \in G\}$.

Conversely, every partition of X canonically determines an equivalence relation G on X. Namely, one may put $(x, y) \in G$ if and only if x and y belong to the same element of the partition.

Let X be an arbitrary set and let G be a binary relation on X. It is said that G is a partial order on X if the following three conditions hold:

(i) $(x, x) \in G$ for each element $x \in X$;

(ii) $(x, y) \in G$ and $(y, z) \in G$ imply $(x, z) \in G$;

(iii) $(x, y) \in G$ and $(y, x) \in G$ imply $x = y$.

Suppose that G is a partial order on a set X. As usual, we write $x \leq y$ iff $(x, y) \in G$. The pair $(X, \leq)$ is called a set equipped with a partial order (or, simply, a partially ordered set). The set X is called the base (ground) set for the given partial order $\leq$.

Let $(X, \leq)$ be a partially ordered set and let x and y be any two elements of X. We say that x and y are comparable (with respect to $\leq$) if $x \leq y$ or $y \leq x$. According to this definition, we say that x and y are incomparable if $x \not\leq y$ and $y \not\leq x$.

Further, we say that elements x and y of X are consistent (in X) if there exists an element z of X such that $z \leq x$ and $z \leq y$. Now, it is clear that elements x and y of X are inconsistent (in X) if there does not exist an element z of X having the above property.

Obviously, if $x \in X$ and $y \in X$ are inconsistent, then they are incomparable. The converse assertion is not true in general.

A subset Y of X is called a chain in X (or a subchain of X) if any two elements of Y are comparable with respect to $\leq$.

A subset Y of X is called an antichain in X if any two distinct elements of Y are incomparable with respect to $\leq$. In this case, Y is also called a free subset of X with respect to $\leq$.

A subset Y of X is called consistent if, for any two elements y_1 and y_2 of Y, there exists an element y of Y such that $y \leq y_1$ and $y \leq y_2$. From this definition it immediately follows, by induction, that if a nonempty set $Y \subset X$ is consistent, then, for every finite set $\{y_1, \ y_2, \ \dots \ , y_n\} \subset Y$, there exists $y \in Y$ satisfying the relations $y \leq y_1$, $y \leq y_2$, ..., $y \leq y_n$.

A subset Y of X is called totally inconsistent if any two distinct elements of Y are inconsistent in X.

The following definition is important for modern set theory and its various applications to general topology and measure theory (see, e.g., [97], [146]).

We say that a partially ordered set $(X, \leq)$ satisfies the countable chain condition (or the Suslin condition) if every totally inconsistent subset of X is at most countable.

We say that a subset Y of a partially ordered set $(X, \leq)$ is coinitial in X if, for each element x of X, there exists an element y of Y such that $y \leq x$.

Martin's Axiom is the following set-theoretical statement:

If $(X, \leq)$ is a partially ordered set satisfying the countable chain condition and $\mathcal{F}$ is a family of coinitial subsets of X, such that $\mathrm{card}(\mathcal{F}) < \mathbf{c}$, then there exists a consistent subset Y of X which intersects every set from the family $\mathcal{F}$, i.e., $(\forall Z \in \mathcal{F})(Y \cap Z \neq \emptyset)$.

Martin's Axiom is usually denoted by **MA**. One can easily show that the Continuum Hypothesis **CH** implies Martin's Axiom. On the other hand, it was established by Martin and Solovay that if the theory **ZFC** is consistent, then the theory

$$\mathbf{ZFC} \ \& \ \mathbf{MA} \ \& \ (\text{the negation of } \mathbf{CH})$$

is consistent, too (see [97] or [146]). Moreover, it was established that Martin's Axiom is a set-theoretical statement much weaker than the Continuum Hypothesis, because it does not bound from above the size of the continuum. At the present time, many applications of Martin's Axiom to the theory of infinite groups, to general topology, to measure theory and real analysis are known (see, e.g., [12], [18], [47], [48], [66], [83], [114], [131], [146], [230]). Some nontrivial applications of **MA** will be discussed below. Notice, in addition, that **MA** can also be formulated in purely topological terms (see [18], [146]).

Let $(X, \leq)$ be again a partially ordered set. We say that $(X, \leq)$ is a linearly ordered set if any two elements of X are comparable with respect to $\leq$. A linearly ordered set is also called a chain.

An element x of a partially ordered set $(X, \leq)$ is called maximal if, for each element y of X, we have the implication $x \leq y \Rightarrow x = y$. In a similar way one can define a minimal element of X.

The next set-theoretical statement is a well-known equivalent of the Axiom of Choice, formulated in terms of partially ordered sets. This statement is usually called the Zorn lemma (or the Kuratowski–Zorn lemma).

Let $(X, \leq)$ be a partially ordered set such that each subchain of X is bounded from above. Then there exists at least one maximal element in X. Moreover, if x is an arbitrary element of X, then there exists at least one maximal element y in X satisfying the inequality $x \leq y$.

Sometimes, it is convenient to apply the Zorn lemma instead of the Axiom of Choice whose standard formulation looks as follows:

For any family $\mathcal{Z}$ of nonempty sets which pairwise have no common elements, there exists a set Z such that every set from $\mathcal{Z}$ has one and only one common element with Z.

The proof of the fact that **AC** is equivalent to the Zorn lemma (within **ZF** theory) is not difficult and can be found in many textbooks (see, e.g., [107] or [153]).

Let $(X, \leq)$ be a partially ordered set and let $x \in X$. We say that x is a smallest (first, least) element of X if $x \leq y$ for all $y \in X$. In a similar way we can define a largest (last, greatest) element of X. It is easy to check that a smallest (respectively, largest) element of X is unique. Moreover, the smallest (respectively, largest) element of X is a unique minimal (respectively, unique maximal) element of X.

We say that a partially ordered set $(X, \leq)$ is well-ordered if any nonempty subset of X, equipped with the induced order, has a smallest element.

Obviously, every well-ordered set is linearly ordered, but not conversely.

Well-ordered sets are very important in the class of all partially ordered sets, because one can directly apply the principle of transfinite induction and the method of transfinite recursion to well-ordered sets.

It is known that, for every well-ordered set $(X, \leq)$, there exists a unique ordinal number α such that $(X, \leq)$ and α are isomorphic as partially ordered sets. Thus, without loss of generality, well-ordered sets can be identified with corresponding ordinal numbers.

Let $(X, \leq)$ be an arbitrary linearly ordered set. It is not difficult to prove, by using the Zorn lemma, that there exists a set $Y \subset X$ well-ordered with respect to the induced order and cofinal in X (i.e., for each element x of X, there is an element y of Y such that $x \leq y$).

This fact immediately implies that the Zorn lemma can be formulated in its weaker version:

Let $(X, \leq)$ be a partially ordered set such that every well-ordered subset of X is bounded from above. Then there exists a maximal element in X. Moreover, if x is an arbitrary element of X, then there exists a maximal element y in X such that $x \leq y$.

Now, we shall consider some simple facts concerning the fundamental notion of a partial function (partial mapping).

Let X and Y be any two sets. Suppose that G is a binary relation between X and Y, i.e., $G \subset X \times Y$. We say that G is a functional graph if the implication $((x, y) \in G \ \& \ (x, y') \in G) \Rightarrow (y = y')$ holds for all elements $x, \ y, \ y'$. It is easy to see that a binary relation $G \subset X \times Y$ is a functional graph if and only if $(\forall x)(\mathrm{card}(G(x)) \leq 1)$.

We say that a triple $g = (G, X, Y)$ is a partial function (or a partial mapping) acting from X into Y if G is a functional graph and $G \subset X \times Y$. In this case, we also say that the set G is the graph of a partial function g. Quite often, it is convenient to identify a partial function with its graph.

Furthermore, we say that a triple $g = (G, X, Y)$ is a function (mapping) acting from X into Y if g is a partial function acting from X into Y and $X = \mathrm{pr}_1(G)$. In this case, we also write $g : X \to Y$. If x is an arbitrary element of X, then the symbol $g(x)$ denotes the unique element y of Y for which $(x, y) \in G$. The element $g(x)$ is called the value of g at x. Hence we can write

$$g \ : \ x \to g(x) \qquad (x \in X, \ g(x) \in Y).$$

One may use a similar notation for any partial function $g = (G, X, Y)$, too. For example, it is sometimes convenient to write $g : X \to Y$ for this partial function. But it should be emphasized that the symbol $g(x)$ can be applied only in the case when $x \in \mathrm{pr}_1(G)$.

Let $g = (G, X, Y)$ be again a partial mapping acting from X into Y.

If $A \subset X$, then we put $g(A) = \{g(x) : x \in \mathrm{pr}_1(G) \cap A\}$. The set $g(A)$ is usually called the image of A under (with respect to) g. Obviously, one can introduce, by the same definition, the set $g(A)$ for an arbitrary set A.

If $B \subset Y$, then we put $g^{-1}(B) = \{x : x \in \mathrm{pr}_1(G),\ g(x) \in B\}$. The set $g^{-1}(B)$ is usually called the pre-image of B under (with respect to) g. Clearly, one can introduce, by the same definition, the set $g^{-1}(B)$ for any set B.

If A is a subset of X, then the symbol $g|A$ denotes the restriction of g to this subset, i.e., $g|A = (G \cap (A \times Y), A, Y)$. Evidently, the same definition can be applied to an arbitrary set A.

We say that a partial function g is an extension of a partial function f if f is a restriction of g.

Let g be a partial function whose graph is G.

The set $\mathrm{pr}_1(G)$ is called the domain of g. It is denoted by $\mathrm{dom}(g)$.

The set $\mathrm{pr}_2(G)$ is called the range of g. It is denoted by $\mathrm{ran}(g)$.

Obviously, we have the equality $\mathrm{ran}(g) = g(\mathrm{dom}(g))$.

We say that a partial function $g = (G, X, Y)$ is an injective partial function (or, simply, injection) if the implication $g(x) = g(x') \Rightarrow x = x'$ holds true for all elements x and x' from the domain of g.

If $g = (G, X, Y)$ is injective, then we can consider a partial function g^{-1} acting from Y into X, whose graph is $G^{-1} = \{(y, x) : (x, y) \in G\}$. This partial function is called the partial function inverse to g.

We say that $g = (G, X, Y)$ is a surjective partial function (or, simply, a surjection) if the equality $\mathrm{ran}(g) = Y$ is fulfilled.

Finally, we say that a function $g = (G, X, Y)$ is a bijective function (or, simply, a bijection) if g is an injection and a surjection simultaneously. In this case, we also say that g is a one-to-one correspondence between the sets X and Y.

A transformation of a set X is an arbitrary bijection acting from X onto X. The set of all transformations of X becomes a group with respect to the natural operation $\circ$ of composition of transformations. This group is called the symmetric group of X and denoted by the symbol $\mathrm{Sym}(X)$.

The group $\mathrm{Sym}(X)$ is universal in the following sense: if $(\Gamma, \cdot)$ is an abstract group such that $\mathrm{card}(\Gamma) \leq \mathrm{card}(X)$, then there exists a subgroup of $\mathrm{Sym}(X)$ isomorphic to $(\Gamma, \cdot)$.

Suppose now that I is a set and g is a function with $\mathrm{dom}(g) = I$. Then we say that $(g(i))_{i \in I}$ (or $\{g(i) : i \in I\}$) is a family of elements indexed by I. In this case, we also say that I is the set of indices of the above-mentioned family.

Moreover, suppose that E is a fixed set and, for each index $i \in I$, the element $g(i)$ coincides with a subset F_i of E. Then we say that $\{F_i : i \in I\}$ is an indexed family of subsets of E. Actually, in such a case, we have a certain mapping $F : I \to \mathcal{P}(E)$ where $\mathcal{P}(E)$ denotes the family of all subsets of E. A mapping of this type is usually called a set-valued mapping (or a multi-valued mapping). As we know, the graph of F is the set

$$\{(i, F(i)) \ : \ i \in I\} \subset I \times \mathcal{P}(E).$$

But if one treats F as a set-valued mapping, then it is sometimes useful to consider another notion of the graph of F. Namely, the graph of a set-valued mapping F is (by definition) the set $\{(i, e) \in I \times E : e \in F(i)\}$.

The concept of a set-valued mapping is more general (in some sense) than the concept of an ordinary mapping.

Indeed, every ordinary mapping $f : X \to Y$ can be regarded as a set-valued mapping $F_f : X \to \mathcal{P}(Y)$ of a special type:

$$F_f(x) = \{f(x)\} \qquad (x \in X).$$

In this way we come to a canonical one-to-one correspondence between all ordinary mappings f acting from X into Y and all set-valued mappings $F : X \to \mathcal{P}(Y)$ satisfying the condition $(\forall x \in X)(\mathrm{card}(F(x)) = 1)$.

Evidently, there are various set-valued mappings canonically associated with the given ordinary mapping $f \ : \ X \to Y$. For instance, we can define a set-valued mapping $F^f \ : \ Y \to \mathcal{P}(X)$ by the following formula:

$$F^f(y) = f^{-1}(y) \qquad (y \in Y).$$

Let $\{X_i : i \in I\}$ be a family of sets. In the usual way, the union

$$\cup\{X_i \ : \ i \in I\} = \{x : (\exists i \in I)(x \in X_i)\}$$

of this family is introduced. If $I \neq \emptyset$, then one may define the intersection

$$\cap\{X_i \ : \ i \in I\} = \{x : (\forall i \in I)(x \in X_i)\}$$

of this family. Further, if J is an arbitrary subset of I, then $\{X_i \ : \ i \in J\}$ is called a subfamily of the family $\{X_i \ : \ i \in I\}$ (in fact, a subfamily of a given family is some restriction of the function determining this family).

We say that a family of elements $\{x_i : i \in I\}$ is a selector of a family of sets $\{X_i : i \in I\}$ if the relation $(\forall i \in I)(x_i \in X_i)$ holds true. A selector $\{x_i : i \in I\}$ is called injective if the corresponding function

$$i \to x_i \qquad (i \in I, \quad x_i \in X_i)$$

is injective.

The set of all selectors of a given family $\{X_i : i \in I\}$ is called the Cartesian product of this family and denoted by the symbol $\prod\{X_i : i \in I\}$.

We recall that a family of sets $\{X_i : i \in I\}$ is disjoint if the equality $X_i \cap X_j = \emptyset$ holds for all indices $i \in I$, $j \in I$, $i \neq j$. Every selector of such a family is trivially injective.

Let X and Y be any two sets. The symbol Y^X denotes the set of all mappings acting from X into Y. Obviously, the set Y^X can be regarded as a particular case of the Cartesian product of a family of sets.

In fact, the Axiom of Choice states that the relation $(\forall i \in I)(X_i \neq \emptyset)$ implies the relation $\prod\{X_i : i \in I\} \neq \emptyset$. Kelley demonstrated in [106] that within **ZF** theory the Axiom of Choice is equivalent to the statement that the product space of an arbitrary family of quasi-compact topological spaces is quasi-compact, too. For more details about this equivalence, see Exercise 6.

The countable form of the Axiom of Choice is the restriction of this axiom to all countable families of sets:

If $\{X_n : n \in \omega\}$ is an arbitrary countable family of nonempty pairwise disjoint sets, then there exists a selector of $\{X_n : n \in \omega\}$.

Obviously, the countable form of the Axiom of Choice is a weak version of this axiom. It is not difficult to show within **ZF** theory that

(*) **DC** implies the countable form of **AC**;

(**) the countable form of **AC** is sufficient to prove the equivalence of the Cauchy and Heine definitions of the continuity of a partial function $f : \mathbf{R} \to \mathbf{R}$ at a point $t \in \mathrm{dom}(f)$;

(***) the countable form of **AC** implies that the union of a countable family of countable sets is a countable set.

Actually, the countable form of the Axiom of Choice is completely sufficient for classical mathematical analysis, Euclidean geometry, and even for elementary topology of point sets in $\mathbf{R}^n$.

Now, we are going to present some notions and facts from general topology (these facts will be needed for our further purposes).

We recall that a topological space is any pair $(E, \mathcal{T})$ where E is a ground set and $\mathcal{T}$ is a topology (or a topological structure) defined on E (see [27], [64], [107]).

If $\mathcal{T}$ is fixed in our considerations, then we simply say that E is a topological space.

Let E be a topological space and let X be a subset of E. We put

$\mathrm{cl}(X)$ = the closure of X;

$\mathrm{int}(X)$ = the interior of X;

bd(X) = the boundary of X.

Consequently, closed subsets of E are all those sets $X \subset E$ for which we have cl$(X) = X$, and open subsets of E are all those sets $Y \subset E$ for which we have int$(Y) = Y$.

We say that a set $X \subset E$ is an F_σ-subset of E if X can be represented as the union of a countable family of closed subsets of E.

We say that a set $Y \subset E$ is a G_δ-subset of E if Y can be represented as the intersection of a countable family of open subsets of E.

The Borel σ-algebra of a topological space E is the σ-algebra of subsets of E, generated by the family of all open sets in E. This σ-algebra is denoted by the symbol $\mathcal{B}(E)$. Obviously, one may say that $\mathcal{B}(E)$ is generated by the family of all closed subsets of E. Elements of $\mathcal{B}(E)$ are called Borel subsets of a space E.

In most cases below we assume (without special pointing out) that, for topological spaces E which are under consideration, the following property is valid: all singletons in E are closed (hence, Borel) in E.

We also recall that a topological space E is Lindelöf if any open covering of E contains a countable subcovering. Evidently, every quasi-compact space (see Exercise 6) is Lindelöf. In addition, any topological space with a countable base is Lindelöf (and, moreover, is hereditarily Lindelöf).

We say that a topological space E is Polish if E is homeomorphic to a complete separable metric space.

We have the following topological characterization of all Polish spaces: a topological space E is Polish if and only if it is homeomorphic to a G_δ-subset of the Hilbert cube $[0,1]^\omega$ (see, e.g., [64], [149], [202]).

We say that a metrizable topological space E is an analytic (or Suslin) space if E can be represented as a continuous image of a Polish space. There is also another definition of analytic spaces starting with the so-called (A)-operation applied to the family of all closed subsets of a Polish space (for the concept of (A)-operation and its basic properties, see [105], [149], [153], [167], [169]).

Let X be a Polish topological space. The family of all analytic subsets of X is denoted by the symbol $\mathcal{A}(X)$. This family is closed under countable unions and countable intersections. Moreover, we have the inclusion $\mathcal{B}(X) \subset \mathcal{A}(X)$. If a given Polish space X is uncountable, then the inclusion mentioned above is proper. This classical result is due to Suslin (for a proof, see [105], [149] or Chapter 1 of the present book).

Another important result, due to Alexandrov and Hausdorff, states that any uncountable analytic set $A \subset X$ contains a subset which is homeomorphic to the Cantor discontinuum $\{0,1\}^\omega$. Consequently, the equality card$(A) = \mathbf{c}$ holds. In particular, for each uncountable Borel set $B \subset X$, we also have the equality card$(B) = \mathbf{c}$ (the proofs of these statements can be found in [105] and [149]).

Let E be a topological space and let μ be a measure given on E.

We recall that μ is a Borel measure (on E) if $\mathrm{dom}(\mu) = \mathcal{B}(E)$.

We also say that μ is a Radon measure on a Hausdorff topological space E if μ is σ-finite, $\mathrm{dom}(\mu) = \mathcal{B}(E)$, and, for each Borel subset B of E, the relation

$$\mu(B) = \sup\{\mu(K) \ : \ K \subset B, \ K \text{ is compact}\}$$

holds true.

Finally, we say that a Hausdorff topological space E is a Radon space if every σ-finite Borel measure on E is Radon.

According to the well-known result from topological measure theory, all Polish spaces turn out to be Radon spaces. More generally, any analytic space is Radon. In addition, if E is a Polish space and A is an arbitrary analytic subset of E, then the space $E \setminus A$ is Radon, too. In other words, all co-analytic subsets of a Polish space are Radon. For more information about Radon measures and Radon spaces, see, e.g., [23], [28], [57], [90], [105], [196]. These measures and spaces play an important role in various questions of mathematical analysis and probability theory.

Let X and Y be any two topological spaces and let f be a mapping acting from X into Y. We say that f is a Borel mapping if, for each Borel subset B of Y, the pre-image $f^{-1}(B)$ is a Borel subset of X.

Clearly, every continuous mapping acting from X into Y is a Borel mapping. Also, the composition of Borel mappings is a Borel mapping.

We say that a mapping $f : X \to Y$ is a Borel isomorphism from X onto Y if f is a bijection and both mappings f and f^{-1} are Borel. In this case, we say that the spaces X and Y are Borel isomorphic.

Obviously, if two topological spaces are homeomorphic, then they are Borel isomorphic. The converse assertion is not true in general.

Let Z be a Borel subset of a Polish topological space, let Y be a metrizable topological space, and let f be an injective Borel mapping acting from Z into Y. Then the image $f(Z)$ is a Borel subset of Y. Consequently, the family of all Borel subsets of a Polish space E is invariant under the family of all injective Borel mappings acting from E into E.

More generally, let Z be a Borel subset of a Polish space, Y be a metrizable topological space, and let $f : Z \to Y$ be a Borel mapping such that

$$(\forall y \in Y)(\mathrm{card}(f^{-1}(y)) \leq \omega).$$

Then the set $f(Z)$ is Borel in Y (see [105], [149], [167], [169]).

This profound result will be essentially used in Chapter 18 of the present book, which is devoted to bad functions given on second category sets.

The above-mentioned facts are rather deep theorems of classical descriptive set theory and their proofs are heavily based on the so-called separation principle of analytic sets (for details, see again [105], [149], [167], [169]).

We also have the following important result:

If X and Y are any two uncountable Borel subsets of a Polish topological space E, then there exists a Borel isomorphism from X onto Y.

Because all infinite countable subsets of a Polish space E are Borel sets, trivially being Borel isomorphic to each other, one can easily deduce from the result mentioned above that, for any Borel subsets X and Y of E, these two conditions are equivalent:

($'$) $\mathrm{card}(X) = \mathrm{card}(Y)$;

($''$) X and Y are Borel isomorphic.

Unfortunately, we do not have (in **ZFC** theory) the analogous nice equivalence for analytic subsets of Polish topological spaces.

Notice also that if Z is a Borel subset of a Polish topological space and f is a Borel mapping acting from Z into a Polish space E, then the image $f(Z)$ is an analytic subset of E.

The theory of Borel subsets and analytic subsets of Polish topological spaces is considered in detail in the fundamental monograph by Kuratowski [149] (see also [105], [167], [169]).

Let E be an arbitrary Polish topological space. We define the classes

$$\mathcal{P}r_0(E), \ \ \mathcal{P}r_1(E), \ \ ..., \ \ \mathcal{P}r_n(E), \ \ ...$$

of subsets of E by recursion. Namely, first of all, we put $\mathcal{P}r_0(E) = \mathcal{B}(E)$. Suppose now that, for a natural number $n > 0$, the class $\mathcal{P}r_{n-1}(E)$ has already been defined. If n is an odd number, then, by definition, $\mathcal{P}r_n(E)$ is the class of all continuous images (in E) of sets from the class $\mathcal{P}r_{n-1}(E)$. If n is an even number, then, by definition, $\mathcal{P}r_n(E)$ is the class of all complements of the sets from the class $\mathcal{P}r_{n-1}(E)$.

Finally, we put $\mathcal{P}r(E) = \cup\{\mathcal{P}r_n(E) : n < \omega\}$.

Sets from the class $\mathcal{P}r(E)$ are called projective subsets of a space E. The notion of a projective set was introduced by Luzin and, independently, by Sierpiński in 1925. At the present time, there are many remarkable works devoted to the theory of projective sets. Elements of this theory are presented in the monograph by Kuratowski mentioned above (a more detailed discussion of logical aspects of this subject can be found in [18], [37], [97], [101], [105], [153], [169]).

We thus see that Borel subsets of E (i.e., sets from the class $\mathcal{P}r_0(E)$) and analytic subsets of E (i.e., sets from the class $\mathcal{P}r_1(E)$) are very special cases of projective sets.

Notice that many natural problems concerning projective sets cannot be solved in **ZFC** theory. For example, it cannot be proved in **ZFC** that each uncountable set from the class $\mathcal{P}r_2(\mathbf{R})$ contains a subset homeomorphic to the Cantor discontinuum, and it cannot be proved in **ZFC** that each set from the class $\mathcal{P}r_3(\mathbf{R})$ is measurable in the Lebesgue sense (or possesses the Baire property).

Now, let us recall some elementary facts about the Baire property of subsets of general topological spaces.

Let E be an arbitrary topological space.

We say that a set $X \subset E$ is nowhere dense (in E) if $\mathrm{int}(\mathrm{cl}(X)) = \emptyset$.

For example, if V is any open subset of E, then the set $\mathrm{bd}(V)$ is nowhere dense in E.

We say that a set $X \subset E$ is a first category subset of E (or X is a meager subset of E) if X can be represented in the form $X = \cup\{X_n : n \in \omega\}$, where all X_n $(n \in \omega)$ are nowhere dense subsets of E.

We say that a set $X \subset E$ is of second category in E if X is not of first category in E.

Finally, we say that a set $X \subset E$ is residual (co-meager) in E if the set $E \setminus X$ is of first category in E.

The family of all first category subsets of E is denoted by the symbol $\mathcal{K}(E)$. If E is not a first category space, then $\mathcal{K}(E)$ forms a certain σ-ideal of subsets of E, which plays an important role in many questions of functional analysis and general topology.

We say that a set $X \subset E$ has (possesses) the Baire property in E if X can be represented in the form $X = (U \cup Y) \setminus Z$, where U is an open subset of E and both Y and Z are first category subsets of E.

It is easy to check that a set $X \subset E$ has the Baire property if and only if X can be represented in the form $X = V \triangle P$, where V is an open subset of E and P is a first category subset of E.

The family of all those subsets of a space E, which possess the Baire property in E, is denoted by the symbol $\mathcal{B}a(E)$.

Obviously, $\mathcal{B}a(E)$ coincides with the σ-algebra of subsets of E generated by the family $\mathcal{T}(E) \cup \mathcal{K}(E)$, where $\mathcal{T}(E)$ is the topology of E (i.e., the family of all open subsets of E). Hence we have the inclusion $\mathcal{B}(E) \subset \mathcal{B}a(E)$. As a rule, this inclusion is proper. But there are some interesting examples of topological spaces E for which this inclusion becomes the equality.

For instance, if E is a classical Luzin subset of $\mathbf{R}$ everywhere dense in $\mathbf{R}$, then we have $\mathcal{K}(E) = [E]^{\leq\omega}$. Consequently, in this case, we trivially obtain the equality $\mathcal{B}(E) = \mathcal{B}a(E)$. Extensive information on Luzin subsets of $\mathbf{R}$ is contained in [105], [145], [149], [184], [192], and [202]. We shall deal with Luzin sets in subsequent sections of this book. At the moment, we only wish to notice that the existence of Luzin subsets of $\mathbf{R}$ cannot be proved in **ZFC** theory. On the other hand, the existence of such subsets of $\mathbf{R}$ easily follows from the Continuum Hypothesis (see Chapter 13).

Another interesting example (within **ZFC** theory) of a topological space E, for which the equality $\mathcal{B}(E) = \mathcal{B}a(E)$ holds, can be obtained if one takes the set of all real numbers equipped with the so-called density topology (see information on this topology in [184], [202], [266]). We shall consider in the sequel some elementary properties of the density topology.

It can be proved that the Baire property and the measurability of sets are preserved under (A)-operation (see, for instance, [105], [149], [225]). This gives, in particular, that all analytic subsets of a Polish space E possess the Baire property and are universally measurable in E (the latter means that they are measurable with respect to the completion of any σ-finite Borel measure on E).

Concerning measure and category, let us mention two important consequences of Martin's Axiom that will be often applied in the present book. Namely, under this axiom, the σ-ideal $\mathcal{K}(\mathbf{R})$ of all first category subsets of $\mathbf{R}$ and the σ-ideal $\mathcal{I}(\lambda)$ of all Lebesgue measure zero subsets of $\mathbf{R}$ are $\mathbf{c}$-additive. In other words, the union of strictly less than $\mathbf{c}$ members of $\mathcal{K}(\mathbf{R})$ (respectively, of $\mathcal{I}(\lambda)$) is again a member of $\mathcal{K}(\mathbf{R})$ (respectively, of $\mathcal{I}(\lambda)$).

This also yields, under the same axiom, that the σ-algebra $\mathcal{B}a(\mathbf{R})$ of subsets of $\mathbf{R}$ with the Baire property and the σ-algebra $\mathrm{dom}(\lambda)$ of all Lebesgue measurable subsets of $\mathbf{R}$ are $\mathbf{c}$-additive (for more details, see [18], [43], [97]).

Let E be again an arbitrary topological space and let X be a subset of E. We say that X has the Baire property in the restricted sense if, for each subspace Y of E, the set $X \cap Y$ has the Baire property in the space Y.

Clearly, the family of all subsets of a space E, having the Baire property in the restricted sense, is a σ-algebra of subsets of E. We denote this σ-algebra by $\mathcal{B}ar(E)$. Obviously, we have the inclusion $\mathcal{B}ar(E) \subset \mathcal{B}a(E)$. It is also easy to check that $\mathcal{B}(E) \subset \mathcal{B}ar(E)$. Moreover, it can be shown that $\mathcal{A}(E) \subset \mathcal{B}ar(E)$, i.e., all analytic subsets of E have the Baire property in the restricted sense (see, for instance, [149]).

Notice that, for the class of all complete metric spaces, the Baire property in the restricted sense is a topological invariant, i.e., if E and E' are two complete metric spaces, $X \subset E$ has the Baire property in the restricted sense, and $Y \subset E'$ is a homeomorphic image of X, then Y also has the

Baire property in the restricted sense (see again [149]).

Let X and Y be any two topological spaces and let f be a mapping acting from X into Y. We say that f has the Baire property if, for each Borel subset B of Y, the set $f^{-1}(B)$ has the Baire property in X.

Evidently, every Borel mapping acting from X into Y has the Baire property.

The composition of two mappings, each of which has the Baire property, can be a mapping without the Baire property. Moreover, it is not difficult to give an example of two functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$, each of which has the Baire property, but their composition $g \circ f$ does not possess this property (the analogous phenomenon can be observed for the composition of two Lebesgue measurable functions acting from $\mathbf{R}$ into $\mathbf{R}$).

However, if X, Y, Z are three topological spaces, $f : X \to Y$ has the Baire property, and $g : Y \to Z$ is a Borel mapping, then $g \circ f$ has the Baire property, too.

In a similar way one can define a mapping with the Baire property in the restricted sense. Namely, one says that $f : X \to Y$ has the Baire property in the restricted sense if, for each Borel subset B of Y, the set $f^{-1}(B)$ is a subset of X having the Baire property in the restricted sense.

It is easy to see that all Borel mappings have the Baire property in the restricted sense.

Let X and Y be topological spaces and let $F : X \to \mathcal{P}(Y)$. We say that F has a closed graph if $G_F = \{(x, y) \in X \times Y \ : \ y \in F(x)\}$ is a closed subset of the product space $X \times Y$.

It is clear that if the given set-valued mapping F has a closed graph, then, for each element $x \in X$, the set $F(x)$ is a closed subset of the space Y. The converse assertion is not true in general.

Set-valued mappings with closed graphs are important in different fields of mathematics, especially in those topics which concern the existence of fixed points of set-valued mappings (we recall that an element $x \in \mathrm{dom}(F)$ is a fixed point for a set-valued mapping F if $x \in F(x)$).

Notice that theorems on the existence of fixed points for set-valued mappings found many interesting applications (see, e.g., [63], [260]).

Let X and Y be again two topological spaces and let $F : X \to \mathcal{P}(Y)$ be a set-valued mapping. We say that F is lower semicontinuous if, for each point $x \in X$, the set $F(x)$ is closed in Y and, for any open set $V \subset Y$, the set $F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ is open in X.

There are certain similarities between set-valued mappings with closed graphs and lower semicontinuous set-valued mappings (cf., for example, Theorem 1 below).

For our further considerations, we need one auxiliary proposition on closed projections. This proposition is due to Kuratowski (see, for instance, [64], [149]) and has numerous applications in general topology and mathematical analysis.

Lemma 1. *Let X be a topological space, let Y be a quasi-compact space, and let pr_1 denote the canonical projection from $X \times Y$ into X, i.e., the mapping $\mathrm{pr}_1 : X \times Y \to X$ is defined by the standard formula*

$$\mathrm{pr}_1((x, y)) = x \qquad ((x, y) \in X \times Y).$$

Then pr_1 is a closed mapping, i.e., for each closed subset A of $X \times Y$, the set $\mathrm{pr}_1(A)$ is closed in X.

Proof. Take any point $x \in X$ such that $U(x) \cap \mathrm{pr}_1(A) \neq \emptyset$ for all neighborhoods $U(x)$ of x. We are going to show that $x \in \mathrm{pr}_1(A)$. For this purpose, it is sufficient to establish that $(\{x\} \times Y) \cap A \neq \emptyset$. Suppose otherwise, i.e., $(\{x\} \times Y) \cap A = \emptyset$. Then, for each point $y \in Y$, there exists an open neighborhood $W(x, y)$ of the point (x, y), satisfying the relation $W(x, y) \cap A = \emptyset$. We may assume that $W(x, y) = U(x) \times V(y)$, where $U(x)$ is an open neighborhood of x and $V(y)$ is an open neighborhood of y. Because the space $\{x\} \times Y$ is quasi-compact, there exists a finite sequence

$$(x, y_1), \quad (x, y_2), \quad \ldots, \quad (x, y_n)$$

of points from $\{x\} \times Y$, such that $\{W(x, y_i) : 1 \leq i \leq n\}$ is a finite covering of $\{x\} \times Y$. Let us put

$$U_i(x) = \mathrm{pr}_1(W(x, y_i)), \qquad O(x) = \cap\{U_i(x) \ : \ 1 \leq i \leq n\}.$$

Then it is easy to check that $O(x)$ is a neighborhood of x satisfying the equality $O(x) \cap \mathrm{pr}_1(A) = \emptyset$. But this is impossible. So we get a contradiction, and the Kuratowski lemma is proved.

Now, we want to give some applications of Lemma 1 to set-valued mappings. For the sake of simplicity (and motivated by the aims of mathematical analysis), we restrict our further considerations to the class of all metric spaces, but it is not difficult to see that the results presented below remain true in more general situations.

Theorem 1. *Let X be a metric space, let Y be a compact metric space, and let $F : X \to \mathcal{P}(Y)$ be a set-valued mapping. Then the following two assertions are equivalent:*

(1) F has a closed graph;

(2) for any point $x \in X$, the set $F(x)$ is closed in Y and, for any closed subset A of Y, the set $F^{-1}(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$ is closed in X.

Proof. Suppose that assertion (1) is valid. Then, obviously, for each element $x \in X$, the set $F(x)$ is closed in Y. Now, let A be any closed subset of Y. It is easy to see that $F^{-1}(A) = \mathrm{pr}_1((X \times A) \cap G_F)$, where G_F denotes the graph of the given set-valued mapping F. Clearly, the set $(X \times A) \cap G_F$ is closed in the product space $X \times Y$, as the intersection of two closed subsets of this space. Therefore, by the Kuratowski lemma proved above, we obtain that $F^{-1}(A)$ is a closed subset of the space X. Thus, assertion (2) holds true, so the implication (1) $\Rightarrow$ (2) is established.

Further, assume that assertion (2) is valid. Let us prove that G_F is a closed subset of the product space $X \times Y$. For this purpose, take an arbitrary sequence $\{(x_i, y_i) : i \in \mathbf{N}\}$ of points of the graph G_F, such that

$$\lim_{i \to +\infty} (x_i, y_i) = (x, y) \in X \times Y.$$

Let us show that the point (x, y) also belongs to G_F. Suppose otherwise, i.e., $(x, y) \notin G_F$. This means, by the definition, that $y \notin F(x)$. Since $F(x)$ is a closed set in Y, there exists a neighborhood $V(y)$ of y, satisfying the relation $V(y) \cap F(x) = \emptyset$. Furthermore, we have

$$\lim_{i \to +\infty} x_i = x, \qquad \lim_{i \to +\infty} y_i = y.$$

Without loss of generality, we may assume that $y_i \in V(y)$ for all indices $i \in \mathbf{N}$. Let us put $A = \{y\} \cup \{y_i : i \in \mathbf{N}\}$. Evidently, A is a closed subset of Y. It is also clear that $A \cap F(x) = \emptyset$. On the other hand, we have $y_i \in F(x_i)$ and $x_i \in F^{-1}(A)$ for all $i \in \mathbf{N}$. Taking into account that $\lim_{i \to +\infty} x_i = x$, and that the set $F^{-1}(A)$ is closed in X, we get $x \in F^{-1}(A)$ and $F(x) \cap A \neq \emptyset$, which yields a contradiction. This contradiction shows that our set-valued mapping F has a closed graph. Thus, the implication (2) $\Rightarrow$ (1) is established and the proof of Theorem 1 is completed.

One can easily see that, in the proof of Theorem 1, only the implication (1) $\Rightarrow$ (2) relies essentially on the Kuratowski lemma on closed projections. The converse implication (2) $\Rightarrow$ (1) does not need this lemma. So we come to the following result.

Let X and Y be any two metric spaces and let $F : X \to \mathcal{P}(Y)$ be a set-valued mapping such that $F(x)$ is a closed subset of Y for each point $x \in X$, and $F^{-1}(A)$ is a closed subset of X for each closed set $A \subset Y$. Then the set-valued mapping F has a closed graph.

As a special case of this result, we also have the following fact.

If X and Y are any two metric spaces and $f : X \to Y$ is a continuous mapping, then the graph $G_f = \{(x, y) \in X \times Y : y = f(x)\}$ of f is a closed subset of the product space $X \times Y$.

Notice that the converse assertion is not true in general. Indeed, it is not difficult to construct an example of a function acting from $\mathbf{R}$ into $\mathbf{R}$, which is discontinuous but has a closed graph. Furthermore, there exists a function $g : \mathbf{R} \to \mathbf{R}$ with a closed graph such that the set of all points of discontinuity of g is a nonempty perfect subset of $\mathbf{R}$ (the function g with this property can be constructed by starting with the classical Cantor subset of $\mathbf{R}$).

On the other hand, it is reasonable to indicate here that any function $f : \mathbf{R} \to \mathbf{R}$ having a closed graph belongs to the first Baire class (see Chapter 2). Therefore, according to a well-known Baire theorem (see, e.g., [8], [64], [107], [149], [202] or Chapter 2), for each nonempty perfect set $A \subset \mathbf{R}$, there exists a point of A at which the restricted function $f|A$ is continuous.

Let us now present a typical application of the Kuratowski lemma to the theory of real functions. We mean here the existence of continuous nowhere differentiable functions. Let $C[0, 1]$ denote, as usual, the family of all continuous real-valued functions defined on the unit segment $[0, 1]$. This family becomes a separable Banach space with respect to the standard sup-norm.

The following celebrated result is due to Banach and Mazurkiewicz (see [14] and [178]).

Theorem 2. *The family of all those functions from the Banach space* $C[0, 1]$*, which are nowhere differentiable on* $[0, 1]$*, is co-meager (i.e., is the complement of a first category set in* $C[0, 1]$*).*

Proof. Let h be a nonzero rational number such that $|h| < 1$. For every natural number $n > 0$, consider the set

$$\Phi_{h,n} = \{f \in C[0, 1] \ : \ (\exists x \in [0, 1])(\forall \delta)(0 < |\delta| < |h| \Rightarrow$$

$$|(f(x + \delta) - f(x))/\delta| \leq n)\}.$$

It is not hard to check that $\Phi_{h,n}$ is a closed subset of the space $C[0, 1]$. Indeed, let us put

$$Z_{h,n} = \{(f, x) \in C[0, 1] \times [0, 1] \ : \ (\forall \delta)(0 < |\delta| < |h| \Rightarrow$$

$$|(f(x + \delta) - f(x))/\delta| \leq n)\}.$$

Then $Z_{h,n}$ is closed in the product space $C[0,1]\times[0,1]$ and $\Phi_{h,n} = \mathrm{pr}_1(Z_{h,n})$, where $\mathrm{pr}_1 : C[0,1] \times [0,1] \to C[0,1]$ denotes the canonical projection onto $C[0,1]$. Taking into account the compactness of the unit segment $[0,1]$ and applying Lemma 1, we immediately obtain that the set $\Phi_{h,n}$ is closed in $C[0,1]$. Simultaneously, $\Phi_{h,n}$ is nowhere dense in $C[0,1]$ (the latter fact is almost trivial from the geometrical point of view). Consequently, the set

$$D = \cup\{\Phi_{h,n} \ : \ h \in \mathbf{Q} \setminus \{0\}, \ |h| < 1, \ n \in \mathbf{N} \setminus \{0\}\}$$

is of first category in $C[0,1]$. Now, it is clear that any function belonging to the set $C[0,1] \setminus D$ is nowhere differentiable on $[0,1]$. This completes the proof of Theorem 2.

The above theorem has natural analogues for many other kinds of derivatives, e.g., for so-called symmetric derivative (in this connection, see Exercise 27). The notion of a symmetric derivative of a function can be regarded as a simple example of the concept of a generalized derivative. In subsequent sections of the book we shall discuss some other types of a generalized derivative. The notion of an approximate derivative (introduced by Khinchin in 1914) is of special interest and will be defined and discussed in Chapter 7. It is well known that this notion plays an important role in various topics of real analysis (for instance, in the theory of generalized integrals). The definition of an approximate derivative relies on the concept of a density point for a given Lebesgue measurable subset of $\mathbf{R}$ and we would like to recall this concept here.

Let λ denote the standard Lebesgue measure on $\mathbf{R}$ and let X be an arbitrary λ-measurable subset of $\mathbf{R}$. We say that $x \in \mathbf{R}$ is a density point of (for) X if

$$\lim_{h\to 0, h>0} \frac{\lambda(X \cap [x-h, x+h])}{2h} \ = \ 1.$$

The classical theorem of real analysis, due to Lebesgue, states that almost all (with respect to λ) points of X are its density points. In order to establish this fact, we need the notion of a Vitali covering of a set lying in $\mathbf{R}$, and the important result of Vitali concerning such coverings. For the sake of completeness, we shall formulate and prove Vitali's result below.

Let $\{D_i \ : \ i \in I\}$ be a family of nondegenerate segments on $\mathbf{R}$ and let Z be a subset of $\mathbf{R}$. We say that this family is a Vitali covering of Z if, for each point $z \in Z$, we have $\inf\{\lambda(D_i) \ : \ i \in I, \ z \in D_i\} = 0$.

The following fundamental statement was obtained by Vitali (cf., e.g., [61], [194], [202], [225]).

Theorem 3. *If Z is a subset of* $\mathbf{R}$ *and* $\{D_i : i \in I\}$ *is a Vitali covering of Z, then there exists a countable set $J \subset I$ such that the partial family* $\{D_j : j \in J\}$ *is disjoint and* $\lambda(Z \setminus \cup\{D_j : j \in J\}) = 0$.

Proof. Without loss of generality we may assume that Z is bounded. Let U be an open bounded set in $\mathbf{R}$ containing Z. We may also assume that $D_i \subset U$ for each index $i \in I$. Define by recursion a disjoint countable subfamily of segments $D_{i(0)}, D_{i(1)}, ..., D_{i(k)}, ...$.

Take $D_{i(0)}$ arbitrarily. Suppose that $D_{i(0)}, D_{i(1)}, ..., D_{i(k)}$ have already been defined and put

$$t(k) = \sup\{\lambda(D_i) \ : \ D_i \subset U \setminus (D_{i(0)} \cup ... \cup D_{i(k)})\}.$$

Let $D_{i(k+1)}$ be a segment from $\{D_i \ : \ i \in I\}$ such that

$$D_{i(k+1)} \subset U \setminus (D_{i(0)} \cup ... \cup D_{i(k)}), \qquad \lambda(D_{i(k+1)}) \geq t(k)/2.$$

Proceeding in this manner, we obtain the disjoint sequence $\{D_{i(k)} \ : \ k \in \mathbf{N}\}$. Notice that

$$\sum_{k \in \mathbf{N}} \lambda(D_{i(k)}) \leq \lambda(U) < +\infty,$$

so we have the equality $\lim_{k \to +\infty} \lambda(D_{i(k)}) = 0$. We are going to show that $\{D_j : j \in J\} = \{D_{i(k)} : k \in \mathbf{N}\}$ is the required subfamily.

For this purpose, denote by $D'_{i(k)}$ the segment in $\mathbf{R}$ whose midpoint coincides with the midpoint of $D_{i(k)}$ and for which $\lambda(D'_{i(k)}) = 5\lambda(D_{i(k)})$. Let us demonstrate that, for every natural number n, the inclusion

$$Z \setminus \cup\{D_{i(k)} \ : \ k \in \mathbf{N}\} \subset \cup\{D'_{i(k)} \ : \ k \in \mathbf{N}, \ k > n\}$$

holds true. Indeed, let z be an arbitrary point from $Z \setminus \cup\{D_{i(k)} : k \in \mathbf{N}\}$. Then $z \in Z \setminus (D_{i(0)} \cup ... \cup D_{i(n)})$. Because $\{D_i : i \in I\}$ is a Vitali covering of Z, there exists a segment D_i for which

$$z \in D_i, \qquad D_i \cap (D_{i(0)} \cup ... \cup D_{i(n)}) = \emptyset.$$

Obviously, we have $\lambda(D_i) > 0$. At the same time, as mentioned above, the relation $\lim_{k \to +\infty} \lambda(D_{i(k)}) = 0$ is valid. So, for some natural numbers k, we must have $D_i \cap D_{i(k+1)} \neq \emptyset$. Let k be the smallest natural number with this property. Evidently, $k \geq n$. Thus, we get

$$D_i \cap D_{i(k+1)} \neq \emptyset, \qquad D_i \cap (D_{i(0)} \cup ... \cup D_{i(k)}) = \emptyset.$$

In addition, we may write $\lambda(D_i) \leq 2\lambda(D_{i(k+1)})$, which immediately implies (in view of the definition of $D'_{i(k+1)}$) the inclusion $D_i \subset D'_{i(k+1)}$. Consequently,

$$z \in D_i \subset D'_{i(k+1)} \subset \cup\{D'_{i(m)} \ : \ m \in \mathbf{N}, \ m > n\}.$$

Finally, for every natural number n, we have the inequality

$$\lambda(\cup\{D'_{i(m)} \ : \ m > n\}) \leq 5 \sum_{m>n} \lambda(D_{i(m)}),$$

from which it is easy to conclude that $\lambda(Z \setminus \cup\{D_j : j \in J\}) = 0$, so the Vitali theorem is proved.

We shall present some standard applications of Theorem 3 in subsequent sections of the book. Here we only want to recall how the Lebesgue result on density points of λ-measurable sets follows from Theorem 3.

Theorem 4. *Let X be an arbitrary λ-measurable set on $\mathbf{R}$ and let*

$$d(X) = \{x \in \mathbf{R} \ : \ x \text{ is a density point of } X\}.$$

Then we have the equality $\lambda(X \setminus (X \cap d(X))) = 0$.

Proof. We may assume, without loss of generality, that X is bounded. For any natural number $n > 0$, define

$$X_n = \{x \in X \ : \ \text{liminf}_{h \to 0, h>0} \lambda(X \cap [x-h, x+h])/2h \ < 1 - 1/n\}.$$

Clearly, it suffices to show that $\lambda^*(X_n) = 0$, where λ^* denotes the outer measure associated with λ. For this purpose, fix $\varepsilon > 0$. Let U be an open subset of $\mathbf{R}$ such that $X_n \subset U$ and $\lambda(U) \leq \lambda^*(X_n) + \varepsilon$. By virtue of the definition of X_n, there exists a Vitali covering $\{D_i : i \in I\}$ of X_n such that

$$(\forall i \in I)(\lambda(D_i \cap X)/\lambda(D_i) < 1 - 1/n).$$

Obviously, we may suppose that $D_i \subset U$ for each index $i \in I$. According to Theorem 3, there exists a disjoint countable subfamily $\{D_j : j \in J\}$ of this covering, for which we have

$$\lambda(X_n \setminus \cup\{D_j \ : \ j \in J\}) = 0.$$

Then we can write

$$\lambda^*(X_n) \leq \sum_{j \in J} \lambda(X \cap D_j) \leq (1 - 1/n) \sum_{j \in J} \lambda(D_j)$$

$$\leq (1 - 1/n)\lambda(U) \leq (1 - 1/n)(\lambda^*(X_n) + \varepsilon).$$

Since $\varepsilon > 0$ was taken arbitrarily, we obtain $\lambda^*(X_n) \leq (1 - 1/n)\lambda^*(X_n)$. Finally, in view of the relation $\lambda^*(X_n) < +\infty$, we conclude that $\lambda^*(X_n) = 0$, and the Lebesgue theorem is proved.

EXERCISES

1*. Demonstrate that the following two assertions are equivalent within **ZF** set theory:

(a) the Axiom of Dependent Choices (**DC**);

(b) no nonempty complete metric space is of first category on itself (the classical theorem of Baire).

Notice that the implication (a) $\Rightarrow$ (b) is widely known in mathematical analysis. In order to establish the converse implication, equip a nonempty set X with the discrete topology and consider the complete metric space X^ω. Further, by starting with a given binary relation G on X satisfying

$$(\forall x \in X)(\exists y \in X)((x, y) \in G),$$

define a certain countable family of everywhere dense open subsets of X^ω and try to obtain with the aid of this family the desired sequence of elements from X.

2. Show that if E is a complete separable metric space, then the Baire theorem for E is valid within **ZF** theory.

Deduce from this fact (again in **ZF** theory) that the real line **R** cannot be represented as the union of countably many finite sets. In particular, one has in **ZF** the classical theorem of Cantor saying that **R** is not countable.

Remark 1. In connection with Exercise 2, let us mention that there are models of **ZF** theory in which **R** is expressible in the form of a countable union of countable sets (for more details, see [18], [89], [97]).

3. Check that the following two assertions are equivalent in **ZF** theory:

(a) there exists a subset of **R** of cardinality ω_1;

(b) there exists a function f acting from $[\mathbf{R}]^{\leq\omega}$ into **R** and satisfying the relation $f(D) \notin D$ for each countable set $D \subset \mathbf{R}$.

Remark 2. Notice that none of these two assertions is provable within **ZF** & **DC** theory (see [210] and [228]; cf. also Chapter 10 of this book).

4. Prove in **ZF** & **DC** theory the implication **CH** $\Rightarrow$ **MA**.

5. Show that there is a canonical one-to-one correspondence between ordinary mappings $f : X \to Y$ and set-valued mappings $F : Y \to \mathcal{P}(X)$ satisfying the condition that the family $\{F(y) : y \in Y\}$ forms a disjoint covering of X.

6*. Recall that a topological space $(E, \mathcal{T})$ is quasi-compact if any open covering of E contains a finite subcovering of E.

Work in **ZF** set theory and demonstrate that these two assertions are equivalent:

(a) the Axiom of Choice (**AC**);

(b) the product space of an arbitrary family of quasi-compact topological spaces is quasi-compact.

For this purpose, notice that the implication (a) $\Rightarrow$ (b) is widely known in general topology (cf. [27], [64], [107], [150]). In order to establish the converse implication, consider any family $\{X_i : i \in I\}$ of nonempty sets and take some element x such that $x \notin \cup\{X_i \ : \ i \in I\}$. Further, for each index $i \in I$, define $X'_i = X_i \cup \{x\}$ and equip X'_i with the topology

$$\mathcal{T}_i = \{\emptyset, X'_i, \{x\}\}.$$

In this way, all X'_i become quasi-compact spaces. Finally, apply (b) to the product space of the family $\{X'_i : i \in I\}$.

Remark 3. A topological space is called compact if it is quasi-compact and Hausdorff simultaneously. At the present time, it is known that the following two assertions are not equivalent in **ZF** theory:

(a) the Axiom of Choice (**AC**);

(b$'$) the product space of an arbitrary family of compact spaces is compact.

However, Sierpiński showed in **ZF** & **DC** theory that the compactness of the product space $\{0, 1\}^{\mathbf{R}}$, where $\{0, 1\}$ is equipped with the discrete topology, implies the existence of a subset of $\mathbf{R}$ nonmeasurable in the Lebesgue sense (cf. [89], [129], [248]).

7. Work in **ZF** set theory and prove the following three assertions:

(a) **DC** implies the countable form of **AC**;

(b) the countable form of **AC** is sufficient to show the equivalence of the Cauchy and Heine definitions of the continuity of a partial function $f : \mathbf{R} \to \mathbf{R}$ at a point $t \in \mathrm{dom}(f)$;

(c) the countable form of **AC** implies that the union of a countable family of countable sets is a countable set.

Further, according to the standard definition, a set X is infinite if, for every natural number n, we have $\mathrm{card}(X) \neq n$.

Demonstrate in the same **ZF** theory that the countable form of **AC** implies that a set X is infinite if and only if there exists an injection from ω into X.

8. Let E and E' be metric spaces and let $f : E \to E'$ be a function. Suppose that E contains an everywhere dense well-ordered subset.

Show that the following two assertions are equivalent within **ZF** theory:

(a) f is continuous (on E) in the Cauchy sense;

(b) f is continuous (on E) in the Heine sense.

Deduce from this fact that if E is separable (in particular, if $E = \mathbf{R}$), then the equivalence of (a) and (b) does not rely on the Axiom of Choice, i.e., this equivalence is provable within **ZF** theory.

Observe that the implication (a) $\Rightarrow$ (b) is also provable in **ZF** for any metric space E.

9. Verify that the following two assertions are valid in **ZF** theory:

(a) every infinite bounded subset of $\mathbf{R}$ has at least one accumulation point;

(b) every continuous (in the Cauchy sense) function $f : [t_1, t_2] \to \mathbf{R}$ is uniformly continuous (where $[t_1, t_2]$ is an arbitrary nondegenerate closed bounded subinterval of $\mathbf{R}$).

Infer from (b) that f is bounded on $[t_1, t_2]$.

10*. Let E be a topological space and let $G = \{g_i : i \in I\}$ be a family of mappings acting from E into E (in general, these mappings are not assumed to be continuous). A set $A \subset E$ is called G-invariant in E if

$$A = \mathrm{cl}(\cup\{g_i(A) : i \in I\}).$$

By using the Zorn lemma or the method of transfinite induction, show that if E is nonempty and quasi-compact, then there exists a subset X of E satisfying the following two conditions:

(a) X is nonempty and G-invariant;

(b) for any G-invariant set $Y \subset E$, either $X \subset Y$ or $X \cap Y = \emptyset$.

11. Let E be a topological space and let X and Y be two subsets of E.

Check that if $X \cap \mathrm{cl}(Y)$ is nowhere dense in $\mathrm{cl}(Y)$, then $X \cap Y$ is nowhere dense in Y.

Deduce from this fact that if $X \cap \mathrm{cl}(Y)$ is of first category in $\mathrm{cl}(Y)$, then $X \cap Y$ is of first category in Y.

12. Let E be a topological space and let X be a subset of E.

Prove that these three assertions are equivalent:

(a) X has the Baire property in the restricted sense;

(b) for any closed set $F \subset E$, the set $F \cap X$ has the Baire property in F;

(c) for any perfect set $P \subset E$, the set $P \cap X$ has the Baire property in P.

13. Let X be a topological space, Y be a topological space with a countable base, and let $f : X \to Y$ be a function.

Show that the following two assertions are equivalent:

(a) f possesses the Baire property;

(b) there exists a first category set $A \subset X$ such that the restricted function $f|(X \setminus A)$ is continuous.

14. Let X be a topological space and let f be a real-valued function on X. Recall that the function f is lower (respectively, upper) semicontinuous if, for any real number t, the set $\{x \in X \ : \ f(x) > t\}$ (respectively, the set $\{x \in X \ : \ f(x) < t\}$) is open in X.

Demonstrate that:

(a) f is lower semicontinuous if and only if $-f$ is upper semicontinuous;

(b) f is lower semicontinuous if and only if the set

$$G^*(f) = \{(x,t) \in X \times \mathbf{R} \ : \ f(x) \leq t\}$$

is closed in the product space $X \times \mathbf{R}$;

(c) f is lower semicontinuous if and only if, for each $x_0 \in X$, one has the equality

$$\mathrm{liminf}_{x \to x_0} f(x) = f(x_0);$$

(d) f is continuous if and only if f is lower and upper semicontinuous;

(e) if f is lower semicontinuous, g is continuous, and $\mathrm{ran}(g) \subset X$, then $f \circ g$ is lower semicontinuous on $\mathrm{dom}(g)$;

(f) if a real t is nonnegative and two functions f and g are lower semicontinuous on X, then the functions tf and $f + g$ are lower semicontinuous on X.

15. Let X be a set and let f be a real-valued function on X. Introduce two set-valued mappings

$$F_{1,f} \ : \ X \to \mathcal{P}(\mathbf{R}), \qquad F_{2,f} \ : \ X \to \mathcal{P}(\mathbf{R})$$

by the following formulas:

$$F_{1,f}(x) = \ ]-\infty, f(x)] \qquad F_{2,f}(x) = [f(x), +\infty[\qquad (x \in X).$$

Supposing that X is a topological space, show that the following two assertions are equivalent:

(a) f is lower semicontinuous (as an ordinary function);
(b) $F_{1,f}$ is lower semicontinuous (as a set-valued function).
Check that the next two assertions are also equivalent:
(c) f is upper semicontinuous (as an ordinary function);
(d) $F_{2,f}$ is lower semicontinuous (as a set-valued function).

16. Let X be a nonempty quasi-compact topological space and let $f : X \to \mathbf{R}$ be a lower semicontinuous function.

Prove that there exists at least one point $x_0 \in X$ satisfying the relation $f(x_0) = \inf_{x \in X} f(x)$.

Formulate and prove an analogous result for upper semicontinuous real-valued functions defined on X.

17. Let X and Y be any two topological spaces and let $f : X \to \mathbf{R}$ and $g : Y \to \mathbf{R}$ be any two lower semicontinuous functions such that

$$f(x) \geq 0, \quad g(y) \geq 0 \qquad (x \in X,\ y \in Y).$$

Define a function $h : X \times Y \to \mathbf{R}$ by the formula

$$h(x, y) = f(x) \cdot g(y) \qquad (x \in X,\ y \in Y).$$

Show that h is also lower semicontinuous.

18*. Let X be a completely regular topological space (for the definition, see, e.g., [64], [107], [149]) and let $f : X \to \mathbf{R}$ be a lower semicontinuous function such that $f(x) \geq 0$ for all $x \in X$.

Prove that there exists a family $(f_i)_{i \in I}$ of functions acting from X into $\mathbf{R}$ and satisfying the following four conditions:
(a) for each $i \in I$, the function f_i is continuous;
(b) for all $i \in I$ and for all $x \in X$, one has $f_i(x) \geq 0$;
(c) $f = \sup_{i \in I} f_i$;
(d) $\mathrm{card}(I) \leq w(X) + \omega$, where $w(X)$ denotes the topological weight of X (i.e., $w(X)$ is the smallest cardinality of a base of X).

In particular, if X has a countable base (i.e., X is metrizable), then f can be represented as a pointwise limit of an increasing sequence of nonnegative real-valued continuous functions defined on X.

19. Let $[a, b]$ be a segment in $\mathbf{R}$ and let $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ be two functions such that $f \geq g$. Suppose, in addition, that f is lower semicontinuous and g is upper semicontinuous.

Demonstrate that there exists a continuous function $h : [a, b] \to \mathbf{R}$ satisfying the inequalities

$$g(x) \leq h(x) \leq f(x) \qquad (x \in [a, b]).$$

Remark 4. The result of Exercise 19 admits a number of generalizations and, actually, is a direct consequence of the well-known Michael theorem on continuous selectors (see [180], [181], [214]).

20. Let X be a second category topological space, $\{f_i : i \in I\}$ be a family of real-valued lower semicontinuous functions on X, and suppose that, for each point $x \in X$, the set $\{f_i(x) : i \in I\}$ is bounded from above.

Show that there exists a nonempty open set $V \subset X$ for which the set $\cup\{f_i(V) : i \in I\}$ is bounded from above, too.

Formulate and prove an analogous result for upper semicontinuous functions.

21*. Let $(G, +)$ be a commutative topological group of second category, $\{f_i : i \in I\}$ be a family of real-valued lower semicontinuous functions on G, and suppose that the following two conditions hold:

(a) for any index $i \in I$, the function f_i is subadditive, i.e., we have

$$f_i(x + y) \leq f_i(x) + f_i(y) \qquad (x \in G,\ y \in G);$$

(b) for each point $x \in G$, the set $\{f_i(x) : i \in I\}$ is bounded from above.

Prove that the given family $\{f_i : i \in I\}$ is locally bounded from above. This means that, for any point $x \in G$, there exists its neighborhood $V(x)$ for which the set $\cup\{f_i(V(x)) : i \in I\}$ is bounded from above.

Formulate and prove an analogous statement for upper semicontinuous functions.

Remark 5. The result presented in Exercise 21 easily implies the well-known Banach–Steinhaus theorem (cf. [17] or [99]).

22. Generalize the notions of lower and upper semicontinuity to those functions which act from a topological space E into the extended real line $\mathbf{R}^* = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$.

Let $f : E \to \mathbf{R}^*$ be a partial function bounded from above (respectively, from below), and suppose that $\mathrm{dom}(f) \neq \emptyset$. For any point x from $\mathrm{cl}(\mathrm{dom}(f))$, define $f^*(x) = \mathrm{limsup}_{y \to x, y \in \mathrm{dom}(f)} f(y)$ (respectively, define $f_*(x) = \mathrm{liminf}_{y \to x, y \in \mathrm{dom}(f)} f(y)$).

Demonstrate that

(a) the function f^* is upper semicontinuous on $\mathrm{cl}(\mathrm{dom}(f))$;

(b) the function f_* is lower semicontinuous on $\mathrm{cl}(\mathrm{dom}(f))$.

Infer from (a) and (b) that

(c) f is upper semicontinuous on $\mathrm{dom}(f)$ if and only if f^* extends f;

(d) f is lower semicontinuous on $\mathrm{dom}(f)$ if and only if f_* extends f.

Observe also that f admits a continuous extension on cl(dom(f)) if and only if the equality $f^* = f_*$ is valid.

23. Let $[0,1]^\omega$ denote the standard Hilbert cube. Introduce two mappings $f_1 : [0,1]^\omega \to [0,1]$ and $f_2 : [0,1]^\omega \to [0,1]$ by putting

$$f_1(x) = \inf(x_n)_{n<\omega}, \qquad f_2(x) = \sup(x_n)_{n<\omega} \qquad (x = (x_n)_{n<\omega} \in [0,1]^\omega).$$

Verify that f_1 is upper semicontinuous and f_2 is lower semicontinuous.

Remark 6. As shown by van Mill and Pol, none of the above two functions is countably continuous (see [183] and Chapter 3 of this book).

24. For a metrizable topological space Y, prove that the property of Y described in Lemma 1 is equivalent to the compactness of Y.

More precisely, demonstrate that, for a metric space Y, the following two assertions are equivalent:

(a) Y is compact;

(b) the canonical projection $\mathrm{pr}_1 : \mathbf{R} \times Y \to \mathbf{R}$ is a closed mapping.

25. Let X be a metric space, Y be a compact metric space, and let f be a mapping acting from X into Y.

Infer from Theorem 1 that the following two conditions are equivalent:

(a) f is a continuous mapping;

(b) the graph G_f is a closed subset of the product space $X \times Y$.

26. Let X and Y be two compact metric spaces and let f be a mapping acting from X into Y.

Deduce from the result of the previous exercise that these two conditions are equivalent:

(a) f is a continuous mapping;

(b) the graph G_f is a compact subset of the product space $X \times Y$.

27. Let f be a function acting from $\mathbf{R}$ into $\mathbf{R}$ and let x be a point of $\mathbf{R}$. Recall that f possesses a symmetric derivative at x if there exists a (finite) limit

$$\lim_{h\to 0, h\neq 0} \frac{f(x+h) - f(x-h)}{2h}.$$

In such a case, this limit is called the symmetric derivative of f at x (denoted by the symbol $f'_s(x)$).

Demonstrate that f can possess a symmetric derivative at a point x being even discontinuous at this point.

Check that if f is differentiable (in the usual sense) at a point x, then there exists a symmetric derivative $f'_s(x)$ and the equality $f'(x) = f'_s(x)$ is fulfilled.

In addition, investigate the question whether an analogue of Theorem 2 holds for the symmetric derivative (instead of the ordinary derivative).

28. For any λ-measurable set $X \subset \mathbf{R}$, verify that the set $d(X)$ introduced in Theorem 4 is a Borel subset of $\mathbf{R}$.

29. Show that if $z \in \mathbf{R}$ is a density point of any two λ-measurable sets X and Y, then z is a density point of the set $X \cap Y$.

Remark 7. The above fact is essential for introducing the so-called density topology on $\mathbf{R}$ which will be discussed in our further considerations (see, e.g., Exercise 9 from Chapter 7). Here we only wish to mention that the density topology on $\mathbf{R}$ can be defined within **ZF** & **DC** theory and is a very special case of a von Neumann topology which is associated with an arbitrary nonzero σ-finite complete measure μ. The theorem on the existence of a von Neumann topology for such a μ is one of the most deep results in measure theory and relies on an uncountable form of the Axiom of Choice (cf. [172], [202], [212], [266], [269]).

30*. Let E be a topological space and let $\{U_i : i \in I\}$ be a family of open subsets of E such that $(\forall i \in I)(U_i$ is a first category set in $E)$.

By using the Zorn lemma, prove that the open set $U = \cup\{U_i : i \in I\}$ is also of first category in E.

Deduce from this fact that if a set $X \subset E$ is locally of first category in E, then X is of first category in E.

Remark 8. The important statement formulated in Exercise 30 is due to Banach (see, e.g., [149] or [202]). It should be underlined that the cardinality of a set I of indices can be arbitrarily large here (in particular, $\mathrm{card}(I)$ can be uncountable).

31. Let X be a λ-measurable subset of $\mathbf{R}$ such that $\lambda(X) > 0$ and, simultaneously, $\lambda(\mathbf{R} \setminus X) > 0$.

Verify that there exists $z \in \mathbf{R}$ such that z is not a density point for X and, simultaneously, z is not a density point for $\mathbf{R} \setminus X$.

Chapter 1
Cantor and Peano type functions

It is widely known that one of the first set-theoretical results of Cantor was his discovery of the existence of a bijection between the set $\mathbf{R}$ of all real numbers and the corresponding product set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ (i.e., the Euclidean plane). For a time, Cantor did not believe that such a bijection exists and even wrote to Dedekind about his doubts in this connection. Of course, Cantor already knew of the existence of a bijection between the set $\mathbf{N}$ of all natural numbers and the product set $\mathbf{N}^2 = \mathbf{N} \times \mathbf{N}$. A simple way to construct such a bijection is the following. We first observe that a function $f : \mathbf{N} \to \mathbf{N} \setminus \{0\}$, defined by the formula $f(n) = n + 1$ for all $n \in \mathbf{N}$, is a bijection between $\mathbf{N}$ and the set of all strictly positive natural numbers. Then, for each integer $n > 0$, we have a unique representation of n in the form $n = 2^k(2l + 1)$, where k and l are some natural numbers. Now, define a function $g : \mathbf{N} \setminus \{0\} \to \mathbf{N} \times \mathbf{N}$ by the formula $g(n) = (k, l)$ for all $n \in \mathbf{N} \setminus \{0\}$. One can immediately check that g is a bijection, which also gives the corresponding bijection between $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$.

Now, the standard partition $(\{0, 2, 4, ...\}, \{1, 3, 5, ...\})$ of $\mathbf{N}$ into two infinite sets shows the existence of a bijection between the two sets $\mathbf{N}$ and $(\mathbf{N} \times \{0\}) \cup (\mathbf{N} \times \{1\})$. By starting with the latter bijection, it is not hard to establish a one-to-one correspondence between the real line $\mathbf{R}$ and the Euclidean plane $\mathbf{R}^2$ (respectively, between the unit segment $[0, 1]$ and the unit square $[0, 1]^2$). Indeed, a simple argument within **ZF** theory yields that the sets $\mathbf{R}$, $[0, 1]$, $2^{\mathbf{N}}$ are equivalent, i.e., there exists a bijective mapping from each of them to any other one. So we only have to verify that the sets $2^{\mathbf{N}}$ and $2^{\mathbf{N}} \times 2^{\mathbf{N}}$ are equivalent, too. But this is obvious since the product set $2^{\mathbf{N}} \times 2^{\mathbf{N}}$ is equivalent with the set $2^{(\mathbf{N} \times \{0\}) \cup (\mathbf{N} \times \{1\})}$, and the latter set is equivalent with $2^{\mathbf{N}}$ because of the existence of a one-to-one correspondence between $\mathbf{N}$ and $(\mathbf{N} \times \{0\}) \cup (\mathbf{N} \times \{1\})$. In a similar way, the existence of a bijection between $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$ implies (in **ZF** theory) the existence of a bijection between $2^{\mathbf{N}}$ and $2^{\mathbf{N} \times \mathbf{N}}$. So, we infer that there is a bijection between $\mathbf{R}$ and $\mathbf{R}^{\mathbf{N}}$. This circumstance also leads to the conclusion that all

Euclidean spaces $\mathbf{R}, \mathbf{R}^2, \mathbf{R}^3, \ldots$ are of the same cardinality.

Keeping in mind these simple constructions, it is reasonable to introduce the following definition.

We say that a mapping f acting from $\mathbf{R}$ into $\mathbf{R}^2$ (respectively, from $[0,1]$ into $[0,1]^2$) is a Cantor type function if f is a bijection.

As mentioned above, Cantor type functions do exist and, in fact, there are many such functions within the framework of **ZF** theory (in this connection, see Exercise 1).

Remark 1. As pointed out earlier, one-to-one correspondences between $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$ (respectively, between $\mathbf{R}$ and $\mathbf{R} \times \mathbf{R}$ or between $[0,1]$ and $[0,1]^2$) can be constructed effectively, i.e., without the aid of the Axiom of Choice. In this context let us recall that, for an arbitrary infinite set X, we also have a bijection between X and $X \times X$, but the existence of such a bijection needs the whole power of the Axiom of Choice. More precisely, according to a classical result of Tarski (cf. [89], [153], [247]), the following two assertions are equivalent in **ZF** theory:

(1) the Axiom of Choice (**AC**);

(2) for any infinite set X, there exists a bijection from X onto $X \times X$.

Exercise 4 of the present chapter explains in more detail the equivalence of (1) and (2).

Now, let f be an arbitrary Cantor type function acting, for example, from $\mathbf{R}$ onto $\mathbf{R}^2$. It is well known that f cannot be continuous. Indeed, suppose for a moment that f is continuous. Then we may write

$$\mathbf{R}^2 = \cup\{f([-n,n]) \ : \ n \in \mathbf{N}\},$$

where each set $f([-n,n])$ $(n \in \mathbf{N})$ is compact (hence closed) in $\mathbf{R}^2$. In accordance with the classical Baire theorem, at least one of these sets has a nonempty interior. Let $k \in \mathbf{N}$ be such that $\mathrm{int}(f([-k,k])) \neq \emptyset$. Then we come to the bijective continuous mapping

$$f|[-k,k] \ : \ [-k,k] \to f([-k,k])$$

that is obviously a homeomorphism between $[-k,k]$ and $f([-k,k])$. But this is impossible since $[-k,k]$ is a one-dimensional space and $f([-k,k])$ is a two-dimensional one. If we want to avoid an argument based on the notion of a dimension of a topological space (and it is reasonable to avoid such an argument because we do not discuss this important notion in our book), we can argue in the following manner. Consider the function

$$f^{-1}|f([-k,k]) \ : \ f([-k,k]) \to [-k,k]$$

that is also a homeomorphism. Let L denote any circle contained in the set $f([-k,k])$, i.e., let L be a subset of $f([-k,k])$ isometric to

$$\{(x,y) \in \mathbf{R}^2 \ : \ x^2 + y^2 = r\},$$

where r is some strictly positive real number (the existence of L is evident since $f([-k,k])$ has a nonempty interior in $\mathbf{R}^2$). We thus see that the function

$$f^{-1}|L \ : \ L \to [-k,k]$$

is injective and continuous. This immediately yields a contradiction since there is no injective continuous function acting from a circle into the real line (cf. Exercise 5).

The following statement is also of some interest in connection with Cantor type functions (see, e.g., [247]).

Theorem 1. *Let f be a function from $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$ continuous with respect to each of the variables $x \in \mathbf{R}$ and $y \in \mathbf{R}$ (separately). Then f is not an injection.*

Proof. Suppose otherwise, i.e., suppose that our f is injective. Denote

$$\phi(x) = f(x,0) \qquad (x \in \mathbf{R}).$$

Then, according to the assumption of the theorem, ϕ is a continuous function acting from $\mathbf{R}$ into $\mathbf{R}$. Let us put $\phi(0) = a$ and $\phi(1) = b$. Because f is injective, we have $a \neq b$. Consequently, either $a < b$ or $b < a$. We may assume, without loss of generality, that $a < b$. The function ϕ, being continuous on the segment $[0,1]$, takes all values from $[\phi(0), \phi(1)] = [a,b]$. In particular, there exists $x_0 \in \ ]0,1[$ such that $\phi(x_0) = (a+b)/2$. Further, let us define a function ψ by putting

$$\psi(y) = f(x_0, y) \qquad (y \in \mathbf{R}).$$

Then ψ is continuous and $\psi(0) = f(x_0, 0) = \phi(x_0) = (a+b)/2$. Hence we get the inequalities $a < \psi(0) < b$ which imply the existence of a neighborhood $U(0)$ of 0 such that $(\forall y \in U(0))(a < \psi(y) < b)$ or, equivalently,

$$(\forall y \in U(0))(a < f(x_0, y) < b).$$

Thus, on the one hand, we can write the inclusion

$$\{f(x_0, y) : y \in U(0) \setminus \{0\}\} \subset \ ]a,b[.$$

On the other hand, we have the relation

$$\{f(x, 0) : 0 \leq x < 1\} \supset [a, b[.$$

Therefore, for some reals $y_0 \neq 0$ and x_1, we get $f(x_0, y_0) = f(x_1, 0)$, which contradicts the injectivity of f. The contradiction obtained finishes the proof of Theorem 1.

It is not difficult to see that Cantor type functions cannot be continuous. (cf. Exercises 6 and 7). In this connection, it is reasonable to ask whether there exist continuous surjections from $\mathbf{R}$ onto $\mathbf{R}^2$ or from $[0, 1]$ onto $[0, 1]^2$. It turned out that such surjections do exist.

The first example of a continuous surjective function acting from $[0, 1]$ onto $[0, 1]^2$ was constructed by Peano. Hence the following definition seems to be natural.

Let $f : [0, 1] \to [0, 1]^2$ be a function. We shall say that f is a Peano type function if f is continuous and surjective.

In order to demonstrate the existence of Peano type functions, we recall the classical Cantor construction of his famous discontinuum. Take the unit segment $[0, 1]$ on the real line $\mathbf{R}$. The first step of Cantor's construction is to remove from this segment the open interval $]1/3, 2/3[$ whose midpoint coincides with the midpoint of $[0, 1]$ and whose length is equal to one-third of the length of $[0, 1]$. After this step we obtain the two segments without common points. Then we apply a similar operation to each of these two segments, etc. After ω-many steps we come to the subset C of $[0, 1]$, which is called the Cantor discontinuum (or the Cantor space). The set C is closed (because we removed open intervals from $[0, 1]$) and, in addition, C is perfect because the removed intervals are disjoint and pairwise have no common end-points. Moreover, the sum of lengths of the removed intervals is equal to 1 (which can easily be checked). So we infer that C is nowhere dense in $\mathbf{R}$ and its Lebesgue measure equals zero. Consequently, C is a small subset of $\mathbf{R}$ from the point of view of the Baire category and from the point of view of the standard Lebesgue measure λ on $\mathbf{R}$. The geometric construction of C described above and due to Cantor is quite visual but, sometimes, other constructions and characterizations of C are needed in order to formulate the corresponding results in a more general form. We present some of such constructions and characterizations of Cantor's discontinuum in Exercises 8 and 9. The abstract characterization of the Cantor space, given in Exercise 9, implies many useful consequences. For instance, by using this characterization, it is not difficult to show that, for each natural number $k \geq 2$, the product space k^ω is homeomorphic to the Cantor discontinuum. Of course, here $k = \{0, 1, ..., k-1\}$ is assumed to be equipped with the discrete topology. For more details, see Exercise 10.

Naturally, the Cantor discontinuum has numerous applications in various branches of mathematics (especially in topology and mathematical analysis). For instance, Exercise 11 of this chapter presents a typical application of C in real analysis.

It immediately follows from the construction of C that, for any clopen set $X \subset C$ and for any real $\varepsilon > 0$, there exists a finite partition of X consisting of clopen subsets of X, each of which has diameter strictly less than ε. It is also easy to check that every zero-dimensional compact metric space possesses the analogous property. At the same time, if E is an arbitrary compact metric space, then for any closed set $X \subset E$ and for any real $\varepsilon > 0$, there exists a finite covering of X consisting of closed subsets of X, each of which has diameter strictly less than ε.

These simple observations lead to the following important statement due to Alexandrov, the proof of which can be carried out within **ZF** & **DC** theory (see, e.g., [59], [64], [107], [150]).

Theorem 2. *Let E be an arbitrary nonempty compact metric space. Then there exists a continuous surjection acting from the Cantor space C onto E.*

Proof. Keeping in mind the preceding remarks, recursively define two sequences $\{(X_{n,k})_{1 \le k \le m(n)} : n \in \omega\}$ and $\{(Y_{n,k})_{1 \le k \le m(n)} : n \in \omega\}$ satisfying these three conditions:

(1) for any $n \in \omega$, the finite family $(X_{n,k})_{1 \le k \le m(n)}$ is a partition of C consisting of clopen subsets of C each of which has diameter strictly less than $1/(n+1)$;

(2) for any $n \in \omega$, the finite family $(Y_{n,k})_{1 \le k \le m(n)}$ is a covering of E by nonempty closed sets each of which has diameter strictly less than $1/(n+1)$;

(3) for any $n \in \omega$ and for any $k \in [1, m(n)]$, there exists a subset $I(n,k)$ of $[1, m(n+1)]$ such that

$$X_{n,k} = \cup\{X_{n+1,i} : i \in I(n,k)\}, \quad Y_{n,k} = \cup\{Y_{n+1,i} : i \in I(n,k)\}.$$

Let now x be an arbitrary point of C. Then x uniquely determines the two sequences $\{k(n) : n \in \omega\}$ and $\{X_{n,k(n)} : n \in \omega\}$ such that

$$(\forall n \in \omega)(1 \le k(n) \le m(n) \ \&\ x \in X_{n,k(n)}).$$

Consider the corresponding sequence of sets $\{Y_{n,k(n)} : n \in \omega\}$. Obviously, we have

$$(\forall n \in \omega)(Y_{n+1,k(n+1)} \subset Y_{n,k(n)}), \quad \lim_{n \to +\infty} \mathrm{diam}(Y_{n,k(n)}) = 0.$$

Hence there exists one and only one point y belonging to all the sets $Y_{n,k(n)}$ $(n \in \omega)$. Let us put $y = f(x)$. In this way we obtain the mapping $f : C \to E$. By starting with the definition of f, it is not hard to verify that f is continuous and surjective. Theorem 2 has thus been proved.

The above theorem immediately implies the existence of Peano type functions. Namely, we have the following statement due to Peano.

Theorem 3. *There exists a Peano type function, i.e., there exists a continuous surjection acting from* $[0,1]$ *onto* $[0,1]^2$.

Proof. It follows from Theorem 2 that there is a continuous surjection $g : C \to [0,1]^2$. Consider any open interval U that was removed from $[0,1]$ during the construction of C. We may add U to the domain of g and extend g in such a way that the extended function would be affine on U and would coincide with g on the end-points of U. Doing this procedure for all removed intervals simultaneously, we come to the function f defined on the whole segment $[0,1]$. The construction of f immediately implies that f is continuous. In addition, since f is an extension of g, we conclude that f is a surjection as well. This completes the proof of Theorem 3.

Furthermore, let us notice that the existence of a Peano type function from $[0,1]$ onto $[0,1]^2$ implies at once the existence of a continuous surjection $h : \mathbf{R} \to \mathbf{R}^2$. Such a function h can also be regarded as a Peano type function (acting from $\mathbf{R}$ onto $\mathbf{R}^2$).

For many interesting properties of Peano type functions, we refer the reader to [78] and [245]. Several exercises of this chapter are also closely connected with the existence of Peano type functions.

As shown by Theorem 2, any nonempty compact metric space is a continuous image of C or, equivalently, of 2^ω. A natural question arises concerning the description of all those Hausdorff topological spaces which are continuous images of C. It turns out that only nonempty compact metrizable spaces are such images (see Exercise 17).

As mentioned earlier, the Cantor set C is small from the topological point of view (i.e., C is a nowhere dense subset of the real line $\mathbf{R}$) and from the measure-theoretical point of view (i.e., C is of Lebesgue measure zero). On the other hand, the operation of vector sum yields the set

$$C + C = \{x + y \ : \ x \in C, \ y \in C\},$$

which is not small at all. Indeed, it can easily be seen that this set contains in itself a nonempty open interval. More precisely, $C + C = [0,2]$. In this connection, see Exercise 19 where a geometric interpretation of the above equality is also given. The result of Exercise 19 shows, in particular, that the operation of vector sum does not preserve the two classical σ-ideals on the real line: the σ-ideal $\mathcal{K}(\mathbf{R})$ of all first category subsets of $\mathbf{R}$ and the σ-ideal $\mathcal{I}(\lambda)$ of all Lebesgue measure zero subsets of $\mathbf{R}$. Let us notice that, at the same time, there exist many σ-ideals on the real line, which are (under some additional set-theoretical axioms) isomorphic to $\mathcal{K}(\mathbf{R})$ and $\mathcal{I}(\lambda)$, are invariant with respect to the group of all translations of $\mathbf{R}$, and are also invariant with respect to the operation of vector sum of sets (see Exercise 20).

By virtue of the preceding considerations, one can conclude that the operation of vector sum is sometimes rather bad from the point of view of preserving σ-ideals on $\mathbf{R}$. Analogously, this operation is bad from the point of view of descriptive set theory. The latter fact may be illustrated, for instance, by various kinds of examples of two G_δ-subsets of $\mathbf{R}$ whose vector sum is not a Borel subset of $\mathbf{R}$ (on the other hand, such a vector sum is always an analytic subset of $\mathbf{R}$). A similar phenomenon can be observed when dealing with the distance set and with the difference set of a given point set lying on the real line or in a finite-dimensional Euclidean space.

We recall that the distance set of a set X lying in a metric space (E, d) is the set of all distances $d(x, y)$ where x and y range over X.

The difference set of a set X lying in a commutative group $(\Gamma, +)$ is the set of all elements of the form $x - y$, where x and y range over X.

Sierpiński was the first mathematician who presented an example of a G_δ-subset of the Euclidean plane, whose distance set is not Borel (see [240]). Much later, several authors constructed two G_δ-subsets of $\mathbf{R}$ whose vector sum is not Borel (see, e.g., [216], [254] and [255]). Moreover, Erdös and Stone constructed in [68] a compact subset of $\mathbf{R}$ and a G_δ-subset of $\mathbf{R}$, such that their vector sum is not a Borel set in $\mathbf{R}$. Here we would like to consider an example of a G_δ-subset of $\mathbf{R}$ whose difference set is not Borel. This example is due to Rogers (see [216]).

First of all, we need to recall the notion of the Hausdorff metric. Let (E, d) be an arbitrary metric space. Denote by the symbol $\mathcal{F}(E)$ the family of all nonempty closed bounded subsets of E. For any two sets $X \in \mathcal{F}(E)$ and $Y \in \mathcal{F}(E)$, we define

$$d'(X, Y) = \inf\{\varepsilon > 0 \ : \ X \subset V_\varepsilon(Y) \ \& \ Y \subset V_\varepsilon(X)\},$$

where $V_\varepsilon(X)$ (respectively, $V_\varepsilon(Y)$) denotes the ε-neighborhood of X (respectively, of Y). It is easy to see that d' is a metric on $\mathcal{F}(E)$, so we get the metric space $(\mathcal{F}(E), d')$. The metric d' is usually called the Hausdorff metric associated with d. If the original metric space (E, d) has good properties, then, sometimes, those properties can be transferred to the space $(\mathcal{F}(E), d')$. For instance, the following assertions are valid:

(*) if (E, d) is complete, then $(\mathcal{F}(E), d')$ is complete, too;

(**) if (E, d) is compact, then $(\mathcal{F}(E), d')$ is compact, too.

Assertions (*) and (**) can be established without any difficulties. On the other hand, let us remark that if the original space (E, d) is separable, then, in general, $(\mathcal{F}(E), d')$ need not be separable. Moreover, it can easily be observed that if (E, d) is bounded but not totally bounded, then $(\mathcal{F}(E), d')$ is necessarily nonseparable.

The Hausdorff metric can be successfully applied in establishing the classical result of Suslin which states the existence of analytic non-Borel subsets of uncountable Polish spaces (see, for instance, [37], [97], [105], [149], [153], [167]). For the sake of completeness, we shall give a short proof of this result, especially taking account of the fact that the proof is essentially based on Theorem 2, which is the main tool for constructing various Peano type functions.

Theorem 4. *Let E be an arbitrary uncountable Polish topological space (or, more generally, an uncountable Borel subset of a Polish space). Then there exists an analytic subset of E which is not Borel.*

Proof. Because all uncountable Borel subsets of Polish spaces are Borel isomorphic, it suffices to show that, in the standard Cantor space $C = 2^\omega$, there exists an analytic subset which is not Borel. In order to do this, let us take the product space $W = 2^\omega \times [0, 1]$ and observe that it is compact. Denote by $\mathcal{F}(W)$ the family of all nonempty closed subsets of W and equip $\mathcal{F}(W)$ with the Hausdorff metric (or, equivalently, with the Vietoris topology that is metrizable by this metric). It is easy to verify that, in such a way, $\mathcal{F}(W)$ becomes a compact metric space (see assertion (**) formulated above). According to Theorem 2, there exists a surjective continuous mapping $h : 2^\omega \to \mathcal{F}(W)$. Further, let Z denote the set of all irrational points of the segment $[0, 1]$. We recall that Z is homeomorphic to the canonical Baire space ω^ω where ω is equipped with the discrete topology. Now, we define a set-valued mapping $\Phi : 2^\omega \to \mathcal{P}(2^\omega)$ by the following formula:

$$\Phi(t) = \mathrm{pr}_1((2^\omega \times Z) \cap h(t)) \qquad (t \in 2^\omega).$$

It is clear that $(2^\omega \times Z) \cap h(t)$ ranges over the family of all closed subsets of the space $2^\omega \times Z$ as t ranges over 2^ω. By starting with this fact, it is not difficult to infer that $\mathrm{ran}(\Phi) = \mathcal{A}(2^\omega)$, where $\mathcal{A}(2^\omega)$ denotes the family of all analytic subsets of 2^ω. Let us put $X = \{t \in 2^\omega : t \notin \Phi(t)\}$ and establish that X is not analytic in 2^ω. Suppose, for a while, that X is analytic. Then, for some $t_0 \in 2^\omega$, we must have the equality $X = \Phi(t_0)$. But, according to the definition of X, we get $t_0 \in X \Leftrightarrow t_0 \notin X$, which obviously yields a contradiction. Consequently, X is not an analytic subset of the Cantor space. On the other hand, let us verify that $2^\omega \setminus X = \{t \in 2^\omega : t \in \Phi(t)\}$ is an analytic subset of the Cantor space. Indeed, $2^\omega \setminus X = \mathrm{pr}_1(D)$ where the set D is defined by the formula

$$D = \{(t, y) \in W \ : \ (t, y) \in h(t) \ \ \& \ \ (t, y) \in 2^\omega \times Z\}.$$

Since h is continuous, the set $D' = \{(t, y) \in W : (t, y) \in h(t)\}$ is closed in W and, therefore, is a G_δ-subset of W. Also, the set

$$D'' = \{(t, y) \in W \ : \ (t, y) \in 2^\omega \times Z\}$$

is a G_δ-subset of W. Hence the intersection $D' \cap D'' = D$ is a G_δ-subset of W, too, and $\mathrm{pr}_1(D)$ is an analytic set in the Cantor space 2^ω. Finally, we easily infer that $\mathrm{pr}_1(D)$ is not Borel because, as has been shown above, the set $X = 2^\omega \setminus \mathrm{pr}_1(D)$ is not analytic. This finishes the proof of Theorem 4.

Example 1. In connection with Theorem 4, it is reasonable to point out that, in classical mathematical analysis, there are many interesting concrete sets of functions which are analytic but not Borel (in an appropriate Polish space). For instance, Mauldin established in [177] that the set of all real-valued continuous functions defined on the unit segment $[0, 1]$ and differentiable at least at one point of $[0, 1]$ is an analytic non-Borel subset of the separable Banach space $C[0, 1]$. Many other such sets can be constructed in the theory of trigonometric series (for more information, see especially [105] and references therein).

Now, keeping in mind Theorem 4, we are going to prove the following statement due to Rogers [216].

Theorem 5. *There exists a G_δ-subset B of $\mathbf{R}$ such that its difference set $B - B$ is not Borel.*

Proof. We begin with some simple observations. First of all, let us introduce two subsets C_1 and C_2 of the unit segment $[0, 1]$. Namely, the set C_1 consists of all real numbers x with decimal expansion $x = 0, t_1t_2t_3...$ where $t_n = 0$ or $t_n = 1$ for every natural index $n \geq 1$; analogously, the set C_2 consists of all real numbers y with decimal expansion $y = 0, t_1t_2t_3...$ where $t_n = 0$ or $t_n = 2$ for every natural index $n \geq 1$.

Obviously, C_1 and C_2 are uncountable closed subsets of the segment $[0, 1]$. Consequently, applying Theorem 4 proved above, we may choose an analytic subset A_1 of C_1 that is not Borel. Also, we can represent the set A_1

as the projection (on the x-axis) of a certain G_δ-subset Z_1 of the Cartesian product $C_1 \times C_2 \subset \mathbf{R}^2$. Now, let us consider a mapping $\phi : \mathbf{R}^2 \to \mathbf{R}$ defined by the formula

$$\phi(x, y) = x + y \qquad ((x, y) \in \mathbf{R}^2).$$

This mapping is continuous. Moreover, by taking account of the definitions of C_1 and C_2, it can easily be checked that the restriction of ϕ to the product set $C_1 \times C_2$ is injective, so is a homeomorphism between $C_1 \times C_2$ and $\phi(C_1 \times C_2)$. Hence $Z_0 = \phi(Z_1) = \{x + y \,:\, (x, y) \in Z_1\}$ is a G_δ-subset of the compact set $\phi(C_1 \times C_2)$. This also implies that Z_0 is a G_δ-subset of $\mathbf{R}$. Now, let us put $B = Z_0 \cup (C_2 - 3)$. Evidently, B is a G_δ-subset of $\mathbf{R}$. We are going to show that the difference set of B is not Borel in $\mathbf{R}$. Denote this difference set by $D = \{x - y : x \in B,\ y \in B\}$. It is enough to establish that the set $D \cap (C_1 + 3)$ is not Borel. First, let us notice that the following inclusions are fulfilled:

$$Z_0 \subset [0, 2], \quad C_2 - 3 \subset [-3, -2], \quad C_1 + 3 \subset [3, 4].$$

Further, it is not difficult to verify that each point of the set $D \cap (C_1 + 3)$ is of the form $u - v$ where

$$u - v \in C_1 + 3, \quad u \in Z_0, \quad v \in C_2 - 3.$$

But the relation $u \in Z_0$ is true if and only if $u = x + y$, where

$$x \in C_1, \quad y \in C_2, \quad (x, y) \in Z_1.$$

Consequently, $D \cap (C_1 + 3)$ is the set of points of the form $x + y - v$ where

$$x + y - v - 3 \in C_1, \ \ x \in C_1, \ \ y \in C_2, \ \ v + 3 \in C_2, \ \ (x, y) \in Z_1.$$

Now, if the points x and y are fixed and $(x, y) \in Z_1$, then there exists one and only one point v satisfying the relations

$$x + y - v - 3 \in C_1, \quad v + 3 \in C_2.$$

Namely, such a point is $v = y - 3$. Hence $D \cap (C_1 + 3)$ coincides with the set $\{x + 3 : (\exists y)((x, y) \in Z_1)\}$. In other words, $D \cap (C_1 + 3) = A_1 + 3$. Because A_1 is not Borel, the set $D \cap (C_1 + 3)$ is not Borel, either. This also shows that the difference set D is not Borel in $\mathbf{R}$. Theorem 5 has thus been proved.

Furthermore, putting $X = B$ and $Y = -B$, we get two G_δ-subsets X and Y of $\mathbf{R}$ for which the vector sum $X + Y$ is not Borel in $\mathbf{R}$.

In order to present another important application of Theorem 2, let us prove the classical Banach–Mazur theorem on the universality of the space $C[0,1]$ for the class of all separable metric spaces.

Let $\mathcal{M}$ be a class of metric spaces and let X be some space from this class. We shall say that X is universal for $\mathcal{M}$ if, for any space Y belonging to $\mathcal{M}$, there exists an isometric embedding of Y into X. In other words, X is universal for $\mathcal{M}$ if $X \in \mathcal{M}$ and all spaces from $\mathcal{M}$ can be realized as isometric copies of appropriate subsets of X.

Let $\mathcal{M}_s$ denote the class of all separable metric spaces. The first example of a space universal for $\mathcal{M}_s$ was constructed by Urysohn. Later, Banach and Mazur discovered that the classical function space $C[0,1]$ is universal for $\mathcal{M}_s$, too. To establish this remarkable fact, we need two simple lemmas.

Lemma 1. *For any metric space (X,d), there exists a Banach space E such that*

(1) X can be isometrically embedded in E;

(2) the weight of E is equal to the weight of X (in particular, if X is separable, then E is separable, too).

Proof. We may assume, without loss of generality, that $X \neq \emptyset$. Let us fix a point $t \in X$. Further, for an arbitrary point $x \in X$, introduce a function $f_x : X \to \mathbf{R}$ by the formula

$$f_x(y) = d(x,y) - d(t,y) \qquad (y \in X).$$

Obviously, f_x is continuous, and the relation

$$|f_x(y)| \leq |d(x,y) - d(t,y)| \leq d(x,t)$$

indicates directly that f_x is bounded. Now, let the symbol $C_b(X)$ denote the Banach space (with respect to the standard sup-norm) of all real-valued bounded continuous functions defined on X. We also introduce a mapping $\phi : X \to C_b(X)$ by the formula $\phi(x) = f_x$ for all $x \in X$. Let us check that this ϕ is an isometric embedding of X into $C_b(X)$. Indeed, for any two elements $x \in X$ and $x' \in X$, we may write

$$||f_x - f_{x'}|| = \sup_{y \in X} |f_x(y) - f_{x'}(y)| = \sup_{y \in X} |d(x,y) - d(x',y)|.$$

It can easily be observed that $\sup_{y \in X}|d(x,y) - d(x',y)| = d(x,x')$. Consequently, we get

$$||\phi(x) - \phi(x')|| = ||f_x - f_{x'}|| = d(x,x'),$$

which shows that ϕ is an isometric embedding. Finally, define E as the closed vector subspace of $C_b(X)$ generated by the set $\phi(X)$. Then E is obviously a Banach space whose weight is equal to the weight of $\phi(X)$ or, equivalently, to the weight of X. We thus conclude that E is the required Banach space, which completes the proof of Lemma 1.

Before formulating the next lemma, we need some auxiliary notions from the theory of topological vector spaces (below, we restrict ourselves to the class of topological vector spaces over the field $\mathbf{R}$).

Let E and E' be any two vector spaces over $\mathbf{R}$. We say that these spaces are in a duality if a bilinear function $\Psi : E \times E' \to \mathbf{R}$ is given such that

(a) for each $x \neq 0$ from E, the partial linear functional

$$y \to \Psi(x, y) \qquad (y \in E')$$

is not identically equal to zero;

(b) for each $y \neq 0$ from E', the partial linear functional

$$x \to \Psi(x, y) \qquad (x \in E)$$

is not identically equal to zero.

In such a situation, it is usually said that Ψ establishes a duality between the given spaces E and E'.

In particular, let E be an arbitrary Banach space (or, more generally, normed vector space). Denote by E^* the vector space of all continuous linear functionals on E. Evidently, a bilinear function $\Phi : E^* \times E \to \mathbf{R}$ defined by the formula

$$\Phi(u, x) = u(x) \qquad (u \in E^*,\ x \in E)$$

establishes a canonical duality between E^* and E.

Let X be a subset of E. We equip E^* with the weakest topology $\sigma(E^*, X)$ for which all linear functionals from the family $(\Phi(\cdot, x))_{x \in X}$ are continuous. Clearly, the pair $(E^*, \sigma(E^*, X))$ is a topological vector space.

Lemma 2. *Let E be a separable normed vector space and let B denote the closed unit ball in E^*. Then B is a compact metric space with respect to the topology induced by $\sigma(E^*, E)$.*

Proof. According to Exercise 23, the topology $\sigma(E^*, E)$ restricted to B coincides with the topology $\sigma(E^*, \{x_n : n \in \omega\})$ restricted to B, where $\{x_n : n \in \omega\}$ is any countable everywhere dense subset of E. Let us define a mapping $h : B \to \mathbf{R}^\omega$ by the formula $h(u) = (u(x_n))_{n \in \omega}$ for all u from

B. Actually, h maps B into the product space $\prod_{n\in\omega}[-||x_n||, ||x_n||]$, which obviously is compact and metrizable.

By starting with the definition of $\sigma(E^*, \{x_n : n \in \omega\})$ and remembering that h is injective, it is not hard to demonstrate that B is homeomorphic to some closed subset of the above-mentioned product space and, consequently, B is compact and metrizable as well. Lemma 2 has thus been proved.

We now are ready to prove the following remarkable result of Banach and Mazur.

Theorem 6. *The space $C[0,1]$ is universal for the class $\mathcal{M}_s$.*

Proof. Let X be an arbitrary separable metric space. According to Lemma 1, there exists a separable Banach space E containing an isometric copy of X. Hence it suffices to show that E can be isometrically embedded in $C[0,1]$. Let us denote by B the closed unit ball in E^*. Because E is separable, the ball B equipped with the topology induced by $\sigma(E^*, E)$ is compact and metrizable (see Lemma 2). According to Theorem 2, there exists a continuous surjection g from the Cantor space $C \subset [0,1]$ onto B. It is easy to see that g can be extended to a continuous mapping acting from $[0,1]$ onto B (cf. the proof of Theorem 3). For the sake of simplicity, the extended mapping will be denoted by the same symbol g. Now, take any element x from E and define a function $f_x : [0,1] \to \mathbf{R}$ by the formula

$$f_x(t) = g(t)(x) \qquad (t \in [0,1]).$$

Because g is continuous, f_x is continuous, too. Moreover, $g(t) \in B$ for each $t \in [0,1]$, and we get $|f_x(t)| \leq ||x||$ whenever $t \in [0,1]$, i.e., f_x is also bounded. If y is another element from E, then, for each $t \in [0,1]$, we may write $|f_x(t) - f_y(t)| = |g(t)(x-y)| \leq ||x-y||$. On the other hand, a simple consequence of the Hahn–Banach theorem says that if $x \neq y$, then there is a continuous linear functional $u : E \to \mathbf{R}$ satisfying the following two relations:

$$||u|| = 1, \quad u(x-y) = ||x-y||.$$

In particular, $u \in B$ and, since g is a surjection from $[0,1]$ onto B, there exists a point $t_0 \in [0,1]$ such that $u = g(t_0)$. Then we have

$$|f_x(t_0) - f_y(t_0)| = |u(x-y)| = ||x-y||,$$

which shows that $||f_x - f_y|| = ||x-y||$ and, consequently, the mapping $x \to f_x$ $(x \in E)$ is an isometric embedding of E into $C[0,1]$. This completes the proof of Theorem 6.

Remark 2. According to the preceding result, the space $C[0,1]$ is universal for the class $\mathcal{M}_s$. It is also known that some proper vector subspaces of $C[0,1]$ are also universal for $\mathcal{M}_s$ (cf. [215]). At the same time, it is obvious that $C[0,1]$ cannot be universal for the class $\mathcal{M}(\mathbf{c})$ consisting of all those metric spaces whose cardinalities are less than or equal to the cardinality of the continuum $\mathbf{c}$ (clearly, we have the proper inclusion $\mathcal{M}_s \subset \mathcal{M}(\mathbf{c})$). The problem of the existence of a metric space $X \in \mathcal{M}(\mathbf{c})$ universal for $\mathcal{M}(\mathbf{c})$ was investigated, with related problems for other infinite cardinals, by Sierpiński (see [246] and [248]).

Evidently, analogous questions about the existence of universal objects can be posed for various mathematical structures: for algebraic structures (e.g., groups), for certain types of binary relations, for topological spaces, and so on. There are some important results in this direction. For instance, let us recall the well-known theorem of Cantor stating that the set $\mathbf{Q}$ equipped with its standard order is universal for the class of all countable linearly ordered sets, i.e., every countable linearly ordered set can be isomorphically embedded in $\mathbf{Q}$. The following example partially describes the situation for linearly ordered sets whose cardinalities are less than or equal to $\mathbf{c}$.

Example 2. Suppose that the Continuum Hypothesis (**CH**) holds. Then there exists a linearly ordered set $(X, \leq)$ with $\mathrm{card}(X) = \mathbf{c}$ such that, for any linearly ordered set $(Y, \leq)$ with $\mathrm{card}(Y) \leq \mathbf{c}$, there is a monomorphism from Y into X. For a proof, see, e.g., [153] or [247] where related results for other infinite cardinals are also discussed.

EXERCISES

1. Denoting by $\Psi(\mathbf{R})$ the family of all bijections acting from $\mathbf{R}$ onto $\mathbf{R}$, check within **ZF** theory that

$$\mathrm{card}(\Psi(\mathbf{R})) = 2^{\mathbf{c}}.$$

By using this fact, establish in the same **ZF** theory that the cardinality of the family of all those Cantor type functions which act from $\mathbf{R}$ onto $\mathbf{R}^2$ is equal to $2^{\mathbf{c}}$.

2. Prove that if $\mathbf{R} = X \cup Y$, then $\mathrm{card}(X) = \mathbf{c}$ or $\mathrm{card}(Y) = \mathbf{c}$.

For this purpose, use the existence of a bijection between $\mathbf{R}$ and $\mathbf{R}^2$.

Analogously, by using the existence of a bijection between $\mathbf{R}$ and $\mathbf{R}^\omega$, prove that if $\mathbf{R} = \cup\{X_n : n < \omega\}$, then there is an index $n_0 < \omega$ such that $\mathrm{card}(X_{n_0}) = \mathbf{c}$.

Observe that both these facts need some form of the Axiom of Choice.

On the other hand, demonstrate in **ZF** theory that $\mathbf{c} \neq \omega_\omega$.

3. Work in **ZF** theory and show that if an infinite set X satisfies the relation $\mathrm{card}((X \times \{0\}) \cup (X \times \{1\})) = \mathrm{card}(X)$, then

$$\mathrm{card}(\mathcal{P}(X)) = \mathrm{card}(\mathcal{P}(X) \times \mathcal{P}(X)).$$

Deduce from this fact that if for an infinite set X we have the equality $\mathrm{card}(X \times X) = \mathrm{card}(X)$, then $\mathrm{card}(\mathcal{P}(X)) = \mathrm{card}(\mathcal{P}(X) \times \mathcal{P}(X))$.

4*. Let X be an arbitrary set.

Prove within **ZF** theory that there exists a well-ordered set $(Y, \leq)$ such that there is no injection from Y into X.

One may suppose, without loss of generality, that $X \cap Y = \emptyset$ for the above-mentioned Y.

Demonstrate (in the same **ZF** theory) that if $\mathrm{card}(X \times Y) \leq \mathrm{card}(X \cup Y)$, then there exists an injection from X into Y and, consequently, X can be made well ordered.

Show also (again in **ZF**) that the relation

$$\mathrm{card}((X \cup Y) \times (X \cup Y)) \leq \mathrm{card}(X \cup Y)$$

implies the inequality $\mathrm{card}(X \times Y) \leq \mathrm{card}(X \cup Y)$.

Deduce from these results that in **ZF** theory the following two assertions are equivalent:

(a) the Axiom of Choice (**AC**);

(b) for any infinite set X, the equality $\mathrm{card}(X \times X) = \mathrm{card}(X)$ is valid.

Remark 3. The equivalence (a) $\Leftrightarrow$ (b) was first obtained by Tarski. In this context, consider the following assertion:

(b′) the equality $\mathrm{card}((X \times \{0\}) \cup (X \times \{1\})) = \mathrm{card}(X)$ holds true for any infinite set X.

It was established that (b′) is strictly weaker than the Axiom of Choice.

5. Let L be an arbitrary circle on the plane $\mathbf{R}^2$ and let $g : L \to \mathbf{R}$ be a continuous mapping. By using the classical Cauchy theorem on intermediate values for continuous functions, prove that there exist two points $z \in L$ and $z' \in L$ satisfying the following relations:

(a) $g(z) = g(z')$;

(b) z and z' are antipodal in L, i.e., the line segment $[z, z']$ in $\mathbf{R}^2$ is a diameter of L.

Conclude that the mapping g cannot be an injection.

Remark 4. The simple result of Exercise 5 admits an important generalization to the case of an n-dimensional sphere (instead of L) and of an n-dimensional Euclidean space (instead of $\mathbf{R}$). The corresponding statement is known as the Borsuk–Ulam theorem on antipodes and plays an essential role in applications of algebraic topology (see, for example, [150] or [193]). In particular, this theorem shows that there are no injective continuous mappings acting from the sphere $\mathbf{S}^n$ into the space $\mathbf{R}^n$.

6. Give an answer to the following question:

Does there exist an injective mapping $f : \mathbf{R}^2 \to \mathbf{R}$ continuous with respect to one of the variables $x \in \mathbf{R}$ and $y \in \mathbf{R}$?

7. Check that there exists a bijection $f : [0,1] \to [0,1]^2$ such that the function $\mathrm{pr}_1 \circ f$ is continuous, where $\mathrm{pr}_1 : [0,1]^2 \to [0,1]$ denotes, as usual, the first canonical projection from $[0,1]^2$ onto $[0,1]$.

More generally, let $f_1 : [0,1] \to [0,1]$ be a function satisfying the relation

$$(\forall x \in [0,1])(\mathrm{card}(f_1^{-1}(x)) = \mathbf{c}).$$

Show that there exists a function $f_2 : [0,1] \to [0,1]$ such that the mapping $f = (f_1, f_2)$ is a bijection between $[0,1]$ and $[0,1]^2$.

8. Take the two-element set $2 = \{0,1\}$ and equip this set with the discrete topology. Equip also the Cartesian product 2^ω with the usual product topology.

Demonstrate that 2^ω is homeomorphic to the classical Cantor discontinuum $C \subset [0,1]$.

9*. Let E be a topological space.

Show that E is homeomorphic to C if and only if the conjunction of the following four relations holds:

(a) E is nonempty and compact;

(b) E has a countable base;

(c) there are no isolated points in E;

(d) E is zero-dimensional, i.e., for each $e \in E$ and for any neighborhood $U(e)$ of e, there exists a neighborhood $V(e)$ of e such that $V(e) \subset U(e)$ and $\mathrm{bd}(V(e)) = \emptyset$, where the symbol $\mathrm{bd}(V(e))$ denotes the boundary of $V(e)$.

Remark 5. Actually, relation (d) of Exercise 9 means that the family of all clopen subsets of E forms a base for E.

10. Demonstrate that, for each natural number $k > 2$, the space k^ω is homeomorphic to the Cantor discontinuum $C \subset [0,1]$.

For this purpose, use Exercise 9.

Verify also that ω^ω is homeomorphic to the space of all irrational real numbers (where ω is equipped with the discrete topology).

What can you say about $C \setminus K$, where K is the set of all end-points of the removed intervals in the construction of C?

11. Indicate a set C' on the unit segment $[0,1]$ such that

(a) C' is the image of the Cantor discontinuum C under some homeomorphism of $[0,1]$ onto itself;

(b) the Lebesgue measure of C' is strictly positive.

Deduce from relations (a) and (b) that the Lebesgue measure λ is not quasi-invariant with respect to the group of all homeomorphisms of $\mathbf{R}$, i.e., this group does not preserve the σ-ideal of all λ-measure zero sets.

12. Let X be an arbitrary nonempty closed subset of the Cantor discontinuum C.

Check that there exists a continuous mapping $f : C \to X$ satisfying the relation $(\forall x \in X)(f(x) = x)$.

In other words, check that each nonempty closed subset of C is a retract of C.

Remark 6. The above simple but useful result follows directly from the well-known Michael theorem concerning the existence of continuous selectors for lower semicontinuous set-valued mappings defined on zero-dimensional paracompact spaces (in this context, see [180], [181], [214]).

13. Using the result of Exercise 12, give another proof of Alexandrov's theorem.

Namely, starting with a canonical continuous surjection $h : 2^\omega \to [0,1]$, show that there exists a continuous surjection $h_1 : 2^\omega \to [0,1]^\omega$. Then, for each closed subset Y of $[0,1]^\omega$, show that there exists a closed subset X of C such that $h_1(X) = Y$. Finally, apply the classical theorem of Urysohn stating that every compact metric space can be realized as a closed subset of the space $[0,1]^\omega$ (see, e.g., [64], [107]).

14. Let X be a nonempty perfect subset of the real line $\mathbf{R}$.

Demonstrate that there exists a continuous surjection from X onto the unit square $[0,1]^2$ and infer from this result that there exists a disjoint family $\{P_j : j \in J\} \subset \mathcal{P}(X)$ of nonempty perfect subsets of $\mathbf{R}$ such that $\mathrm{card}(J) = \mathbf{c}$.

15. Prove that, for any Peano type function $f = (f_1, f_2) : \mathbf{R} \to \mathbf{R}^2$, the relation

$$\mathbf{R} \neq \{x \in \mathbf{R} : f_1'(x) \text{ exists}\} \cup \{x \in \mathbf{R} : f_2'(x) \text{ exists}\}$$

holds true (cf. Exercise 7 from Chapter 17).

16. By starting with the existence of a Peano type function acting from the segment $[0,1]$ onto the square $[0,1]^2$, demonstrate that there exists an injective continuous mapping $g : [0,1] \to \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ satisfying the following two conditions:

(a) the orthogonal projection of $g([0,1])$ on the plane $\mathbf{R} \times \mathbf{R} \times \{0\}$ coincides with $[0,1]^2$;

(b) every plane in $\mathbf{R}^3$ parallel to $\mathbf{R} \times \mathbf{R} \times \{0\}$ has at most one common point with $g([0,1])$.

17*. Let E be a topological space. A family $\mathcal{S}$ of subsets of E is called a net in E (in the sense of Archangelskii) if, for any open set $U \subset E$, there exists a subfamily of $\mathcal{S}$ whose union coincides with U.

Clearly, every base of E is a net but the converse assertion does not hold in general.

Now, let E be a compact space and let κ be an infinite cardinal number.

Show that if E admits a net of cardinality not exceeding κ, then E possesses also a base of cardinality not exceeding κ.

Deduce from this fact that if E' and E'' are two compact spaces and $g : E' \to E''$ is a continuous surjection, then the topological weight of E'' is less than or equal to the topological weight of E'. In particular, if E' has a countable base, then E'' has a countable base, too.

Conclude from this result that if a Hausdorff topological space Y is a continuous image of Cantor's discontinuum C, then Y is compact and metrizable.

18*. Let κ be an arbitrary infinite cardinal number. Equip the set 2^κ with the product topology (where the two-element set $2 = \{0,1\}$ is endowed with the discrete topology). The space 2^κ is usually called the generalized Cantor discontinuum (of weight κ). This space can also be regarded as a commutative compact topological group with respect to the addition operation modulo 2. Hence there exists a Haar probability measure μ on 2^κ which is invariant under the group of all translations of 2^κ.

By starting with the fact that $\mu(U) > 0$ for each nonempty open subset U of 2^κ, show that 2^κ satisfies the so-called Suslin condition or countable chain condition, i.e., any disjoint family of nonempty open subsets of 2^κ is at most countable.

Deduce from this fact that if a topological space E is a continuous image of 2^κ, then E satisfies the countable chain condition.

Conclude that there exists a nonempty compact topological space X such that, for all $\kappa \geq \omega$, there is no continuous surjection from 2^κ onto X.

On the other hand, demonstrate that, for any compact topological space Y of weight κ, there exists a closed subset Z of 2^κ such that Y is a continuous image of Z.

Remark 7. The results presented in Exercise 18 are essentially due to Szpilrajn (Marczewski).

19. Prove that Cantor's set C satisfies the relation $C + C = [0, 2]$ and, by using the canonical continuous mapping

$$(x, y) \to x + y \qquad ((x, y) \in C \times C),$$

give a simple geometrical interpretation of the above relation.

For this purpose, represent $C \times C$ in the form $C \times C = \cap\{K_n : n < \omega\}$, where $\{K_n : n < \omega\}$ is a decreasing (by inclusion) sequence of compact subsets of $[0, 1]^2$, and the orthogonal projection of each K_n on the line $\{(x, y) \in \mathbf{R}^2 : x = y\}$ is equal to the line segment with end-points $(0, 0)$ and $(1, 1)$.

20. Consider on the Euclidean plane $\mathbf{R}^2$ the family of all straight lines parallel to the line $\{0\} \times \mathbf{R}$. Let $\mathcal{J}$ denote the σ-ideal of subsets of the plane, generated by this family.

Verify that

(a) $\mathcal{J}$ is invariant under the group of all translations of $\mathbf{R}^2$;

(b) $\mathcal{J}$ is invariant under the operation of vector sum of subsets of $\mathbf{R}^2$.

Remark 8. Notice that, by assuming the Continuum Hypothesis (or, more generally, by assuming Martin's Axiom), it can be shown that $\mathcal{J}$ is isomorphic to each of the σ-ideals $\mathcal{K}(\mathbf{R}^2)$ and $\mathcal{I}(\lambda_2)$, where λ_2 denotes the standard two-dimensional Lebesgue measure on $\mathbf{R}^2$. This fact follows directly from the Sierpiński–Erdös Duality Principle (see, for instance, [43], [192], [202]). Let us also mention that the σ-ideals $\mathcal{K}(\mathbf{R})$ and $\mathcal{K}(\mathbf{R}^2)$ (respectively, $\mathcal{I}(\lambda)$ and $\mathcal{I}(\lambda_2)$) are isomorphic to each other, and the corresponding isomorphisms can be constructed within **ZF** & **DC** theory (see, e.g., [43]).

21. Taking into account the fact that the additive group $(\mathbf{R}, +)$ is isomorphic to the additive group $(\mathbf{R}^2, +)$ (see, e.g., Chapter 11 of this book), demonstrate that there exists a σ-ideal $\mathcal{I}$ of subsets of $\mathbf{R}$, satisfying these three conditions:

(a) $\mathcal{I}$ is invariant with respect to the group of all translations of $\mathbf{R}$;

(b) $\mathcal{I}$ is invariant with respect to the operation of vector sum of subsets of $\mathbf{R}$;

(c) under the Continuum Hypothesis (or, more generally, under Martin's Axiom), $\mathcal{I}$ is isomorphic to each of the σ-ideals $\mathcal{K}(\mathbf{R})$ and $\mathcal{I}(\lambda)$.

22. Let $\lambda : \mathcal{F}([0,1]) \to [0,1]$ denote the standard Lebesgue measure restricted to the family of all nonempty closed subsets of the unit segment $[0,1]$. Equipping this family with the Hausdorff metric, one obtains the compact metric space $\mathcal{F}([0,1])$.

Check that the function λ is upper semicontinuous on this space.

23. Let X be an everywhere dense subset of a normed vector space E (over the field $\mathbf{R}$) and let $B = \{u \in E^* \ : \ ||u|| \leq 1\}$.

Verify that the topologies on B induced by $\sigma(E^*, X)$ and $\sigma(E^*, E)$, respectively, are identical.

24*. Starting with the fact that $(\mathbf{Q}, \leq)$ is universal for the class of all countable linearly ordered sets, prove in **ZF** theory that there exists a partition $\{A_\xi : \xi < \omega_1\}$ of $\mathbf{R}$.

Remark 9. The result of Exercise 24 is due to Lebesgue (see [161]). As was shown by Luzin and Sierpiński in their work [170], within the same **ZF** theory there exists a partition $\{B_\xi : \xi < \omega_1\}$ of $\mathbf{R}$ such that all the sets B_ξ $(\xi < \omega_1)$ are Borel in $\mathbf{R}$. Moreover, it is known that an arbitrary non-Borel analytic (co-analytic) set $X \subset \mathbf{R}$ admits a canonical representation $X = \cup\{X_\xi : \xi < \omega_1\}$, where all the sets X_ξ $(\xi < \omega_1)$ are nonempty, pairwise disjoint, and Borel in $\mathbf{R}$ (for details, see [97], [105], [149], [153]).

25. Using the existence of $\{A_\xi : \xi < \omega_1\}$ mentioned in Exercise 24, prove within **ZF** theory that the set $\mathbf{R}$ is uncountable and that the power set $\mathcal{P}(\mathbf{R})$ cannot be represented as the union of a countable family of countable sets.

In addition, deduce from the existence of $\{A_\xi : \xi < \omega_1\}$ that the inequality $2^{\omega_1} \leq 2^{\mathbf{c}}$ holds true within **ZF** theory. Consequently, one has in **ZF** the relation $\omega_1 < 2^{\omega_1} \leq 2^{\mathbf{c}}$.

Remark 10. The result of Exercise 25 is due to Tarski. It is useful to compare this result with the theorem of **ZF** theory, stating that $\mathbf{R}$ cannot be represented as the union of a countable family of finite sets (see Exercise 2 from Chapter 0). Notice also that the inequality $\omega_1 \leq \mathbf{c}$ cannot be established even within **ZF** & **DC** theory, because $\omega_1 \leq \mathbf{c}$ implies the existence of a subset of $\mathbf{R}$ nonmeasurable in the Lebesgue sense (the result of Shelah [228] and Raisonnier [210]).

Chapter 2
Functions of first Baire class

Let E be a topological space and let $f : E \to \mathbf{R}$ be a function.

We shall say that f is of Baire zero class on E if f is continuous at all points of E (i.e., in short, f is continuous on E).

The family of all continuous functions acting from E into $\mathbf{R}$ will be denoted by the symbol $C(E, \mathbf{R})$. In accordance with the definition above, we will also use the notation $Ba_0(E, \mathbf{R})$ for the same family of functions. Thus, we have $Ba_0(E, \mathbf{R}) = C(E, \mathbf{R})$.

By the standard definition due to Baire (see [8], [149], [202]), a function $f : E \to \mathbf{R}$ is of first Baire class (or is of Baire one class) on E if there exists a sequence $\{f_n : n < \omega\} \subset Ba_0(E, \mathbf{R})$ such that $\lim_{n \to +\infty} f_n(x) = f(x)$ for all $x \in E$. In other words, $f : E \to \mathbf{R}$ belongs to the first Baire class if and only if f can be represented as a pointwise limit of a sequence of functions belonging to the class $Ba_0(E, \mathbf{R})$.

It is well known that functions of first Baire class play a significant role in various topics of real analysis. The following simple but important example emphasizes this circumstance.

Example 1. Let $E = \mathbf{R}$ and let $f : E \to \mathbf{R}$ be a derivative, i.e., there exists a function $g : E \to \mathbf{R}$ such that $g'(x) = f(x)$ for every $x \in E$. Define a sequence $\{g_n : 1 \le n < \omega\}$ of real-valued functions on $\mathbf{R}$ by the formula

$$g_n(x) = n(g(x + 1/n) - g(x)) \qquad (x \in \mathbf{R},\ n = 1, 2, ...).$$

Obviously, we have $\lim_{n \to +\infty} g_n(x) = f(x)$ for all $x \in \mathbf{R}$. Because each g_n is a continuous function on $\mathbf{R}$, we conclude that f belongs to the first Baire class.

For any topological space E, the family of all functions $f : E \to \mathbf{R}$ belonging to the first Baire class will be denoted by $Ba_1(E, \mathbf{R})$.

Notice that the other Baire classes $Ba_\xi(E, \mathbf{R})$ of real-valued functions on E can be naturally introduced by iterating the limit process and using the method of transfinite recursion on $\xi < \omega_1$ (in this connection, see [8], [149],

[194] and Chapter 8 of the present book). Here we are mainly interested in various properties of functions belonging to $Ba_1(E, \mathbf{R})$.

The following simple properties follow directly from the definition of the class $Ba_1(E, \mathbf{R})$:

(1) $Ba_1(E, \mathbf{R})$ is an algebra over the field $\mathbf{R}$, i.e., if $\{f, g\} \subset Ba_1(E, \mathbf{R})$ and $\{a, b\} \subset \mathbf{R}$, then $af + bg \in Ba_1(E, \mathbf{R})$ and $f \cdot g \in Ba_1(E, \mathbf{R})$;

(2) if $f \in Ba_1(E, \mathbf{R})$, $g \in Ba_1(E, \mathbf{R})$, and $g(x) \neq 0$ for all $x \in E$, then $f/g \in Ba_1(E, \mathbf{R})$;

(3) if $f \in Ba_1(E, \mathbf{R})$ and $\phi : \ ]a, b[\ \to \mathbf{R}$ is a continuous function such that $\mathrm{ran}(f) \subset \ ]a, b[$, then $\phi \circ f \in Ba_1(E, \mathbf{R})$.

Let us consider some other, less trivial, properties of $Ba_1(E, \mathbf{R})$.

Lemma 1. *Let E be a topological space, let $\{a_n : 1 \leq n < \omega\}$ be a sequence of strictly positive real numbers such that $\sum_{1 \leq n < \omega} a_n < +\infty$, and let $\{f_n : 1 \leq n < \omega\} \subset Ba_1(E, \mathbf{R})$ be a sequence of functions such that*

$$|f_n(x)| < a_n \qquad (x \in E, \ 1 \leq n < \omega).$$

Define a function $f : E \to \mathbf{R}$ by $f(x) = f_1(x) + f_2(x) + ... + f_n(x) + ...$ for all $x \in E$. Then f also belongs to $Ba_1(E, \mathbf{R})$.

Proof. First, notice that the function f is well defined since the series

$$f_1(x) + f_2(x) + ... + f_n(x) + ... \qquad (x \in E)$$

converges uniformly on E in view of $|f_n(x)| < a_n$ $(x \in E, \ n = 1, 2, ...)$. Further, because $f_n \in Ba_1(E, \mathbf{R})$ for any natural number $n \geq 1$, we can write

$$f_n(x) = \lim_{k \to +\infty} f_{n,k}(x) \qquad (x \in E),$$

where $f_{n,k}$ $(k = 1, 2, ...)$ are real-valued continuous functions on E. Without loss of generality, we may assume that

$$|f_{n,k}(x)| \leq a_n \qquad (x \in E, \ n = 1, 2, ..., \ k = 1, 2, ...).$$

Now, let us put $h_k(x) = f_{1,k}(x) + f_{2,k}(x) + ... + f_{k,k}(x)$ for each $x \in E$ and $k = 1, 2, ...$ Clearly, all functions h_k are continuous on E and it suffices to show that

$$f(x) = \lim_{k \to +\infty} h_k(x) \qquad (x \in E).$$

For this purpose, fix a real $\varepsilon > 0$. There exists a natural number m such that $a_{m+1} + a_{m+2} + ... + a_i + ... < \varepsilon/3$. Consequently, we have

$$|f_{m+1}(x)| + |f_{m+2}(x)| + ... + |f_i(x)| + ... < \varepsilon/3 \qquad (x \in E),$$

$$|f_{m+1,k}(x)| + |f_{m+2,k}(x)| + ... + |f_{i,k}(x)| + ... \leq \varepsilon/3 \qquad (x \in E,\ k = 1, 2, ...).$$

Further, for any $x \in E$ and $k > m$, we may write the inequalities

$$|f(x) - h_k(x)| \leq |f_1(x) - f_{1,k}(x)| + |f_2(x) - f_{2,k}(x)| + ... + |f_m(x) - f_{m,k}(x)| +$$

$$|f_{m+1}(x)| + |f_{m+2}(x)| + ... + |f_i(x)| + ... +$$

$$|f_{m+1,k}(x)| + |f_{m+2,k}(x)| + ... + |f_{k,k}(x)|$$

$$\leq |f_1(x) - f_{1,k}(x)| + |f_2(x) - f_{2,k}(x)| + ... + |f_m(x) - f_{m,k}(x)| + 2\varepsilon/3.$$

If $x \in E$ is fixed, then we can find $k_0 < \omega$ so large that for all natural numbers $k > k_0$ the relation

$$|f_1(x) - f_{1,k}(x)| + |f_2(x) - f_{2,k}(x)| + ... + |f_m(x) - f_{m,k}(x)| < \varepsilon/3$$

would be satisfied. This yields the inequality $|f(x) - h_k(x)| < \varepsilon$ for all integers $k > k_0$. Therefore, we get $\lim_{k \to +\infty} h_k(x) = f(x)$ for any $x \in E$, which completes the proof of Lemma 1.

We need this lemma in order to prove the following result due to Baire.

Theorem 1. *Let E be a topological space and let $\{f_n : n < \omega\}$ be a sequence of functions from $Ba_1(E, \mathbf{R})$ uniformly convergent to a function $f : E \to \mathbf{R}$. Then we have $f \in Ba_1(E, \mathbf{R})$.*

Proof. According to the assumption, for any natural number k, there exists a natural number n_k such that $|f(x) - f_{n_k}(x)| < 1/2^{k+1}$ for all $x \in E$. Evidently, we may assume that $n_0 < n_1 < ... < n_k < ...$. Let us consider the series of functions $(f_{n_1} - f_{n_0}) + (f_{n_2} - f_{n_1}) + ... + (f_{n_{k+1}} - f_{n_k}) + ...$. Since the inequalities

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq |f_{n_{k+1}}(x) - f(x)| + |f_{n_k}(x) - f(x)| < \frac{1}{2^{k+2}} + \frac{1}{2^{k+1}} < \frac{1}{2^k}$$

hold for all $x \in E$, we can apply Lemma 1 to the above series. So we obtain that the function $g = (f_{n_1} - f_{n_0}) + (f_{n_2} - f_{n_1}) + ... + (f_{n_{k+1}} - f_{n_k}) + ...$ belongs to $Ba_1(E, \mathbf{R})$. Also, $g = \lim_{k \to +\infty} f_{n_{k+1}} - f_{n_0} = f - f_{n_0}$. According to property (1), we finally get $f = g + f_{n_0} \in Ba_1(E, \mathbf{R})$. The theorem has thus been proved.

Remark 1. Theorem 1 implies that, for $E \neq \emptyset$, the family of all bounded functions from $Ba_1(E, \mathbf{R})$ is a Banach space with respect to the norm of uniform convergence or, equivalently, with respect to the standard sup-norm

$$||f|| = \sup_{x \in E} |f(x)|.$$

Remark 2. Theorem 1 can be directly generalized to the case of the Baire class $Ba_\xi(E, \mathbf{R})$, where ξ is an arbitrary ordinal number strictly less than ω_1. The proof essentially remains the same as above (cf. [8], [149]).

Lemma 2. *Let E be a topological space and let $g \in Ba_1(E, \mathbf{R})$. Then $g^{-1}(]-\infty, t[) = \{x \in E : g(x) < t\}$ and $g^{-1}(]t, +\infty[) = \{x \in E : g(x) > t\}$ are F_σ-subsets of E for every $t \in \mathbf{R}$.*

Proof. Take any $t \in \mathbf{R}$. Because $g \in Ba_1(E, \mathbf{R})$, there exists a sequence $\{g_n : n < \omega\} \subset Ba_0(E, \mathbf{R})$ such that $\lim_{n \to +\infty} g_n(x) = g(x)$ for each $x \in E$. It is not difficult to verify the following two relations:

$$g(x) < t \Leftrightarrow (\exists k < \omega)(\exists n < \omega)(\forall m \in [n, \omega[)(g_m(x) \leq t - 1/k),$$

$$g(x) > t \Leftrightarrow (\exists k < \omega)(\exists n < \omega)(\forall m \in [n, \omega[)(g_m(x) \geq t + 1/k).$$

These relations yield at once that the above-mentioned sets are F_σ-subsets of E, and the lemma is proved.

Lemma 3. *Let E be a normal topological space, $g : E \to \mathbf{R}$ be a function and suppose that* $\mathrm{ran}(g) = \{t_1, t_2, ..., t_k\}$. *If, for any integer $i \in [1, k]$, the set $X_i = \{x \in E : g(x) = t_i\} = g^{-1}(t_i)$ is an F_σ-subset of E, then $g \in Ba_1(E, \mathbf{R})$.*

Proof. Obviously, we can write $E = X_1 \cup X_2 \cup ... \cup X_k$ and

$$X_i = F_{i,0} \cup F_{i,1} \cup ... \cup F_{i,n} \cup ... \qquad (i = 1, 2, ..., k),$$

where all $F_{i,n}$ $(1 \leq i \leq k,\ n < \omega)$ are closed subsets of E. Moreover, we may assume without loss of generality that $F_{i,0} \subset F_{i,1} \subset ... \subset F_{i,n} \subset ...$. For $n = 0, 1, 2, ...$ introduce the set $F_n = F_{1,n} \cup F_{2,n} \cup ... \cup F_{k,n}$ and define a function $g_n : F_n \to \mathbf{R}$ by putting $g_n(x) = t_i$ iff $x \in F_{i,n}$.

Since the finite family of closed sets $\{F_{1,n}, F_{2,n}, ..., F_{k,n}\}$ is disjoint, the function g_n is continuous on the set F_n. By the Tietze–Urysohn theorem (see, e.g., [64] or [149]), g_n admits a continuous extension $g_n^* : E \to \mathbf{R}$. Now, it is easy to check that $\lim_{n \to +\infty} g_n^*(x) = g(x)$ for all $x \in E$. This finishes the proof of the lemma.

Lemma 4. *Let E be a topological space in which every open set is an F_σ-subset of E (or, equivalently, in which every closed set is a G_δ-subset of E). Let $X \subset E$ be representable in the form $X = A_1 \cup A_2 \cup ... \cup A_k$, where all A_j $(j = 1, 2, ..., k)$ are F_σ-subsets of E. Then X is representable in the form $X = B_1 \cup B_2 \cup ... \cup B_k$, where $B_j \subset A_j$ $(j = 1, 2, ..., k)$, all B_j are also F_σ-subsets of E, and, in addition, they are pairwise disjoint.*

Proof. Obviously, we have the equality $X = F_1 \cup F_2 \cup ... \cup F_i \cup ...$, where all sets F_i $(1 \le i < \omega)$ are closed in E and each F_i is contained in some set $A_{j(i)}$. Let us put

$$C_1 = F_1,\ C_2 = F_2 \setminus F_1, ...,\ C_i = F_i \setminus (F_1 \cup ... \cup F_{i-1}), ...$$

The family of sets $\{C_i : 1 \le i < \omega\}$ is disjoint and, in view of our assumption on E, all C_i are F_σ-subsets of E. Moreover, $X = C_1 \cup C_2 \cup ... \cup C_i \cup ...$. Now, for any natural number $j \in [1, k]$, define the set B_j by the formula

$$B_j = \cup\{C_i : j \text{ is the smallest number for which } C_i \subset A_j\}.$$

Clearly, the family $\{B_1, B_2, ..., B_k\}$ is disjoint, all sets B_j $(j = 1, 2, ..., k)$ are F_σ-subsets of E, and

$$X = B_1 \cup B_2 \cup ... \cup B_k, \qquad B_j \subset A_j \qquad (j = 1, 2, ..., k).$$

Lemma 4 has thus been proved.

Recall that a topological space is perfectly normal if E is normal and each open set in E is an F_σ-subset of E. For such spaces the following important statement due to Lebesgue is true.

Theorem 2. *Let E be a perfectly normal space and let $f : E \to \mathbf{R}$ be a function. Then these three assertions are equivalent:*

(1) $f \in Ba_1(E, \mathbf{R})$;

(2) for any $t \in \mathbf{R}$, both sets $\{x \in E : f(x) < t\}$ and $\{x \in E : f(x) > t\}$ are F_σ-subsets of E;

(3) for any open set $U \subset \mathbf{R}$, the set $f^{-1}(U)$ is an F_σ-subset of E.

Proof. The equivalence (2) $\Leftrightarrow$ (3) is almost trivial and the implication (1) $\Rightarrow$ (2) was established by Lemma 2 (even for an arbitrary topological space E). Consequently, it remains to prove the implication (2) $\Rightarrow$ (1).

Suppose that (2) is valid and suppose first that $\mathrm{ran}(f) \subset\ ]0, 1[$. For any integer $n \ge 1$, consider the sequence $\{t_0, t_1, ..., t_n\}$ of points of $\mathbf{R}$ determined by the following conditions:

$$t_0 = 0, \quad t_{j+1} - t_j = 1/n \qquad (j = 0, ..., n-1).$$

In particular, we have $t_n = 1$. Further, introduce the sets

$$A_0 = \{x \in E : f(x) < t_1\}, \quad A_n = \{x \in E : f(x) > t_{n-1}\},$$

$$A_j = \{x \in E : t_{j-1} < f(x) < t_{j+1}\} \qquad (j = 1, ..., n-1).$$

Obviously, we have the equality $E = A_0 \cup A_1 \cup ... \cup A_n$ and all the sets A_j $(j = 0, 1, ..., n)$ are F_σ-subsets of E. Applying Lemma 4, we get another representation $E = B_0 \cup B_1 \cup ... \cup B_n$, where all the sets B_j are also F_σ-subsets of E, are pairwise disjoint, and $B_j \subset A_j$ for $j = 0, 1, ..., n$. Now, define a function $f_n : E \to \mathbf{R}$ by putting $f_n(x) = t_j$ iff $x \in B_j$. According to Lemma 3, the function f_n belongs to $Ba_1(E, \mathbf{R})$.

Take an arbitrary point $x \in E$. Then $x \in B_j$ for some integer $j \in [0, n]$. If $j = 0$, then we have

$$t_0 < f(x) < t_1, \quad f_n(x) = t_0, \quad |f(x) - f_n(x)| < 1/n.$$

If $j = n$, then we have

$$t_{n-1} < f(x) < t_n, \quad f_n(x) = t_n, \quad |f(x) - f_n(x)| < 1/n.$$

Finally, if $1 \leq j \leq n - 1$, then we have

$$t_{j-1} < f(x) < t_{j+1}, \quad f_n(x) = t_j, \quad |f(x) - f_n(x)| < 2/n.$$

These relations show that $\lim_{n\to+\infty} f_n(x) = f(x)$ uniformly with respect to $x \in E$. By virtue of Theorem 1, we obtain that $f \in Ba_1(E, \mathbf{R})$.

Suppose now that $f : E \to \mathbf{R}$ is an arbitrary function satisfying (2). Fix any increasing homeomorphism $\phi : \mathbf{R} \to \ ]0, 1[$ and consider the function $\phi \circ f$. This function also satisfies (2) and $\mathrm{ran}(\phi \circ f) \subset \ ]0, 1[$. As demonstrated above, $\phi \circ f \in Ba_1(E, \mathbf{R})$. Consequently, we may write

$$f = \phi^{-1} \circ (\phi \circ f) \in Ba_1(E, \mathbf{R}),$$

which completes the proof of Theorem 2.

Example 2. Let E be a perfectly normal space and let X be a subset of E. Denote by f_X the characteristic function (i.e., indicator) of X. It can readily be verified that if X is closed in E, then relation (2) of Theorem 2 is satisfied for $f = f_X$. Therefore, we get $f_X \in Ba_1(E, \mathbf{R})$. Now, if Y is an open subset of E, then, taking into account the equality $f_Y = 1 - f_{E \setminus Y}$, we see that $f_Y \in Ba_1(E, \mathbf{R})$, too. The above-mentioned facts follow also from Lemma 3.

Example 3. Let E be a subinterval of $\mathbf{R}$ and let $f : E \to \mathbf{R}$ be a monotone function. Then, for any $t \in \mathbf{R}$, both sets $\{x \in E : f(x) < t\}$ and $\{x \in E : f(x) > t\}$ are some subintervals of E. Because each interval in E is an F_σ-subset of E, we infer (in view of Theorem 2) that $f \in Ba_1(E, \mathbf{R})$. The same conclusion is true for those $f : E \to \mathbf{R}$ which are of finite variation on

E. Indeed, as is well known, all such functions are representable in the form of the difference of two increasing functions on E (see, e.g., [194], [225]).

Example 4. Let E be a perfectly normal space and let $f : E \to \mathbf{R}$ be an upper semicontinuous function. According to the definition of upper semicontinuous functions, for any $t \in \mathbf{R}$, the set $\{x \in E : f(x) < t\}$ is open in E and, hence, is an F_σ-subset of E. At the same time, the set

$$\{x \in E : f(x) > t\} = \cup_{n<\omega}\{x \in E : f(x) \geq t + 1/(n+1)\}$$

is the union of countably many closed sets, i.e., is also an F_σ-subset of E. Applying again Theorem 2, we deduce that $f \in Ba_1(E, \mathbf{R})$. From this fact it immediately follows that $g \in Ba_1(E, \mathbf{R})$ for any lower semicontinuous function $g : E \to \mathbf{R}$. Actually, the characteristic function f_X of a closed set $X \subset E$ (see Example 2) is upper semicontinuous.

Recall that a topological space E is Baire if no nonempty open subset of E is of first category in E. For such an E, an important result was established by Baire within **ZF** & **DC** theory. It yields an essential information about the structure of the set $D(f)$ of discontinuity points of an arbitrary function $f : E \to \mathbf{R}$ belonging to the class $Ba_1(E, \mathbf{R})$.

Theorem 3. *Let E be a Baire space and let $f \in Ba_1(E, \mathbf{R})$. Then the set $D(f)$ of all discontinuity points of f is of first category in E. In particular, for the set $C(f)$ of all continuity points of f, we have the relation $C(f) \cap U \neq \emptyset$ whenever U is a nonempty open subset of E.*

Proof. As is well known (see, e.g., Exercise 3), the relation

$$C(f) = E \setminus D(f) = \bigcap_{1 \leq n < \omega} \{x \in E : \Omega_f(x) < 1/n\}$$

is valid, where all sets $\{x \in E : \Omega_f(x) < 1/n\}$ are open in E. So it suffices to demonstrate that all these sets are everywhere dense in E. In other words, it suffices to show that, for any real $\varepsilon > 0$ and for any nonempty open set $U \subset E$, there exists a nonempty open set $W \subset U$ such that

$$(\forall x \in W)(\forall y \in W)(|f(x) - f(y)| < \varepsilon).$$

Keeping in mind the circumstance that $f \in Ba_1(E, \mathbf{R})$, choose a sequence $\{f_k : k < \omega\} \subset Ba_0(E, \mathbf{R})$ such that $f(x) = \lim_{k \to +\infty} f_k(x)$ for all $x \in E$. Further, for any natural number k, introduce the set

$$X_k = \{x \in E : (\forall i \geq k)(\forall j \geq k)(|f_i(x) - f_j(x)| \leq \varepsilon/3)\}.$$

All these sets X_k $(k < \omega)$ are closed in E and

$$(\forall k < \omega)(X_k \subset X_{k+1}), \quad E = \cup\{X_k : k < \omega\}.$$

Consequently, we have $U = (U \cap X_0) \cup (U \cap X_1) \cup ... \cup (U \cap X_k) \cup ...$. Since E is a Baire space, there is a natural number n such that $\mathrm{int}(U \cap X_n) \neq \emptyset$. Let $V \subset U \cap X_n$ be a nonempty open subset of E. If x is an arbitrary point of V, then $(\forall i \geq n)(\forall j \geq n)(|f_i(x) - f_j(x)| \leq \varepsilon/3)$. Putting $j = n$ and tending i to $+\infty$, we get $(\forall x \in V)(|f(x) - f_n(x)| \leq \varepsilon/3)$. Therefore, we can write

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \leq$$

$$2\varepsilon/3 + |f_n(y) - f_n(x)|$$

for any two points x and y from V. Finally, because f_n is a continuous function, there exists a nonempty open set $W \subset V$ such that

$$(\forall x \in W)(\forall y \in W)(|f_n(y) - f_n(x)| < \varepsilon/3).$$

This gives at once the relation $(\forall x \in W)(\forall y \in W)(|f(y) - f(x)| < \varepsilon)$, which completes the proof of the Baire theorem.

Remark 3. The proof presented above is based on the classical argument due to Baire (cf. also [7], [8], [194], [202]).

Remark 4. More general versions of the Baire theorem (with further information about it) can be found in [149].

For a while, let us turn our attention to real-valued functions of two variables and let us briefly discuss their descriptive structure from the viewpoint of descriptive properties of the corresponding partial functions of one variable.

We restrict our further consideration to real-valued functions defined on the topological product of two metric spaces.

Let X and Y be metric spaces and let $h : X \times Y \to \mathbf{R}$ be a function of two variables. In many cases, it is important to know whether h belongs to the first Baire class if we have some information about the descriptive structure of all partial functions

$$h(x, \cdot) : Y \to \mathbf{R} \quad (x \in X), \qquad h(\cdot, y) : X \to \mathbf{R} \quad (y \in Y).$$

Here is a simple (but useful) sufficient condition which enables us to assert that $h \in Ba_1(X \times Y, \mathbf{R})$.

Theorem 4. *If the partial functions $h(x,\cdot)$ and $h(\cdot,y)$ are continuous for all $x \in X$ and $y \in Y$, then h is of first Baire class.*

Proof. Take any nonempty closed subset A of $\mathbf{R}$ and, for every natural number $n \geq 1$, denote $V_n(A) = \{t \in \mathbf{R} : \inf\{|t-a| : a \in A\} < 1/n\}$. It is easy to verify that

$$h(x,y) \in A \Leftrightarrow (\forall n \geq 1)(\exists x' \in X)(d(x,x') < 1/n \;\&\; h(x',y) \in V_n(A)),$$

where d stands for the metric in X. This relation directly implies the equality

$$h^{-1}(A) = \cap_{n\geq 1}(\cup_{x'\in X}\{(x,y) \in X\times Y : d(x',x) < 1/n \,\&\, h(x',y) \in V_n(A)\}).$$

Our assumption on h yields at once that, for any $x' \in X$, the set

$$\{(x,y) \in X \times Y : d(x',x) < 1/n \;\&\; h(x',y) \in V_n(A)\}$$

is open in the product space $X \times Y$. Consequently, $h^{-1}(A)$ is a G_δ-subset of $X \times Y$. Therefore, for any open set $B \subset \mathbf{R}$, the pre-image $h^{-1}(B)$ turns out to be an F_σ-subset of $X \times Y$. In view of Theorem 2, we conclude that h is of first Baire class. This completes the proof.

Remark 5. Theorem 4 can be generalized to those functions h acting from $X \times Y$ into $\mathbf{R}$, for which all $h(x,\cdot)$ $(x \in X)$ are continuous and all $h(\cdot,y)$ $(y \in Y)$ belong to the class $Ba_\xi(X,\mathbf{R})$ where ξ is a fixed ordinal number strictly less than ω_1. Such a generalization is presented, e.g., in [149]. It turns out that any function h with the above-mentioned property belongs to the Baire class $Ba_{\xi+1}(X\times Y,\mathbf{R})$. In this context, it is reasonable to underline that the so-called Montgomery operation plays an essential role for obtaining the generalized result just indicated (for details, see [149]).

From Theorem 4 we easily get the following useful statement.

Theorem 5. *If a function $h : X \times Y \to \mathbf{R}$ is continuous with respect to each of its variables $x \in X$ and $y \in Y$, then h is continuous almost everywhere on $X \times Y$ in the sense of the Baire category.*

Proof. Indeed, by virtue of Theorem 4, our function h belongs to the class $Ba_1(X\times Y,\mathbf{R})$. It remains to apply the Baire theorem on the structure of the set of discontinuity points of functions belonging to the first Baire class (see Theorem 3 and Exercise 6 of this chapter).

Remark 6. The question naturally arises whether a given function $h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous with respect to each of its variables is continuous almost everywhere in the sense of the two-dimensional Lebesgue measure λ_2 on $\mathbf{R}^2$. Tolstov [268] answered this question negatively, namely, he constructed an example of a function $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying the following relations:

(1) g is continuous with respect to each of its variables;

(2) the set $D(g)$ has a strictly positive λ_2-measure.

A useful discussion of the topic concerning the separate and joint continuity of functions of several variables is presented in the work by Piotrowski [208].

Remark 7. Also, it is natural to ask about descriptive properties of a function $h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, all corresponding partial functions of which belong to the first Baire class. It turns out that the descriptive structure of such a function can be very complicated. For instance, we will see in Chapter 19 that there exists a function $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying the following relations:

(1) g is lower semicontinuous with respect to each of its variables;

(2) g is not measurable in the Lebesgue sense (i.e., g is not measurable with respect to λ_2).

On the other hand, it should be mentioned that if a given function $h : X \times Y \to \mathbf{R}$ is continuous with respect to $y \in Y$ and is measurable (in some sense) with respect to $x \in X$, then h turns out to be measurable (in an appropriate sense) on the product space $X \times Y$. Actually, these two properties of h are fundamental for functions of two variables and play a significant role in various questions of mathematical analysis, the theory of ordinary differential equations, optimization theory, probability, and stochastic processes (cf. Chapters 21 and 22).

Let E be a topological space, $\{f_n : n < \omega\}$ be a sequence of real-valued continuous functions on E, and suppose that $f(x) = \lim_{n\to+\infty} f_n(x)$ for each $x \in E$. According to the Baire definition, we have $f \in Ba_1(E, \mathbf{R})$. Naturally, one can ask about necessary and sufficient conditions under which this f is also continuous (i.e., $f \in Ba_0(E, \mathbf{R})$). A convenient sufficient condition is well known from the standard course of mathematical analysis. Namely, if $\{f_n : n < \omega\}$ converges uniformly on E, then f is continuous. However, this condition is very far from being necessary (see Exercise 15).

We are going to give here one necessary and sufficient condition for the continuity of a limit function (cf. [149]).

Theorem 6. *Let E be a topological space, $\{f_n : n < \omega\}$ be a sequence of real-valued continuous functions on E, and let $f(x) = \lim_{n\to+\infty} f_n(x)$ for all $x \in E$. Then the following two assertions are equivalent:*

(1) f is continuous on E;

(2) for any real $\varepsilon > 0$ and for any natural number m, there exists a natural number $n \geq m$ such that the set $\{x \in E : |f(x) - f_n(x)| < \varepsilon\}$ is open in E.

Proof. Let us check (1) $\Rightarrow$ (2). Suppose that relation (1) is valid. Then, for any real $\varepsilon > 0$ and for any $m < \omega$, the set $\{x \in E : |f(x) - f_m(x)| < \varepsilon\}$ is open in E. We thus see that, in this case, relation (2) holds automatically.

Let us verify (2) $\Rightarrow$ (1). Suppose that (2) is valid. We must show that f is continuous at each point $x_0 \in E$. Take an arbitrary real $\varepsilon > 0$. In view of (2), there exists a strictly increasing sequence $\{n_1, n_2, ..., n_k, ...\}$ of natural numbers such that all the sets $E_k = \{x \in E : |f(x) - f_{n_k}(x)| < \varepsilon/3\}$ are open in E. Moreover, since $\{f_n : n < \omega\}$ converges to f pointwise, we have the equality $E = \cup\{E_k : 1 \leq k < \omega\}$. Therefore, there exists a natural number k_0 such that $x_0 \in E_{k_0}$. We thus get that E_{k_0} is an open neighborhood of x_0. Now, for any point $x \in E_{k_0}$, we may write

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_{k_0}}(x)| + |f_{n_{k_0}}(x) - f_{n_{k_0}}(x_0)| + |f_{n_{k_0}}(x_0) - f(x_0)|$$

$$\leq 2\varepsilon/3 + |f_{n_{k_0}}(x) - f_{n_{k_0}}(x_0)|.$$

Because the function $f_{n_{k_0}}$ is continuous, there exists a neighborhood $U(x_0)$ of x_0 such that

$$U(x_0) \subset E_{k_0}, \quad (\forall x \in U(x_0))(|f_{n_{k_0}}(x) - f_{n_{k_0}}(x_0)| < \varepsilon/3).$$

This immediately implies $|f(x) - f(x_0)| < \varepsilon$ for all points $x \in U(x_0)$, which shows the continuity of f at x_0. Theorem 6 has thus been proved.

Remark 8. The above theorem is a particular case of a more general result concerning functions of the class $Ba_\xi(E, \mathbf{R})$, where ξ is an arbitrary ordinal strictly less than ω_1. Actually, if we have $\{f_n : n < \omega\} \subset Ba_\xi(E, \mathbf{R})$ and the relation $\lim_{n \to +\infty} f_n(x) = f(x)$ holds true for each $x \in E$, then an appropriate generalization of Theorem 6 yields necessary and sufficient conditions for the validity of the relation $f \in Ba_\xi(E, \mathbf{R})$. Further details may be found, e.g., in [149].

EXERCISES

1. Suppose that E is a separable topological space (i.e., E contains a countable everywhere dense subset).

Show that $\mathrm{card}(Ba_1(E, \mathbf{R})) \leq \mathbf{c}$ (in fact, show that if E is nonempty and separable, then $\mathrm{card}(Ba_1(E, \mathbf{R})) = \mathbf{c}$).

2. Check that the set $\mathbf{Q} \subset \mathbf{R}$ of all rational numbers is not a G_δ-subset of $\mathbf{R}$ and deduce from this circumstance that the characteristic function $f_{\mathbf{Q}}$, the so-called Dirichlet function, does not belong to $Ba_1(\mathbf{R}, \mathbf{R})$ (actually, $f_{\mathbf{Q}}$ is of second Baire class).

More generally, prove that if E is an uncountable Polish space without isolated points and X is a countable everywhere dense subset of E, then the characteristic function f_X does not belong to $Ba_1(E, \mathbf{R})$.

3. Let E be a topological space and let $f : E \to \mathbf{R}$ be a function. For any $x \in E$, define $\Omega_f(x) = \inf_{V \in \mathcal{V}(x)} \mathrm{diam}(f(V))$, where $\mathcal{V}(x)$ denotes the filter of all neighborhoods of x and $\mathrm{diam}(f(V))$ stands for the diameter of the set $f(V)$. Denote also by $D(f)$ the set of all discontinuity points of f.

Verify that

(a) $\Omega_f(x) = 0$ if and only if f is continuous at x (equivalently, $\Omega_f(x) > 0$ if and only if x is a discontinuity point of f);

(b) for any $t \in \mathbf{R}$, the set $\{x \in E : \Omega_f(x) \geq t\}$ is closed in E;

(c) $D(f) = E_1 \cup E_2 \cup ... \cup E_n \cup ...$, where

$$E_n = \{x \in E : \Omega_f(x) \geq 1/n\} \qquad (n = 1, 2, ...).$$

Conclude from (b) and (c) that $D(f)$ is an F_σ-subset of E (therefore, the set $C(f)$ of all continuity points of f is a G_δ-subset of E).

Let $\mathcal{B}$ be a base of open sets in $\mathbf{R}$ and let $\mathcal{F} = \{Y \subset \mathbf{R} : \mathbf{R} \setminus Y \in \mathcal{B}\}$.

Prove that $D(f) = \cup\{\mathrm{cl}(f^{-1}(Y)) \setminus f^{-1}(Y) : Y \in \mathcal{F}\}$.

Generalize these results to an arbitrary function $f : E \to E'$, where E' is a metric space.

Remark 9. Some applications of the above facts will be presented later (see, for instance, Chapter 8).

4*. Let E be a topological space. This E is called resolvable if it admits a representation in the form $E = A \cup B$, where A and B are some disjoint everywhere dense subsets of E (this notion was first introduced by Hewitt).

As a rule, topological spaces used in various topics of mathematical analysis turn out to be resolvable. In particular, prove that

(i) every locally compact topological space without isolated points is resolvable;

(ii) every Hausdorff topological vector space over $\mathbf{R}$, whose dimension is not equal to zero, is resolvable.

Fix a resolvable space E and let X be an F_σ-subset of E.

Show that there exists a function $f : E \to \mathbf{R}$ such that X coincides with the set $D(f)$ of all discontinuity points of f.

To do this, first represent X in the form $X = F_1 \cup F_2 \cup ... \cup F_n \cup ...$, where all sets F_n are closed in E and $F_n \subset F_{n+1}$ for each natural number $n \geq 1$. Further, put $F_0 = \emptyset$ and, for any integer $n \geq 1$, define some function $f_n : E \to \{0, 1\}$ satisfying the following two relations:

(a) f_n is equal to zero at all points of the set $E \setminus F_n$;

(b) $\Omega_{f_n}(x) = 1$ if $x \in F_n$.

Now, take a sequence $\{a_n : n \geq 1\}$ of strictly positive real numbers, such that $a_{n+1} + a_{n+2} + ... + a_k + ... < a_n$ for $n = 1, 2, ...$. For example, it suffices to put $a_n = 1/3^n$ for $n = 1, 2, ...$. Finally, consider the function $f = a_1 f_1 + a_2 f_2 + ... + a_n f_n + ...$. This function is well-defined since the series on the right-hand side of the above equality converges uniformly on E.

Verify that

(c) f is continuous at all points of the set $E \setminus X$;

(d) for any integer $n \geq 1$ and for all points $x \in F_n \setminus F_{n-1}$, one has

$$\Omega_f(x) \geq a_n - \sum_{k>n} a_k > 0.$$

Conclude from (c) and (d) that $D(f) = X$.

5. Let E be a perfectly normal topological space and let $f : E \to \mathbf{R}$ be a function whose graph is closed in the product space $E \times \mathbf{R}$.

Applying the Kuratowski lemma on closed projections (i.e., Lemma 1 from Chapter 0), show that f is of first Baire class.

Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ whose graph is closed in $\mathbf{R} \times \mathbf{R}$ and whose discontinuity points constitute a nonempty perfect set in $\mathbf{R}$.

6*. Let E be a topological space and let $f \in Ba_1(E, \mathbf{R})$.

Demonstrate that the set $D(f)$ is of first category in E (for this purpose, use Exercise 3 and Lemma 2).

Another way to show this fact is based on the Banach statement (see Exercise 30 from Chapter 0) which leads to a representation of E in the form $E = E' \cup E''$, where E' is an open Baire subspace of E and E'' is a first category closed subset of E.

By applying Theorem 3 to the set $D(f|E')$, obtain the required result.

Deduce from this result that if $f \in Ba_1(E, \mathbf{R})$ and X is an arbitrary subspace of E, then the set $D(f|X)$ is of first category in X.

Conclude that if E is a complete metric space, $f \in Ba_1(E, \mathbf{R})$, and X is a nonempty closed subspace of E, then there exists a point in X at which $f|X$ is continuous.

7. According to the definition of Luzin, a topological space X is always of first category (or X is perfectly meager) if each nonempty dense in itself subset of X is of first category in X.

Let E be a topological space and let X be a subspace of E such that, for every perfect set $P \subset E$, the set $X \cap P$ is of first category in P.

Show that the space X is always of first category (apply Exercise 11 from Chapter 0).

Remark 10. Luzin proved that there exists an uncountable subspace X of $\mathbf{R}$ which is always of first category (see [149], [169]). Other constructions of uncountable universally small sets can be found in [207] and [283] (cf. also Chapter 13 of this book).

8. Let E be a hereditarily Lindelöf topological space always of first category and let $f : E \to \mathbf{R}$ be a function.

Demonstrate that, for each subspace X of E, the set $D(f|X)$ is of first category in X.

For this purpose, begin with establishing the fact that X admits a representation in the form $X = Y \cup Z$, where Y is dense in itself, Z is at most countable, and $Y \cap Z = \emptyset$. Then verify that $D(f|X) \subset Y \cup (X' \cap Z)$, where X' denotes the set of all accumulation points of X (in E). Finally, observe that both sets Y and $X' \cap Z$ are of first category in X.

9*. Let E be a subspace of $\mathbf{R}$ satisfying the following two relations:

(a) $\mathrm{card}(E) = \mathbf{c}$;

(b) E is always of first category.

The existence of such a subspace of $\mathbf{R}$ is demonstrated in Chapter 13 under Martin's Axiom (see Theorem 7 of that chapter).

Prove that there exists a function $f : E \to \mathbf{R}$ having the following properties:

(c) f is not Borel (consequently, f does not belong to $Ba_1(E, \mathbf{R})$);

(d) for any subspace Z of E, the set $D(f|Z)$ is of first category in Z.

Remark 11. This classical result is due to Luzin. It shows that property (d) of f does not imply the relation $f \in Ba_1(E, \mathbf{R})$. It is useful to compare the above-mentioned result with Exercise 12 given below.

10. Let E be a hereditarily Lindelöf space and let $\{F_\xi : \xi < \omega_1\}$ be a decreasing (with respect to the inclusion relation) ω_1-sequence of closed subsets of E.

Prove that there exists $\alpha < \omega_1$ such that $(\forall \xi \in [\alpha, \omega_1[)(F_\xi = F_\alpha)$.

This result is known as the Cantor–Baire stationarity principle.

In particular, take an arbitrary closed subset X of E and define by transfinite recursion an ω_1-sequence $\{X_\xi : \xi < \omega_1\}$ in the following manner:

$X_0 = X$;

$X_{\xi+1} = (X_\xi)'$ for any $\xi < \omega_1$, where $(X_\xi)'$ denotes the set of all accumulation points of X_ξ;

$X_\xi = \cap\{X_\zeta : \zeta < \xi\}$ for any limit ordinal $\xi < \omega_1$.

Applying the Cantor–Baire stationarity principle to $\{X_\xi : \xi < \omega_1\}$, show that X admits a representation in the form $X = Y \cup Z$, where Y and Z are disjoint, Y is perfect in E, and Z is at most countable (the Cantor–Bendixson theorem).

Give another proof of the same result that does not use the method of transfinite induction.

For this purpose, consider the set of all condensation points of X and take it as Y. Then define the set Z by the equality $Z = X \setminus Y$.

Remark 12. Notice that a certain generalization of the Cantor–Baire stationarity principle was obtained by Luzin for decreasing ω_1-sequences of F_σ-subsets of $\mathbf{R}$ (in this connection, see [169]).

11*. Let E be a topological space. This space is called scattered if E does not contain a nonempty dense in itself subset.

Demonstrate that the following two assertions are equivalent:

(a) E is scattered;

(b) $E = \{e_\xi : \xi < \alpha\}$, where α is some ordinal, $\{e_\xi : \xi < \alpha\}$ is an injective α-sequence, and, for any $\xi < \alpha$, the element e_ξ is an isolated point of the set $\{e_\zeta : \xi \leq \zeta < \alpha\}$.

Notice that the implication (b) $\Rightarrow$ (a) is trivial.

Supposing now that (a) is valid, use the method of transfinite recursion for obtaining the required representation $\{e_\xi : \xi < \alpha\}$ of E.

Finally, demonstrate that every topological space X can be represented in the form $X = Y \cup Z$, where Y is a perfect subset of X, Z is a scattered subset of X, and $Y \cap Z = \emptyset$ (this classical result is due to Cantor).

12*. Let E be a separable metric space and let $g : E \to \mathbf{R}$ be a function such that, for every nonempty closed set $F \subset E$, there exists a point $x \in F$ at which the restricted function $g|F$ is continuous.

Prove that g is of first Baire class (this remarkable result is due to Baire).

The following argument enables one to establish the above-mentioned result.

Take any $a \in \mathbf{R}$ and $b \in \mathbf{R}$ such that $a < b$, and denote

$$A = \{x \in E : g(x) > a\}, \quad B = \{x \in E : g(x) < b\}.$$

Clearly, one has the equality $E = A \cup B$. Further, construct by transfinite recursion an ω_1-sequence $F_0 \supset F_1 \supset ... \supset F_\xi \supset ...$ of closed subsets of E.

Put $F_0 = E$. Suppose that, for an ordinal number $\xi < \omega_1$, the partial family $\{F_\zeta : \zeta < \xi\}$ has already been defined.

If ξ is a limit ordinal, then put $F_\xi = \cap\{F_\zeta : \zeta < \xi\}$.

If $\xi = \eta + 1$, consider the set F_η. Only two cases are possible.

1. $F_\eta = \emptyset$. In this case, define $F_\xi = F_\eta = \emptyset$.

2. $F_\eta \neq \emptyset$. In this case, there exists a point $x \in F_\eta$ at which the function $g|F_\eta$ is continuous. Consequently, there exists an open neighborhood $V(x)$ of x such that $F_\eta \cap V(x) \subset A$ or $F_\eta \cap V(x) \subset B$. Then define $F_\xi = F_\eta \setminus V(x)$.

By proceeding in this manner, it will be possible to construct all the sets F_ξ $(\xi < \omega_1)$.

Observe now that, for each ordinal $\xi < \omega_1$, one has $F_\xi \setminus F_{\xi+1} \subset A$ or $F_\xi \setminus F_{\xi+1} \subset B$, and, according to the Cantor–Baire stationarity principle, for some $\alpha < \omega_1$, the equalities

$$\emptyset = F_\alpha = F_{\alpha+1} = ... = F_\xi = ... \quad (\alpha \leq \xi < \omega_1)$$

are valid. Deduce from these facts that there exist two sets A' and B' such that

$$A' \subset A, \quad B' \subset B, \quad A' \cup B' = E, \quad A' \cap B' = \emptyset,$$

and both A' and B' are F_σ-subsets of E.

Now, let $\{b_n : n < \omega\}$ be a strictly decreasing sequence of real numbers satisfying the relation $\lim_{n \to +\infty} b_n = a$. For every natural number n, define $B_n = \{x \in E : g(x) < b_n\}$. As above, show the existence of F_σ-sets A'_n and B'_n such that

$$A'_n \subset A, \quad B'_n \subset B_n, \quad A'_n \cup B'_n = E, \quad A'_n \cap B'_n = \emptyset.$$

Finally, denote $X = \cup\{A'_n : n < \omega\}$ and verify that $X = A$. Therefore, A is an F_σ-subset of E.

By using a similar argument, demonstrate that B is an F_σ-subset of E, too. Conclude, in view of the Lebesgue theorem (i.e., Theorem 2 of this chapter), that the function g is of first Baire class.

Remark 13. More general versions of the result presented above can be found in [149] where a different argument is utilized. Namely, it is proved therein that if E is a complete metric space and $g : E \to \mathbf{R}$ is a function, then the following two assertions are equivalent:

(1) $g \in Ba_1(E, \mathbf{R})$;

(2) for any nonempty closed set $F \subset E$, there exists a point $x \in F$ at which the restricted function $g|F$ is continuous.

Theorem 3 and the preceding exercise establish the equivalence (1) $\Leftrightarrow$ (2) in the case of a Polish space E. For a nonseparable complete metric space E, the proof of (1) $\Leftrightarrow$ (2) relies on properties of the so-called Montgomery operation (see again [149]). Notice that this operation needs uncountable forms of the Axiom of Choice.

In the case $E = \mathbf{R}$ for any function $g : \mathbf{R} \to \mathbf{R}$, the following two assertions are effectively equivalent, i.e., are equivalent within **ZF** theory:

(1) g is of first Baire class;

(2) for every nonempty closed set $F \subset \mathbf{R}$, there are points of F at which the function $g|F$ is continuous.

In other words, we have a certain effective characterization of functions acting from $\mathbf{R}$ into $\mathbf{R}$ and belonging to the first Baire class. Unfortunately, we do not have an analogous nice characterization of derivatives on $\mathbf{R}$ which form an important proper subclass of $Ba_1(\mathbf{R}, \mathbf{R})$ (see Example 1).

13. Let E be a Polish topological space and let $f : E \to \mathbf{R}$ be a function whose set of discontinuity points is at most countable.

Show, by applying Exercise 12, that $f \in Ba_1(E, \mathbf{R})$.

Infer from this fact that any function $g : [a, b] \to \mathbf{R}$ of finite variation on a segment $[a, b]$ belongs to the first Baire class (another way to establish this result was indicated in Example 3).

14. Let C denote the classical Cantor discontinuum on $[0, 1]$. Define two functions $f : [0, 1] \to \mathbf{R}$ and $g : [0, 1] \to \mathbf{R}$ by the following formulas:

$f(x) = 1$ if $x \in C$ and $f(x) = 0$ if $x \in [0, 1] \setminus C$;

$g(x) = 1$ if x is not an end-point of a removed interval for C, and $g(x) = 0$ if x is an end-point of some removed interval for C.

Verify that

(a) f is upper semicontinuous and, hence, is of first Baire class;

(b) $g|C$ is discontinuous at all points of C, hence g is not of first Baire class;

(c) $D(f) = D(g) = C$.

The last relation shows, in particular, that the sets of continuity points of f and g are identical, but f and g are of substantially different descriptive structure.

15. Give an example of a sequence $\{f_n : n < \omega\}$ of real-valued continuous functions on $[0, 1]$ which converges pointwise to some continuous function $f : [0, 1] \to \mathbf{R}$, but this convergence is not uniform on $[0, 1]$.

16. Deduce from Theorem 6 the following classical result of Arzelá on the continuity of a limit function.

Let E be a quasi-compact topological space, $\{f_n : n < \omega\} \subset Ba_0(E, \mathbf{R})$, and let $f(x) = \lim_{n\to+\infty} f_n(x)$ for all $x \in E$. Then these two assertions are equivalent:

(a) f is continuous on E;

(b) $\{f_n : n < \omega\}$ converges quasi-uniformly to f; in other words, for any real $\varepsilon > 0$, there exist a finite family $\{V_1, V_2, ..., V_k\}$ of open sets in E and a finite family $\{n_1, n_2, ..., n_k\}$ of natural numbers, such that

$$E = V_1 \cup V_2 \cup ... \cup V_k, \quad V_i \subset \{x \in E : |f(x) - f_{n_i}(x)| < \varepsilon\} \quad (i = 1, 2, ..., k).$$

Notice that in (b) it is not required that the numbers $n_1, n_2, ..., n_k$ would be arbitrarily large.

17. Deduce from the above-mentioned result of Arzelá another classical theorem due to Dini.

Namely, let E be a quasi-compact topological space, let $\{f_n : n < \omega\}$ be a monotone sequence of real-valued continuous functions on E, and suppose that $f(x) = \lim_{n\to+\infty} f_n(x)$ for all $x \in E$. Then the following two assertions are equivalent:

(a) f is continuous on E;

(b) $\{f_n : n < \omega\}$ converges uniformly to f.

18. Let E be a topological space, X be a subset of E, and let $e \in E$ be an accumulation point of X. Suppose, in addition, that a sequence $\{f_n : n < \omega\}$ of real-valued functions on X is given such that

(a) the series $\sum_{n<\omega} f_n(x)$ converges to some $f(x) \in \mathbf{R}$ uniformly with respect to $x \in X$;

(b) for each $n < \omega$, there exists a $\lim_{x\to e} f_n(x)$ and $a_n = \lim_{x\to e} f_n(x)$.

Demonstrate that the series $\sum_{n<\omega} a_n$ converges to some $a \in \mathbf{R}$ and the equality $\lim_{x\to e} f(x) = a$ holds true.

19. Let $\{f_n : n < \omega\}$ be a sequence of real-valued differentiable functions defined on a segment $T \subset \mathbf{R}$. Suppose that the series $\sum_{n<\omega} f_n$ is convergent at some point t_0 of T and that the series of derivatives $\sum_{n<\omega} f'_n$ is uniformly convergent on T.

By using the result of Exercise 18, verify the validity of the following three assertions:

(a) the series $\sum_{n<\omega} f_n$ converges uniformly on T;

(b) the function $f = \sum_{n<\omega} f_n$ is differentiable on T;

(c) for all points $t \in T$, we have $f'(t) = \sum_{n<\omega} f'_n(t)$.

Infer from these facts that the family of all bounded derivatives on T is a Banach subspace of $Ba_1(T, \mathbf{R})$ with respect to the norm of uniform convergence (i.e., with respect to the standard sup-norm).

Chapter 3
Semicontinuous functions that are not countably continuous

Luzin's theorem on the structure of Lebesgue measurable functions acting from $\mathbf{R}$ into itself is one of the most fundamental statements in real analysis and has numerous applications. Let us recall the formulation of this classical theorem. It is convenient for us to give the formulation in terms of partial functions (cf. Chapter 0). As usual, we denote by the symbol λ $(= \lambda_1)$ the Lebesgue measure on $\mathbf{R}$ $(= \mathbf{R}^1)$.

Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. According to the well-known definition, f is measurable in the Lebesgue sense (or, simply, f is λ-measurable) if the set $f^{-1}(U)$ is Lebesgue measurable for every open subset U of $\mathbf{R}$.

It immediately follows from this definition that the domain of an arbitrary Lebesgue measurable partial function f is a λ-measurable subset of $\mathbf{R}$ (because of the equality $\mathrm{dom}(f) = f^{-1}(\mathbf{R})$).

Theorem 1. *For any partial function $g : \mathbf{R} \to \mathbf{R}$ which is measurable in the Lebesgue sense and for any real $\varepsilon > 0$, there exists a closed set $F \subset \mathbf{R}$ such that $F \subset \mathrm{dom}(g)$, $\lambda(\mathrm{dom}(g) \setminus F) < \varepsilon$, and the restriction $g|F$ is continuous.*

We omit the standard proof of Theorem 1 (see, for example, [28], [168], [194], [196], [202]). Notice only that there are various generalizations and extensions of this classical result (see, e.g., Exercise 1 from Chapter 8, Exercises 5 and 6 from Chapter 15).

Let us indicate one of the direct consequences of the Luzin theorem.

Theorem 2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a Lebesgue measurable function. Then there exists a disjoint countable covering $\{A_n : n < \omega\}$ of $\mathbf{R}$ such that $\lambda(A_0) = 0$, and, for each integer $n \geq 1$, the set A_n is closed in $\mathbf{R}$ and the function $f|A_n$ is continuous.*

In connection with some delicate moments in Theorem 2, see Exercise 1. Taking into account this theorem and Exercise 1, it is natural to pose the following question:

Let $f : \mathbf{R} \to \mathbf{R}$ be a Borel function. Does there exist a countable covering $\{A_n : 1 \leq n < \omega\}$ of $\mathbf{R}$ such that the function $f|A_n$ is continuous for each natural number $n \geq 1$?

This question was originally raised by Luzin many years ago. Adian and Novikov [2] gave a negative answer to it (see also Sierpiński [243]).

Moreover, the above-mentioned authors were able to construct an upper semicontinuous function $f : \mathbf{R} \to \mathbf{R}$ that does not admit a decomposition into countably many continuous partial functions (in other words, f is not countably continuous).

We are going to present here the construction of Adian and Novikov. Then we will consider some related results which are due to other authors and are also motivated by the Luzin problem posed above.

First, let us introduce some preliminary notions.

Let E be a topological space and let $\mathcal{I}$ be a σ-ideal of subsets of E.

We shall say that a function $f : E \to \mathbf{R}$ is countably continuous $(\mathrm{mod}(\mathcal{I}))$ if there exists a covering $\{A_n : n < \omega\}$ of E such that $A_0 \in \mathcal{I}$ and the restriction $f|A_n$ is continuous for each natural number $n \geq 1$.

Example 1. Let E be a topological space of second category on itself and let $\mathcal{K}(E)$ denote the σ-ideal of all first category subsets of E. We know that, for any function $f : E \to \mathbf{R}$ possessing the Baire property, there exists a first category set $X \subset E$ such that $f|(E \setminus X)$ is continuous (see Exercise 13 from Chapter 0). Thus, f is countably continuous $(\mathrm{mod}(\mathcal{K}(E)))$.

Example 2. In view of Theorem 2, every Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$ is countably continuous $(\mathrm{mod}(\mathcal{I}(\lambda)))$, where $\mathcal{I}(\lambda)$ denotes the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$.

If a σ-ideal $\mathcal{I}$ is trivial, i.e., $\mathcal{I} = \{\emptyset\}$, and a function $f : E \to \mathbf{R}$ is countably continuous $(\mathrm{mod}(\mathcal{I}))$, then we simply say that f is countably continuous.

To give the construction of Adian and Novikov, we need several auxiliary propositions.

Lemma 1. *Let E be a topological space, let $\{f_n : n < \omega\}$ be a sequence of real-valued upper (respectively, lower) semicontinuous functions on E, and suppose that $f(x) = \lim_{n \to +\infty} f_n(x)$ uniformly with respect to $x \in E$. Then f is also an upper (respectively, a lower) semicontinuous function on E.*

Proof. Obviously, it suffices to consider the case where all f_n $(n < \omega)$ are upper semicontinuous. Take any real $\varepsilon > 0$ and fix $x_0 \in E$. For each

$n < \omega$ and for each $x \in E$, we can write

$$f(x) - f(x_0) \leq |f(x) - f_n(x)| + (f_n(x) - f_n(x_0)) + |f_n(x_0) - f(x_0)|.$$

According to our assumption, there exists $m < \omega$ such that

$$|f(x) - f_n(x)| < \varepsilon/3 \qquad (x \in E,\ n = m, m+1, m+2, ...).$$

Consequently, we have $f(x)-f(x_0) \leq (f_m(x)-f_m(x_0))+2\varepsilon/3$ for all $x \in E$. Since f_m is upper semicontinuous at x_0, there exists a neighborhood $V(x_0)$ of x_0 such that $f_m(x)-f_m(x_0) < \varepsilon/3$ for every $x \in V(x_0)$. This immediately implies the relation $f(x)-f(x_0) < \varepsilon$ for every $x \in V(x_0)$. Actually, we have proved that if all functions f_n $(n < \omega)$ are upper semicontinuous at $x_0 \in E$ and the sequence $\{f_n : n < \omega\}$ converges uniformly to f, then f is also upper semicontinuous at x_0.

Exercise 2 for this chapter presents a more general result.

The next two lemmas provide some easy facts concerning extensions of semicontinuous partial functions.

Lemma 2. *Let E be a topological space and let $f : X \to \mathbf{R}$ be a bounded from above (from below) upper (lower) semicontinuous function on some $X \subset E$. Then there exists a function $f^* : \mathrm{cl}(X) \to \mathbf{R}$ which extends f, is upper (respectively, lower) semicontinuous, and satisfies the equality $\sup_{x\in X} f(x) = \sup_{x\in \mathrm{cl}(X)} f^*(x)$ (respectively, satisfies the equality $\inf_{x\in X} f(x) = \inf_{x\in \mathrm{cl}(X)} f^*(x)$).*

Proof. We shall consider only the case of a bounded from above upper semicontinuous function $f : X \to \mathbf{R}$. Let z be any point from $\mathrm{cl}(X)$. Put $f^*(z) = \mathrm{limsup}_{x\to z, x\in X} f(x)$. Obviously, the function f^* extends f (since f is upper semicontinuous). Let us verify that f^* is also upper semicontinuous at all points of $\mathrm{cl}(X)$.

Take again an arbitrary point $z \in \mathrm{cl}(X)$ and fix a real number t such that $f^*(z) < t$. Then, for sufficiently small real numbers $\varepsilon > 0$, we may write $f^*(z) \leq t - \varepsilon$. Consider any such ε. By the definition of f^*, we infer that there exists an open neighborhood $V(z)$ of z for which

$$\sup_{x\in V(z)\cap X} f(x) \leq t - \varepsilon/2.$$

In view of the same definition, we get $\sup_{x\in V(z)\cap \mathrm{cl}(X)} f^*(x) \leq t - \varepsilon/2$ or, equivalently, $f^*(x) \leq t - \varepsilon/2 < t$ for all $x \in V(z) \cap \mathrm{cl}(X)$. This establishes that the set $\{x \in \mathrm{cl}(X) : f^*(x) < t\}$ is open in $\mathrm{cl}(X)$, i.e., f^* is upper semicontinuous on $\mathrm{cl}(X)$. Finally, the equality

$$\sup_{x\in X} f(x) = \sup_{x\in \mathrm{cl}(X)} f^*(x)$$

also follows from the definition of f^*. Lemma 2 has thus been proved (cf. Exercise 22 of Chapter 0).

Lemma 3. *Let E be a normal topological space, X be a subset of E, and let $[a, b]$ be a segment in $\mathbf{R}$. Suppose that $f : X \to [a, b]$ is an upper (respectively, lower) semicontinuous function. Then there exists an upper (respectively, lower) semicontinuous function $f^* : E \to [a, b]$ which extends the original function f.*

Proof. It suffices to consider only the case of a lower semicontinuous function $f : X \to [a, b]$. By virtue of Lemma 2, we may assume without loss of generality that X is a closed subset of E. Further, because X is completely regular, we have

$$f(x) = \sup_{i \in I} f_i(x) \qquad (x \in X),$$

where $\{f_i : i \in I\}$ is some family of real-valued continuous functions on X and $\mathrm{ran}(f_i) \subset [a, b]$ for all indices $i \in I$ (cf. Exercise 18 from Chapter 0). By the Tietze–Urysohn theorem, for an arbitrary index $i \in I$, there exists a continuous function $f_i^* : E \to [a, b]$ extending f_i. Now, define $f^*(x) = \sup_{i \in I} f_i^*(x)$ for each $x \in E$. Clearly, f^* is the required lower semicontinuous extension of f. This completes the proof of Lemma 3.

Lemma 4. *Let E be a topological space, $f : E \to \mathbf{R}$ be a function and let X be a subset of E. If the restriction $f|X$ is not countably continuous, then f is not countably continuous on E.*

The trivial proof of this lemma is left to the reader.

Lemma 5. *Suppose that there exists a real-valued bounded upper semicontinuous function on the classical Cantor set C which is not countably continuous. Then an analogous function exists on an arbitrary uncountable Polish space E.*

Proof. It is well known that E contains a topological copy of C (see, e.g., [64], [105], [107], [149]), so we may assume that $C \subset E$. Because C is compact, it can be regarded as a closed subset of E. Let $f : C \to \mathbf{R}$ be a bounded upper semicontinuous function which is not countably continuous. According to Lemma 3, there exists a bounded upper semicontinuous function $f^* : E \to \mathbf{R}$ extending f. By virtue of Lemma 4, f^* is not countably continuous. This completes the proof of Lemma 5.

Lemma 6. *Let E be a nonempty Polish topological space without isolated points. There exists a disjoint countable family $\{C_n : n < \omega\}$ of subsets of E such that*

(1) each C_n is homeomorphic to the Cantor discontinuum C;
(2) each C_n is nowhere dense in E;
(3) the set $\cup\{C_n : n < \omega\}$ is everywhere dense in E.

Proof. Denote by $\{U_n : n < \omega\}$ a base of nonempty open sets in E. We are going to construct the required family by using ordinary recursion. Suppose that, for a given natural number n, the partial family $\{C_i : i < n\}$ has already been defined. Consider the set $U = U_n \setminus \cup\{C_i : i < n\}$. Clearly, U is a nonempty open set in E without isolated points. It can easily be shown that there exists a countable set $D \subset U$ dense in U and such that $U \setminus D$ is also dense in U. Now, since $U \setminus D$ is a G_δ-subset of E, we infer that $U \setminus D$ is a Polish space without isolated points. Consequently, there exists a homeomorphic copy X of C contained in $U \setminus D$. It suffices to put $C_n = X$. Proceeding in this manner, we will be able to construct by recursion the required family $\{C_n : n < \omega\}$.

The reader can easily check the validity of relations (1), (2), and (3) for this family.

Lemma 7. *Let E be a nonempty Polish space without isolated points. There exists a countable family*

$$\{C_{n_1,n_2,\ldots,n_k} : n_1 \geq 1,\ n_2 \geq 1,\ \ldots\ ,\ n_k \geq 1,\ k < \omega\}$$

of subsets of E satisfying the following six conditions:

(1) $C_\emptyset = E$;

(2) for all integers $k \geq 1, n_1 \geq 1, n_2 \geq 1, ..., n_k \geq 1$, the set $C_{n_1,n_2,\ldots,n_k}$ is homeomorphic to the Cantor discontinuum C;

(3) $C_{n_1,n_2,\ldots,n_k,n_{k+1}} \subset C_{n_1,n_2,\ldots,n_k}$ and $C_{n_1,n_2,\ldots,n_k,n_{k+1}}$ is nowhere dense in $C_{n_1,n_2,\ldots,n_k}$;

(4) the set $\cup\{C_{n_1,n_2,\ldots,n_k,n} : 1 \leq n < \omega\}$ is everywhere dense in the set $C_{n_1,n_2,\ldots,n_k}$;

(5) if $(n_1, n_2, ..., n_k) \neq (m_1, m_2, ..., m_k)$, then the sets $C_{n_1,n_2,\ldots,n_k}$ and $C_{m_1,m_2,\ldots,m_k}$ are disjoint;

(6) for any natural number $k \geq 1$, we have $\mathrm{diam}(C_{n_1,n_2,\ldots,n_k}) < 1/2^k$.

Proof. This lemma can readily be deduced from Lemma 6 by using the method of induction. The details are left to the reader. Let us only remark that all sets $C(k) = \cup\{C_{n_1,n_2,\ldots,n_k} : n_1 \geq 1, n_2 \geq 1, ..., n_k \geq 1\}$, where $k < \omega$, are everywhere dense in E. This simple fact will be substantially utilized below.

Lemma 8. *Under the notation of the previous lemma and of its proof, the set $R = \cap\{C(k) : k < \omega\}$ is everywhere dense in E. Moreover, the same R is everywhere dense in each set $C_{n_1,n_2,\ldots,n_k}$.*

Proof. Let us establish the density of R in E. We use a fairly standard argument. Take a closed ball B in E. Since $\cup\{C_n : n \geq 1\}$ is dense in E, there exists some C_{n_1} such that $\text{int}(B) \cap C_{n_1} \neq \emptyset$. Analogously, because $\cup\{C_{n_1,n} : n \geq 1\}$ is dense in C_{n_1}, there exists some C_{n_1,n_2} such that $\text{int}(B) \cap C_{n_1,n_2} \neq \emptyset$. Proceeding in this manner, we will be able to define by recursion the sequence of sets $C_{n_1} \supset C_{n_1,n_2} \supset ... \supset C_{n_1,n_2,...,n_k} \supset ...$ satisfying the relations

$$\text{int}(B) \cap C_{n_1,n_2,...,n_k} \neq \emptyset \qquad (k < \omega).$$

It follows from this fact that $B \cap C_{n_1} \cap C_{n_1,n_2} \cap ... \cap C_{n_1,n_2,...,n_k} \cap ... \neq \emptyset$. In view of the inclusion $C_{n_1} \cap C_{n_1,n_2} \cap ... \cap C_{n_1,n_2,...,n_k} \cap ... \subset R$, we immediately obtain $B \cap R \neq \emptyset$, which shows that R is everywhere dense in E.

By using a similar argument, one can establish the density of R in any set $C_{n_1,n_2,...,n_k}$ (actually, it suffices to replace E by the last set). Lemma 8 has thus been proved.

Lemma 9. *We preserve the notation of the two preceding lemmas. For each natural number $k \geq 1$, define a function $f_k : E \to [0,1]$ by the formula*

$$f_k(x) = 0 \text{ if } x \notin C(k), \quad f_k(x) = 1/2^{n_1+n_2+...+n_k} \text{ if } x \in C_{n_1,n_2,...,n_k}.$$

Then the following two relations hold:
(1) $0 \leq f_k(x) \leq 1/2^k$ for any $x \in E$;
(2) f_k is upper semicontinuous on E.

Proof. Relation (1) is almost trivial. Indeed, in view of the inequalities $n_1 \geq 1,\ n_2 \geq 1,\ \ldots,\ n_k \geq 1$, we have

$$0 \leq f_k(x) \leq \frac{1}{2^{n_1+n_2+...+n_k}} \leq \frac{1}{2^k}.$$

Let us establish the validity of relation (2). Taking any point $x \in E$, we need to show that f_k is upper semicontinuous at x. Assume first that $x \in C_{n_1,n_2,...,n_k}$. We must verify that $\text{limsup}_{i\to+\infty} f_k(x_i) \leq f_k(x)$ whenever a sequence $\{x_i : i < \omega\} \subset E$ converges to x. Suppose to the contrary that $\text{limsup}_{i\to+\infty} f_k(x_i) > f_k(x)$ for some sequence $\{x_i : i < \omega\} \subset E$ converging to x. Then among the members of $\{x_i : i < \omega\}$ there are infinitely many points $x_i \in C_{m_1,m_2,...,m_k}$ where $m_1 + m_2 + ... + m_k < n_1 + n_2 + ... + n_k$. Hence there exists a fixed $(m_1, m_2, ..., m_k)$ satisfying the above-mentioned inequality and such that infinitely many points x_i belong to $C_{m_1,m_2,...,m_k}$. By virtue of the relation $\lim_{i\to+\infty} x_i = x$, we obtain

$$x \in C_{m_1,m_2,...,m_k} \cap C_{n_1,n_2,...,n_k},$$

whence it follows that $C_{m_1,m_2,...,m_k} \cap C_{n_1,n_2,...,n_k} \neq \emptyset$, which is impossible because of $(m_1, m_2, ..., m_k) \neq (n_1, n_2, ..., n_k)$. The contradiction obtained shows the validity of the inequality $\limsup_{i\to+\infty} f_k(x_i) \leq f_k(x)$.

The case when $x \notin C(k)$ can be considered analogously. This finishes the proof of the lemma.

Lemma 10. *Preserve the same notation as in Lemma 9 and define a function $f : E \to \mathbf{R}$ by $f(x) = f_1(x) + f_2(x) + ... + f_k(x) + ...$ for all $x \in E$. Then the following four relations are satisfied:*

(1) f is upper semicontinuous on E;

(2) $0 \leq f(x) \leq 1$ for all points $x \in E$;

(3) if $x \in C_{n_1,n_2,...,n_k} \setminus C(k+1)$, then

$$f(x) = \frac{1}{2^{n_1}} + \frac{1}{2^{n_1+n_2}} + ... + \frac{1}{2^{n_1+n_2+...+n_k}};$$

(4) if $x \in C_{n_1} \cap C_{n_1,n_2} \cap ... \cap C_{n_1,n_2,...,n_k} \cap ...$, then

$$f(x) = \frac{1}{2^{n_1}} + \frac{1}{2^{n_1+n_2}} + ... + \frac{1}{2^{n_1+n_2+...+n_k}} + ... \,.$$

Proof. Notice first that our f is well defined because the function series $\sum_{k\geq 1} f_k$ converges uniformly on E. Further, in view of Lemmas 1 and 9, f is upper semicontinuous on E. Relation (2) also follows from Lemma 9. Relations (3) and (4) are more or less trivial and their checking is left to the reader. Lemma 10 has thus been proved.

Now, we are able to present the following result (essentially due to Adian and Novikov).

Theorem 3. *Let E be a topological copy of the Cantor space C. Under the same notation, the function f defined in the previous lemma is not countably continuous.*

Proof. Take any countable covering $\{D_n : n \geq 1\}$ of the space E and, for any natural number $n \geq 1$, denote $A_n = R \cap D_n$. In this way we come to the countable covering $\{A_n : n \geq 1\}$ of the set R.

First, we shall show that there exist an integer $k \geq 1$ and a k-sequence $(n_1, n_2, ..., n_k)$ of natural numbers, such that the set A_k is not nowhere dense in the set $C_{n_1,n_2,...,n_k}$, i.e., $\mathrm{int}(\mathrm{cl}(A_k \cap C_{n_1,n_2,...,n_k})) \neq \emptyset$, where the standard topological operations int and cl are restricted to the family of all subsets of $C_{n_1,n_2,...,n_k}$.

Suppose otherwise and fix $n_1 \geq 1$. According to our assumption, for A_1 there exists a nonempty clopen set B_1 in C_{n_1} such that $A_1 \cap B_1 = \emptyset$. Because

$\cup\{C_{n_1,n} : n \geq 1\}$ is dense in C_{n_1}, there exist C_{n_1,n_2} and a nonempty clopen set $B_2 \subset C_{n_1,n_2}$ such that $B_2 \subset C_{n_1,n_2} \cap B_1$ and $A_2 \cap B_2 = \emptyset$. Continuing this process by recursion, we obtain two infinite sequences

$$\{n_1, n_2, ..., n_k, ...\}, \quad \{B_1, B_2, ..., B_k, ...\},$$

where $n_1, n_2, ..., n_k, ...$ are some natural numbers and

$$B_1 \supset B_2 \supset ... \supset B_k \supset ..., \quad A_k \cap B_k = \emptyset \quad (k \geq 1).$$

Since all the sets B_k are compacts, we have $\cap\{B_k : k \geq 1\} \neq \emptyset$. Also, we can write $\cap\{B_k : k \geq 1\} \subset R = \cup\{A_k : k \geq 1\}$, which yields a contradiction with the relation $(\cap\{B_k : k \geq 1\}) \cap (\cup\{A_k : k \geq 1\}) = \emptyset$. The contradiction obtained shows that, for some A_k and $C_{n_1,n_2,...,n_k}$, the set $A_k \cap C_{n_1,n_2,...,n_k}$ is not nowhere dense in $C_{n_1,n_2,...,n_k}$. Choose any nonempty clopen set $B \subset C_{n_1,n_2,...,n_k}$ in which A_k is dense. Obviously,

$$B \cap A_k =$$

$$(\cup\{B \cap A_k \cap C_{n_1,n_2,...,n_k,i} : i \leq 2m\}) \cup (\cup\{B \cap A_k \cap C_{n_1,n_2,...,n_k,i} : i \geq 2m+1\})$$

for any natural number $m \geq 1$. The set $\cup\{B \cap A_k \cap C_{n_1,n_2,...,n_k,i} : i \leq 2m\}$ is nowhere dense in $B \cap A_k$ and the set $\cup\{B \cap A_k \cap C_{n_1,n_2,...,n_k,i} : i \geq 2m+1\}$ is everywhere dense in $B \cap A_k$.

Let us establish that $f|A_k$ is discontinuous at each point of $B \cap A_k$. For this purpose, take any $x_0 \in B \cap A_k$. Because $x_0 \in R$, we have

$$f(x_0) = \frac{1}{2^{n_1}} + \frac{1}{2^{n_1+n_2}} + ... + \frac{1}{2^{n_1+n_2+...+n_k}} + \frac{1}{2^{n_1+n_2+...+n_k+m}} + ...,$$

where $m \geq 1$ is some natural number. Take $\varepsilon_0 = 1/2^{n_1+n_2+...+n_k+2m}$. As was mentioned above, every neighborhood of x_0 contains points x belonging to the set $\cup\{B \cap A_k \cap C_{n_1,n_2,...,n_k,i} : i \geq 2m+1\}$. Consequently, for all such x, we may write

$$f(x) \leq \frac{1}{2^{n_1}} + ... + \frac{1}{2^{n_1+n_2+...+n_k}} + \frac{1}{2^{n_1+n_2+...+n_k+2m}} \leq f(x_0) - \varepsilon_0,$$

which yields at once that $f|A_k$ is discontinuous at x_0. In this manner, we have established that $f|A_k$ is not continuous and hence the original function f cannot be countably continuous. The proof of Theorem 3 is finished.

The result just presented is essentially based on specific properties of the classical Cantor space C. However, by taking into account Lemma 5, one can easily conclude that the following more general statement is also true.

Theorem 4. *Let E be an uncountable Polish space. Then there exists an upper (lower) semicontinuous bounded function $f : E \to \mathbf{R}$ that is not countably continuous.*

After Theorem 4 was proved, a number of publications appeared and extended the investigation of those real-valued functions which possess more or less good descriptive properties, but are not countably continuous.

In particular, Jackson and Mauldin proved in [94] that the Lebesgue measure $\lambda : \mathcal{F}([0,1]) \to [0,1]$ restricted to the family $\mathcal{F}([0,1])$ consisting of all nonempty compact subsets of $[0,1]$ and equipped with the Hausdorff metric is not countably continuous. Notice, in this context, that λ is upper semicontinuous (see Exercise 22 from Chapter 1).

Another proof of the fact that λ is not countably continuous was given by van Mill and Pol in their joint paper [183]. In the same paper they considered many other examples and constructions of semicontinuous functions which fail to be countably continuous. One of their results states that in the Banach space V of bounded functions defined on $[0,1]$ and belonging to $Ba_1([0,1],\mathbf{R})$, the family of all countably continuous functions is a set of first category. An analogous result holds true for the Banach space V_d of all bounded derivatives on $[0,1]$ (cf. Exercise 19 from Chapter 2, where it is indicated that the family of all bounded derivatives forms a Banach space with respect to the standard norm of uniform convergence).

Remark 1. Laczkovich obtained a natural generalization of Theorem 4 to functions of higher Baire classes, which act from an uncountable Polish space into $\mathbf{R}$ (unpublished manuscript). His result was reproved and generalized by Cichoń and Morayne in [45]. Their approach is essentially based on the classical techniques of so-called universal functions (cf. Exercise 6 from Chapter 8).

Further information about this topic can be found, e.g., in works by Cichoń, Morayne, Pawlikowski, Solecki [46] and Darji [54], [55] (cf. also [47]).

EXERCISES

1. Using the notation of Theorem 2, show that if $A_0 = \emptyset$, then f is a function of first Baire class (see Chapter 2) and, in particular, f is a Borel function.

Conclude from this fact that, for a general Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$, the set A_0 cannot be ignored.

2. Let E be a topological space, x_0 be an arbitrary point from E, and let $\{f_n : n < \omega\}$ be a sequence of real-valued functions on E. Suppose that this sequence converges quasi-uniformly to a function f (see Exercise 16 from Chapter 2) and that all f_n $(n < \omega)$ are upper (respectively, lower) semicontinuous at the point x_0.

Prove that f is also upper (respectively, lower) semicontinuous at x_0.

3. Let E be a Polish topological space and let μ be a nonzero σ-finite Borel measure on E vanishing at all singletons in E.

Show that there exists a subset P of E satisfying the following relations:
(a) P is homeomorphic to the Cantor discontinuum C;
(b) P is nowhere dense in E;
(c) $\mu(P) > 0$.

Remark 2. In fact, the preceding exercise can be regarded as a natural generalization of Exercise 11 from Chapter 1.

4. Let C denote the classical Cantor discontinuum on $[0,1]$, let $[0,1]^\omega$ denote the Hilbert cube, and let $\phi : C \to [0,1]^\omega$ be a Peano type function (see, e.g., Chapter 1). Define two functions

$$f_1 : [0,1]^\omega \to C, \qquad f_2 : [0,1]^\omega \to C$$

by the following formulas:

$$f_1(x) = \min(\phi^{-1}(x)), \quad f_2(x) = \max(\phi^{-1}(x)) \qquad (x \in [0,1]^\omega).$$

Demonstrate that
(1) both compositions $\phi \circ f_1$ and $\phi \circ f_2$ coincide with the identical transformation of $[0,1]^\omega$ (in particular, both f_1 and f_2 are injections);
(2) f_1 is lower semicontinuous;
(3) f_2 is upper semicontinuous.

5*. Preserve the notation of Exercise 4. By using the Hurewicz theorem stating that the Hilbert cube $[0,1]^\omega$ is not a countable union of its zero-dimensional subspaces (see, for instance, [149] or [182]) and that any nonempty subspace of C is zero-dimensional, prove that neither f_1 nor f_2 is countably continuous.

Remark 3. The result presented in Exercise 5 is due to van Mill and Pol [183]. In Chapter 13 we shall return to real-valued semicontinuous functions which are not countably continuous. Moreover, several other constructions of such functions will be given in that chapter and some interesting connections with so-called Luzin sets on $\mathbf{R}$ will be indicated.

Chapter 4
Singular monotone functions

This chapter is devoted to some elementary properties of monotone functions acting from **R** into **R**, and to some widely known examples of strange monotone functions.

Let $f : \mathbf{R} \to \mathbf{R}$ be a partial function. We recall that f is said to be increasing (respectively, strictly increasing) if, for any two points x and y from $\mathrm{dom}(f)$, the relation $x \le y$ (respectively, $x < y$) implies the relation $f(x) \le f(y)$ (respectively, $f(x) < f(y)$).

Analogously the notion of a decreasing (respectively, strictly decreasing) partial function can be introduced. It is easy to see that f is increasing (strictly increasing) if and only if $-f$ is decreasing (strictly decreasing).

A partial function acting from **R** into **R** is said to be monotone (respectively, strictly monotone) if it is either increasing or decreasing (respectively, either strictly increasing or strictly decreasing).

We shall consider below only increasing partial functions (this, of course, does not restrict the generality of our considerations).

Let $f : \mathbf{R} \to \mathbf{R}$ be an increasing bounded partial function and suppose that $\mathrm{dom}(f) \neq \emptyset$. Fix a point t from $\mathrm{dom}(f)$. For any $x \le t$, put

$$f^*(x) = \inf\{f(z) : z \in \mathrm{dom}(f),\ x \le z \le t\}$$

and, for any $x \ge t$, put

$$f^*(x) = \sup\{f(z) : z \in \mathrm{dom}(f),\ t \le z \le x\}.$$

It can readily be demonstrated that f^* is an increasing bounded function acting from **R** into **R** and extending f.

We thus see that every increasing bounded partial function admits an increasing bounded extension defined on the whole of **R**. A similar argument shows that every increasing partial function $f : \mathbf{R} \to \mathbf{R}$ can be extended to an increasing function defined on some subinterval of **R**. In view of this circumstance, we primarily will be dealing with increasing functions acting from a subinterval of **R** into **R**.

If $f : \mathbf{R} \to \mathbf{R}$ is an increasing function and $x \in \mathbf{R}$, then there exist limits

$$\lim_{y \to x, y > x} f(y) = f(x+), \quad \lim_{y \to x, y < x} f(y) = f(x-),$$

and we have the obvious inequalities $f(x-) \leq f(x) \leq f(x+)$. Clearly, f is continuous at x if and only if $f(x-) = f(x) = f(x+)$. Thus, x is a discontinuity point of f if and only if

$$f(x-) < f(x) \quad \vee \quad f(x) < f(x+).$$

More generally, let $g : \mathbf{R} \to \mathbf{R}$ be an arbitrary function and let $x \in \mathbf{R}$.

We say that x is a simple discontinuity point for g if there exist both limits

$$\lim_{y \to x, y < x} g(y) = g(x-), \quad \lim_{y \to x, y > x} g(y) = g(x+)$$

and, at the same time,

$$g(x-) \neq g(x) \quad \vee \quad g(x+) \neq g(x).$$

Evidently, if g is a monotone function, then all discontinuity points for g are its simple discontinuity points.

We have the following useful result (see, e.g., [221]).

Theorem 1. *Let f be an arbitrary function acting from $\mathbf{R}$ into $\mathbf{R}$. Then the set of all simple discontinuity points for f is at most countable.*

Proof. Denote by E the set of all simple discontinuity points for f. Denote also

$$E_1 = \{x \in E \ : \ f(x-) < f(x+)\},$$
$$E_2 = \{x \in E \ : \ f(x-) > f(x+)\},$$
$$E_3 = \{x \in E \ : \ f(x) < f(x-) = f(x+)\},$$
$$E_4 = \{x \in E \ : \ f(x) > f(x-) = f(x+)\}.$$

Then we can write $E = E_1 \cup E_2 \cup E_3 \cup E_4$, and it suffices to demonstrate that each set E_i $(i = 1, 2, 3, 4)$ is at most countable. Because the argument is similar for all E_i, we shall only show that $\mathrm{card}(E_1) \leq \omega$. For this purpose, let us define a mapping

$$x \to (p(x), q(x), r(x)) \qquad (x \in E_1)$$

such that $\{p(x), q(x), r(x)\} \subset \mathbf{Q}$ and

$$x \in \ ]q(x), r(x)[,$$

$$(\forall t \in \ ]q(x), x[)(f(t) < p(x)), \quad (\forall t \in \ ]x, r(x)[)(f(t) > p(x)).$$

The existence of such a mapping is obvious. Now, it is not hard to check that this mapping is injective. Indeed, suppose to the contrary that, for some distinct points x_1 and x_2 from $\mathbf{R}$, we have the equality

$$(p(x_1), q(x_1), r(x_1)) = (p(x_2), q(x_2), r(x_2)).$$

Without loss of generality we may assume that $x_1 < x_2$. Choose any point t from $]x_1, x_2[$. Then we must have simultaneously $f(t) < p(x_2) = p(x_1)$ and $f(t) > p(x_1) = p(x_2)$, which is impossible. The contradiction obtained establishes that the described mapping is injective. This trivially implies the relation

$$\mathrm{card}(E_1) \leq \mathrm{card}(\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}) = \omega.$$

Theorem 1 has thus been proved.

As a direct consequence of the above theorem, we obtain that, for any monotone function f acting from $\mathbf{R}$ into $\mathbf{R}$, the set $D(f)$ of all discontinuity points of f is at most countable.

We now are going to present the classical Lebesgue theorem concerning the differentiation of monotone functions. For this purpose, we need three simple lemmas (cf. [194], [225]).

First, let us recall the notion of a derived number for partial functions acting from $\mathbf{R}$ into $\mathbf{R}$.

Suppose that $[a, b]$ is a segment of $\mathbf{R}$ and that $f : [a, b] \to \mathbf{R}$ is a function. Let $x \in [a, b]$. We shall say that $t \in \mathbf{R} \cup \{-\infty, +\infty\}$ is a derived number (or a Dini derived number) of f at x if there exists a sequence $\{x_n : n \in \mathbf{N}\}$ of points from $[a, b]$ tending to x, such that

$$(\forall n \in \mathbf{N})(x_n \neq x), \quad \lim_{n \to +\infty}(f(x_n) - f(x))/(x_n - x) = t.$$

In this case, we shall write $t \in f'_D(x)$.

One more remark. For any two real numbers t_1 and t_2, it will be convenient to denote below by the symbol $[t_1, t_2]$ the segment of $\mathbf{R}$ with the end-points t_1 and t_2. Thus, we do not assume in this notation that $t_1 < t_2$.

Lemma 1. *Let $f : [a, b] \to \mathbf{R}$ be a strictly increasing function, let q be a positive real number, and let X be a subset of $[a, b]$ such that, for any point $x \in X$, there exists at least one $t \in f'_D(x)$ satisfying $t \leq q$. Then we have the inequality $\lambda^*(f(X)) \leq q\lambda^*(X)$, where λ is the standard Lebesgue measure on $\mathbf{R}$ and λ^* denotes the outer measure associated with λ.*

Proof. Fix an arbitrary real $\varepsilon > 0$ and take an open subset G of $[a, b]$ such that $X \subset G$ and $\lambda(G) \leq \lambda^*(X) + \varepsilon$. Consider the family of nondegenerate segments

$$\{[f(x), f(x+h)] : x \in X,\ h \neq 0,\ [x, x+h] \subset G, (f(x+h) - f(x))/h \leq q + \varepsilon\}$$

and denote it by $\mathcal{V}$. Clearly, $\mathcal{V}$ forms a Vitali covering for the set $f(X)$. Consequently, by Vitali's theorem (see Chapter 0), there exists a disjoint countable family

$$\{[f(x_n), f(x_n + h_n)] \ : \ n \in \mathbf{N}\} \subset \mathcal{V}$$

such that

$$\lambda(f(X) \setminus \cup\{[f(x_n), f(x_n + h_n)] \ : \ n \in \mathbf{N}\}) = 0.$$

Since the function f is strictly increasing, the countable family of nondegenerate segments $\{[x_n, x_n + h_n] \ : \ n \in \mathbf{N}\}$ is disjoint, too, and the union of this family is contained in G. So we can write

$$\lambda^*(f(X)) \leq \sum_{n \in \mathbf{N}} |f(x_n + h_n) - f(x_n)| \leq (q + \varepsilon) \sum_{n \in \mathbf{N}} |h_n|$$
$$\leq (q + \varepsilon)\lambda(G) \leq (q + \varepsilon)(\lambda^*(X) + \varepsilon).$$

Taking account of the fact that $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\lambda^*(f(X)) \leq q\lambda^*(X)$, and the proof of Lemma 1 is complete.

Lemma 2. *Let $f : [a, b] \to \mathbf{R}$ be a strictly increasing function, let q be a positive real number, and let X be a subset of $[a, b]$ such that, for any point $x \in X$, there exists at least one $t \in f'_D(x)$ satisfying $t \geq q$. Then we have the inequality $\lambda^*(f(X)) \geq q\lambda^*(X)$.*

Proof. As we know, the set of all discontinuity points for f is at most countable. Keeping in mind this fact, we may assume without loss of generality that f is continuous at each point belonging to the given set X. Now, if $q = 0$, then there is nothing to prove. So let us suppose that $q > 0$. Pick an arbitrary real number $\varepsilon > 0$ for which $q - \varepsilon > 0$. There exists an open set $G \subset \mathbf{R}$ such that $f(X) \subset G$ and $\lambda(G) \leq \lambda^*(f(X)) + \varepsilon$. Consider the family of nondegenerate segments

$$\{[x, x+h] : x \in X,\ h \neq 0,\ [f(x), f(x+h)] \subset G,\ (f(x+h) - f(x))/h \geq q - \varepsilon\}$$

and denote it by $\mathcal{V}$. Obviously, $\mathcal{V}$ forms a Vitali covering for the set X. Consequently, by the Vitali theorem, there exists a disjoint countable family

$$\{[x_n, x_n + h_n] \ : \ n \in \mathbf{N}\} \subset \mathcal{V}$$

for which we have

$$\lambda(X \setminus \cup\{[x_n, x_n + h_n] \ : \ n \in \mathbf{N}\}) = 0.$$

Again, since f is strictly increasing, the countable family of nondegenerate segments $\{[f(x_n), f(x_n + h_n)] \ : \ n \in \mathbf{N}\}$ must be disjoint, too, and the union of this family is contained in G. Hence we may write

$$(q - \varepsilon)\lambda^*(X) \leq (q - \varepsilon) \sum_{n \in \mathbf{N}} |h_n| \leq \sum_{n \in \mathbf{N}} |f(x_n + h_n) - f(x_n)|$$

$$\leq \lambda(G) \leq \lambda^*(f(X)) + \varepsilon.$$

Taking account of the fact that $\varepsilon > 0$ can be arbitrarily small, we come to the desired inequality $q\lambda^*(X) \leq \lambda^*(f(X))$. Lemma 2 has thus been proved.

Lemma 3. *Let $f : [a, b] \to \mathbf{R}$ be a strictly increasing function and let*

$$X = \{x \in [a, b] \ : \ \text{there exist two distinct derived numbers of } f \text{ at } x\}.$$

Then X is a set of λ-measure zero.

Proof. For any two rational numbers p and q satisfying the inequalities $0 \leq p$ and $p < q$, introduce the notation

$$X_{p,q} = \{x \in [a, b] \ : \ \text{there exists a derived number of } f \text{ at } x \text{ less}$$

$$\text{than } p, \text{ and there exists a derived number of } f \text{ at } x \text{ greater than } q\}.$$

Clearly, we have the equality

$$X = \cup\{X_{p,q} \ : \ 0 \leq p < q, \ p \in \mathbf{Q}, \ q \in \mathbf{Q}\}.$$

So it suffices to show that each set $X_{p,q}$ is of λ-measure zero. Indeed, according to Lemma 1, we may write $\lambda^*(f(X_{p,q})) \leq p\lambda^*(X_{p,q})$. At the same time, according to Lemma 2, we have $\lambda^*(f(X_{p,q})) \geq q\lambda^*(X_{p,q})$. These two inequalities yield $q\lambda^*(X_{p,q}) \leq p\lambda^*(X_{p,q})$ or, equivalently,

$$0 \leq (p - q)\lambda^*(X_{p,q}).$$

Because $p - q < 0$, we must have $\lambda^*(X_{p,q}) = 0$. This finishes the proof of Lemma 3.

Now, we are ready to present the classical Lebesgue theorem on the differentiability (almost everywhere) of monotone functions.

Theorem 2. *Let $f : [a,b] \to \mathbf{R}$ be a monotone function. Then f is differentiable at almost all (with respect to λ) points of $[a,b]$.*

Proof. Obviously, we may suppose that f is increasing. Moreover, because the set of all differentiability points for f coincides with the set of all differentiability points for f_1, where

$$f_1(x) = f(x) + x \qquad (x \in [a,b]),$$

and f_1 is strictly increasing, we may assume without loss of generality that the original function f is also strictly increasing. Now, in view of Lemma 3, it suffices to demonstrate that the set

$$X = \{x \in [a,b] \ : \ (\forall n \in \mathbf{N})(\exists t \in f'_D(x))(t \geq n)\}$$

is of λ-measure zero. But this follows directly from Lemma 2, because, in conformity with this lemma, we may write

$$n\lambda^*(X) \leq \lambda^*(f(X)) \leq f(b) - f(a), \quad \lambda^*(X) \leq (f(b) - f(a))/n$$

for every natural number $n \geq 1$. This immediately yields the required equality $\lambda^*(X) = 0$, and completes the proof of Theorem 2.

It follows at once from the above theorem that a nowhere differentiable real-valued function f defined on a segment $[a,b]$ is simultaneously nowhere monotone on $[a,b]$, i.e., there does not exist a nondegenerate subinterval of $[a,b]$ on which f is monotone.

For our further considerations, we need the following useful result due to Fubini (cf. [225]).

Theorem 3. *Let $\{F_n \ : \ n \in \mathbf{N}\}$ be a sequence of positive (i.e., non-negative) increasing functions given on a segment $[a,b] \subset \mathbf{R}$, such that the series $\sum\{F_n(x) : n \in \mathbf{N}\}$ converges for each point $x \in [a,b]$, and let*

$$F(x) = \sum\{F_n(x) : n \in \mathbf{N}\} \qquad (x \in [a,b]).$$

Then, for almost all (with respect to the Lebesgue measure λ restricted to $[a,b]$) points $x \in [a,b]$, the equality $F'(x) = \sum\{F'_n(x) : n \in \mathbf{N}\}$ is satisfied.

Proof. Clearly, F is an increasing function on $[a,b]$ and we may write

$$F'(x) \geq F'_0(x) + F'_1(x) + ... + F'_n(x) + ...$$

for all $x \in [a,b]$ at which the derivatives $F'(x), F'_0(x), F'_1(x), ..., F'_n(x), ...$ exist simultaneously. In view of Theorem 2, the series of functions $\sum_{n \in \mathbf{N}} F'_n$ is convergent almost everywhere on $[a,b]$. Now, denote

$$S_n(x) = \sum\{F_m(x) : m \leq n\}$$

and, for any natural number k, choose an index $n(k)$ such that

$$F(b) - S_{n(k)}(b) \leq 1/2^k.$$

Since all F_n are positive and increasing, we have

$$0 \leq F(x) - S_{n(k)}(x) \leq 1/2^k$$

for each $x \in [a, b]$. This implies that the series of increasing functions

$$\sum_{k \in \mathbf{N}} (F(x) - S_{n(k)}(x))$$

converges uniformly on $[a, b]$ to some increasing function. Applying Theorem 2 once more, we easily infer that the series

$$\sum_{k \in \mathbf{N}} (F(x) - S_{n(k)}(x))'$$

converges at almost all points $x \in [a, b]$. From this fact we also deduce that

$$\lim_{k \to +\infty} (F(x) - S_{n(k)}(x))' = 0$$

for almost all $x \in [a, b]$, so

$$F'(x) = \lim_{k \to +\infty} (F'_0(x) + F'_1(x) + \ldots + F'_{n(k)}(x))$$

for almost all $x \in [a, b]$. But this immediately yields that $F' = \sum_{n \in \mathbf{N}} F'_n$ almost everywhere on $[a, b]$. Theorem 3 has thus been proved.

Exercise 6 of this chapter provides an application of the above theorem to the differentiation of an indefinite Lebesgue integral.

We now turn our attention to the construction of a continuous strictly increasing function $g : \mathbf{R} \to \mathbf{R}$ whose derivative vanishes almost everywhere. Our considerations below will be essentially based on Theorem 3.

Let us recall that the first step of the construction of Cantor's classical discontinuum on $\mathbf{R}$ is that we remove from the segment $[0, 1]$ the open interval $]1/3, 2/3[$. Let us put at this step

$$f(x) = 0 \text{ if } x \leq 0, \quad f(x) = 1 \text{ if } x \geq 1, \quad f(x) = 1/2 \text{ if } x \in \ ]1/3, 2/3[.$$

Suppose that on the n-th step of the construction we have already defined the function f for all those points which belong to the union of the removed (at this and earlier steps) intervals. Obviously, we obtain a finite family

$\{[a_i, b_i] : 1 \leq i \leq m\}$ of pairwise disjoint segments on $[0, 1]$. It is easy to check that $m = 2^n$, but we do not need this fact for our further purposes. Pick any segment $[a_i, b_i]$ from the above-mentioned family. Taking into account the inductive assumption, we may put $f(x) = (f(a_i-) + f(b_i+))/2$ for all points $x \in\ ](2a_i+b_i)/3, (2b_i+a_i)/3[$. So we have defined our function f for all points belonging to the union of all intervals removed at the $(n+1)$-th step. Continuing the process in this manner, we will be able to construct f on the set $\mathbf{R} \setminus C$, where C denotes the Cantor set. From the definition of f immediately follows that f is increasing and continuous on its domain. Moreover, it is easily seen that f can be uniquely extended to an increasing continuous function $f : \mathbf{R} \to [0, 1]$. Since f is constant on each removed interval, we obviously get $f'(x) = 0$ for all $x \in \mathbf{R} \setminus C$, so the derivative of f vanishes almost everywhere on $\mathbf{R}$.

Thus, we have shown that there exists an increasing bounded continuous function f from $\mathbf{R}$ into $\mathbf{R}$, which is not constant and whose derivative is zero almost everywhere on $\mathbf{R}$.

Now, let p and q be any two points of $\mathbf{R}$ such that $p < q$. Because f is not constant, there are some points x and y from $\mathbf{R}$ such that $f(x) < f(y)$. Evidently, $x < y$ and there exists a homothety (or translation) h of the plane $\mathbf{R}^2$, for which $h((x, 0)) = (p, 0)$ and $h((y, 0)) = (q, 0)$. Let f^* denote the function from $\mathbf{R}$ into $\mathbf{R}$, whose graph coincides with the image of the graph of f with respect to h. Then we may assert that f^* is also an increasing bounded continuous function whose derivative vanishes almost everywhere, and $f^*(p) < f^*(q)$.

By virtue of the remarks just made, we can formulate and prove the following classical result concerning the existence of strictly increasing continuous singular functions.

Theorem 4. *There exists a function* $g : \mathbf{R} \to \mathbf{R}$ *satisfying these three conditions:*

(1) g *is continuous and strictly increasing;*

(2) $(\forall x \in \mathbf{R})(0 \leq g(x) \leq 1)$*;*

(3) the derivative of g *is zero almost everywhere on* $\mathbf{R}$*.*

Proof. Let $\{(p_n, q_n) : n \in \mathbf{N}\}$ denote the countable family of all pairs of rational numbers, such that $p_n < q_n$. According to the argument presented above, for each natural index n, there exists a function $g_n : \mathbf{R} \to [0, 1]$ such that

(a) g_n is continuous and increasing;

(b) $0 \leq g_n(x) \leq 1/2^{n+1}$ for all $x \in \mathbf{R}$;

(c) the inequality $g_n(p_n) < g_n(q_n)$ holds true;

(d) the derivative of g_n vanishes almost everywhere.

It follows from (b) that the series $\sum_{n \in \mathbf{N}} g_n$ is uniformly convergent. So we may consider the function $g = \sum_{n \in \mathbf{N}} g_n$, which is continuous and increasing because of (a). Evidently, $\mathrm{ran}(g) \subset [0, 1]$. In accordance with (c), we also have $g(p_n) < g(q_n)$ for all $n \in \mathbf{N}$, whence it immediately follows that g is a strictly increasing function. Finally, taking into account condition (d) and applying Theorem 3, we conclude that the derivative of g equals zero almost everywhere on $\mathbf{R}$. Theorem 4 has thus been proved.

EXERCISES

1. Recall that a function $f : \mathbf{R} \to \mathbf{R}$ possesses the Darboux property if, for each subinterval $[a, b]$ of $\mathbf{R}$, the range of f contains the segment with the end-points $f(a)$ and $f(b)$.

Demonstrate that any function with the Darboux property has no simple discontinuity points.

In particular, infer from this fact that if f is the derivative of some function acting from $\mathbf{R}$ into $\mathbf{R}$, then f has no simple discontinuity points.

2. Let $E = \{x_n \ : \ n \in \mathbf{N}\}$ be an arbitrary countable subset of $\mathbf{R}$ and let $\{r_n \ : \ n \in \mathbf{N}\}$ be a countable family of strictly positive real numbers, such that $\sum_{n \in \mathbf{N}} r_n < +\infty$. For any point $x \in \mathbf{R}$, put

$$f(x) = \sum_{n \in N(x)} r_n,$$

where $N(x) = \{n \in \mathbf{N} : x_n < x\}$. In this way, a certain function f acting from $\mathbf{R}$ into $\mathbf{R}$ is defined.

Show that

(a) f is increasing;

(b) f is continuous at each point from $\mathbf{R} \setminus E$;

(c) for any natural index n, one has

$$f(x_n+) - f(x_n-) = r_n,$$

and, in particular, f is discontinuous at each point of the given set E.

Deduce from this result that if E is everywhere dense in $\mathbf{R}$ (for example, if $E = \mathbf{Q}$), then the function f constructed above has an everywhere dense set of its discontinuity points.

3*. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function.

Demonstrate that the following two assertions are equivalent:

(a) f is injective;

(b) f is strictly monotone.

Verify that there exist continuous functions $g : \mathbf{R} \to \mathbf{R}$ which cannot be represented in the form $g = g_1 + g_2$, where g_1 and g_2 are monotone functions acting from $\mathbf{R}$ into $\mathbf{R}$.

On the other hand, by using the method of transfinite induction, prove that any function $h : \mathbf{R} \to \mathbf{R}$ is representable in the form $h = h_1 + h_2$, where both functions $h_1 : \mathbf{R} \to \mathbf{R}$ and $h_2 : \mathbf{R} \to \mathbf{R}$ are injective.

Remark 1. It should be noticed, in connection with Exercise 3, that if h is Lebesgue measurable, then both h_1 and h_2 can be chosen to be Lebesgue measurable, too (see [173]).

4. Let $f : [a, b] \to \mathbf{R}$ be an increasing continuous function.

Show that $\int_a^b f'(t)dt \leq f(b) - f(a)$.

Give an example where this inequality is strict (cf. Theorem 4 of this chapter).

In addition, demonstrate that if $\int_a^b f'(t)dt = f(b) - f(a)$, then the function f is absolutely continuous on the line segment $[a, b]$.

5. Let λ denote, as usual, the standard Lebesgue measure on $\mathbf{R}$ and let X be an arbitrary Lebesgue measure zero subset of $\mathbf{R}$. As is well known, there exists a sequence $\{U_n : n \in \mathbf{N}\}$ of open subsets of $\mathbf{R}$, such that $X \subset U_n$ and $\lambda(U_n) < 1/2^n$ for all $n \in \mathbf{N}$. For any $n \in \mathbf{N}$, introduce the function

$$f_n(x) = \lambda(U_n \cap \;]-\infty, x]) \qquad (x \in \mathbf{R}).$$

Observe that f_n is increasing, continuous, and $0 \leq f_n(x) \leq 1/2^n$ for all $x \in \mathbf{R}$. Further, define

$$f_X(x) = \sum_{n \in \mathbf{N}} f_n(x) \qquad (x \in \mathbf{R}).$$

Show that the function f_X is increasing, continuous, and, for any point $x \in X$, the equality

$$\lim_{y \to x, y \neq x} (f_X(y) - f_X(x))/(y - x) = +\infty$$

holds true.

6*. Let f be a positive lower semicontinuous function given on a line segment $[a, b]$. We recall (see Exercise 18 from Chapter 0) that f can be represented in the form $f = \sup\{f_n : n \in \mathbf{N}\}$, where all functions f_n are positive and continuous.

Derive from this fact that $f = \sum_{n\in\mathbf{N}} g_n$, where all functions g_n are positive and continuous.

Let now f be a positive, Lebesgue integrable, lower semicontinuous function on $[a,b]$ and let

$$F(x) = \int_a^x f(t)dt \qquad (x \in [a,b]).$$

Prove, by applying Theorem 3 of this chapter and the fact formulated above, that $F'(x) = f(x)$ for almost all (with respect to the Lebesgue measure λ) points $x \in [a,b]$.

Let g be a positive bounded Lebesgue measurable function on $[a,b]$.

Show that, for each real $\varepsilon > 0$, there exists a bounded lower semicontinuous function f on $[a,b]$, such that

$$g \le f, \qquad \int_a^b g(t)dt + \varepsilon > \int_a^b f(t)dt.$$

Deduce from this fact that there exists a sequence $\{f_n : n \in \mathbf{N}\}$ of bounded lower semicontinuous functions such that

(a) $f_{n+1} \le f_n$ for any $n \in \mathbf{N}$;

(b) $g \le f_n$ for any $n \in \mathbf{N}$;

(c) $\lim_{n\to+\infty} f_n(x) = g(x)$ for almost all points $x \in [a,b]$.

In particular, the equality $(f_0 - g) = \sum_{n\in\mathbf{N}}(f_n - f_{n+1})$ holds true almost everywhere on $[a,b]$. Observe that

$$f_0 - g \ge 0, \quad f_n - f_{n+1} \ge 0 \qquad (n \in \mathbf{N}).$$

Putting

$$F(x) = \int_a^x (f_0 - g)(t)dt \qquad (x \in [a,b]),$$

$$F_n(x) = \int_a^x (f_n - f_{n+1})(t)dt \qquad (x \in [a,b])$$

and applying Theorem 3 once more, demonstrate that

$$(\int_a^x g(t)dt)' = g(x)$$

for almost all points $x \in [a,b]$.

Finally, prove the Lebesgue theorem stating that if h is any real-valued Lebesgue integrable function on $[a,b]$, then

$$(\int_a^x h(t)dt)' = h(x)$$

for almost all points $x \in [a, b]$.

Remark 2. The argument presented in Exercise 6 shows that the classical Lebesgue theorem concerning the differentiation of a function H defined by

$$H(x) = \int_a^x h(t)dt \qquad (x \in [a, b]),$$

can be logically deduced from Theorem 3. However, this approach has a weak side, because it does not yield the description of the set of those points $x \in [a, b]$ at which $H'(x) = h(x)$.

7. Let f be an arbitrary continuous increasing function acting from $\mathbf{R}$ into $\mathbf{R}$, whose derivative is equal to zero almost everywhere on $\mathbf{R}$. For each half-open subinterval $[a, b[$ of $\mathbf{R}$, let us put $\mu([a, b[) = f(b) - f(a)$.

Demonstrate that μ can be uniquely extended to a σ-finite Borel measure on $\mathbf{R}$ (denoted by the same symbol μ) that is diffused (i.e., vanishes on all one-element subsets of $\mathbf{R}$) and is singular with respect to the standard Lebesgue measure λ. The latter means that there exists a Borel subset X of $\mathbf{R}$ for which $\lambda(X) = 0$ and $\mu(\mathbf{R} \setminus X) = 0$.

Formulate the converse assertion and prove it by utilizing the Vitali covering theorem.

8. Verify that

(a) an increasing function $f : \mathbf{R} \to \mathbf{R}$ is continuous from the right if and only if f is upper semicontinuous;

(b) an increasing function $f : \mathbf{R} \to \mathbf{R}$ is continuous from the left if and only if f is lower semicontinuous.

Starting with these facts, construct a monotone function $g : \mathbf{R} \to \mathbf{R}$ that is upper (lower) semicontinuous and whose discontinuity points form an everywhere dense subset of $\mathbf{R}$ (cf. Exercise 2 of this chapter).

9. Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary bounded from above function. Define a new function $f^* : \mathbf{R} \to \mathbf{R}$ by putting

$$f^*(x) = \sup_{y<x} f(y) \qquad (x \in \mathbf{R}).$$

Check that f^* is increasing and lower semicontinuous.

In addition, suppose that the original function f is increasing.

Show that f and f^* have the same set of continuity points and, in general, f^* does not coincide with f.

Finally, check that f^* coincides with f if and only if f is continuous from the left.

10. Let $\{t_n : n < \omega\}$ be any sequence of real numbers.

Prove that at least one of the following three assertions is valid:

(a) $\{t_n : n < \omega\}$ contains an infinite strictly increasing subsequence;

(b) $\{t_n : n < \omega\}$ contains an infinite strictly decreasing subsequence;

(c) $\{t_n : n < \omega\}$ contains infinitely many terms which are equal to each other.

Give a straightforward proof of this result. On the other hand, show that the same result is a direct consequence of the Ramsey combinatorial theorem for countable graphs (see, e.g., [81], [211]).

Conclude from the above facts that any partial function $f : \mathbf{R} \to \mathbf{R}$ satisfying the relation $\mathrm{card}(\mathrm{dom}(f)) \geq \omega$ either is strictly monotone on some infinite subset of $\mathbf{R}$ or is constant on some infinite subset of $\mathbf{R}$.

Remark 3. The last result cannot be extended to uncountable subsets of $\mathrm{dom}(f)$ (within the theory **ZFC**). Indeed, we shall see in our further considerations that under the Continuum Hypothesis (**CH**) there exists a function $g : \mathbf{R} \to \mathbf{R}$, which is not monotone on any uncountable subset of $\mathbf{R}$. The construction of such a function is very similar to the classical construction of a Sierpiński–Zygmund function (cf. Chapter 8) or to the classical construction of a Luzin set on $\mathbf{R}$ (cf. Chapter 13).

11*. Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary continuous function.

Prove that there exists a nonempty perfect set $P \subset \mathbf{R}$ such that the restriction $f|P$ is monotone on P.

This can be done by using the following (fairly standard) argument. Suppose that there exists no nondegenerate subinterval of $\mathbf{R}$ on which f is decreasing. In this case, construct by recursion a dyadic system

$$(T_{i_1 i_2 \ldots i_k})_{i_1 \in \{0,1\}, i_2 \in \{0,1\}, \ldots, i_k \in \{0,1\}, k \geq 0}$$

of nondegenerate compact subintervals of $\mathbf{R}$ satisfying the relations

(a) $T_{i_1 i_2 \ldots i_k i_{k+1}} \subset T_{i_1 i_2 \ldots i_k}$;

(b) $T_{i_1 i_2 \ldots i_k 0} \cap T_{i_1 i_2 \ldots i_k 1} = \emptyset$;

(c) $\mathrm{diam}(T_{i_1 i_2 \ldots i_k}) < 1/2^k$;

(d) if $(i_1, i_2, ..., i_k) \prec (j_1, j_2, ..., j_k)$, then $x < y$ and $f(x) < f(y)$ for all points $x \in T_{i_1 i_2 \ldots i_k}$ and $y \in T_{j_1 j_2 \ldots j_k}$ (here $\preceq$ denotes the standard lexicographical ordering of the set of all k-sequences whose terms belong to $\{0, 1\}$).

Finally, put $P = \cap_{k \geq 0}(\cup\{T_{i_1 i_2 \ldots i_k} : (i_1, i_2, ..., i_k) \in \{0, 1\}^k\})$ and verify that f is increasing on the nonempty perfect set P.

Establish the same result for those functions $f : \mathbf{R} \to \mathbf{R}$ which are Lebesgue measurable or possess the Baire property (reduce this more general situation to the case of a continuous real-valued function given on a nonempty perfect subset of $\mathbf{R}$).

Moreover, let $\{f_1, f_2, ..., f_n\}$ be a finite family of real-valued functions on $\mathbf{R}$, all of which are Lebesgue measurable (or, respectively, possess the Baire property).

Show that there exists a nonempty perfect set $P \subset \mathbf{R}$ such that all the restrictions $f_1|P$, $f_2|P$, ..., $f_n|P$ are monotone on P.

On the other hand, construct an infinite sequence $\{f_n : n \in \mathbf{N}\}$ of real-valued continuous functions on $[0,1]$ such that there is no uncountable set $X \subset [0,1]$ for which all the restrictions $\{f_n|X : n \in \mathbf{N}\}$ are monotone.

Remark 4. In connection with the previous exercise, see also [20].

12. Give an example of a Peano type mapping

$$f = (f_1, f_2) : [0,1] \to [0,1]^2$$

such that both coordinate functions f_1 and f_2 are differentiable at almost all points of $[0,1]$ (in the sense of the Lebesgue measure λ) and the equalities $f_1' = f_2' = 0$ hold almost everywhere on $[0,1]$ (in the same sense).

Remark 5. It is useful to compare the above exercise with Exercise 15 from Chapter 1.

13. Let E be a topological space and let $\mathcal{F}_l(E, \mathbf{R})$ denote the family of all real-valued lower semicontinuous functions on E.

Verify that if $f \in \mathcal{F}_l(E, \mathbf{R})$ and $g \in \mathcal{F}_l(E, \mathbf{R})$, then

(a) $tf + rg \in \mathcal{F}_l(E, \mathbf{R})$ for any two real numbers $t \geq 0$ and $r \geq 0$;

(b) $\min(f, g) \in \mathcal{F}_l(E, \mathbf{R})$ and $\max(f, g) \in \mathcal{F}_l(E, \mathbf{R})$.

Chapter 5
A characterization of constant functions via Dini's derived numbers

In this chapter we again will be dealing with Dini's derived numbers for continuous real-valued functions defined on various nondegenerate subintervals of the real line **R**.

In terms of Dini's derived numbers, a necessary and sufficient condition will be given for a continuous function to be constant on a subinterval of **R**.

One of the main theorems of Calculus states that if $f : [a, b] \to \mathbf{R}$ is differentiable on $[a, b]$ and its derivative is identically equal to zero, then f is constant.

There are further extensions and generalizations of this fundamental statement. It turns out that one of such generalizations can be formulated in terms of Dini's derived numbers. Here we would like to discuss in more detail Dini's derived numbers with their application to some generalized version of the above-mentioned basic statement of Calculus.

For extensive information about Dini's derived numbers and their properties, we refer the reader to [33], [194], and [225] (see also the previous chapter where certain differentiability properties of singular monotone functions were briefly considered).

In what follows the symbol **N** stands, as usual, for the set of all natural numbers. Also, instead of the real line **R**, it will be convenient in this chapter to deal with the extended real line $\mathbf{R}^*$, which is defined by

$$\mathbf{R}^* = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$$

and to which the standard structures of **R** are extended in a natural way.

Let $[a, b]$ be a nondegenerate closed subinterval of **R**, let $f : [a, b] \to \mathbf{R}$ be a function, and let $x \in [a, b[$.

Recall that $t \in \mathbf{R}^*$ is Dini's right derived number of f at x if there exists a sequence of points $\{x_n : n \in \mathbf{N}\} \subset \ ]x, b]$ such that

$$x = \lim_{n\to+\infty} x_n, \qquad t = \lim_{n\to+\infty}(f(x_n) - f(x))/(x_n - x).$$

In a similar manner the notion of Dini's left derived number is usually introduced. Namely, $\tau \in \mathbf{R}^*$ is Dini's left derived number of f at $y \in\ ]a,b]$ if there exists a sequence of points $\{y_n : n \in \mathbf{N}\} \subset [a,y[$ such that

$$y = \lim_{n\to+\infty} y_n, \qquad \tau = \lim_{n\to+\infty}(f(y_n) - f(y))/(y_n - y).$$

Accordingly, one says that $r \in \mathbf{R}^*$ is Dini's derived number of f at $x \in [a,b]$ if r is either Dini's right derived number of f at x or Dini's left derived number of f at x.

At first sight, the above definition of Dini's derived numbers seems to be different from the definition given in Chapter 4. However, it is not difficult to check that these definitions are equivalent for all points belonging to the open interval $]a,b[$.

Since Dini's derived numbers are defined in terms of limits of sequences, they look simpler (and, possibly, more convenient) than the usual concept of derivative which relies on the operation of taking limits along uncountable point sets.

It is easy to see that, for any function $f : [a,b] \to \mathbf{R}$ and for each point $x \in\ ]a,b[$, there exist Dini's right and left derived numbers of f at x.

Further, f is differentiable at x if and only if all Dini's derived numbers of f at x are finite and mutually coincide.

A lot of continuous functions on $[a,b]$ can be indicated which are not differentiable at some points of $]a,b[$ but have the uniformly bounded Dini's right and left derived numbers at all points of $]a,b[$.

Lemma 1. *Let L be a nonnegative real number and let $f : [a,b] \to \mathbf{R}$ be a continuous function. Suppose that this f satisfies the following condition:*

(*) there are countably many points $\{x_n : n \in \mathbf{N}\} \subset [a,b]$ such that, for any point $x \in [a,b[\ \setminus \{x_n : n \in \mathbf{N}\}$, there exists Dini's right derived number of f at x, which does not exceed L.

Then one may assert that $f(x) - f(a) \leq L(x-a)$ for all points x belonging to $[a,b]$.

Proof. Take any real $\varepsilon > 0$ and choose a sequence $\{\varepsilon_n : n \in \mathbf{N}\}$ of strictly positive real numbers such that $\sum\{\varepsilon_n : n \in \mathbf{N}\} < \varepsilon$. It suffices to show that, for every point $x \in [a,b]$, the inequality

$$f(x) - f(a) \leq (L+\varepsilon)(x-a) + \sum\{\varepsilon_n : x_n < x\}$$

holds true. This can be done in the following manner. Fix $x \in\ ]a,b]$ and denote

$$Y = \{a \leq y < x : f(y) - f(a) \leq (L+\varepsilon)(y-a) + \sum\{\varepsilon_n : x_n < y\}\}.$$

The set Y is nonempty, because of the relation $a \in Y$. Now, let us put $z = \sup(Y)$. By virtue of the continuity of f, we have $z \in Y$, i.e.,

$$f(z) - f(a) \leq (L + \varepsilon)(z - a) + \sum\{\varepsilon_n : x_n < z\}.$$

It remains to demonstrate that $z = x$. Suppose to the contrary that $z < x$. Only two cases are possible.

1. $z = x_k$ for some natural number k.

In this case, using the continuity of f at x_k, we may write

$$f(y') - f(z) \leq \varepsilon_k + (L + \varepsilon)(y' - z)$$

for some point $y' \in \,]z, x[$. So we readily deduce that

$$f(y') - f(a) \leq (L + \varepsilon)(y' - a) + \sum\{\varepsilon_n : x_n < y'\},$$

which yields a contradiction with the definition of z.

2. z differs from all points x_k ($k \in \mathbf{N}$).

In this case, there exists Dini's right derived number of f at z, which does not exceed L. This circumstance implies that

$$f(y'') - f(z) \leq (L + \varepsilon)(y'' - z)$$

for some point $y'' \in \,]z, x[$, whence it follows that

$$f(y'') - f(a) \leq (L + \varepsilon)(y'' - a) + \sum\{\varepsilon_n : x_n < y''\},$$

and we again obtain a contradiction with the definition of z.

Thus, under the condition (*), we have the inequality

$$f(x) - f(a) \leq (L + \varepsilon)(x - a) + \sum\{\varepsilon_n : x_n < x\}$$

and hence the inequality

$$f(x) - f(a) \leq L(x - a) \qquad (x \in [a, b]),$$

which completes the proof of Lemma 1.

Keeping in mind that in the above inequality L is an arbitrary non-negative real number, consider the particular case when $L = 0$. In this case $f(x) \leq f(a)$. Applying the natural analogue of this inequality to any subinterval $[y, x]$ of $[a, b]$, we immediately get $f(x) \leq f(y)$. So we come to the following auxiliary statement.

Lemma 2. *If a given function $f : [a, b] \to \mathbf{R}$ is continuous and possesses Dini's nonpositive right derived number at each point of a co-countable subset of $[a, b]$, then f is decreasing (in general, not strictly decreasing) on $[a, b]$.*

This fact yields at once a generalized version of the basic statement of Calculus.

Theorem 1. *If a function $f : [a, b] \to \mathbf{R}$ is continuous and possesses Dini's zero right derived number at each point of a co-countable subset of $[a, b]$, then f is constant on $[a, b]$.*

Proof. As was shown above, both functions f and $-f$ must be decreasing on $[a, b]$ or, equivalently, f must be simultaneously decreasing and increasing on $[a, b]$, which implies that f is constant.

Remark 1. The assumption of continuity of f on $[a, b]$ is essential in the formulation of Theorem 1. Indeed, an arbitrary real-valued step-function g on $[a, b]$ has a derivative at every point of a co-countable subset of $[a, b]$, but g does not need to be constant.

Remark 2. In connection with Theorem 1, it makes sense to recall the classical result of Lebesgue stating that any continuous monotone function on $[a, b]$ is differentiable at almost all (in the sense of Lebesgue measure) points of $[a, b]$ (see, e.g., [33], [194], [225], and Chapter 4 of this book).

Taking into account the above-mentioned result of Lebesgue, one readily obtains the next statement.

Theorem 2. *If a function $f : [a, b] \to \mathbf{R}$ is continuous and possesses Dini's nonnegative (respectively, nonpositive) right derived number at each point of a co-countable subset of $[a, b]$, then f is increasing (respectively, decreasing) and, consequently, is differentiable at almost all points of $[a, b]$ in the sense of Lebesgue measure.*

There are many functions which are continuous and strictly increasing (strictly decreasing) on $[a, b]$, whose derivatives are zero at almost all points of $[a, b]$ in the sense of Lebesgue measure (cf. [78], [168], [225], and Chapter 4). Usually, they are called singular functions of Cantor type. In this context, the following problem naturally arises:

Given a continuous function $f : [a, b] \to \mathbf{R}$ such that its derivative is zero at almost all points of $[a, b]$, find a necessary and sufficient condition for f to be a constant function.

In order to resolve this problem (even in a more general setting), let us establish two auxiliary propositions.

The first of them is very similar to Lemma 1. In its formulation below, the symbol λ stands, as usual, for the standard Lebesgue measure on $[a, b]$. Here we do not need any profound properties of λ. We only will use the fact that, for every λ-measurable set $X \subset [a, b]$ and for an arbitrary real number $\varepsilon > 0$, there exists an open set $G \subset [a, b]$ containing X and satisfying the inequality $\lambda(G \setminus X) < \varepsilon$.

Lemma 3. *Let $L \geq 0$ be a real number and let $f : [a, b] \to \mathbf{R}$ be a continuous function for which there exists a λ-measurable set $D \subset [a, b]$ such that*

(1) at each point of D, the function f possesses Dini's right derived number not exceeding L;

(2) at each point of the set $[a, b] \setminus D$, there exists Dini's right derived number of f which is equal to zero.

Then the inequality $f(x) - f(a) \leq L\lambda(D \cap [a, x])$ holds true for any point $x \in [a, b]$.

Proof. Let ε be an arbitrary strictly positive real number and let G be an open subset of $[a, b]$ containing D and satisfying the relation $\lambda(G \setminus D) < \varepsilon$. It suffices to show that

$$f(x) - f(a) \leq (L + \varepsilon)\lambda(G \cap [a, x]) + \varepsilon(x - a)$$

for any point $x \in [a, b]$. We may proceed analogously to the argument presented earlier (cf. the proof of Lemma 1). Namely, consider the set

$$Y = \{a \leq y < x : f(y) - f(a) \leq (L + \varepsilon)\lambda(G \cap [a, y]) + \varepsilon(y - a)\}$$

and observe that $a \in Y$. Further, denote $z = \sup(Y)$. In view of the continuity of f, we get

$$f(z) - f(a) \leq (L + \varepsilon)\lambda(G \cap [a, z]) + \varepsilon(z - a).$$

We now claim that $z = x$. Indeed, suppose otherwise, i.e., $z < x$. Only two cases are possible.

1. $z \in D$. In this case, we have $z \in G$ and there exists $y' \in\ ]z, x[$ such that $[z, y'] \subset G$ and

$$f(y') - f(z) \leq (L + \varepsilon)(y' - z).$$

So we easily obtain the inequality

$$f(y') - f(a) \leq (L + \varepsilon)\lambda(G \cap [a, y']) + \varepsilon(y' - a),$$

which contradicts the definition of z.

2. $z \notin D$. In this case, there exists Dini's right derived number of f at z, which is equal to zero. Consequently, we may write

$$f(y'') - f(z) \leq \varepsilon(y'' - z)$$

for some point $y'' \in \ ]z, x[$. This inequality readily implies the relation

$$f(y'') - f(a) \leq (L + \varepsilon)\lambda(G \cap [a, y'']) + \varepsilon(y'' - a),$$

and once again we come to a contradiction with the definition of z. Lemma 3 has thus been proved.

If in the formulation of Lemma 3 we have $L = 0$, then likewise as earlier we can conclude that the function f is decreasing on $[a, b]$ (in general, not strictly decreasing).

Also, if in the same lemma $\lambda(D) = 0$, then f again turns out to be decreasing on $[a, b]$.

Lemma 4. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function, D be a λ-measure zero subset of $[a, b]$, and let the following two conditions be fulfilled:*

(1) for any point $x \in [a, b] \setminus D$, there exists Dini's right derived number of f at x which is equal to zero;

(2) there is a constant $L \geq 0$ such that the absolute values of all Dini's right derived numbers at all points from D do not exceed L.

Then the function f is constant on $[a, b]$.

Proof. According to Lemma 3, both functions f and $-f$ must be decreasing or, equivalently, f must be decreasing and increasing simultaneously, which implies at once that f is a constant function.

The statement formulated below gives a characterization of constant functions via Dini's derived numbers.

Theorem 3. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function satisfying the following two conditions:*

(1) there is some λ-measure zero set $D \subset [a, b]$ such that, at each point of $[a, b] \setminus D$, there exists Dini's right derived number of f equal to zero;

(2) at any point of $[a, b]$ all Dini's derived numbers of f are finite.

Then the function f is constant on $[a, b]$.

Proof. Suppose to the contrary that f is not constant on $[a, b]$. We may assume, without loss of generality, that $f(a) \neq f(b)$. According to Lemma 4, the set of all absolute values of Dini's right derived numbers of f on D is unbounded. Clearly, this circumstance implies the existence of a nondegenerate segment $[a_1, b_1] \subset [a, b]$ such that $|f(b_1) - f(a_1)| > b_1 - a_1$. In addition, we may suppose that $b_1 - a_1 < (b - a)/2$. Further, since we have $f(a_1) \neq f(b_1)$, the set of all absolute values of Dini's right derived numbers of f on $D \cap [a_1, b_1]$ is also unbounded. This implies the existence of a nondegenerate segment $[a_2, b_2] \subset [a_1, b_1[$ such that

$$|f(b_2) - f(a_2)| > 2(b_2 - a_2), \qquad b_2 - a_2 < (b_1 - a_1)/2.$$

Proceeding in this manner by recursion, we finally obtain a decreasing (by inclusion) sequence $\{[a_n, b_n] : n \in \mathbf{N} \setminus \{0\}\}$ of nondegenerate closed subintervals of $[a, b]$ satisfying the relations

$$b_{n+1} - a_{n+1} < (b_n - a_n)/2, \qquad |f(b_n) - f(a_n)| > n(b_n - a_n)$$

for all natural numbers $n \geq 1$. Let now x be a unique point belonging to all of these subintervals. Then a simple geometric argument shows that at least one of the following two equalities holds:

$$\lim_{n\to+\infty} \frac{|f(a_n) - f(x)|}{a_n - x} = -\infty, \quad \lim_{n\to+\infty} \frac{|f(b_n) - f(x)|}{b_n - x} = +\infty.$$

This yields a contradiction with condition (2) and completes the proof of the theorem.

At the end of this chapter it makes sense to present one direct corollary of Theorem 3.

Theorem 4. *Let $f : [a, b] \to \mathbf{R}$ be a continuous function for which there is a λ-measure zero set $D \subset [a, b]$ having the property that at every point of the set $[a, b] \setminus D$ there exists Dini's right derived number of f equal to zero. Then these two relations are equivalent:*

(1) f is differentiable at all points of $]a, b[$;

(2) f is constant on $[a, b]$.

The proof of this last statement is left to the reader.

Remark 3. The equivalence of the relations (1) and (2) in Theorem 4 perfectly explains why the singularity of certain continuous monotone functions on $[a, b]$ is closely connected with the absence of (finite) derivatives of such functions at some points of $]a, b[$.

EXERCISES

1. Check that the definition of Dini's derived number introduced in Chapter 4 is equivalent to the definition of Dini's derived number introduced in the present chapter.

2. Give an example of a continuous function on a nondegenerate segment $[a, b]$, which is not differentiable at some points of the open interval $]a, b[$ but has the uniformly bounded Dini's right and left derived numbers at all points of $]a, b[$.

3. Give a proof of Theorem 4.

4. Let $f : [a, b] \to \mathbf{R}$ be a continuous function. Denote

$$X = \{x \in [a, b] : \text{all Dini's derived numbers of } f \text{ at } x \text{ are finite}\}.$$

Demonstrate that X is a Borel subset of $[a, b]$.

5*. Preserving the notation of Exercise 4 and assuming that f is not constant and almost everywhere on $[a, b]$ there are Dini's right derived numbers of f equal to zero, show that the set $[a, b] \setminus X$ is of cardinality continuum $\mathbf{c}$.

For this purpose, keep in mind the Alexandrov–Hausdorff theorem stating that the cardinality of any uncountable Borel subset of $[a, b]$ is equal to $\mathbf{c}$ (see [97], [105], [149], [167]).

6. A function $f : [a, b] \to \mathbf{R}$ is called weakly smooth (or smooth in the Riemann sense) at a point $x_0 \in \;]a, b[$ if

$$\lim_{h \to 0, h \neq 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h} = 0.$$

Verify that if f is differentiable at x_0, then f is also weakly smooth at x_0.

7*. Let $f : [a, b] \to \mathbf{R}$ be a continuous function weakly smooth at all points of $]a, b[$. Denote $X = \{x \in [a, b] : \text{there exists } f'(x)\}$.

Demonstrate that, for every nondegenerate segment $[c, d] \subset [a, b]$, the set $X \cap [c, d]$ is of cardinality continuum.

Argue as follows. Let $\phi : [a, b] \to \mathbf{R}$ be an affine function such that

$$\phi(c) = f(c), \qquad \phi(d) = f(d).$$

Then the function $g = f - \phi$ is continuous, weakly smooth, and satisfies the equalities $g(c) = g(d) = 0$. Let $x_0 \in \;]c, d[$ be a point at which g has a local minimum.

Check that there exists $g'(x_0) = 0$, so $f'(x_0) = (f(d) - f(c))/(d - c)$.

Finally, changing arbitrarily a point $t \in \;]c, d[$ and applying the above argument to the segment $[c, t]$, obtain the required result.

Remark 4. We will see in our further considerations that there exist everywhere discontinuous weakly smooth functions on $\mathbf{R}$ which are also non-measurable with respect to the standard Lebesgue measure λ (cf. Chapter 11).

Chapter 6
Everywhere differentiable nowhere monotone functions

It follows from the results discussed in Chapter 4 that if a continuous function $f : \mathbf{R} \to \mathbf{R}$ is nowhere differentiable, then f is nowhere monotone, i.e., there does not exist a nondegenerate subinterval of $\mathbf{R}$ on which f is monotone. The present chapter is devoted to some constructions of functions also acting from $\mathbf{R}$ into $\mathbf{R}$, differentiable everywhere but nowhere monotone. The question of the existence of the above-mentioned functions is obviously typical for classical mathematical analysis. In this connection, it should be noticed that many mathematicians of the end of the 19th century and of the beginning of the 20th century tried to give various constructions of functions of such a kind. As a rule, their constructions were either incorrect or, at least, incomplete. As pointed out in [104], the first explicit construction of such a function was suggested by Köpcke in 1889. Another example was given by Pereno in 1897 (this example is presented in [92]). In addition, Denjoy presented in his extensive work [58] a proof of the existence of an everywhere differentiable nowhere monotone function, as a consequence of his profound investigations concerning trigonometric series and their convergence. Afterwards, a number of distinct proofs of the existence of everywhere differentiable nowhere monotone functions were given by several authors (see, e.g., [79], [104], [156], [275]).

We begin with the discussion of the construction described in [104]. This construction is completely elementary and belongs to classical mathematical analysis. We need some easy auxiliary propositions.

Lemma 1. *Let r and s be two strictly positive real numbers. The following assertions are valid:*

(1) if $r > s$, then $(r - s)/(r^2 - s^2) < 2/r$;

(2) if $r > 1$ and $s > 1$, then $(r + s - 2)/(r^2 + s^2 - 2) < 2/s$.

Proof. Clearly, we have the relation

$$(r-s)/(r^2-s^2) = 1/(r+s) < 1/r < 2/r.$$

Thus, (1) holds true. Further, it can easily be checked that the inequality of (2) is equivalent to the inequality

$$(r-s)^2 + (r-1)(s-1) + r^2 + r + 3s > 5$$

which, obviously, is fulfilled under our assumptions $r > 1$ and $s > 1$. This completes the proof of Lemma 1.

Lemma 2. *Let ϕ be a function acting from $\mathbf{R}$ into $\mathbf{R}$ and defined by the formula $\phi(x) = (1+|x|)^{-1/2}$ for all $x \in \mathbf{R}$, and let a and b be any two distinct real numbers. Then we have*

$$(1/(b-a))\int_a^b \phi(x)dx < 4\min(\phi(a), \phi(b)).$$

Proof. Without loss of generality we may assume that $a < b$. Only three cases are possible.

1. $0 \le a < b$. In this case, taking into account (1) of Lemma 1, we can write

$$(1/(b-a))\int_a^b \phi(x)dx = 2((1+b)^{1/2} - (1+a)^{1/2})/((1+b)-(1+a))$$

$$< 4/(1+b)^{1/2} = 4\min(\phi(a), \phi(b)).$$

2. $a < b \le 0$. This case can be reduced to the previous one, because of the evenness of our function ϕ.

3. $a < 0 < b$. In this case, taking into account (2) of Lemma 1, we can write

$$(1/(b-a))\int_a^b \phi(x)dx = 2((1+b)^{1/2} + (1-a)^{1/2} - 2)/((1+b)+(1-a)-2)$$

$$< 4\min(\phi(a), \phi(b)).$$

This finishes the proof of Lemma 2.

Lemma 3. *Let $n > 0$ be a natural number and let $\psi : \mathbf{R} \to \mathbf{R}$ be a function of the form*

$$\psi(x) = \sum_{1 \le k \le n} c_k \phi(d_k(x - t_k)) \qquad (x \in \mathbf{R}),$$

where ϕ is the function from Lemma 2, $c_1, c_2, ..., c_n, d_1, d_2, ..., d_n$ are strictly positive real numbers, and $t_1,\ t_2,\ \ldots,\ t_n$ are any real numbers. Then

$$(1/(b-a))\int_a^b \psi(x)dx < 4\min(\psi(a), \psi(b))$$

for all distinct reals a and b.

Proof. The assertion follows immediately from Lemma 2, by taking into account the fact that

$$(1/(b-a))\int_a^b \phi(d(x-t))dx = (1/(d(b-t)-d(a-t)))\int_{d(a-t)}^{d(b-t)} \phi(x)dx$$

for every real number t and any $d > 0$.

Lemma 4. *Let $(\psi_n)_{n\geq 1}$ be a sequence of functions as in Lemma 3. For each point $x \in \mathbf{R}$ and for each integer $n \geq 1$, let us define*

$$\Psi_n(x) = \int_0^x \psi_n(z)dz$$

and suppose that, for some $a \in \mathbf{R}$, the series $\sum_{n\geq 1}\psi_n(a)$ is convergent. Denote

$$\sum_{n\geq 1}\psi_n(a) = s < +\infty.$$

Then we have

(1) the series $F(x) = \sum_{n\geq 1}\Psi_n(x)$ converges uniformly on every bounded subinterval of $\mathbf{R}$;

(2) the function F is differentiable at the point a and $F'(a) = s$.

In particular, if $f(z) = \sum_{n\geq 1}\psi_n(z) < +\infty$ for each point $z \in \mathbf{R}$, then the function F is differentiable everywhere on $\mathbf{R}$ and the equality $F' = f$ holds true.

Proof. Take any $b \in \mathbf{R}$ satisfying the relation $b \geq |a|$. In view of Lemma 3, for all points $x \in [-b, b]$ and for all integers $n \geq 1$, we may write

$$|\Psi_n(x)| \leq |\int_0^a \psi_n(z)dz| + |\int_a^x \psi_n(z)dz| \leq$$

$$4|a|\psi_n(a) + 4|x-a|\psi_n(a) \leq 12b\psi_n(a).$$

This shows the uniform convergence of the series $\sum_{n\geq 1}\Psi_n(x)$ with respect to $x \in [-b, b]$.

Further, let $\varepsilon > 0$ be given. Pick a natural number $k > 0$ such that

$$10 \cdot \sum_{n>k} \psi_n(a) < \varepsilon.$$

Because all functions ψ_n are continuous on the whole $\mathbf{R}$ (in particular, at a), there exists some real $\delta > 0$ such that

$$|(1/h)\int_a^{a+h} \psi_n(z)dz - \psi_n(a)| < \varepsilon/2k,$$

whenever $0 < |h| < \delta$ and $1 \leq n \leq k$. Consequently, assuming $0 < |h| < \delta$ and applying Lemma 3 once again, we get

$$|(F(a+h) - F(a))/h - s| = |\sum_{n\geq 1}((1/h)\int_a^{a+h} \psi_n(z)dz - \psi_n(a))| \leq$$

$$\sum_{1\leq n\leq k} |(1/h)\int_a^{a+h} \psi_n(z)dz - \psi_n(a)| + \sum_{n>k}((1/h)\int_a^{a+h} \psi_n(z)dz + \psi_n(a))$$

$$< \varepsilon/2 + \sum_{n>k} 5\psi_n(a) < \varepsilon.$$

We thus conclude that $F'(a) = s$, and the lemma is proved.

Lemma 5. *Let $n > 0$ be a natural number, let I_1, ..., I_n be pairwise disjoint nondegenerate segments on $\mathbf{R}$, and let t_k denote the midpoint of I_k for each natural number $k \in [1, n]$. Fix any strictly positive real numbers*

$$\varepsilon,\ y_1,\ \ldots,\ y_n.$$

Then there exists a function ψ as in Lemma 3 such that, for all natural numbers $k \in [1, n]$, the following three relations are fulfilled:
(1) $\psi(t_k) > y_k$;
(2) $(\forall x \in I_k)(\psi(x) < y_k + \varepsilon)$;
(3) $(\forall x \in \mathbf{R} \setminus (I_1 \cup ... \cup I_n))(\psi(x) < \varepsilon)$.

Proof. For each natural number $k \in [1, n]$, denote $c_k = y_k + \varepsilon/2$ and define

$$\phi_k(x) = c_k\phi(d_k(x - t_k)) \qquad (x \in \mathbf{R}),$$

where a real $d_k > 0$ is chosen so large that $(\forall x \in \mathbf{R} \setminus I_k)(\phi_k(x) < \varepsilon/2n)$. Finally, put $\psi = \phi_1 + \ \ldots \ + \phi_n$. Then, taking into account the fact that

$$\max_{x\in\mathbf{R}}\phi_k(x) = \phi_k(t_k) = c_k \qquad (1 \leq k \leq n),$$

it is easy to check that the function ψ satisfies relations (1), (2), (3).

Lemma 6. *Let any two disjoint countable subsets*

$$\{t_k \ : \ k \in \mathbf{N}, \ k \geq 1\}, \quad \{r_k \ : \ k \in \mathbf{N}, \ k \geq 1\}$$

of $\mathbf{R}$ *be given. Then there exists a function* $F : \mathbf{R} \to \mathbf{R}$ *such that*
(1) F *is differentiable everywhere on* $\mathbf{R}$*;*
(2) $0 < F'(x) \leq 1$ *for all* $x \in \mathbf{R}$*;*
(3) $F'(t_k) = 1$ *for each* $k \in \mathbf{N} \setminus \{0\}$*;*
(4) $F'(r_k) < 1$ *for each* $k \in \mathbf{N} \setminus \{0\}$.

Proof. We shall construct by recursion a sequence $\{\psi_n : n \geq 1\}$ of functions as in Lemma 3 with some additional properties. Namely, denoting $f_n = \sum_{1 \leq k \leq n} \psi_k$, we desire the following conditions would be satisfied:
(i) for any integer $n \geq 1$ and for all integers $k \in [1, n]$, we have

$$f_n(t_k) > 1 - 1/n;$$

(ii) for any integer $n \geq 1$ and for each point $x \in \mathbf{R}$, we have

$$f_n(x) < 1 - 1/(n+1);$$

(iii) for any integer $n \geq 1$ and for all integers $k \in [1, n]$, we have

$$\psi_n(r_k) < 1/(2n \cdot 2^n).$$

At the first step we choose a nondegenerate segment I_1 with midpoint t_1, such that $r_1 \notin I_1$, and apply Lemma 5 with $\varepsilon = 1/4$ and $y_1 = 1/4$. Evidently, we obtain ψ_1 and $f_1 = \psi_1$ such that relations (i), (ii), and (iii) are fulfilled for $n = 1$.

Suppose now that, for a natural number $n > 1$, we have already defined the functions $\psi_1, ..., \psi_k, ..., \psi_{n-1}$ satisfying the corresponding analogues of (i) – (iii) for $n - 1$. Pick disjoint nondegenerate segments $I_1, \ldots, I_n$ in such a way that
(a) t_k is the midpoint of I_k for each integer $k \in [1, n]$;
(b) $I_k \cap \{r_1, \ \ldots, \ r_n\} = \emptyset$ for each integer $k \in [1, n]$;
(c) for any integer $k \in [1, n]$ and for any point $x \in I_k$, we have the inequality $f_{n-1}(x) < f_{n-1}(t_k) + \delta$, where

$$\delta = 1/(n(n+1)) \ - \ 1/(2n \cdot 2^n).$$

We now can apply Lemma 5 with $\varepsilon = 1/(2n \cdot 2^n)$ and

$$y_k = 1 - (1/n) - f_{n-1}(t_k) \qquad (1 \leq k \leq n).$$

Applying the above-mentioned lemma, we get the function ψ_n. Clearly,

$$\psi_n(r_k) < \varepsilon = 1/(2n \cdot 2^n)$$

for all natural numbers $k \in [1, n]$, so relation (iii) holds true. Further, for any natural number $k \in [1, n]$, we also have

$$f_n(t_k) = f_{n-1}(t_k) + \psi_n(t_k) > f_{n-1}(t_k) + y_k = 1 - 1/n,$$

which shows that relation (i) holds true, too. Finally, in order to verify (ii), fix any point $x \in \mathbf{R}$. If, for some integer $k \in [1, n]$, the point x belongs to I_k, then we may write

$$f_n(x) = f_{n-1}(x) + \psi_n(x) < f_{n-1}(t_k) + \delta + y_k + \varepsilon =$$

$$1 - 1/n + 1/(n(n+1)) = 1 - 1/(n+1).$$

If x does not belong to $I_1 \cup \ ... \ \cup I_n$, then

$$f_n(x) = f_{n-1}(x) + \psi_n(x) < 1 - 1/n + \varepsilon < 1 - 1/(n+1).$$

Thus, relation (ii) holds true, too. Proceeding in this manner, we are able to construct the required sequence $\{\psi_n : n \geq 1\}$. Putting

$$f = \sum_{n \geq 1} \psi_n = \lim_{n \to +\infty} f_n$$

and denoting

$$F(x) = \int_0^x f(z)dz \qquad (x \in \mathbf{R}),$$

we obtain the function $F : \mathbf{R} \to \mathbf{R}$. In view of Lemma 4, we also get

$$F'(x) = f(x) \qquad (x \in \mathbf{R}).$$

Further, the definition of F immediately implies

$$(\forall x \in \mathbf{R})(0 < F'(x) \leq 1), \quad F'(t_k) = 1 \quad (k \in \mathbf{N},\ k \geq 1).$$

Now, fix an integer $k \geq 1$ and let n be a natural number strictly greater than k. Then

$$F'(r_k) = f_{n-1}(r_k) + \sum_{m \geq n} \psi_m(r_k) <$$

$$1 - 1/n + \sum_{m \geq n} 1/(2m \cdot 2^m) < 1 - 1/n + 1/2n = 1 - 1/2n < 1.$$

This completes the proof of Lemma 6.

Theorem 1. *There exists a function $H : \mathbf{R} \to \mathbf{R}$ such that*
(1) H is differentiable everywhere on $\mathbf{R}$;
(2) H' is bounded on $\mathbf{R}$;
(3) H is monotone on no nondegenerate subinterval of $\mathbf{R}$.

Proof. Denote by

$$\{t_n \ : \ n \in \mathbf{N}, \ n > 0\}, \quad \{r_n \ : \ n \in \mathbf{N}, \ n > 0\}$$

some two disjoint countable everywhere dense subsets of $\mathbf{R}$. Using the previous lemma, take any two everywhere differentiable functions $F : \mathbf{R} \to \mathbf{R}$ and $G : \mathbf{R} \to \mathbf{R}$ satisfying the relations

(a) $0 < F'(x) \leq 1$ and $0 < G'(x) \leq 1$ for all $x \in \mathbf{R}$;
(b) $F'(t_n) = 1$ and $F'(r_n) < 1$ for each natural number $n \geq 1$;
(c) $G'(r_n) = 1$ and $G'(t_n) < 1$ for each natural number $n \geq 1$.

Now, define $H = F - G$. Obviously, we have

$$H'(t_n) > 0, \quad H'(r_n) < 0 \qquad (n \in \mathbf{N}, \ n \geq 1).$$

Because both the sets $\{t_n : n \in \mathbf{N}, \ n > 0\}$ and $\{r_n : n \in \mathbf{N}, \ n > 0\}$ are everywhere dense in $\mathbf{R}$, we infer that H cannot be monotone on any subinterval of $\mathbf{R}$. Also, the relation

$$-1 < H'(x) < 1 \qquad (x \in \mathbf{R})$$

implies that H' is bounded, and the theorem has thus been proved.

In fact, the preceding argument establishes the existence of many functions $f : \mathbf{R} \to \mathbf{R}$ which are everywhere differentiable, nowhere monotone and such that f' is bounded on $\mathbf{R}$. Let us mention several other interesting and extraordinary properties of any such function f.

1. f has a point of a local maximum and a point of a local minimum in every nonempty open subinterval of $\mathbf{R}$. Actually, for each nondegenerate segment $[a, b] \subset \mathbf{R}$, we can find two points x_1 and x_2 satisfying the relations

$$a < x_1 < x_2 < b, \quad f'(x_1) > 0, \quad f'(x_2) < 0.$$

Let us denote $M = \sup_{t \in [x_1, x_2]} f(t)$. Then, for some $\tau \in [x_1, x_2]$, we must have $f(\tau) = M$, and it is clear that τ must be in the interior of $[x_1, x_2]$. In other words, τ is a point of a local maximum for our f.

Applying a similar argument, we can also find a point of a local minimum of f on the same nondegenerate segment $[a, b]$.

2. Because f' is bounded, the function f satisfies the so-called Lipschitz condition, i.e., for some real constant $d \geq 0$, we have

$$|f(x) - f(y)| \leq d|x - y| \qquad (x \in \mathbf{R},\ y \in \mathbf{R}).$$

Notice that, in the latter relation, we may put $d = \sup_{t \in \mathbf{R}} |f'(t)|$. In particular, f is absolutely continuous. This also implies that f' is Lebesgue integrable on each bounded subinterval of $\mathbf{R}$.

3. The function f' is not integrable in the Riemann sense on any nondegenerate segment $[a, b] \subset \mathbf{R}$. To see this, suppose otherwise, i.e., suppose that f' is Riemann integrable on $[a, b]$. Then, according to a well-known theorem of mathematical analysis, f' must be continuous at almost all (with respect to the standard Lebesgue measure) points of $[a, b]$ (see, e.g., [194]). Taking into account the fact that f' changes its sign on each nonempty open subinterval of $\mathbf{R}$, we infer that f' must be equal to zero at almost all points of $[a, b]$. Consequently, f must be constant on $[a, b]$, which is impossible. The contradiction obtained yields the desired result.

4. Being a derivative, the function f' belongs to the first Baire class, i.e., f' can be represented as a pointwise limit of a sequence of continuous functions (see Example 1 from Chapter 2). Hence, by virtue of the classical Baire theorem (see, e.g., [149], [194], [202] or Theorem 3 from Chapter 2), the set of all those points of $\mathbf{R}$ at which f' is continuous is residual (co-meager), i.e., is the complement of a first category subset of $\mathbf{R}$. This fact is in contrast to the circumstance mentioned in 3.

5. Let us introduce the notation

$$X = \{t \in \mathbf{R} \ :\ f'(t) > 0\}, \quad Y = \{t \in \mathbf{R} \ :\ f'(t) < 0\}.$$

The sets X and Y are disjoint, Lebesgue measurable, and have the property that, for each nondegenerate segment $[a, b] \subset \mathbf{R}$, the relations

$$\lambda(X \cap [a, b]) > 0, \quad \lambda(Y \cap [a, b]) > 0$$

are valid (where λ denotes, as usual, the Lebesgue measure on $\mathbf{R}$). In order to demonstrate this fact, suppose, for example, that $\lambda(Y \cap [a, b]) = 0$. Then we get $f'(t) \geq 0$ for almost all points $t \in [a, b]$. But this immediately implies that f, being of the form

$$f(x) = \int_0^x f'(t)dt + f(0) \qquad (x \in \mathbf{R}),$$

is increasing on $[a, b]$, which contradicts the definition of f.

Now, we are going to present an essential generalization of Theorem 1 due to Weil (see [275]). We will see that Weil's argument uses the classical Baire theorem on category.

We recall that a function $f : \mathbf{R} \to \mathbf{R}$ is a derivative if there exists at least one everywhere differentiable function $F : \mathbf{R} \to \mathbf{R}$ satisfying the relation

$$(\forall x \in \mathbf{R})(F'(x) = f(x)).$$

Let us consider the set $D = \{f : f \text{ is a derivative and } f \text{ is bounded}\}$. Obviously, D is a vector space over $\mathbf{R}$. We may equip this set with a metric d defined by the formula

$$d(f, g) = \sup_{x \in \mathbf{R}} |f(x) - g(x)|.$$

Clearly, the metric d produces the topology of uniform convergence. In view of the well-known theorem of mathematical analysis, a uniform limit of a sequence of bounded derivatives is a bounded derivative (cf. Exercise 19 from Chapter 2). This shows, in particular, that the pair (D, d) is a Banach space (it can easily be seen that it is nonseparable). Take any function $f \in D$ and consider the set $f^{-1}(0)$. We assert that this set is a G_δ-subset of $\mathbf{R}$. Indeed, we may write

$$f^{-1}(0) = \{x \in \mathbf{R} \ : \ \lim_{n \to +\infty} n(F(x + 1/n) - F(x)) = 0\},$$

where a function $F : \mathbf{R} \to \mathbf{R}$ is such that $F'(x) = f(x)$ for all $x \in \mathbf{R}$. Equivalently, we have

$$f^{-1}(0) = \bigcap_{1 \leq n < \omega} \Big(\bigcup_{n \leq m < \omega} \{x \in \mathbf{R} \ : \ m(F(x + 1/m) - F(x)) < 1/n\} \Big).$$

This formula yields at once the desired result.

Let us put

$$D_0 = \{f \in D \ : \ \text{the set } f^{-1}(0) \text{ is everywhere dense in } \mathbf{R}\}.$$

We need the following simple fact.

Lemma 7. *The set D_0 is a closed vector subspace of the space D. Consequently, D_0 is a Banach space, as well.*

Proof. First, we show that D_0 is closed in D. Let $\{f_k \ : \ k < \omega\}$ be a sequence of functions from D_0, converging (in metric d) to some function $f \in D$. For each $k < \omega$, define $Z_k = f_k^{-1}(0)$. Then all the sets Z_k are everywhere dense G_δ-subsets of $\mathbf{R}$. Therefore, the set $Z = \cap\{Z_k \ : \ k < \omega\}$

is an everywhere dense G_δ-subset of $\mathbf{R}$, too, and the inclusion $Z \subset f^{-1}(0)$ is valid. Thus, we obtain that $f \in D_0$.

Now, let us demonstrate that D_0 is a vector subspace of D. Clearly, if $f \in D_0$ and $t \in \mathbf{R}$, then $tf \in D_0$. Further, take any two functions $g \in D_0$ and $h \in D_0$ and consider the sets

$$Z_g = g^{-1}(0), \quad Z_h = h^{-1}(0), \quad Z_{g+h} = (g+h)^{-1}(0).$$

Then $Z_g \cap Z_h$ is an everywhere dense G_δ-subset of $\mathbf{R}$, and it is evident that $Z_g \cap Z_h \subset Z_{g+h}$. This shows that D_0 is a vector space. Lemma 7 has thus been proved.

Notice now that the space D_0 is nontrivial, i.e., contains nonzero functions. For instance, this fact follows directly from Theorem 1 (and it can also be proved by using another argument).

Theorem 2. *Let E denote the set of all those functions $f \in D_0$ for which there exists a nondegenerate subinterval of* $\mathbf{R}$ *(depending on f) where f preserves its sign.*

Then the set E is of first category in the space D_0.

Proof. Let $\{I_n : n \in \mathbf{N}\}$ be an enumeration of all nondegenerate subintervals of $\mathbf{R}$ with rational end-points. For each $n \in \mathbf{N}$, put

$$A_n = \{f \in D_0 : (\forall x \in I_n)(f(x) \geq 0)\},$$

$$B_n = \{f \in D_0 : (\forall x \in I_n)(f(x) \leq 0)\}.$$

Clearly, we have $E = \cup_{n \in \mathbf{N}}(A_n \cup B_n)$, so it suffices to demonstrate that each of the sets A_n and B_n is closed and nowhere dense. We shall establish this fact only for A_n (for B_n, the argument is analogous). The closedness of A_n is trivial. In order to prove that A_n contains no ball in D_0, take any $f \in D_0$ and fix an arbitrary real $\varepsilon > 0$. Because $f \in D_0$, there exists a point $x \in I_n$ such that $f(x) = 0$. Now, by starting with the existence of a nonzero bounded derivative belonging to D_0, it is easy to show that there is a function $h \in D_0$ for which

$$h(x) < 0, \quad \sup_{y \in \mathbf{R}} |h(y)| < \varepsilon.$$

Let us define $g = f + h$. Then the function g belongs to the ball of D_0 with center f and radius ε. At the same time, g does not belong to A_n because

$$g(x) = f(x) + h(x) = h(x) < 0.$$

This establishes that A_n is nowhere dense in D_0. Theorem 2 has thus been proved.

Remark 1. In the next chapter, we shall consider one more proof of the existence of everywhere differentiable nowhere monotone functions, by applying some properties of the so-called density topology on $\mathbf{R}$.

EXERCISES

1. Give a direct construction of two disjoint Lebesgue measurable subsets X and Y of $\mathbf{R}$, such that, for any nonempty open interval $I \subset \mathbf{R}$, the inequalities $\lambda(I \cap X) > 0$ and $\lambda(I \cap Y) > 0$ hold true.

More generally, show that there exists a partition $\{X_n : n < \omega\}$ of $\mathbf{R}$ consisting of Lebesgue measurable sets and such that $\lambda(I \cap X_n) > 0$ for each nonempty open subinterval I of $\mathbf{R}$ and for every natural number n.

Infer from the latter fact that there exists a Lebesgue measurable function $g : \mathbf{R} \to \mathbf{R}$ which is nonintegrable on any nonempty open subinterval of $\mathbf{R}$.

2. Denote by $\mathcal{E}$ the family of all Lebesgue measurable subsets of the unit segment $[0, 1]$. For any two sets $X \in \mathcal{E}$ and $Y \in \mathcal{E}$, put

$$d(X, Y) = \lambda(X \triangle Y).$$

Identifying all those X and Y for which $d(X, Y) = 0$, we come to the metric space $(\mathcal{E}, d)$.

Check that $(\mathcal{E}, d)$ is complete and separable, i.e., is a Polish space.

Further, let $\mathcal{E}'$ be a subspace of $\mathcal{E}$ consisting of all sets $X \in \mathcal{E}$ such that

$$\lambda(X \cap I) > 0, \quad \lambda(([0, 1] \setminus X) \cap I) > 0$$

for each nondegenerate subinterval I of $[0, 1]$.

Show that $\mathcal{E}'$ is the complement of a first category subset of $\mathcal{E}$. Hence, according to the Baire theorem, we have $\mathcal{E}' \neq \emptyset$.

Remark 2. It is useful to compare this fact with Exercise 1.

3. Verify that the Banach space (D, d) introduced in this chapter is nonseparable (moreover, its topological weight is equal to $\mathbf{c}$).

4. Give a direct proof (i.e., without the aid of Theorem 1) that the set D_0 contains nonzero elements.

5*. Let E be an arbitrary topological space. Recall that a family $\mathcal{N}$ of subsets of E is a net in E if every open subset of E can be represented as the union of some subfamily of $\mathcal{N}$ (recall also that the concept of a net, for topological spaces, was introduced by Archangelskii; obviously, it generalizes the concept of a base of a topological space). We denote by the symbol $\mathrm{nw}(E)$ the smallest cardinality of a net in E.

Let now $f : E \to \mathbf{R}$ be a function. We put

$\mathrm{lmaxv}(f)$ = the set of all those $t \in \mathbf{R}$ for which there exists a nonempty open subset U of E such that $t = \sup(f(U))$ and, in addition, there is a point $e \in U$ such that $f(e) = \sup(f(U))$;

$\mathrm{lminv}(f)$ = the set of all those $t \in \mathbf{R}$ for which there exists a nonempty open subset V of E such that $t = \inf(f(V))$ and, in addition, there is a point $e \in V$ such that $f(e) = \inf(f(V))$.

Check that

$$\mathrm{card}(\mathrm{lmaxv}(f)) \leq \mathrm{nw}(E) + \omega, \quad \mathrm{card}(\mathrm{lminv}(f)) \leq \mathrm{nw}(E) + \omega.$$

In particular, if E possesses a countable net, then the above-mentioned subsets of $\mathbf{R}$ are at most countable.

Further, let us introduce the notation

$\mathrm{slmax}(f)$ = the set of all points $e \in E$ having the property that there exists a neighborhood $U(e)$ such that $f(e) > f(x)$ for each $x \in U(e) \setminus \{e\}$;

$\mathrm{slmin}(f)$ = the set of all points $e \in E$ having the property that there exists a neighborhood $V(e)$ such that $f(e) < f(x)$ for each $x \in V(e) \setminus \{e\}$.

Verify that

$$\mathrm{card}(\mathrm{slmax}(f)) \leq \mathrm{nw}(E) + \omega, \quad \mathrm{card}(\mathrm{slmin}(f)) \leq \mathrm{nw}(E) + \omega.$$

In particular, if E possesses a countable net, then the sets $\mathrm{slmax}(f)$ and $\mathrm{slmin}(f)$ are at most countable.

Finally, for $E = \mathbf{R}$, give an example of a continuous function f for which the latter two sets are everywhere dense in E.

6*. Let X be a λ-measurable set in $[0,1]$ such that

$$\lambda(X \cap I) > 0, \qquad \lambda(([0,1] \setminus X) \cap I) > 0$$

for each nondegenerate subinterval of $[0,1]$. Denote by f_X the characteristic function of X and define

$$F(x) = \int_0^x f_X(t)dt \qquad (x \in [0,1]).$$

Show that the absolutely continuous function $F : [0,1] \to \mathbf{R}$ is not differentiable at all points of some co-meager subset of $[0,1]$.

Chapter 7
Continuous nowhere approximately differentiable functions

The first example of a nowhere approximately differentiable function belonging to the Banach space $C[0,1]$ is due to Jarník (see [95]). Moreover, he showed that such functions are typical, i.e., they constitute a set whose complement is of first category in $C[0,1]$.

In this chapter we present one precise construction of a continuous function acting from $\mathbf{R}$ into $\mathbf{R}$ which is nowhere approximately differentiable. Actually, this construction is due to Malý [174] (cf. also [49] and [50]). It is not difficult and, at the same time, is quite visual from the geometrical point of view.

We begin with some preliminary notions and facts.

Let λ denote the standard Lebesgue measure on $\mathbf{R}$ and let X be a λ-measurable subset of $\mathbf{R}$. We recall that a point $x \in \mathbf{R}$ is said to be a density point for (of) X if

$$\lim_{h \to 0,\ h>0} \lambda(X \cap [x-h, x+h])/2h = 1.$$

According to the classical Lebesgue theorem (see, e.g., Chapter 0), almost all points of X are its density points.

The notion of a density point turned out to be very deep and fruitful not only for real analysis but also for general topology, probability theory and some other domains of mathematics. For example, starting with this notion the important concept of the density topology on $\mathbf{R}$ was introduced and investigated by several authors (Pauc, Goffman, Waterman, Nishiura, Neugebauer, Tall, and others). This topology was studied, with its further generalizations, from different points of view (see, e.g., [80], [202], [212], and [266]). We shall deal with the density topology (and with some of its natural analogues) in our considerations below.

Now, let $f : \mathbf{R} \to \mathbf{R}$ be a function and let $x \in \mathbf{R}$.

We recall that f is said to be approximately continuous at x if there exists a λ-measurable set X such that

(1) x is a density point of X;

(2) the function $f|(X \cup \{x\})$ is continuous at x.

Exercises 3 and 4 of this chapter show that all Lebesgue measurable functions on $\mathbf{R}$ can be described in terms of approximate continuity.

Again, let $f : \mathbf{R} \to \mathbf{R}$ be a function and let $x \in \mathbf{R}$.

We say that f is approximately differentiable at x if there exist a λ-measurable set Y, for which x is a density point, and a limit

$$\lim_{y \to x,\ y \in Y,\ y \neq x} \frac{f(y) - f(x)}{y - x}.$$

This limit is denoted by $f'_{ap}(x)$ and is called the approximate derivative of f at the point x.

For our purposes below, we need two simple auxiliary propositions.

Lemma 1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function, let x be a point of $\mathbf{R}$, and suppose that f is approximately differentiable at x. Then, for any real number $M_1 > f'_{ap}(x)$, we have*

$$\lim_{h \to 0+} \lambda(\{y \in [x-h, x+h] \setminus \{x\} : (f(y) - f(x))/(y - x) \geq M_1\})/2h = 0.$$

Similarly, for any real number $M_2 < f'_{ap}(x)$, we have

$$\lim_{h \to 0+} \lambda(\{y \in [x-h, x+h] \setminus \{x\} : (f(y) - f(x))/(y - x) \leq M_2\})/2h = 0.$$

Proof. Because the argument in both cases is completely analogous, we shall consider only the case of M_1. There exists a λ-measurable set X such that x is a density point of X and

$$\lim_{y \to x, y \neq x, y \in X} (f(y) - f(x))/(y - x) = f'_{ap}(x).$$

Fix a real number $\varepsilon > 0$ for which $f'_{ap}(x) + \varepsilon < M_1$. Then there exists a real number $\delta > 0$ such that, for any strictly positive real $h < \delta$, we have

$$(\forall y \in X \cap [x-h, x+h] \setminus \{x\})((f(y) - f(x))/(y - x) \leq f'_{ap}(x) + \varepsilon).$$

But, if a real $\delta > 0$ is sufficiently small, then

$$\lambda(X \cap [x-h, x+h])/2h \geq 1 - \varepsilon$$

for all strictly positive reals $h < \delta$. So we obtain the relation

$$\lambda(\{y \in [x-h, x+h] \setminus \{x\} : (f(y) - f(x))/(y - x) \geq M_1\})/2h \leq \varepsilon,$$

and the lemma is proved.

Actually, in our further considerations we need only the following auxiliary assertion, which is an immediate consequence of Lemma 1.

Lemma 2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function, let x be a point of $\mathbf{R}$ and suppose that, for every strictly positive real number M, the relation*

$$\mathrm{liminf}_{h \to 0+} \frac{\lambda(\{y \in [x-h, x+h] \setminus \{x\} \ : \ |f(y) - f(x)|/|y-x| \geq M\})}{2h} > 0$$

holds true. Then f is not approximately differentiable at x.

In particular, suppose that two sequences

$$\{h_k \ : \ k \in \mathbf{N}\}, \quad \{M_k \ : \ k \in \mathbf{N}\}$$

of real numbers are given, satisfying the following conditions:

(1) $h_k > 0$ and $M_k > 0$ for all integers $k \geq 0$;

(2) $\lim_{k \to +\infty} h_k = 0$ and $\lim_{k \to +\infty} M_k = +\infty$;

(3) the lower limit

$$\mathrm{liminf}_{k \to +\infty} \lambda(\{y \in [x - h_k, x + h_k] \setminus \{x\} \ :$$
$$|f(y) - f(x)|/|y - x| \geq M_k\})/2h_k$$

is strictly positive.

Then we can assert (by virtue of Lemma 2) that our function f is not approximately differentiable at the point x.

After these simple preliminary remarks, we are able to begin the construction of a continuous nowhere approximately differentiable function.

First of all, let us put

$$f_1(0/9) = 0, \quad f_1(1/9) = 1/3, \quad f_1(2/9) = 0, \quad f_1(3/9) = 1/3,$$
$$f_1(4/9) = 2/3, \quad f_1(5/9) = 1/3, \quad f_1(6/9) = 2/3, \quad f_1(7/9) = 3/3,$$
$$f_1(8/9) = 2/3, \quad f_1(9/9) = 3/3$$

and extend (uniquely) this partial function to a continuous function

$$f_1 \ : \ [0,1] \to [0,1]$$

in such a way that f_1 becomes affine on each segment $[k/9, (k+1)/9]$ where $k = 0, 1, ..., 8$. We shall start with this function f_1. In our further construction, we also need an analogous function g acting from the segment $[0, 9]$ into the segment $[0, 3]$. Namely, we put

$$g(x) = 3f_1(x/9) \qquad (x \in [0, 9]).$$

Obviously, g is continuous and affine on each segment $[k, k+1]$, where $k = 0, 1, ..., 8$. Also, another function similar to g will be useful in the construction. Namely, we denote by g^* the function from $[0, 9]$ into $[0, 3]$, whose graph is symmetric with the graph of g with respect to the straight line

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} \ : \ y = 3/2\}.$$

In other words, we put

$$g^*(x) = 3 - g(x)$$

for all $x \in [0, 9]$. Suppose now that, for a natural number $n \geq 1$, the function

$$f_n : [0, 1] \to [0, 1]$$

has already been defined so that

(a) f_n is continuous;

(b) for each segment of the form $[k/9^n, (k+1)/9^n]$, where

$$k \in \{0, 1, ..., 9^n - 1\},$$

the function f_n is affine on it, and the image of this segment with respect to f_n is some segment of the form $[j/3^n, (j+1)/3^n]$, where

$$j \in \{0, 1, ..., 3^n - 1\}.$$

Let us construct a function

$$f_{n+1} \ : \ [0, 1] \to [0, 1].$$

Obviously, for this purpose it suffices to define f_{n+1} on any segment of the form $[k/9^n, (k+1)/9^n]$ where $k \in \{0, 1, ..., 9^n - 1\}$. Here only two cases are possible.

1. f_n is increasing on $[k/9^n, (k+1)/9^n]$. In this case, let us consider the following two sets of points of the plane:

$$\{(0, 0), \ (0, 3), \ (9, 3), \ (9, 0)\},$$

$$\{(k/9^n, f_n(k/9^n)), \quad (k/9^n, f_n((k+1)/9^n)),$$

$$((k+1)/9^n, f_n((k+1)/9^n)), \quad ((k+1)/9^n, f_n(k/9^n))\}.$$

Because we have here the vertices of two rectangles, there exists a unique affine transformation $h : \mathbf{R}^2 \to \mathbf{R}^2$ satisfying the conditions

$$h(0, 0) = (k/9^n, f_n(k/9^n)), \quad h(0, 3) = (k/9^n, f_n((k+1)/9^n)),$$

$$h(9,3) = ((k+1)/9^n, f_n((k+1)/9^n)), \quad h(9,0) = ((k+1)/9^n, f_n(k/9^n)).$$

Let the graph of the restriction of f_{n+1} to the segment $[k/9^n, (k+1)/9^n]$ coincide with the image of the graph of g with respect to h.

2. f_n is decreasing on $[k/9^n, (k+1)/9^n]$. In this case, let us consider the following two sets of points of the plane:

$$\{(0,0),\ (0,3),\ (9,3),\ (9,0)\},$$

$$\{(k/9^n, f_n((k+1)/9^n)),\quad (k/9^n, f_n(k/9^n)),$$
$$((k+1)/9^n, f_n(k/9^n)),\quad ((k+1)/9^n, f_n((k+1)/9^n))\}.$$

Here we also have the vertices of two rectangles, so there exists a unique affine transformation $h^* : \mathbf{R}^2 \to \mathbf{R}^2$ satisfying the relations

$$h^*(0,0) = (k/9^n, f_n((k+1)/9^n)), \quad h^*(0,3) = (k/9^n, f_n(k/9^n)),$$

$$h^*(9,3) = ((k+1)/9^n, f_n(k/9^n)), \quad h^*(9,0) = ((k+1)/9^n, f_n((k+1)/9^n)).$$

Let the graph of the restriction of f_{n+1} to the segment $[k/9^n, (k+1)/9^n]$ coincide with the image of the graph of g^* with respect to h^*.

The function f_{n+1} has thus been determined.

From the above construction immediately follows that the corresponding analogues of the conditions (a) and (b) hold true for the function f_{n+1}, too. In other words, f_{n+1} is continuous and, for each segment of the form $[k/9^{n+1}, (k+1)/9^{n+1}]$, where

$$k \in \{0, 1, ..., 9^{n+1} - 1\},$$

the function f_{n+1} is affine on it, and the image of this segment with respect to f_{n+1} is some segment of the form $[j/3^{n+1}, (j+1)/3^{n+1}]$, where

$$j \in \{0, 1, ..., 3^{n+1} - 1\}.$$

Moreover, the described construction shows that

$$(\forall x \in [0,1])(|f_{n+1}(x) - f_n(x)| \leq 1/3^n).$$

In addition, let

$$[u, v] = [k/9^n, (k+1)/9^n]$$

be an arbitrary segment on which f_n is affine. Then it is not hard to check that

$$f_{n+1}([u, (2u+v)/3]) = f_n([u, (2u+v)/3]),$$

$$f_{n+1}([(2u+v)/3,(2v+u)/3]) = f_n([(2u+v)/3,(2v+u)/3]),$$
$$f_{n+1}([(u+2v)/3,v]) = f_n([(u+2v)/3,v]).$$

Proceeding in this way, we come to the sequence of functions

$$\{f_1,\ f_2,\ \ldots,\ f_n,\ \ldots\}$$

uniformly convergent to some continuous function f also acting from $[0,1]$ into $[0,1]$. We assert that f is nowhere approximately differentiable on the segment $[0,1]$. In order to demonstrate this fact, let us take an arbitrary point $x \in [0,1]$ and fix a natural number $n \geq 1$.

Clearly, there exists a number $k \in \{0,1,...,9^n-1\}$ such that

$$x \in [k/9^n,(k+1)/9^n].$$

Therefore, we have

$$f_n(x) \in [j/3^n,(j+1)/3^n]$$

for some number $j \in \{0,1,...,3^n-1\}$. For the sake of simplicity, denote

$$[u,v] = [k/9^n,(k+1)/9^n], \quad [p,q] = [j/3^n,(j+1)/3^n].$$

From the remarks made above it immediately follows that, for all natural numbers $m > n$, we have

$$f_m(x) \in [p,q]$$

and, consequently, $f(x) \in [p,q]$, too. Further, we may assume without loss of generality that f_n is increasing on $[u,v]$ (the case when f_n is decreasing on $[u,v]$ can be considered completely analogously).

Suppose first that $f(x) \leq (p+q)/2$ and put $D_1 = [(2v+u)/3,v]$. Then, for each point $y \in D_1$, we may write

$$f(y) \in [(2q+p)/3,q].$$

Hence, we get

$$(f(y)-f(x))/(y-x) \geq ((2q+p)/3-(p+q)/2)/(v-u) = (1/6)(3^n).$$

Suppose now that $f(x) \geq (p+q)/2$ and denote $D_2 = [u,(2u+v)/3]$. In this case, for any point $y \in D_2$, we may write

$$f(y) \in [p,(2p+q)/3].$$

Hence, we get

$$(f(x)-f(y))/(x-y) \geq ((p+q)/2-(2p+q)/3)/(v-u) = (1/6)(3^n).$$

Thus, in both of these cases, we have

$$\lambda(\{y \in [x - 1/9^n, x + 1/9^n] \setminus \{x\} \ : \\ |f(y) - f(x)|/|y - x| \geq (1/6)(3^n)\}) \geq (1/3)(1/9^n)$$

or, equivalently,

$$\lambda(\{y \in [x - 1/9^n, x + 1/9^n] \setminus \{x\} \ : \\ |f(y) - f(x)|/|y - x| \geq (1/6)(3^n)\}) \geq (1/6)\lambda([x - 1/9^n, x + 1/9^n]).$$

The latter relation immediately yields that the function f is not approximately differentiable at x (see Lemma 2 and the comments after it). Thus, we get the following statement.

Theorem 1. *There exists a continuous nowhere approximately differentiable function acting from* $[0, 1]$ *into* $[0, 1]$.

Remark 1. The function f constructed above has a number of other interesting properties (for more information concerning f, see [174] and [50]).

Now, starting with an arbitrary continuous nowhere approximately differentiable function acting from $[0, 1]$ into $[0, 1]$, one can easily obtain an analogous function for $\mathbf{R}$. We thus come to the following classical result (first obtained by Jarník in 1934).

Theorem 2. *There exist continuous bounded functions acting from* $\mathbf{R}$ *into* $\mathbf{R}$ *which are nowhere approximately differentiable.*

Remark 2. Actually, Jarník proved that almost all (in the sense of the Baire category) functions from the Banach space $C[0, 1]$ are nowhere approximately differentiable. Clearly, this result generalizes the corresponding result of Banach and Mazurkiewicz for the usual differentiability of real-valued continuous functions on $[0, 1]$ (see Theorem 2 from Chapter 0). Further investigations showed that analogous statements hold true for many kinds of generalized derivatives. The main tool for obtaining such statements is the notion of porosity of a subset X of $\mathbf{R}$ at a given point $x \in \mathbf{R}$. However, this interesting topic is out of the scope of the present book. So we only refer the reader to the fundamental paper [34] where several category analogues of Theorem 1 for generalized derivatives are discussed from this position.

In Chapter 19 we will give an application of a continuous nowhere approximately differentiable function to the question concerning some relationships between the sup-measurability and weak sup-measurability of functions acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$.

Because the concept of an approximate derivative relies essentially on the notion of a density point, it is reasonable to mention here once more the concept of the density topology on $\mathbf{R}$. For a precise definition and some elementary properties of this topology, see Exercise 9 of the present chapter.

EXERCISES

1. Let $(t_n)_{n\in\mathbf{N}}$ be a sequence of strictly positive real numbers, such that

$$\lim_{n\to+\infty} t_n = 0, \qquad \lim_{n\to+\infty} t_n/t_{n+1} = 1.$$

Let X be a Lebesgue measurable subset of $\mathbf{R}$ and let $x \in \mathbf{R}$.

Prove that the following two assertions are equivalent:

(1) x is a density point of X;

(2) $\lim_{n\to+\infty} \lambda(X \cap [x - t_n, x + t_n])/2t_n = 1$.

2. Let X be a Lebesgue measurable subset of $\mathbf{R}$ and let $x \in \mathbf{R}$.

Show that the following two assertions are equivalent:

(1) x is a density point of X;

(2) $\lim_{h\to0+,k\to0+} \lambda(X \cap [x - h, x + k])/(h + k) = 1$.

3. Let $g : \mathbf{R} \to \mathbf{R}$ be a function, let ε be a strictly positive real number, and suppose that, for any λ-measurable set X with $\lambda(X) > 0$, there exists a λ-measurable set $Y \subset X$ with $\lambda(Y) > 0$ such that

$$(\forall x \in Y)(\forall y \in Y)(|g(x) - g(y)| < \varepsilon).$$

Demonstrate that there exists a λ-measurable function $h : \mathbf{R} \to \mathbf{R}$ having the property

$$(\forall x \in \mathbf{R})(|g(x) - h(x)| < \varepsilon).$$

Infer from this fact that if the given function g satisfies the above condition for any $\varepsilon > 0$, then g is measurable in the Lebesgue sense.

4*. Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary function.

By applying the result of Exercise 3 and utilizing Luzin's classical theorem on the structure of Lebesgue measurable functions (see, e.g., [168], [194], [202] or Theorem 1 from Chapter 3 of this book), show that the following two assertions are equivalent:

(a) the function f is measurable in the Lebesgue sense;

(b) for almost all (with respect to λ) points $x \in \mathbf{R}$, the function f is approximately continuous at x.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be a locally bounded Lebesgue measurable function and let

$$F(x) = \int_0^x f(t)dt \qquad (x \in \mathbf{R}).$$

Prove that, for any point $x \in \mathbf{R}$ at which the function f is approximately continuous, $F'(x) = f(x)$.

Also, check that the local boundedness of f is essential for the validity of this result.

6. Demonstrate that if a function $f : \mathbf{R} \to \mathbf{R}$ is approximately differentiable at $x \in \mathbf{R}$, then f is also approximately continuous at x.

7. Check that the approximate derivative of a function $f : \mathbf{R} \to \mathbf{R}$ at a point $x \in \mathbf{R}$ is uniquely determined, i.e., it does not depend on the choice of a Lebesgue measurable set Y for which x is a density point.

For this purpose, apply Exercise 29 from Chapter 0.

Check also that the family of all functions acting from $\mathbf{R}$ into $\mathbf{R}$ and approximately differentiable at x forms a vector space over $\mathbf{R}$.

8. Verify that if a function $f : \mathbf{R} \to \mathbf{R}$ is differentiable (in the usual sense) at a point $x \in \mathbf{R}$, then f is approximately differentiable at x and $f'_{ap}(x) = f'(x)$.

Give an example showing that the converse assertion is not true, in general.

9. For any Lebesgue measurable subset X of $\mathbf{R}$, denote

$$d(X) = \{x \in \mathbf{R} \ : \ x \text{ is a density point for } X\}.$$

Further, denote by $\mathcal{T}_d$ the family of all those Lebesgue measurable sets $Y \subset \mathbf{R}$, for which $Y \subset d(Y)$.

Demonstrate that

(a) $\mathcal{T}_d$ is a topology on $\mathbf{R}$ strictly extending the standard Euclidean topology of $\mathbf{R}$;

(b) the topological space $(\mathbf{R}, \mathcal{T}_d)$ is a Baire space and satisfies the Suslin condition (i.e., no nonempty open set in $(\mathbf{R}, \mathcal{T}_d)$ is of first category and each disjoint family of nonempty open sets in $(\mathbf{R}, \mathcal{T}_d)$ is at most countable);

(c) every first category set in $(\mathbf{R}, \mathcal{T}_d)$ is nowhere dense and closed (in particular, the family of all subsets of $(\mathbf{R}, \mathcal{T}_d)$ having the Baire property coincides with the Borel σ-algebra of $(\mathbf{R}, \mathcal{T}_d)$);

(d) a set $X \subset \mathbf{R}$ is Lebesgue measurable if and only if X has the Baire property in $(\mathbf{R}, \mathcal{T}_d)$;

(e) a set $X \subset \mathbf{R}$ is of Lebesgue measure zero if and only if X is a first category subset of $(\mathbf{R}, \mathcal{T}_d)$;

(f) the space $(\mathbf{R}, \mathcal{T}_d)$ is not separable.

Remark 3. The above-mentioned topology $\mathcal{T}_d$ is usually called the density topology on $\mathbf{R}$. In a similar way, the density topology can be introduced for the Euclidean space $\mathbf{R}^n$ $(n \geq 2)$ equipped with the n-dimensional Lebesgue measure λ_n.

10. Let $f : \mathbf{R} \to \mathbf{R}$ be a function and let $x \in \mathbf{R}$.

Prove that the following two assertions are equivalent:

(a) f is approximately continuous at x;

(b) f regarded as a mapping from $(\mathbf{R}, \mathcal{T}_d)$ into $\mathbf{R}$ is continuous at x.

11*. By starting with the result of the previous exercise, show that the topological space $(\mathbf{R}, \mathcal{T}_d)$ is connected.

For this purpose, suppose to the contrary that there exists a partition $\{A, B\}$ of $\mathbf{R}$ into two nonempty sets $A \in \mathcal{T}_d$ and $B \in \mathcal{T}_d$. Then define a function

$$f \ : \ \mathbf{R} \to \mathbf{R}$$

by putting $f(x) = 1$ for all $x \in A$, and $f(x) = -1$ for all $x \in B$. Obviously, f is a bounded continuous mapping acting from $(\mathbf{R}, \mathcal{T}_d)$ into $\mathbf{R}$ and hence, according to Exercise 10, f is approximately continuous at each point of $\mathbf{R}$. Further, define

$$F(x) = \int_0^x f(t)dt \qquad (x \in \mathbf{R}).$$

By applying Exercise 5 of this chapter, demonstrate that the function F is differentiable everywhere on $\mathbf{R}$ and

$$F'(x) = 1 \quad \vee \quad F'(x) = -1$$

for each $x \in \mathbf{R}$. This yields a contradiction with the Darboux property of any derivative.

Remark 4. One of the most interesting results concerning the density topology states that $(\mathbf{R}, \mathcal{T}_d)$ is a completely regular topological space (see, for instance, [202] and [266]). This property of $\mathcal{T}_d$ implies some nontrivial consequences in real analysis. The next exercise serves as an illustration and presents a proof of the existence of everywhere differentiable nowhere monotone functions by applying the complete regularity of $\mathcal{T}_d$ (notice that this approach is due to Goffman [79]).

12*. Consider any two disjoint countable sets

$$A = \{a_n \ : \ n \in \mathbf{N}\} \subset \mathbf{R}, \quad B = \{b_n \ : \ n \in \mathbf{N}\} \subset \mathbf{R},$$

each of which is everywhere dense in $\mathbf{R}$. Taking into account Exercise 10 and the fact that $(\mathbf{R}, \mathcal{T}_d)$ is completely regular, one can find, for any $n \in \mathbf{N}$, an approximately continuous function $f_n : \mathbf{R} \to [0, 1]$ satisfying the relations

$$0 \le f_n(x) \le 1 \qquad (x \in \mathbf{R}),$$

$$f_n(a_n) = 1, \quad (\forall x \in B)(f_n(x) = 0).$$

Analogously, for any $n \in \mathbf{N}$, there exists an approximately continuous function $g_n : \mathbf{R} \to [0, 1]$ such that

$$0 \le g_n(x) \le 1 \qquad (x \in \mathbf{R}),$$

$$g_n(b_n) = 1, \quad (\forall x \in A)(g_n(x) = 0).$$

Now, define a function $h : \mathbf{R} \to \mathbf{R}$ by the formula

$$h = \sum_{n \in \mathbf{N}} (1/2^n)(f_n - g_n)$$

and verify that

(a) h is bounded and approximately continuous;

(b) $h(a) > 0$ for all $a \in A$ and $h(b) < 0$ for all $b \in B$.

Also, denoting by H an indefinite integral of h, show that

(i) H is everywhere differentiable on $\mathbf{R}$ and $H'(x) = h(x)$ for each $x \in \mathbf{R}$;

(ii) H is nowhere monotone.

Conclude that, with the aid of the density topology on $\mathbf{R}$, it is possible to give another proof of the existence of everywhere differentiable nowhere monotone functions acting from $\mathbf{R}$ into $\mathbf{R}$ (cf. the proof presented in Chapter 6).

Remark 5. The density topology on $\mathbf{R}$ can be regarded as a very special case of the so-called von Neumann topology. Let $(E, \mathcal{S}, \mu)$ be a space with a complete probability measure (or, more generally, with a complete nonzero σ-finite measure). Then, in conformity with a deep theorem of von Neumann and Maharam (see, e.g., [172], [202], [212], [269]), there exists a topology $\mathcal{T} = \mathcal{T}(\mu)$ on E such that

(1) $(E, \mathcal{T})$ is a Baire space satisfying the Suslin condition;

(2) the family of all subsets of $(E, \mathcal{T})$ having the Baire property coincides with the σ-algebra $\mathcal{S}$;

(3) a set $X \subset E$ is of μ-measure zero if and only if X is of first category in $(E, \mathcal{T})$.

We say that $\mathcal{T} = \mathcal{T}(\mu)$ is a von Neumann topology associated with the measure space $(E, \mathcal{S}, \mu)$. Notice that $\mathcal{T}$, in general, is not unique. This fact is not so surprising, because the proof of the existence of $\mathcal{T}$ is essentially based on the Axiom of Choice. There are very nontrivial applications of a von Neumann topology in various branches of contemporary mathematics (for instance, some applications to the general theory of stochastic processes can be found in [212]).

Remark 6. For the real line $\mathbf{R}$, an interesting analogue of the density topology, formulated in terms of category and the Baire property, was introduced and considered by Wilczyński in [278]. Wilczyński's topology was then investigated by many authors. An extensive survey devoted to properties of this topology and to functions continuous with respect to it, is given in [50] (see also the list of references presented in the same work).

Remark 7. There are some invariant extensions of the Lebesgue measure λ for which an analogue of the classical Lebesgue theorem on density points does not hold. For example, there exist a measure μ on $\mathbf{R}$ and a μ-measurable set $X \subset \mathbf{R}$, such that

(1) μ is an extension of λ;

(2) μ is invariant under the group of all isometric transformations of $\mathbf{R}$;

(3) there is only one μ-density point for X, i.e., there exists a unique point $x \in \mathbf{R}$ for which

$$\lim_{h \to 0+} \frac{\mu(X \cap [x-h, x+h])}{2h} = 1.$$

A more detailed account of the measure μ and its other extraordinary properties can be found in [118].

Chapter 8
Blumberg's theorem and Sierpiński–Zygmund functions

In various questions of mathematical analysis, one often needs to consider some nontrivial restrictions of a given function (e.g., acting from $\mathbf{R}$ into $\mathbf{R}$), which have nice additional descriptive properties. In general, these properties do not hold for the original function but may be valid for its restrictions to certain non-small subsets of its domain.

In order to illustrate this circumstance, let us recall two widely known statements from the theory of real functions. The first of them is the classical theorem of Luzin concerning the structure of an arbitrary Lebesgue measurable function from $\mathbf{R}$ into $\mathbf{R}$. Undoubtedly, this theorem plays the most fundamental role in real analysis and topological measure theory.

Let λ be the Lebesgue measure on $\mathbf{R}$ and let $f : \mathbf{R} \to \mathbf{R}$ be any function measurable in the Lebesgue sense. Then, according to the Luzin theorem (see, e.g., [194], [202] or Theorem 2 from Chapter 3), there exists a sequence $\{D_n : n \in \mathbf{N}\}$ of closed subsets of $\mathbf{R}$, such that

$$\lambda(\mathbf{R} \setminus \cup\{D_n : n \in \mathbf{N}\}) = 0$$

and, for each $n \in \mathbf{N}$, the restricted function $f|D_n : D_n \to \mathbf{R}$ is continuous.

It immediately follows from this important statement that, for the same $f : \mathbf{R} \to \mathbf{R}$, there exists a continuous function $g : \mathbf{R} \to \mathbf{R}$ such that

$$\lambda(\{x \in \mathbf{R} : f(x) = g(x)\}) > 0.$$

Indeed, it suffices to take a set D_n with $\lambda(D_n) > 0$ and then to extend the function $f|D_n$ to a continuous function g acting from $\mathbf{R}$ into $\mathbf{R}$ (obviously, we are dealing here with a very special case of the classical Tietze–Urysohn theorem on the existence of a continuous extension of a continuous real-valued function defined on a closed subset of a normal topological space). In particular, we have the equality

$$\mathrm{card}(\{x \in \mathbf{R} : f(x) = g(x)\}) = \mathbf{c},$$

where $\mathbf{c}$ denotes, as usual, the cardinality of the continuum.

Also, we may formulate the corresponding analogue of Luzin's theorem for real-valued functions possessing the Baire property. This analogue is essentially due to Baire (see, for instance, [149], [202] or Exercise 13 from Chapter 0).

Let $f : \mathbf{R} \to \mathbf{R}$ be a function having the Baire property. Then there exists a subset D of $\mathbf{R}$ such that

(1) the set $\mathbf{R} \setminus D$ is of first category;

(2) the function $f|D$ is continuous.

In particular, because $\mathrm{card}(D) = \mathbf{c}$ and $\mathrm{cl}(D) = \mathbf{R}$, we conclude that the restriction of f to some everywhere dense subset of $\mathbf{R}$ having cardinality $\mathbf{c}$ turns out to be continuous.

It can easily be observed that Luzin's theorem and its analogue for the Baire property hold true in much more general situations (see Exercises 1 and 2 of this chapter).

The preceding results show the nice behavior of Lebesgue measurable functions from $\mathbf{R}$ into $\mathbf{R}$ (respectively, of functions from $\mathbf{R}$ into $\mathbf{R}$ with the Baire property) on some subsets of $\mathbf{R}$ that are not small (in a certain sense). Namely, as mentioned above, an immediate consequence of Luzin's theorem is that, for any Lebesgue measurable function f acting from $\mathbf{R}$ into $\mathbf{R}$, there exists a closed subset A of $\mathbf{R}$ with strictly positive measure, such that the restriction $f|A$ is continuous. The corresponding analogue for the Baire property states even more: for any function g acting from $\mathbf{R}$ into $\mathbf{R}$ and possessing the Baire property, there exists a co-meager subset B of $\mathbf{R}$ such that the restriction $g|B$ is continuous, too.

In this connection, the following natural question arises: what can be said about an arbitrary function acting from $\mathbf{R}$ into $\mathbf{R}$?

In other words, if an arbitrary function $f : \mathbf{R} \to \mathbf{R}$ is given, is it true that $f|D$ is continuous on some set $D \subset \mathbf{R}$, which is not small (in a certain sense)?

The first topological property of D that may be considered in this respect is the everywhere density of D in $\mathbf{R}$. It turns out that this property is completely sufficient for solving the above question in a positive way. Namely, Blumberg established in [22] that there always exists an everywhere dense subset D of $\mathbf{R}$ for which the restriction $f|D$ is continuous.

This chapter of the book is devoted to Blumberg's theorem and to some strange functions that naturally appear when one tries to generalize his theorem in various directions. Such strange functions were first constructed by Sierpiński and Zygmund (see [249]). They are extremely discontinuous. More precisely, the restrictions of such functions to all subsets of $\mathbf{R}$ having cardinality **c** are always discontinuous.

It should be noticed that Blumberg's theorem is a result of **ZF** & **DC** theory (cf. the proof of this theorem given below).

At the same time, the construction of a Sierpiński–Zygmund function cannot be carried out within **ZF** & **DC** and essentially relies on an appropriate uncountable form of the Axiom of Choice.

We begin with the following auxiliary notion.

Let f be a function acting from $\mathbf{R}$ into $\mathbf{R}$ and let x be a point of $\mathbf{R}$.

We shall say that x is a pleasant point with respect to f (or, briefly, f-pleasant point) if, for each real $\varepsilon > 0$, there exists a neighborhood $V(x, \varepsilon)$ of x such that the set

$$Y(f, x, \varepsilon) = \{y \in \mathbf{R} : |f(y) - f(x)| < \varepsilon\}$$

is categorically dense in $V(x, \varepsilon)$ (i.e., the intersection of this set with any nonempty open interval contained in $V(x, \varepsilon)$ is of second category in that interval).

In accordance with the definition above, we shall say that a point $x \in \mathbf{R}$ is unpleasant with respect to f (or, briefly, f-unpleasant point) if x is not f-pleasant.

The following key lemma shows that, for any function $f : \mathbf{R} \to \mathbf{R}$, the set of all f-unpleasant points is small in the sense of the Baire category.

Lemma 1. *Let f be a function acting from $\mathbf{R}$ into $\mathbf{R}$. Then the set of all f-unpleasant points is of first category in $\mathbf{R}$. Consequently, the set of all f-pleasant points is co-meager on every nonempty open subinterval of $\mathbf{R}$.*

Proof. Suppose to the contrary that the set

$$A = \{x \in \mathbf{R} \ : \ x \text{ is unpleasant with respect to } f\}$$

is not of first category. For each point $x \in A$, there exists a strictly positive real number $\varepsilon(x)$ such that, for any neighborhood V of x, the set

$$Y(f, x, \varepsilon) = \{y \in \mathbf{R} \ : \ |f(y) - f(x)| < \varepsilon(x)\}$$

is not categorically dense in V. Let us pick two rational numbers $r(x)$ and $s(x)$ satisfying the inequalities

$$f(x) - \varepsilon(x)/2 < r(x) < f(x) < s(x) < f(x) + \varepsilon(x)/2.$$

Further, for any pair (r, s) of rational numbers, let us put

$$A_{r,s} = \{x \in A \ : \ r(x) = r, \ s(x) = s\}.$$

Evidently, we have the equality $A = \cup_{(r,s)\in \mathbf{Q}\times\mathbf{Q}} A_{r,s}$. By our assumption A is not of first category, so there exists a pair $(r_0, s_0) \in \mathbf{Q} \times \mathbf{Q}$ such that the set A_{r_0,s_0} is not of first category, either. Consequently, there exists a nonempty open subinterval $]a, b[$ of $\mathbf{R}$ such that the set A_{r_0,s_0} is categorically dense in $]a, b[$. Choose any point $x_0 \in\ ]a, b[\ \cap\ A_{r_0,s_0}$. For this point, we may write

$$f(x_0) - \varepsilon(x_0)/2 < r_0 < f(x_0) < s_0 < f(x_0) + \varepsilon(x_0)/2.$$

Analogously, for each point $y \in A_{r_0,s_0}$, we have

$$f(y) - \varepsilon(y)/2 < r_0 < f(y) < s_0 < f(y) + \varepsilon(y)/2.$$

Therefore, the inequalities

$$|f(y) - f(x_0)| < s_0 - r_0 < \varepsilon(x_0)$$

hold true. In other words, we get the inclusion

$$A_{r_0,s_0} \subset \{y \in \mathbf{R}\ :\ |f(y) - f(x_0)| < \varepsilon(x_0)\},$$

which immediately implies that the set

$$Y(f, x_0, \varepsilon) = \{y \in \mathbf{R}\ :\ |f(y) - f(x_0)| < \varepsilon(x_0)\}$$

is categorically dense in $]a, b[$, contradicting the definition of $\varepsilon(x_0)$. The contradiction obtained finishes the proof of Lemma 1.

We are now ready to prove the classical theorem of Blumberg.

Theorem 1. *Let f be an arbitrary function acting from $\mathbf{R}$ into $\mathbf{R}$. Then there exists an everywhere dense subset X of $\mathbf{R}$ such that the function $f|X$ is continuous.*

Proof. Since $\mathbf{R}$ is homeomorphic to the unit interval $]0, 1[$, it suffices to establish Theorem 1 for any function $f\ :\ ]0, 1[\ \to\ ]0, 1[$. Let $\Gamma(f)$ denote the graph of f. Starting with Lemma 1, we can recursively construct two sequences of sets $\{Z_n\ :\ n \in \mathbf{N}\}$ and $\{D_n\ :\ n \in \mathbf{N}\}$ satisfying the following five conditions:

(1) for each natural number n, the set Z_n can be represented in the form

$$Z_n = \cup\{]a_i, b_i[\times]c_i, d_i[\ :\ i \in I(n)\}$$

where $I(n)$ is a countable set, the intervals of the family $\{]a_i, b_i[\ :\ i \in I(n)\}$ are contained in $]0, 1[$ and are pairwise disjoint, the set $\mathrm{pr}_1(Z_n \cap \Gamma(f))$ is categorically dense in $]0,1[$, and, for any $i \in I(n)$, the length of $]c_i, d_i[$ is strictly less than $1/(n+1)$;

(2) for each natural number n, the set D_n is a finite $(1/(n+1))$-net of $]0,1[$, i.e., for any point t from $]0,1[$, there exists a point $d \in D_n$ such that $|t-d| < 1/(n+1)$;

(3) the sequence of sets $\{Z_n \ : \ n \in \mathbf{N}\}$ is decreasing by inclusion;

(4) the sequence of sets $\{D_n \ : \ n \in \mathbf{N}\}$ is increasing by inclusion;

(5) for each natural number n, we have $D_n \subset \mathrm{pr}_1(Z_n \cap \Gamma(f))$ and any point from D_n is f-pleasant.

We leave all details of this construction to the reader, because they are not difficult. As soon as the above-mentioned sequences are defined, we put

$$D = \cup\{D_n : n \in \mathbf{N}\}.$$

Then condition (2) implies that the set D is everywhere dense in $]0,1[$, and it can easily be verified, by using conditions (1), (3), (4), (5), that the restriction of f to D is continuous. Blumberg's theorem has thus been proved.

Some stronger versions of Blumberg's theorem may be obtained by using additional set-theoretical hypotheses. For instance, in [12] the situation is discussed when Martin's Axiom (or certain of its consequences) is assumed. Actually, Blumberg's theorem was analyzed and generalized in many directions. Moreover, the concept of a Blumberg topological space was introduced and investigated. Here we only formulate the corresponding definition.

Namely, we say that a topological space E is a Blumberg space if, for any function $g : E \to \mathbf{R}$, there exists an everywhere dense subset X of E such that the restriction $g|X$ is continuous.

The class of all Blumberg spaces turns out to be sufficiently wide and possesses a number of interesting properties. Let us point out that any Blumberg space must be a Baire space (the reader can easily verify this simple fact). A useful survey of results concerning Blumberg's theorem and its generalizations is presented in [29] (see also [30], [96], [230], [276]).

We thus see that any function (acting from $\mathbf{R}$ into $\mathbf{R}$) restricted to an appropriate countable everywhere dense subset of $\mathbf{R}$ becomes continuous. In this connection, the question arises whether that subset can be chosen to be uncountable. A partial negative answer to this question is given by the Sierpiński–Zygmund function constructed in [249], with the aid of an uncountable form of the Axiom of Choice. This function has the property that its restriction to each subset of $\mathbf{R}$ of cardinality continuum $\mathbf{c}$ is discontinuous. Consequently, if the Continuum Hypothesis (**CH**) holds, then the restriction of the same function to any uncountable subset of $\mathbf{R}$ is discontinuous, too.

We shall describe below the construction of Sierpiński–Zygmund type functions. Moreover, we shall give a slightly more general construction of a Sierpiński–Zygmund type function possessing some additional topological properties.

For this purpose, we need several auxiliary notions and statements.

Let E be a Hausdorff topological space. We recall that E is normal if, for any two disjoint closed sets $X \subset E$ and $Y \subset E$, there exist two open sets $U \subset E$ and $V \subset E$, such that

$$X \subset U, \quad Y \subset V, \quad U \cap V = \emptyset.$$

In other words, E is normal if and only if any two disjoint closed subsets of E can be separated by disjoint open subsets of E.

We also recall the well-known Tietze–Urysohn theorem already mentioned at the beginning of this chapter (see [64], [107], [149]). It states that if E is a normal space, X is a closed subset of E, and $f : X \to \mathbf{R}$ is a continuous function, then there exists a continuous function $f^* : E \to \mathbf{R}$ extending f. Furthermore, if, for the original function f, we have the relation $\mathrm{ran}(f) \subset [a, b] \subset \mathbf{R}$, then the extended function f^* may be chosen to satisfy the analogous relation $\mathrm{ran}(f^*) \subset [a, b]$. Actually, the property of the extendability of continuous real-valued functions defined on closed subsets of E is equivalent to the normality of E.

Recall that a normal topological space E is perfectly normal if each closed set in E is a G_δ-subset of E (or, equivalently, if each open set in E is an F_σ-subset of E).

Exercise 3 of this chapter yields another definition of perfectly normal spaces.

Exercise 4 indicates one important property of any perfectly normal space E, concerning the structure of its Borel σ-algebra $\mathcal{B}(E)$. In fact, the result presented in Exercise 4 allows us to define Borel subsets of a perfectly normal space E in the following manner. First, we put

$\mathcal{B}^*_0(E)$ = the class of all closed subsets of E.

Suppose now that, for a nonzero ordinal $\xi < \omega_1$, all the classes of sets $\mathcal{B}^*_\zeta(E)$ $(\zeta < \xi)$ have already been defined.

If ξ is an odd ordinal number, then we put

$\mathcal{B}^*_\xi(E)$ = the class of all those sets which can be represented in the form $\cup\{X_n \ : \ n \in \mathbf{N}\}$, where $\{X_n \ : \ n \in \mathbf{N}\}$ is some increasing (with respect to inclusion) sequence of sets belonging to $\cup\{\mathcal{B}^*_\zeta(E) \ : \ \zeta < \xi\}$.

If ξ is an even ordinal number, then we put

$\mathcal{B}^*_\xi(E)$ = the class of all those sets which can be represented in the form $\cap\{Y_n : n \in \mathbf{N}\}$, where $\{Y_n : n \in \mathbf{N}\}$ is some decreasing (with respect to inclusion) sequence of sets belonging to $\cup\{\mathcal{B}^*_\zeta(E) : \zeta < \xi\}$.

Finally, we define

$$\mathcal{B}^*(E) = \cup\{\mathcal{B}^*_\xi(E) : \xi < \omega_1\}.$$

Then, by virtue of Exercise 4, one may assert that

$$\mathcal{B}^*(E) = \mathcal{B}(E).$$

For any set $X \in \mathcal{B}(E)$, we say that X is of order $\xi < \omega_1$ if

$$X \in \mathcal{B}^*_\xi(E) \setminus \cup\{\mathcal{B}^*_\zeta(E) : \zeta < \xi\}.$$

Let now E be an arbitrary topological space.

We denote by the symbol $B(E, \mathbf{R})$ the family of all Borel mappings acting from E into $\mathbf{R}$. Furthermore, for any ordinal number $\xi < \omega_1$, we define by transfinite recursion the class $Ba_\xi(E, \mathbf{R})$ of functions also acting from E into $\mathbf{R}$.

First of all, we put $Ba_0(E, \mathbf{R}) = C(E, \mathbf{R})$, where $C(E, \mathbf{R})$ is the family of all continuous real-valued functions on E (see Chapter 2).

Suppose that, for a nonzero ordinal $\xi < \omega_1$, the classes of functions

$$Ba_\zeta(E, \mathbf{R}) \qquad (\zeta < \xi)$$

have already been defined. Let us denote by $Ba_\xi(E, \mathbf{R})$ the class of all those functions $f : E \to \mathbf{R}$ which satisfy the following condition: there exists a sequence

$$\{f_n : n \in \mathbf{N}\} \subset \cup\{Ba_\zeta(E, \mathbf{R}) : \zeta < \xi\}$$

(certainly, depending on f) for which the relation

$$f(x) = \lim_{n \to +\infty} f_n(x) \qquad (x \in E)$$

holds true. In other words, $Ba_\xi(E, \mathbf{R})$ consists of all pointwise limits of sequences of functions belonging to $\cup\{Ba_\zeta(E, \mathbf{R}) : \zeta < \xi\}$.

Continuing in this manner, we are able to define all the classes of functions $Ba_\xi(E, \mathbf{R})$ $(\xi < \omega_1)$. Finally, we put

$$Ba(E, \mathbf{R}) = \cup\{Ba_\xi(E, \mathbf{R}) : \xi < \omega_1\}.$$

For every ordinal $\xi < \omega_1$, the family of functions

$$Ba_\xi(E, \mathbf{R}) \setminus \cup\{Ba_\zeta(E, \mathbf{R}) : \zeta < \xi\}$$

is called the Baire class of those functions which are of order ξ.

A real-valued function given on E is called measurable in the Baire sense if it belongs to $Ba(E, \mathbf{R})$.

Exercises 5 and 6 of this chapter give some necessary information about Baire measurable functions.

As usual, the symbol $B(E, \mathbf{R})$ denotes the family of all Borel measurable (in short, Borel) functions acting from E into $\mathbf{R}$.

Lemma 2. *For any perfectly normal space E, the equality*

$$Ba(E, \mathbf{R}) = B(E, \mathbf{R})$$

is fulfilled. Consequently, this equality holds true for an arbitrary metric space E (in particular, for $E = \mathbf{R}$).

Proof. Obviously, it suffices to demonstrate that every bounded real-valued Borel function on E belongs to the class $Ba(E, \mathbf{R})$. Because any such function is uniformly approximable by linear combinations of characteristic functions of Borel subsets of E, it is enough to show that the characteristic function of each Borel subset of E belongs to $Ba(E, \mathbf{R})$.

Let X be an arbitrary closed subset of E. Taking account of the perfect normality of E and applying the Tietze–Urysohn extension theorem, we can easily verify that the characteristic function f_X is a pointwise limit of a sequence of continuous functions on E whose ranges are contained in $[0, 1]$ (cf. Exercise 3).

Let now ξ be an ordinal from the interval $]0, \omega_1[$ and suppose that our assertion is valid for the characteristic functions of all Borel subsets of E belonging to the Borel classes of order strictly less than ξ. Take any Borel set $X \subset E$ of order ξ. According to the result of Exercise 4, we may write $X = \cup\{X_n : n \in \mathbf{N}\}$ (or, respectively, $X = \cap\{X_n : n \in \mathbf{N}\}$) where $\{X_n : n \in \mathbf{N}\}$ is an increasing (respectively, a decreasing) sequence of Borel sets in E belonging to some Borel classes of strictly lower orders. But, in both of these cases, we have

$$f_X = \lim_{n \to +\infty} f_{X_n}.$$

In conformity with our inductive assumption, all characteristic functions f_{X_n} belong to $Ba(E, \mathbf{R})$. Hence the characteristic function f_X belongs to $Ba(E, \mathbf{R})$, too (cf. Exercise 5 of the present chapter). This completes the proof of Lemma 2.

Let X be a topological space, Y be a metric space, and let Z be a subset of X. Suppose that a function $f : Z \to Y$ is given, and let x be an arbitrary point from X. We denote

$$\Omega_f(x) = \inf_{U \in \mathcal{B}(x)} \mathrm{diam}(f(U)),$$

where $\mathcal{B}(x)$ is a local base of X at x (i.e., $\mathcal{B}(x)$ is a fundamental system of open neighborhoods of x). The value $\Omega_f(x) \geq 0$ is usually called the oscillation of a function f at a point x (cf. Exercise 3 from Chapter 2).

It can easily be observed that if x does not belong to the closure of Z, then $\Omega_f(x) = 0$.

Also, for a given function $g : X \to Y$, the following two assertions are equivalent:

(1) g is continuous on X;

(2) for each point $x \in X$, we have $\Omega_g(x) = 0$.

The next auxiliary statement is due to Lavrentieff (see [158]). It has many important applications in general topology, descriptive set theory and mathematical analysis (cf., for instance, [64], [105], [149]).

Lemma 3. *Let X be a metric space, Y be a complete metric space, and let Z be a subset of X. Suppose that a continuous function $f : Z \to Y$ is given. Then there exist a set $Z^* \subset X$ and a function $f^* : Z^* \to Y$, satisfying these three relations:*

(1) $Z \subset Z^*$;

(2) Z^* is a G_δ-subset of X;

(3) f^* is a continuous extension of f.

Proof. Let $\mathrm{cl}(Z)$ denote the closure of Z in X. We put

$$Z^* = \{z \in \mathrm{cl}(Z) \ : \ \Omega_f(z) = 0\}.$$

Since the original function f is continuous, we have $(\forall z \in Z)(\Omega_f(z) = 0)$. Consequently, the inclusion $Z \subset Z^*$ is valid. Now, let z be an arbitrary point of Z^*. Then there exists a sequence of points $\{z_n : n \in \mathbf{N}\} \subset Z$ such that $\lim_{n\to+\infty} z_n = z$. Taking account of the equality $\Omega_f(z) = 0$, we see that $\{f(z_n) : n \in \mathbf{N}\}$ is a Cauchy sequence in Y. But Y is complete, so the above-mentioned sequence converges to some point $y \in Y$. In addition, it can easily be shown that y does not depend on the choice of $\{z_n \ : \ n \in \mathbf{N}\}$. So we may put $f^*(z) = y$. In this way we get the mapping $f^* : Z^* \to Y$, which is continuous because, according to the definition of f^*, we have

$$(\forall z \in Z^*)(\Omega_{f^*}(z) = 0).$$

Thus, it remains to demonstrate that Z^* is a G_δ-subset of X. Obviously, the equality

$$Z^* = (\cap\{V_n \ : \ n \in \mathbf{N}\}) \cap \mathrm{cl}(Z)$$

holds, where, for each $n \in \mathbf{N}$, the set V_n is defined as follows:

$$V_n = \{x \in X \ : \ \Omega_f(x) < 1/(n+1)\}.$$

Since all sets V_n ($n \in \mathbf{N}$) are open in X (cf. Exercise 3 from Chapter 2) and the closed set $\mathrm{cl}(Z)$ is a G_δ-subset of X, we infer that Z^* is a G_δ-subset of X, too. This completes the proof of Lemma 3.

Lemma 4. *Let X be an arbitrary metric space, Z be a subset of X, and let $f : Z \to \mathbf{R}$ be a Borel mapping. Then there exist a set $Z^* \subset X$ and a mapping $f^* : Z^* \to \mathbf{R}$, satisfying the following four relations:*
(1) $Z \subset Z^*$;
(2) Z^* is a Borel subset of X;
(3) f^* is a Borel mapping;
(4) f^* is an extension of f.

Proof. Taking into account the equalities

$$B(Z, \mathbf{R}) = Ba(Z, \mathbf{R}) = \cup\{Ba_\xi(Z, \mathbf{R}) \ : \ \xi < \omega_1\},$$

it is natural to apply here the method of transfinite induction.

If our mapping f belongs to the class $Ba_0(Z, \mathbf{R})$, then we use the result formulated in Lemma 3.

Now, let ξ be a nonzero ordinal number strictly less than ω_1, and suppose that the assertion is true for all functions from $\cup\{Ba_\zeta(Z, \mathbf{R}) : \zeta < \xi\}$. Let f be an arbitrary function belonging to the class $Ba_\xi(Z, \mathbf{R})$. According to the definition of $Ba_\xi(Z, \mathbf{R})$, there exist two sequences

$$\{\xi_n \ : \ n \in \mathbf{N}\}, \qquad \{f_n \ : \ n \in \mathbf{N}\}$$

of ordinal numbers and functions, respectively, such that

(a) for each $n \in \mathbf{N}$, we have $\xi_n < \xi$ and $f_n \in Ba_{\xi_n}(Z, \mathbf{R})$;

(b) for any point $z \in Z$, we have $f(z) = \lim_{n \to +\infty} f_n(z)$.

By the inductive assumption, for every $n \in \mathbf{N}$, there exist a Borel set $Z_n^* \subset X$ and a Borel function $f_n^* : Z_n^* \to \mathbf{R}$ extending f_n (in particular, $Z \subset Z_n^*$). Let us denote

$$Z' = \cap\{Z_n^* : n \in \mathbf{N}\},$$

$$Z^* = \{z \in Z' : \text{there exists a } \lim_{n \to +\infty} f_n^*(z)\}.$$

Evidently, the set Z^* is Borel in X and $Z \subset Z^*$. Further, for any point $z \in Z^*$, we may put

$$f^*(z) = \lim_{n \to +\infty} f_n^*(z).$$

Then the mapping $f^* : Z^* \to \mathbf{R}$ defined in this manner is Borel, too, and extends the original mapping f. This finishes the proof of Lemma 4.

Remark 1. A slightly more precise formulation of Lemma 4 can be found, e.g., in monograph [149] (the proof remains almost the same). But, for our further purposes, Lemma 4 is completely enough.

Now, we are ready to establish the result of Sierpiński and Zygmund (in a more general form, not only for continuous restrictions of functions but also for Borel restrictions).

Theorem 2. *There exists a function $f : \mathbf{R} \to \mathbf{R}$ such that, for any set $Z \subset \mathbf{R}$ of cardinality continuum, the restriction of f to Z is not a Borel mapping (from Z into $\mathbf{R}$). In particular, this restriction is not continuous.*

Proof. Let Φ denote the family of all partial Borel mappings (from $\mathbf{R}$ into $\mathbf{R}$) defined on uncountable Borel subsets of $\mathbf{R}$. Clearly, we have the equality $\mathrm{card}(\Phi) = \mathbf{c}$. Consequently, we may enumerate this family in the form $\Phi = \{\phi_\beta \ : \ \beta < \alpha\}$, where α is the smallest ordinal number with $\mathrm{card}(\alpha) = \mathbf{c}$. Also, we can analogously enumerate the set of all points of $\mathbf{R}$, i.e., represent $\mathbf{R}$ in the form $\mathbf{R} = \{x_\beta \ : \ \beta < \alpha\}$. Now, for each $\beta < \alpha$, take the family $\{\Gamma(\phi_\xi) : \xi \leq \beta\}$ of graphs of functions from $\{\phi_\xi : \xi \leq \beta\}$ and consider the set

$$(\{x_\beta\} \times \mathbf{R}) \setminus \cup\{\Gamma(\phi_\xi) \ : \ \xi \leq \beta\}.$$

Clearly, the latter set is not empty (moreover, it is of cardinality $\mathbf{c}$). So we may pick a point (x_β, y_β) from this set. Let us put

$$f(x_\beta) = y_\beta \qquad (\beta < \alpha).$$

Evidently, we finally obtain some mapping $f : \mathbf{R} \to \mathbf{R}$. Let us demonstrate that f is the desired function.

Indeed, it immediately follows from the definition of f that, for any partial Borel function $\phi \ : \ \mathbf{R} \to \mathbf{R}$ defined on an uncountable Borel subset of $\mathbf{R}$, the inequality

$$\mathrm{card}(\{x \in \mathrm{dom}(\phi) \ : \ \phi(x) = f(x)\}) < \mathbf{c}$$

is fulfilled. Suppose for a while that there exists a set $Z \subset \mathbf{R}$ of cardinality $\mathbf{c}$, such that the restriction $f|Z$ is Borel. Then, in conformity with Lemma 4, there exists a partial Borel mapping $f^* : \mathbf{R} \to \mathbf{R}$ extending $f|Z$ and defined on some uncountable Borel subset of $\mathbf{R}$. Consequently, f^* belongs to the family Φ, and

$$Z \subset \{x \in \mathrm{dom}(f^*) \ : \ f^*(x) = f(x)\},$$

$$\mathrm{card}(\{x \in \mathrm{dom}(f^*) \ : \ f^*(x) = f(x)\}) = \mathbf{c},$$

which yields an obvious contradiction. Theorem 2 has thus been proved.

Actually, the proof of Theorem 2 presented above shows that there are many functions of Sierpiński–Zygmund type (for instance, the cardinality of the family of all such functions is equal to $2^{\mathbf{c}}$).

Some interesting extensions and generalizations of Theorem 2 are possible (see, e.g., Exercise 9 of the present chapter and Exercise 16 from Chapter 14).

Remark 2. In a certain sense, we may say that any Sierpiński–Zygmund function (defined on $\mathbf{R}$) is totally discontinuous with respect to the family of all subsets of $\mathbf{R}$ having the cardinality of the continuum. In other words, such a function is very bad from the point of view of continuity of its restrictions to large subsets of $\mathbf{R}$ (here "large" means that the cardinality of those subsets must be equal to $\mathbf{c}$). We shall see in Chapter 10 that a Sierpiński–Zygmund function is also bad from the points of view of the Lebesgue measurability and the Baire property, i.e., such a function is not measurable in the Lebesgue sense and does not possess the Baire property. Actually, this fact follows directly from Luzin's theorem on the structure of all Lebesgue measurable functions and from its corresponding analogue for all functions having the Baire property. Thus, we may conclude that the Sierpiński–Zygmund construction yields an example of a function which simultaneously is not measurable in the Lebesgue sense and does not possess the Baire property. Of course, there are many other constructions of such pathological functions (they are necessarily based on uncountable forms of the Axiom of Choice). The best known constructions are due to Vitali [272] and Bernstein [19]. We shall examine their constructions in our further considerations (see Chapter 10).

Remark 3. In connection with the Sierpiński–Zygmund result presented above, the following question arises naturally: does there exist a function $f : \mathbf{R} \to \mathbf{R}$ such that, for any uncountable subset Z of $\mathbf{R}$, the restriction $f|Z$ is not continuous? As mentioned earlier, under the Continuum Hypothesis (**CH**), any Sierpiński–Zygmund function yields a positive answer to this question. Nevertheless, the question cannot be resolved within **ZFC** set theory. Moreover, Shinoda demonstrated in [230] that if Martin's Axiom with the negation of the Continuum Hypothesis holds and g is an arbitrary function acting from $\mathbf{R}$ into $\mathbf{R}$, then, for each uncountable set $X \subset \mathbf{R}$, there always exists an uncountable set $Y \subset X$ such that the restriction $g|Y$ is continuous (for further details, see [230]).

Several related questions concerning the existence of a good restriction of a given function to some non-small subset of its domain are discussed in [47] where the corresponding references can also be found.

EXERCISES

1. Let E be a Hausdorff topological space, let μ be a finite Radon measure on E, and let μ' denote the usual completion of μ.

Prove that, for any μ'-measurable function $f : E \to \mathbf{R}$ and for every real number $\varepsilon > 0$, there exists a compact set $K \subset E$ for which these two conditions are fulfilled:

(a) $\mu(E \setminus K) < \varepsilon$;

(b) the restriction of f to K is continuous.

Deduce from this result that there exists an F_σ-subset Y of $\mathbf{R}$ such that $Y \subset f(E)$ and $\mu'(E \setminus f^{-1}(Y)) = 0$.

Remark 4. In other words, Exercise 1 states that every Radon measure is perfect (the notion of a perfect measure was introduced by Gnedenko and Kolmogorov). Let us underline that perfect probability measures play an essential role in the theory of random (stochastic) processes (cf. [212]).

2. Let E be a Baire topological space and let E' be a topological space with a countable base.

Demonstrate that, for any mapping $f : E \to E'$ having the Baire property, there exists a set $D \subset E$ satisfying the following relations:

(a) if U is an arbitrary nonempty open subset of E, then the set $D \cap U$ is co-meager in U;

(b) the function $f|D$ is continuous.

Infer from (a) and (b) that the restriction of f to some everywhere dense subset of E is continuous.

3. Let E be an arbitrary normal space.

Show that these two assertions are equivalent:

(a) E is perfectly normal;

(b) for any closed set $X \subset E$, there is a continuous function $f : E \to [0, 1]$ such that $f^{-1}(\{0\}) = X$.

4. Let E be a topological space such that every closed subset of E is a G_δ-set in E. Let $\mathcal{M}$ be some class of subsets of E, satisfying the following three conditions:

(a) the family of all closed subsets of E is contained in $\mathcal{M}$;

(b) if $\{X_n : n \in \mathbf{N}\}$ is an increasing (by inclusion) sequence of sets belonging to $\mathcal{M}$, then the set $\cup\{X_n : n \in \mathbf{N}\}$ belongs to $\mathcal{M}$, too;

(c) if $\{Y_n \ : \ n \in \mathbf{N}\}$ is a decreasing (by inclusion) sequence of sets belonging to $\mathcal{M}$, then the set $\cap\{Y_n \ : \ n \in \mathbf{N}\}$ belongs to $\mathcal{M}$, too.

Prove, by applying the method of transfinite induction, that $\mathcal{B}(E) \subset \mathcal{M}$, i.e., each Borel subset of E belongs to $\mathcal{M}$.

5. Let E be a topological space. Prove, by using the method of transfinite induction, that

(a) for any $\xi < \zeta < \omega_1$, the inclusion $Ba_\xi(E, \mathbf{R}) \subset Ba_\zeta(E, \mathbf{R})$ is fulfilled;

(b) $Ba(E, \mathbf{R})$ is a vector space over $\mathbf{R}$;

(c) if $f \in Ba(E, \mathbf{R})$ and $g \in Ba(E, \mathbf{R})$, then $f \cdot g \in Ba(E, \mathbf{R})$;

(d) if $f \in Ba(E, \mathbf{R})$, then $|f| \in Ba(E, \mathbf{R})$;

(e) if f and g belong to $Ba(E, \mathbf{R})$, then

$$\max(f, g) \in Ba(E, \mathbf{R}), \qquad \min(f, g) \in Ba(E, \mathbf{R});$$

(f) if f and g belong to $Ba(E, \mathbf{R})$ and $g(x) \neq 0$ for all $x \in E$, then f/g also belongs to $Ba(E, \mathbf{R})$;

(g) if a sequence $\{f_n : n \in \mathbf{N}\} \subset Ba(E, \mathbf{R})$ is given such that there exists a pointwise limit $f(x) = \lim_{n \to +\infty} f_n(x)$ for all $x \in E$, then f belongs to $Ba(E, \mathbf{R})$;

(h) if $g \in Ba(E, \mathbf{R})$ is such that $\mathrm{ran}(g) \subset \ ]a, b[$ for some open interval $]a, b[$, then for any continuous function $\phi \ : \ ]a, b[\ \to \mathbf{R}$, the function $\phi \circ g$ belongs to $Ba(E, \mathbf{R})$;

(i) every function belonging to $Ba(E, \mathbf{R})$ is a Borel mapping from E into $\mathbf{R}$; in other words, one has the inclusion $Ba(E, \mathbf{R}) \subset B(E, \mathbf{R})$.

In some cases, the above inclusion can be proper. To see this, equip the ordinal number ω_1 with its order topology and check that

$$Ba(\omega_1, \mathbf{R}) \neq B(\omega_1, \mathbf{R}).$$

6*. Let Φ be a class of functions acting from $[0, 1]$ into $\mathbf{R}$.

We say that a function $h : [0, 1]^2 \to \mathbf{R}$ is universal for Φ if, for any function $\phi \in \Phi$, there exists a point $y = y(\phi)$ of $[0, 1]$ such that

$$\phi(x) = h(x, y) \qquad (x \in [0, 1]).$$

By starting with the existence of continuous mappings of Peano type (see Chapter 1) and applying the method of transfinite induction, show that, for each ordinal $\xi < \omega_1$, there exists a function $h_\xi : [0, 1]^2 \to \mathbf{R}$ satisfying the following two conditions:

(a) h_ξ is a Borel mapping;

(b) h_ξ is universal for the class $Ba_\xi([0, 1], \mathbf{R})$.

Deduce from this fact that, for any ordinal $\xi < \omega_1$, the set

$$Ba_\xi([0,1],\mathbf{R}) \setminus \cup\{Ba_\zeta([0,1],\mathbf{R}) \ : \ \zeta < \xi\}$$

is not empty (i.e., there are Baire functions of order ξ) and conclude that the family of all Baire classes $\{Ba_\xi([0,1],\mathbf{R}) : \xi < \omega_1\}$ is strictly increasing by inclusion.

Remark 5. This remarkable result was first obtained by Lebesgue (see, e.g., [149], [160], [161], [194]).

7*. Let X and Y be two complete metric spaces, let A be a subset of X, and let B be a subset of Y. Suppose also that $f : A \to B$ is a homeomorphism between A and B.

By starting with the result of Lemma 3, show that there exist two sets $A^* \subset X$ and $B^* \subset Y$ and a mapping $f^* : A^* \to B^*$ satisfying these four relations:

(a) $A \subset A^*$ and $B \subset B^*$;
(b) A^* is a G_δ-subset of X and B^* is a G_δ-subset of Y;
(c) f^* is a homeomorphism between A^* and B^*;
(d) f^* is an extension of f.

Remark 6. This celebrated result was obtained by Lavrentieff [158] and is known as Lavrentieff's theorem on extensions of homeomorphisms. It found numerous applications in topology and, especially, in descriptive set theory (see, e.g., [149] where this theorem is applied in order to prove the topological invariance of Borel classes in complete metric spaces).

8. Let $[0,1]^\omega$ denote, as usual, the Hilbert cube (or the Tychonoff cube of weight ω).

By applying the result of Lemma 4, show that if X is a metric space, Z is a subset of X, and a mapping $g : Z \to [0,1]^\omega$ is Borel, then there exist a set $Z^* \subset X$ and a mapping $g^* \ : \ Z^* \to [0,1]^\omega$ satisfying the following four conditions:

(a) $Z \subset Z^*$;
(b) Z^* is a Borel subset of X;
(c) g^* is a Borel mapping;
(d) g^* is an extension of g.

9. Give a generalization of Theorem 2 for uncountable Polish spaces.

In other words, prove that if E is an arbitrary uncountable Polish topological space, then there exists a function $f : E \to \mathbf{R}$ such that, for every set $Z \subset E$ of cardinality $\mathbf{c}$, the restriction of f to Z is not a Borel mapping.

10. Demonstrate that any Sierpiński–Zygmund function $f : \mathbf{R} \to \mathbf{R}$ possesses the following property: for each set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction $f|X$ is not monotone on X (use the fact that if g is an arbitrary monotone partial function acting from $\mathbf{R}$ into $\mathbf{R}$, then the set of all discontinuity points of g is at most countable).

Deduce from this result that, under the Continuum Hypothesis, the restriction of f to each uncountable set $Y \subset \mathbf{R}$ is not monotone on Y (cf. Exercise 10 from Chapter 4).

Chapter 9
The cardinality of first Baire class

Let E be a nonempty Polish topological space. In Chapter 2 we were concerned with the class $Ba_1(E, \mathbf{R})$ of functions of first Baire class (in short, Baire one functions), which play a key role in various topics of mathematical analysis.

Recall, for example, that any real-valued function on E whose set of discontinuity points is at most countable belongs to this class (see Exercise 13 from Chapter 2).

In particular, if a function $g : \mathbf{R} \to \mathbf{R}$ is such that all discontinuity points of g are simple, then $g \in Ba_1(\mathbf{R}, \mathbf{R})$.

Indeed, this fact directly follows from the theorem stating that the set of all simple discontinuity points of any function $h : \mathbf{R} \to \mathbf{R}$ is at most countable (see Theorem 1 from Chapter 4).

According to the definition of the class $Ba_1(E, \mathbf{R})$, if a given function f belongs to $Ba_1(E, \mathbf{R})$, then there exists a sequence $\{f_n : n \in \mathbf{N}\}$ of continuous functions on E such that

$$f(x) = \lim_{n \to +\infty} f_n(x) \qquad (x \in E).$$

Since the class $Ba_0(E, \mathbf{R})$ of all real-valued continuous functions on E is of cardinality continuum $\mathbf{c}$ and since the equality $\mathbf{c}^\omega = \mathbf{c}$ holds true, one can easily infer from the definition of $Ba_1(E, \mathbf{R})$ that

$$\mathrm{card}(Ba_1(E, \mathbf{R})) = \mathbf{c}.$$

However, the above simple argument substantially relies on the Axiom of Choice (**AC**). Moreover, it is not difficult to check that this argument exploits an uncountable version of **AC**. So the natural question arises whether it is possible to establish the equality $\mathrm{card}(Ba_1(E, \mathbf{R})) = \mathbf{c}$ within **ZF** set theory.

The answer to the formulated question turns out to be positive, and here we intend to give a detailed proof of this quite profound result that goes back to Baire and Kuratowski (cf. [8], [148], [149]).

In order to carry out our plan, we need several auxiliary notions and facts.

In accordance with our purpose, we will work in **ZF** set theory. In other words, all objects, reasonings, and constructions below will be effective. By the way, we are going to show in this chapter that Alexandrov's celebrated theorem on the topological completeness of all G_δ-subsets of a complete metric space is a result of **ZF** theory.

Lemma 1. *Let $\{(E_n, \rho_n) : n \in \mathbf{N}\}$ be a sequence of complete metric spaces and let the product set*

$$E = \prod\{E_n : n \in \mathbf{N}\}$$

be equipped with the ordinary product topology. Then there exists an effectively defined metric ρ on E such that

(1) ρ is compatible with the topology of E, i.e., the topology on E produced by ρ coincides with the product topology of E;

(2) (E, ρ) is a complete metric space.

Proof. We may assume, without loss of generality, that all the given metrics ρ_n $(n \in \mathbf{N})$ are bounded by 1. Indeed, supposing otherwise, we can define the new equivalent metrics ρ'_n $(n \in \mathbf{N})$ by

$$\rho'_n(x, y) = \rho_n(x, y) \text{ if } \rho_n(x, y) < 1, \text{ and } \rho'_n(x, y) = 1 \text{ if } \rho_n(x, y) \geq 1,$$

where x and y are any two points from E_n.

Further, in the standard way we introduce the metric ρ on E by putting

$$\rho(x, y) = \sum\{\frac{1}{2^{n+1}}\rho_n(x_n, y_n) : n \in \mathbf{N}\} \qquad (x \in E, y \in E).$$

It is not hard to verify that ρ is bounded by 1, too, and satisfies conditions (1) and (2) of the lemma. The reader may carry out all the details of such a verification.

The following statement is a classical result of Alexandrov (see, e.g., [64], [107], [149], [202]).

Theorem 1. *Let (E, ρ) be a complete metric space and let X be a G_δ-subset of E. There exists a canonical metric ρ' on the topological subspace X of E which induces the topology of X and, in addition, (X, ρ') is a complete metric space.*

Proof. Let U be an arbitrary open set in E. In the product space $\mathbf{R} \times E$ consider the following set:

$$Z = \{(t, x) : t\rho(x, E \setminus U) = 1\}.$$

This Z is trivially closed in $\mathbf{R} \times E$, so may be regarded as a complete metric space (by virtue of Lemma 1). Further, the projection

$$\mathrm{pr}_2 : \mathbf{R} \times E \to E$$

restricted to Z turns out to be a bijection between Z and U. Since the same projection is a continuous and open mapping, we conclude that U is homeomorphic to Z. Consequently, U is topologically complete. Actually, we have a canonical complete metric ρ' on U produced by the metric on $\mathbf{R} \times E$.

Let now $X = \cap\{U_n : n \in \mathbf{N}\}$, where all U_n $(n \in \mathbf{N})$ are open subsets of E. We may equip every U_n with the canonical complete metric ρ'_n. So the product space

$$V = \prod\{U_n : n \in \mathbf{N}\} \subset E^{\mathbf{N}}$$

is also complete (see again Lemma 1). Let $\triangle(E^{\mathbf{N}})$ denote the diagonal in the product space $E^{\mathbf{N}}$. Clearly, this diagonal is a closed subset of $E^{\mathbf{N}}$. Now, the set

$$X' = V \cap \triangle(E^{\mathbf{N}})$$

is closed in V and is homeomorphic to X. This circumstance directly implies that X is metrizable by a canonical complete metric ρ'. The proof of Theorem 1 is finished.

Lemma 2. *Let (E, ρ) be a complete separable metric space, $\{G_n : n \in \mathbf{N}\}$ be a sequence of open subsets of E, and let*

$$X = \cap\{G_n : n \in \mathbf{N}\} \neq \emptyset.$$

Then some point $x \in X$ can be indicated effectively.

Proof. Fix a countable base $\mathcal{B}$ of the topology of E, consisting of open balls. We may assume, without loss of generality, that

$$\mathcal{B} = \{B_k : k \in \mathbf{N}\}.$$

Take the nonempty open set $U_0 = G_0$. Obviously, there exists a least natural number $k(0)$ such that

$$\mathrm{cl}(B_{k(0)}) \subset U_0, \quad \mathrm{rad}(B_{k(0)}) < 1, \quad B_{k(0)} \cap X \neq \emptyset,$$

where $\mathrm{cl}(\cdot)$ denotes the closure operation and the symbol $\mathrm{rad}(B_{k(0)})$ stands for the radius of $B_{k(0)}$.

Then take the nonempty open set $U_1 = G_1 \cap B_{k(0)}$. There exists a least natural number $k(1)$ for which the relations

$$\mathrm{cl}(B_{k(1)}) \subset U_1, \quad \mathrm{rad}(B_{k(1)}) < 1/2, \quad B_{k(1)} \cap X \neq \emptyset$$

hold true. We put $U_2 = G_2 \cap B_{k(1)}$.

Proceeding by recursion, we get two effective sequences $\{U_n : n \in \mathbf{N}\}$ and $\{B_{k(n)} : n \in \mathbf{N}\}$ of nonempty open sets and open balls, respectively, such that

$$U_{n+1} = G_{n+1} \cap B_{k(n)} \qquad (n \in \mathbf{N}),$$

$$B_{k(n)} \cap X \neq \emptyset, \quad \mathrm{cl}(B_{k(n)}) \subset U_n, \quad \mathrm{rad}(B_{k(n)}) < 1/2^n \qquad (n \in \mathbf{N}).$$

Now, there exists a unique point x in $\cap\{\mathrm{cl}(B_{k(n)}) : n \in \mathbf{N}\}$, and it is easy to see that $x \in X$. Lemma 2 has thus been proved.

Lemma 3. *Let (E, ρ) be a complete separable metric space and let $\mathcal{G}$ denote the family of all those nonempty subsets of E which effectively are of type G_δ. Then there exists an effective choice function $\Phi : \mathcal{G} \to E$, i.e., we have the relation $\Phi(X) \in X$ for any set $X \in \mathcal{G}$.*

In particular, the domain of Φ contains the family of all nonempty closed subsets of E.

Proof. Actually, this lemma follows from Lemma 2, keeping in mind the circumstance that every closed subset Y of E admits a canonical representation in the form

$$Y = \cap\{V_n(Y) : n \in \mathbf{N} \setminus \{0\}\},$$

where, as usual, $V_n(Y) = \{x \in E : \rho(x, Y) < 1/n\}$.

In the sequel we also need an effective version of the Tietze–Urysohn theorem (cf. [59]).

Lemma 4. *Let (E, ρ) be a metric space, F be a nonempty closed subset of E, and let $g : F \to \mathbf{R}$ be a continuous bounded mapping. Then there exists a continuous mapping $g^* : E \to \mathbf{R}$ which extends g and for which the relations*

$$\sup_{x \in E} g^*(x) = \sup_{x \in F} g(x), \qquad \inf_{x \in E} g^*(x) = \inf_{x \in F} g(x)$$

are satisfied.

Proof. If g is constant, then there is nothing to prove. So suppose that f is not a constant function. By using an appropriate affine transformation of $\mathbf{R}$ having the form

$$t \to at + b \qquad (t \in \mathbf{R}),$$

where $a > 0$, we may assume without loss of generality that

$$\sup_{x \in F} g(x) = 2, \quad \inf_{x \in F} g(x) = 1.$$

Now, we define $g^* : E \to \mathbf{R}$ by putting $g^*(x) = g(x)$ if $x \in F$, and

$$g^*(x) = \inf_{y \in F} \frac{g(y)\rho(x, y)}{\rho(x, F)}$$

if $x \in E \setminus F$. It can be verified (within **ZF** theory) that the function g^* is as required. The details of this checking are left to the reader.

The next auxiliary proposition may be treated as a certain separation principle for two disjoint G_δ-sets in a complete separable metric space.

Lemma 5. *Let E be a Polish topological space, X and Y be any two G_δ-subsets of E without common points. Then there exists an effective set $Z \subset E$ such that*

(1) Z is simultaneously of type F_σ and of type G_δ;

(2) $X \subset Z$ and $Y \cap Z = \emptyset$.

Proof. Fix a countable base $\{B_n : n \in \mathbf{N}\}$ of topology of E. We shall construct by transfinite recursion a certain decreasing (by inclusion) ω_1-sequence $\{F_\xi : \xi < \omega_1\}$ of closed subsets of E.

First of all, we put $F_0 = E$.

Suppose that, for an ordinal number $\xi < \omega_1$, the partial family $\{F_\zeta : \zeta < \xi\}$ has already been defined.

If ξ is a limit ordinal, then we simply put $F_\xi = \cap\{F_\zeta : \zeta < \xi\}$.

If $\xi = \eta + 1$, then either $F_\eta = \emptyset$ or $F_\eta \neq \emptyset$. In the first case, we put $F_\xi = \emptyset$. In the second case, we consider the following two sets:

$$X_\eta = X \cap F_\eta, \quad Y_\eta = Y \cap F_\eta.$$

Obviously, both X_η and Y_η are of type G_δ in the Polish space F_η, and

$$X_\eta \cap Y_\eta = \emptyset.$$

By virtue of the Baire theorem (see Exercise 2 from Chapter 0), one of these two sets is not everywhere dense in F_η. Therefore, there exists a least natural index n such that $B_n \cap F_\eta \neq \emptyset$ but

$$B_n \cap X_\eta = \emptyset \quad \vee \quad B_n \cap Y_\eta = \emptyset.$$

We then put $F_\xi = F_\eta \setminus B_n$.

Proceeding in this manner, we finally come to the ω_1-sequence $\{F_\xi : \xi < \omega_1\}$.

Now, according to the Cantor–Baire stationarity principle that is valid for E within **ZF** theory (see Exercise 10 of Chapter 2 and Exercise 2 of this chapter), there exists a least ordinal number $\alpha < \omega_1$ for which $F_\alpha = \emptyset$. Thus, we may write

$$E = \cup\{F_\xi \setminus F_{\xi+1} : \xi < \alpha\}.$$

Further, let us introduce the notation

$$\Xi = \{\xi < \alpha : (F_\xi \setminus F_{\xi+1}) \cap X \neq \emptyset\}, \qquad \Xi' = [0, \alpha[\ \setminus \Xi,$$

$$Z = \cup\{F_\xi \setminus F_{\xi+1} : \xi \in \Xi\}, \qquad Z' = \cup\{F_\xi \setminus F_{\xi+1} : \xi \in \Xi'\}.$$

Clearly, we have the relations

$$Z \cap Z' = \emptyset, \qquad Z \cup Z' = E.$$

Keeping in mind the method of constructing the sets F_ξ ($\xi < \omega_1$), we readily conclude that $X \subset Z$ and $Y \subset Z'$ (hence $Y \cap Z = \emptyset$). Exercise 4 for this chapter shows that both Z and Z' are effectively of type F_σ. Lemma 5 is thus proved.

Lemma 6. *Let E be a Polish space and let $f : E \to [0, 1]$ be a function of first Baire class. Then there exists an effective sequence $\{f_n : n \in \mathbf{N} \setminus \{0\}\}$ of functions on E such that*

(1) $\mathrm{ran}(f_n) \subset \{0, 1/n, 2/n, ..., (n-1)/n, 1\}$ for each natural number $n \geq 1$;

(2) all functions f_n ($n \geq 1$) are of first Baire class;

(3) the sequence $\{f_n : n \in \mathbf{N} \setminus \{0\}\}$ converges uniformly to f.

Proof. Fix a natural number $n \geq 1$ and, for an arbitrary natural index $k \in \{0, 1, 2, ..., n\}$, introduce the notation

$$X_k = \{x \in E : |f(x) - k/n| \leq 1/n\}, \quad Y_k = \{x \in E : |f(x) - k/n| \geq 2/n\}.$$

Observe that the sets X_k and Y_k are of type G_δ in E (see Lemma 2 from Chapter 2). Since $X_k \cap Y_k = \emptyset$, we may apply to these sets the previous lemma and effectively find a set $Z_k \subset E$ which is simultaneously of type F_σ and of type G_δ, and satisfies the relations

$$X_k \subset Z_k, \qquad Z_k \cap Y_k = \emptyset.$$

Moreover, the equality $E = \cup\{Z_k : k \in \{0, 1, 2, ..., n\}\}$ is easily verified.

We now define a function $f_n : E \to [0, 1]$ by putting

$$f_n(x) = k/n \qquad (x \in Z_k \setminus \cup\{Z_i : i < k\})$$

for $k = 0, 1, 2, ..., n$. Using Lemma 4, it is not difficult to check that f_n is of first Baire class. Finally, since

$$|f(x) - f_n(x)| < 2/n \qquad (x \in E),$$

the sequence of functions $\{f_n : n \in \mathbf{N} \setminus \{0\}\}$ uniformly converges to f, which finishes the proof of Lemma 6.

Lemma 7. *Let E be a Polish space and let $f : E \to [0, 1]$ be a function of first Baire class. If $\mathrm{ran}(f)$ is finite, then a sequence $\{\phi_n : n \in \mathbf{N}\}$ of continuous functions from E into $[0, 1]$ can be effectively indicated which converges pointwise to f.*

Proof. According to our assumption, $\mathrm{ran}(f) = \{t_1, t_2, ..., t_m\} \subset [0, 1]$, where all t_i ($i = 1, 2, ..., m$) are pairwise distinct. Let us denote

$$X_k = f^{-1}(t_k) \qquad (k \in \{1, 2, ..., m\}).$$

All sets X_k are simultaneously of types G_δ and F_σ. By virtue of Lemma 5 and Exercise 4 of the present chapter, we infer that all these sets are effectively of type F_σ. Finally, by applying once again Lemma 4, the required sequence of functions $\{\phi_n : n \in \mathbf{N}\}$ can readily be constructed, and we come to the desired result.

Lemma 8. *Let E be a Polish space, and let $\{f_n : n \in \mathbf{N}\}$ and $\{g_n : n \in \mathbf{N}\}$ be two sequences of continuous functions from E into $[0,1]$ pointwise converging to the functions f and g, respectively. Suppose also that*

$$(\forall x \in E)(|f(x) - g(x)| < a)$$

for some real $a > 0$. Then there exists effectively a sequence $\{h_n : n \in \mathbf{N}\}$ of continuous functions from E into $[0,1]$, pointwise converging to g and satisfying the relation

$$\sup_{x\in E}|h_n(x) - f_n(x)| \leq a \qquad (n \in \mathbf{N}).$$

Proof. First, for any real-valued function ϕ on E, we need to recall the notation $\phi|^a_{-a}$. Namely, the symbol $\phi|^a_{-a}$ is an abbreviation for a real-valued function on E defined by the formula

$\phi|^a_{-a}(x) = \phi(x)$ if $|\phi(x)| \leq a$,

$\phi|^a_{-a}(x) = a$ if $\phi(x) > a$,

$\phi|^a_{-a}(x) = -a$ if $\phi(x) < -a$.

Obviously, if ϕ is continuous on E, then $\phi|^a_{-a}$ is continuous on E, too.

Using the above notation, it suffices to define h_n $(n \in \mathbf{N})$ by the equality

$$h_n = f_n + (g_n - f_n)|^a_{-a}.$$

A careful verification of the fact that $\{h_n : n \in \mathbf{N}\}$ is a required sequence of functions is left to the reader.

We now are ready to establish the main result of this chapter (within **ZF** theory).

Theorem 2. *Let E be a Polish space and let $f : E \to [0,1]$ be a function of first Baire class. There exists effectively a sequence of continuous functions from E into $[0,1]$, pointwise convergent to f.*

Proof. By virtue of Lemma 6, a sequence $\{f_m : m \in \mathbf{N}\}$ of functions from E into $[0,1]$ can be effectively defined, uniformly converging to f and such that all f_m have finite ranges and are of first Baire class. We may assume without loss of generality that

$$(*) \quad |f_{m+1}(x) - f_m(x)| < 1/2^m \qquad (x \in E,\ m \in \mathbf{N}).$$

For every $m \in \mathbf{N}$, there exists effectively a sequence $\{f_{m,n} : n \in \mathbf{N} \setminus \{0\}\}$ of continuous functions from E into $[0,1]$, pointwise convergent to f_m (in this connection, see Lemma 7).

We now are going to define by recursion a double sequence

$$\{h_{m,n} \ : \ m \in \mathbf{N}, \ n \in \mathbf{N} \setminus \{0\}\}$$

of real-valued continuous functions on E so that

$$\lim_{n \to +\infty} h_{m,n}(x) = f_m(x) \qquad (m \in \mathbf{N}, \ x \in E),$$

$$|h_{m+1,n}(x) - h_{m,n}(x)| \leq 1/2^m \qquad (m \in \mathbf{N}, \ n \in \mathbf{N} \setminus \{0\}, \ x \in E).$$

For this purpose, we first put

$$\{h_{0,n} : n \in \mathbf{N} \setminus \{0\}\} = \{f_{0,n} : n \in \mathbf{N} \setminus \{0\}\}.$$

Further, suppose that the sequences

$$\{h_{0,n} : n \in \mathbf{N} \setminus \{0\}\}, \ \ \{h_{1,n} : n \in \mathbf{N} \setminus \{0\}\}, \ \ldots, \ \ \{h_{m,n} : n \in \mathbf{N} \setminus \{0\}\}$$

have already been constructed. Taking into account (*) and the relations

$$f_m(x) = \lim_{n \to +\infty} h_{m,n}(x) \qquad (x \in E),$$

$$f_{m+1}(x) = \lim_{n \to +\infty} f_{m+1,n}(x) \qquad (x \in E),$$

we may apply Lemma 8 to the functions f_m and f_{m+1}, to $a = 1/2^m$, and to the sequences

$$\{h_{m,n} : n \in \mathbf{N} \setminus \{0\}\}, \quad \{f_{m+1,n} : n \in \mathbf{N} \setminus \{0\}\}.$$

As a result, we effectively get the sequence $\{h_{m+1,n} : n \in \mathbf{N} \setminus \{0\}\}$ of continuous functions from E into $[0, 1]$, such that

$$\lim_{n \to +\infty} h_{m+1,n}(x) = f_{m+1}(x) \qquad (x \in E),$$

$$|h_{m+1,n}(x) - h_{m,n}(x)| \leq 1/2^m \qquad (n \in \mathbf{N} \setminus \{0\}, \ x \in E).$$

It remains to demonstrate that

$$\lim_{n \to +\infty} h_{n,n}(x) = f(x) \quad (x \in E).$$

For this purpose, fix a point $x_0 \in E$ and take an arbitrary real $\varepsilon > 0$. Let a natural number m be so large that

$$1/2^{m-1} < \varepsilon/3, \quad |f_m(x_0) - f(x_0)| < \varepsilon/3.$$

Further, choose a natural number $n_0 > m$ such that

$$|h_{m,n}(x_0) - f_m(x_0)| < \varepsilon/3 \qquad (n > n_0).$$

Then, for all natural indices $n > n_0$, we have

$$|h_{n,n}(x_0) - f(x_0)| \leq$$

$$|h_{n,n}(x_0) - h_{n-1,n}(x_0)| + |h_{n-1,n}(x_0) - h_{n-2,n}(x_0)| + ... + |h_{m+1,n}(x_0) - h_{m,n}(x_0)| +$$

$$|h_{m,n}(x_0) - f_m(x_0)| + |f_m(x_0) - f(x_0)| \leq$$

$$1/2^{n-1} + 1/2^{n-2} + ... + 1/2^m + \varepsilon/3 + \varepsilon/3 \leq 1/2^{m-1} + 2\varepsilon/3 < \varepsilon,$$

which completes the proof.

From Theorem 2 one can easily deduce the following slightly more general statement (essentially due to Baire).

Theorem 3. *Let E be a Polish space and let $f \in Ba_1(E, \mathbf{R})$. Then there is an effective sequence $\{f_n : n \in \mathbf{N}\} \subset Ba_0(E, \mathbf{R})$ pointwise convergent to f.*

Proof. Let us fix an arbitrary (effective) homeomorphism $\phi : \mathbf{R} \to \ ]0, 1[$ and consider the function $\phi \circ f$. Obviously, $\phi \circ f$ belongs to the first Baire class and its range is contained in $[0, 1]$. According to Theorem 2, there exists an effective sequence

$$\{g_n : n \in \mathbf{N}\} \subset Ba_0(E, [0, 1])$$

which converges pointwise to $\phi \circ f$. It is now clear that the sequence

$$\{\phi^{-1} \circ g_n : n \in \mathbf{N}\} \subset Ba_0(E, \mathbf{R})$$

is also effectively defined and converges pointwise to f. So, the proof of Theorem 3 is finished.

Remark 1. It follows from the above theorem that, for a nonempty Polish space E, the equality

$$\mathrm{card}(Ba_1(E, \mathbf{R})) = \mathbf{c}$$

holds true effectively, i.e., within **ZF** theory.

Remark 2. A radically different situation is for the second Baire class $Ba_2(E, \mathbf{R})$. Indeed, if E is a nonempty Polish space, then the equality

$$\mathrm{card}(Ba_2(E, \mathbf{R})) = \mathbf{c}$$

can be established in **ZFC** set theory without any difficulties. However, even the inequality

$$\mathrm{card}(Ba_2(\mathbf{R}, \mathbf{R})) \leq \mathbf{c}$$

cannot be proved within the framework of **ZF** & **DC** theory, because it implies the existence of a Lebesgue nonmeasurable subset of $\mathbf{R}$ (in this connection, see Exercise 23 from Chapter 10).

EXERCISES

1. Let E be a Polish topological space and let X be a closed subset of E.

Indicate effectively a countable subset of X which is everywhere dense in X.

For this purpose, use Lemma 3 of the present chapter.

2. Let E be an arbitrary topological space with a countable base.

Work in **ZF** theory and verify that the Cantor–Baire stationarity principle holds true for E in the following form:

There exists no strictly decreasing family $\{F_\xi : \xi < \omega_1\}$ of nonempty closed subsets of E.

3. Check in detail that the function g^* defined in the proof of Lemma 4 is a continuous extension of g with the required properties.

4*. Preserving the notation of the proof of Lemma 5, show that the ordinal α is effectively equinumerous with some subset of $\mathbf{N}$, so is at most countable.

For this purpose, take any ordinal $\xi < \alpha$ and find the least $n(\xi) \in \mathbf{N}$ such that $B_{n(\xi)} \cap F_\xi \neq \emptyset$ and $B_{n(\xi)} \cap F_{\xi+1} = \emptyset$. Then infer that the mapping $\xi \to n(\xi)$ $(\xi < \alpha)$ is injective.

Keeping in mind this result and taking into account the fact that all sets $F_\xi \setminus F_{\xi+1}$ $(\xi < \alpha)$ are effectively of type F_σ, conclude that both sets Z and Z' in Lemma 5 are effectively of type F_σ, too.

5. Let E be an arbitrary nonempty topological space.

Show that the family $\mathcal{D}(E)$ of all those sets in E, which simultaneously are of type G_δ and of type F_σ, forms an algebra with respect to the standard set-theoretical operations.

Check also that, in general, $\mathcal{D}(E)$ need not to be a σ-algebra (for example, $\mathcal{D}(\mathbf{R})$ is not a σ-algebra).

6*. Let E be an arbitrary uncountable Polish topological space.

Work in **ZFC** theory and deduce the equality

$$\mathrm{card}(Ba_2(E, \mathbf{R}) \setminus Ba_1(E, \mathbf{R})) = \mathbf{c}.$$

7. Let a sequence $\{f_n : n \in \mathbf{N}\}$ of polynomials over $\mathbf{R}$ be pointwise convergent on the unit segment $[0, 1]$ and let the degrees of all f_n $(n \in \mathbf{N})$ not exceed a fixed natural number k.

Is it true that the function $f = \lim_{n \to +\infty} f_n$ is also a polynomial of degree less than or equal to k?

Chapter 10
Lebesgue nonmeasurable functions and functions without the Baire property

This chapter is devoted to some well-known constructions of a function acting from $\mathbf{R}$ into $\mathbf{R}$ and nonmeasurable in the Lebesgue sense (respectively, of a function acting from $\mathbf{R}$ into $\mathbf{R}$ and lacking the Baire property). Obviously, the existence of such a function is equivalent to the existence of a subset of $\mathbf{R}$ nonmeasurable in the Lebesgue sense (respectively, of a subset of $\mathbf{R}$ without the Baire property). Since the fundamental concept of the Lebesgue measure on $\mathbf{R}$ (respectively, the concept of the Baire property) was introduced, it has been extremely useful in various problems of mathematical analysis. The natural question arose whether all subsets of $\mathbf{R}$ are measurable in the Lebesgue sense (respectively, whether all subsets of $\mathbf{R}$ possess the Baire property). Very soon, two essentially different constructions of extraordinary point sets in $\mathbf{R}$ were discovered which gave simultaneously negative answers to these two questions. The first construction is due to Vitali [272] and the second one was carried out by Bernstein [19]. Both of them were heavily based on an uncountable form of the Axiom of Choice, so it was reasonable to ask whether it is possible to construct a Lebesgue nonmeasurable subset of $\mathbf{R}$ (or a subset of $\mathbf{R}$ without the Baire property) by using some weak forms of the Axiom of Choice which are enough for most domains of classical mathematical analysis (for instance, the Axiom of Dependent Choices). Almost all outstanding mathematicians working at that time in mathematical analysis and particularly in the theory of real functions (Borel, Lebesgue, Hausdorff, Luzin, Sierpiński, etc.) believed that there is no effective construction of a Lebesgue nonmeasurable subset of $\mathbf{R}$. However, only after long-term developments in mathematical logic and axiomatic set theory and, especially, after the creation (in 1963) of the forcing method by Cohen, did it become possible to establish the needed result. We shall return to this theme in our further considerations and touch upon some related problems that are also interesting from the logical point of view. But first, we wish to discuss more thoroughly analytic aspects of the problem of the existence of Lebesgue nonmeasurable point sets (respectively, of point sets without the Baire property).

Let us indicate one important common feature of Lebesgue measurable sets and of sets with the Baire property. We shall demonstrate that not all subsets of $\mathbf{R}$ have the above-mentioned feature. Obviously, such an approach will give us the existence of required bad subsets of $\mathbf{R}$.

In order to carry out this plan, we first introduce the following definition.

Let X be a subset of $\mathbf{R}$. We say that X has the Steinhaus property if there exists a real $\varepsilon > 0$ such that

$$(\forall h \in \mathbf{R})(|h| < \varepsilon \Rightarrow (X + h) \cap X \neq \emptyset).$$

In other words, a set $X \subset \mathbf{R}$ has the Steinhaus property if the corresponding difference set $X - X = \{x' - x'' : x' \in X,\ x'' \in X\}$ is a neighborhood of point $0 \in \mathbf{R}$.

It turns out that, as a rule, all good subsets of $\mathbf{R}$ are either of Lebesgue measure zero, or of first category, or have the Steinhaus property. In this connection, it is reasonable to indicate here that Steinhaus himself observed that all Lebesgue measurable sets on $\mathbf{R}$ with strictly positive measure have his property (see [261]). Some years later, it was also established that an analogous result is true for second category subsets of $\mathbf{R}$ possessing the Baire property.

Let λ denote, as usual, the standard Lebesgue measure on $\mathbf{R}$.

We now formulate and prove the following classical result.

Theorem 1. *Let X be a subset of $\mathbf{R}$ satisfying at least one of these two assumptions:*

(1) $X \in \mathrm{dom}(\lambda)$ and $\lambda(X) > 0$;

(2) $X \in \mathcal{B}a(\mathbf{R}) \setminus \mathcal{K}(\mathbf{R})$.

Then X has the Steinhaus property.

Proof. Suppose first that assumption (1) holds. Let x be a density point of X (see Theorem 4 from Chapter 0) and let $]a, b[$ be an open interval containing x, for which we have $\lambda(X \cap\ ]a, b[) > 2(b-a)/3$. Obviously, there exists a real $\varepsilon > 0$ such that

$$(\forall h \in \mathbf{R})(|h| < \varepsilon \Rightarrow \lambda(]a + h, b + h[\ \cup\ ]a, b[) \leq 4(b - a)/3).$$

Take an arbitrary $h \in \mathbf{R}$ with $|h| < \varepsilon$. We assert that

$$(X + h) \cap X \neq \emptyset.$$

Indeed, assuming for a while that $(X + h) \cap X = \emptyset$, we must have

$$\lambda(]a + h, b + h[\ \cup\ ]a, b[) \geq \lambda(((X \cap\ ]a, b[) + h) \cup (X \cap\ ]a, b[)) =$$

$$2\lambda(X \cap\ ]a, b[) > 4(b - a)/3,$$

which is impossible. Thus X has the Steinhaus property.

Suppose now that assumption (2) holds. Then X can be represented in the form $X = U \triangle X_1$, where U is a nonempty open set in $\mathbf{R}$ and X_1 is a first category subset of $\mathbf{R}$. Evidently, there exists a real $\varepsilon > 0$ such that

$$(\forall h \in \mathbf{R})(|h| < \varepsilon \Rightarrow (U + h) \cap U \neq \emptyset).$$

Let us fix any $h \in \mathbf{R}$ with $|h| < \varepsilon$. It is easy to check the inclusion

$$((U + h) \cap U) \setminus ((X_1 + h) \cup X_1) \subset (X + h) \cap X.$$

Taking account of the fact that $(U + h) \cap U$ is a nonempty open subset of $\mathbf{R}$ and $(X_1 + h) \cup X_1$ is a first category subset of $\mathbf{R}$, we infer that

$$((U + h) \cap U) \setminus ((X_1 + h) \cup X_1) \neq \emptyset.$$

Consequently, we obtain $(X + h) \cap X \neq \emptyset$, and this finishes the proof of Theorem 1.

The following statement is an easy consequence of Theorem 1 but, sometimes, is more useful in practice.

Theorem 2. *Let X and Y be subsets of $\mathbf{R}$ such that at least one of these two conditions holds:*

(1) $\{X, Y\} \subset \mathrm{dom}(\lambda)$, $\lambda(X) > 0$, $\lambda(Y) > 0$;

(2) $\{X, Y\} \subset \mathcal{B}a(\mathbf{R}) \setminus \mathcal{K}(\mathbf{R})$.

Then the vector sum $X + Y = \{x + y\ :\ x \in X,\ y \in Y\}$ has nonempty interior.

Proof. Clearly, under assumption (1), there exists an element $t \in \mathbf{R}$ such that $\lambda((X + t) \cap Y) > 0$. Actually, this relation follows from the metrical transitivity of the measure λ (also, from the Lebesgue theorem on density points).

Similarly, under assumption (2), there exists an element $r \in \mathbf{R}$ such that

$$(X + r) \cap Y \in \mathcal{B}a(\mathbf{R}) \setminus \mathcal{K}(\mathbf{R}).$$

In fact, here we have an analogue of the metrical transitivity for the Baire property.

Let us put $Z = (X + t) \cap Y$ in the first case, and $Z = (X + r) \cap Y$ in the second one.

It suffices to show that the set $Z + Z$ has nonempty interior. If Z is symmetric with respect to zero, then we may directly apply Theorem 1. Generally, we can find $z \in \mathbf{R}$ such that $\lambda((Z + z) \cap (-Z)) > 0$ in the first case, and $(Z+z) \cap (-Z) \in \mathcal{B}a(\mathbf{R}) \setminus \mathcal{K}(\mathbf{R})$ in the second one. Further, define $Z' = (Z + z/2) \cap (-Z - z/2)$. The set Z' is symmetric with respect to zero and $Z' \subset Z + z/2$. Moreover, we see that $\lambda(Z') = \lambda((Z + z) \cap (-Z)) > 0$ in the first case, and $Z' \in \mathcal{B}a(\mathbf{R}) \setminus \mathcal{K}(\mathbf{R})$ in the second one. Applying Theorem 1 to Z' and taking into account the relation

$$Z' - Z' = Z' + Z' \subset Z + Z + z,$$

we come to the required result.

Exercises 1 and 2 for this chapter show that Theorems 1 and 2 admit analogues in much more general situations.

Now, we are ready to present the first classical construction of a subset of the real line, nonmeasurable in the Lebesgue sense and without the Baire property. As mentioned earlier, this construction is due to Vitali (see [272]).

Let us consider a binary relation $V(x, y)$ on $\mathbf{R}$ defined by the formula

$$x \in \mathbf{R} \ \& \ y \in \mathbf{R} \ \& \ x - y \in \mathbf{Q},$$

where $\mathbf{Q}$ denotes, as usual, the set of all rational numbers. Since $\mathbf{Q}$ is a subgroup of the additive group $(\mathbf{R}, +)$, we infer that $V(x, y)$ is an equivalence relation on $\mathbf{R}$. Consequently, we obtain the partition of $\mathbf{R}$ canonically associated with $V(x, y)$.

This partition is usually called the Vitali partition of $\mathbf{R}$ and will be denoted by $\mathbf{R}/\mathbf{Q}$.

Any selector of the Vitali partition is called a Vitali set in $\mathbf{R}$.

Theorem 3. *There exist Vitali subsets of* $\mathbf{R}$. *If X is an arbitrary Vitali subset of* $\mathbf{R}$, *then X is Lebesgue nonmeasurable and does not possess the Baire property (equivalently, the characteristic function of X is Lebesgue nonmeasurable and does not possess the Baire property).*

Proof. The existence of Vitali sets follows directly from the Axiom of Choice applied to the Vitali partition (since $\mathbf{R}/\mathbf{Q}$ is uncountable, here a certain uncountable form of **AC** should be utilized). Now, let X be a Vitali set and suppose for a moment that X is either Lebesgue measurable or possesses the Baire property. Then, keeping in mind the relation

$$\mathbf{R} = \cup\{X + q \ : \ q \in \mathbf{Q}\},$$

we infer that X must be of strictly positive measure (respectively, of second category). But this immediately yields a contradiction. Indeed, for each rational number $q \neq 0$, we have $(X + q) \cap X = \emptyset$, because X is a selector of $\mathbf{R}/\mathbf{Q}$. Here q can be taken arbitrarily small. In other words, we see that our X does not have the Steinhaus property. This contradicts Theorem 1 and finishes the proof of Theorem 3.

We thus obtained that Vitali sets are very bad from the points of view of the Lebesgue measure and Baire property. However, these sets may be rather good for other nonzero σ-finite invariant measures given on $\mathbf{R}$. In this connection, see Exercise 3. On the other hand, Exercise 4 shows that any Vitali set remains nonmeasurable with respect to every translation invariant extension of the Lebesgue measure λ.

It is not hard to see that the argument used in the Vitali construction heavily relies on the assumption of the invariance of the Lebesgue measure λ with respect to translations of $\mathbf{R}$. This argument does not work for a nonzero σ-finite measure μ on $\mathbf{R}$ which is only quasi-invariant (i.e., μ is defined on a σ-algebra of subsets of $\mathbf{R}$, invariant under translations, and the σ-ideal of all μ-measure zero sets is preserved by translations, too). So the following question arises: how to prove the existence of nonmeasurable sets with respect to such a measure μ. We shall consider this question in the next chapter of the book. Namely, we shall show therein that a more general algebraic construction is possible which yields the existence of nonmeasurable sets with respect to μ. The main role in that construction will be played by so-called Hamel bases of $\mathbf{R}$.

Now, we want to turn our attention to another classical construction of a Lebesgue nonmeasurable set (and of a set without the Baire property). As pointed out earlier, this construction is due to Bernstein (see [19]). First, let us introduce one useful notion closely related to the Bernstein construction.

Let E be a topological space and let X be a subset of E.

We say that X is totally imperfect in E if X contains no nonempty perfect subset of E.

We say that X is a Bernstein subset of E if X and $E \setminus X$ are totally imperfect in E. Equivalently, X is a Bernstein subset of E if, for each nonempty perfect set $P \subset E$, we have $P \cap X \neq \emptyset$ and $P \cap (E \setminus X) \neq \emptyset$.

It immediately follows from this definition that $X \subset E$ is a Bernstein set if and only if its complement $E \setminus X$ is a Bernstein set.

Clearly, each subset of $\mathbf{R}$ with cardinality strictly less than the cardinality of the continuum $\mathbf{c}$ is totally imperfect. The question concerning the existence of totally imperfect subsets of $\mathbf{R}$, having cardinality $\mathbf{c}$, turns out to be nontrivial. For its solution, we again need an uncountable version of the Axiom of Choice (cf. Exercise 5).

There are many examples of totally imperfect subsets of the Euclidean space $\mathbf{R}^n$. A wide class of such sets was introduced and investigated by Marczewski (see [263]).

Let E be a Polish topological space and let X be a subset of E.

We say that X is a Marczewski subset of E if, for each nonempty perfect set $P \subset E$, there exists a nonempty perfect set $P' \subset E$ such that $P' \subset P$ and $P' \cap X = \emptyset$.

It immediately follows from this definition that every Marczewski set is totally imperfect in E, and that any subset of a Marczewski set is a Marczewski set, too. Also, it can easily be observed that any set $Y \subset E$ with $\mathrm{card}(Y) < \mathbf{c}$ is a Marczewski set. Indeed, let us take an arbitrary nonempty perfect set $P \subset E$. Then, as we know (cf. Exercise 14 from Chapter 1), there exists a disjoint family $\{P_i : i \in I\}$ consisting of nonempty perfect sets in E and satisfying the relations $\mathrm{card}(I) = \mathbf{c}$ and $(\forall i \in I)(P_i \subset P)$. Now, because of $\mathrm{card}(Y) < \mathrm{card}(I)$, it is clear that there exists at least one index $i_0 \in I$ such that $P_{i_0} \cap Y = \emptyset$, and thus Y is a Marczewski set.

Let us recall the classical result of Alexandrov and Hausdorff stating that every uncountable Borel set in a Polish topological space contains a subset homeomorphic to the Cantor discontinuum (hence contains a nonempty perfect subset). Taking this result into account, we can give another, equivalent, definition of Marczewski sets.

Namely, we may say that a set X lying in a Polish space E is a Marczewski set if, for each uncountable Borel subset B of E, there exists an uncountable Borel set $B' \subset E$ such that $B' \subset B$ and $B' \cap X = \emptyset$.

In some situations, the second definition is more convenient. For instance, let E_1 and E_2 be two Polish spaces and let $f : E_1 \to E_2$ be a Borel isomorphism between them. Then, for a set $X \subset E_1$, the following two assertions are equivalent:

(1) X is a Marczewski set in E_1;

(2) $f(X)$ is a Marczewski set in E_2.

In other words, the Borel isomorphism f yields a one-to-one correspondence between Marczewski sets in the spaces E_1 and E_2. This fact is helpful in many situations. For instance, suppose that we need to construct a Marczewski subset of E_1 having some additional properties which are invariant under Borel isomorphisms. Sometimes, it turns out that such a set can much more easily be constructed in E_2. Let us denote it by X'. Then we apply the Borel isomorphism f^{-1} to X' and obtain the required Marczewski set $f^{-1}(X')$ in the space E_1.

Later, we shall demonstrate the usefulness of this idea. Namely, we shall show that there exist Marczewski subsets of $\mathbf{R}$ nonmeasurable in the Lebesgue sense (respectively, without the Baire property).

One simple (but important) fact concerning Marczewski sets is presented in the next statement.

Lemma 1. *Let $\{X_k : k < \omega\}$ be a countable family of Marczewski subsets of a Polish topological space E. Then $\cup\{X_k : k < \omega\}$ is a Marczewski set, too. In particular, if the space E is uncountable, then the family of all Marczewski subsets of E forms a σ-ideal in the Boolean algebra of all subsets of E.*

Proof. Fix a nonempty perfect set $P \subset E$. Since X_0 is a Marczewski set, there exists a nonempty perfect set $P_0 \subset E$ such that

$$P_0 \subset P, \quad P_0 \cap X_0 = \emptyset, \quad \mathrm{diam}(P_0) < 1.$$

Further, because X_1 is a Marczewski set, too, there exist nonempty perfect sets $P_{00} \subset E$ and $P_{01} \subset E$ such that

$$P_{00} \subset P_0, \quad P_{01} \subset P_0, \quad \mathrm{diam}(P_{00}) < 1/2, \quad \mathrm{diam}(P_{01}) < 1/2,$$

$$P_{00} \cap X_1 = \emptyset, \quad P_{01} \cap X_1 = \emptyset, \quad P_{00} \cap P_{01} = \emptyset.$$

Proceeding in this manner, we will be able to define a dyadic system

$$\{P_{j_1 \dots j_k} : j_1 = 0, \ j_2 \in \{0,1\}, ..., j_k \in \{0,1\}, \ 1 \leq k < \omega\}$$

of nonempty perfect sets in E whose diameters converge to zero, and

$$P_{j_1 \dots j_k j} \subset P_{j_1 \dots j_k}, \quad P_{j_1 \dots j_k 0} \cap P_{j_1 \dots j_k 1} = \emptyset,$$

$$P_{j_1 \dots j_k} \cap X_{k-1} = \emptyset$$

for each natural number $k \geq 1$. Now, putting

$$D_k = \cup\{P_{j_1 \dots j_k} : j_1 = 0, j_2 \in \{0,1\}, ..., j_k \in \{0,1\}\},$$

$$D = \cap\{D_k : 1 \leq k < \omega\},$$

we obtain a nonempty perfect set $D \subset P$ satisfying the relation

$$D \cap (\cup\{X_k : k < \omega\}) = \emptyset.$$

This shows that $\cup\{X_k : k < \omega\}$ is a Marczewski subset of E, and the proof is finished.

Remark 1. Lemma 1 shows that, in an uncountable Polish topological space E, the family of all Marczewski subsets of E forms a σ-ideal. It is usually called Marczewski's σ-ideal in E and plays an essential role in classical point set theory (cf. [31]). As mentioned above, Marczewski subsets of E can be regarded as a certain type of small sets in E. In our further considerations, we shall also deal with some other types of small sets which generate proper σ-ideals in E. For instance, we shall deal with the σ-ideal generated by all Luzin subsets of $E = \mathbf{R}$ (respectively, by all Sierpiński subsets of $\mathbf{R}$). In addition, we shall consider the σ-ideal of so-called universal measure zero subsets of $\mathbf{R}$ and the σ-ideal of strong measure zero subsets of $\mathbf{R}$. Various properties of these subsets will be discussed in subsequent chapters of the book (notice that valuable information about different kinds of small sets in Polish spaces can be found in [31], [149], [184], [207], and [283]).

Let us return to Bernstein sets. We now formulate and prove the classical Bernstein result on the existence of such sets.

Theorem 4. *There exists a Bernstein subset X of $\mathbf{R}$. This X is Lebesgue nonmeasurable and does not possess the Baire property (equivalently, the characteristic function of X is Lebesgue nonmeasurable and does not possess the Baire property).*

Proof. Let α denote the least ordinal for which $\mathrm{card}(\alpha) = \mathbf{c}$. We know that the family of all nonempty perfect subsets of $\mathbf{R}$ is of cardinality $\mathbf{c}$. So we may denote this family by $\{P_\xi \ : \ \xi < \alpha\}$. Moreover, we may assume without loss of generality that each of the two partial families

$$\{P_\xi : \xi < \alpha,\ \xi \text{ is an even ordinal}\}, \quad \{P_\xi : \xi < \alpha,\ \xi \text{ is an odd ordinal}\}$$

also consists of all nonempty perfect subsets of $\mathbf{R}$. Now, applying the method of transfinite recursion, we define an α-sequence $\{x_\xi : \xi < \alpha\} \subset \mathbf{R}$ of points satisfying the following two conditions:

(1) if $\xi < \zeta < \alpha$, then $x_\xi \neq x_\zeta$;

(2) for every $\xi < \alpha$, we have $x_\xi \in P_\xi$.

Suppose that, for $\beta < \alpha$, the partial β-sequence $\{x_\xi : \xi < \beta\}$ has already been defined. Take the set P_β. Obviously,

$$\mathrm{card}(P_\beta) = \mathbf{c}, \qquad \mathrm{card}(\{x_\xi \ : \ \xi < \beta\}) \leq \mathrm{card}(\beta) < \mathbf{c}.$$

Hence we can write $P_\beta \setminus \{x_\xi : \xi < \beta\} \neq \emptyset$. Choose an arbitrary element x from the above-mentioned nonempty difference of sets and put $x_\beta = x$. Continuing in this manner, we will be able to construct the required family

$\{x_\xi : \xi < \alpha\}$ of points of $\mathbf{R}$, satisfying conditions (1) and (2). Further, we put

$$X = \{x_\xi \ : \ \xi < \alpha, \ \xi \text{ is an even ordinal number}\}.$$

It immediately follows from our construction that X is a Bernstein subset of $\mathbf{R}$ (because both sets X and $\mathbf{R} \setminus X$ are totally imperfect in $\mathbf{R}$).

It remains to demonstrate that X is not Lebesgue measurable and does not possess the Baire property.

Suppose first that X is measurable in the Lebesgue sense. Then the set $\mathbf{R} \setminus X$ is Lebesgue measurable, too, and at least one of these two sets must be of strictly positive measure. We may assume without loss of generality that $\lambda(X) > 0$. Then a well-known regularity property of λ implies that there exists a closed set $F \subset \mathbf{R}$ contained in X and also having strictly positive measure. Because λ is a diffused measure, we obtain $\mathrm{card}(F) > \omega$ and hence $\mathrm{card}(F) = \mathbf{c}$. Denote by F_0 the set of all condensation points of F. Obviously, F_0 is a nonempty perfect subset of $\mathbf{R}$ contained in X. But this contradicts the circumstance that X is a Bernstein set in $\mathbf{R}$.

Suppose now that X possesses the Baire property. Then the set $\mathbf{R} \setminus X$ possesses the Baire property, too, and at least one of these two sets must be of second category. We may assume without loss of generality that X is of second category. Consequently, we have a representation of X in the form

$$X = V \triangle Y = (V \setminus Y) \cup (Y \setminus V),$$

where V is a nonempty open subset of $\mathbf{R}$ and Y is a first category subset of $\mathbf{R}$. Applying the classical Baire theorem, we see that the set $V \setminus Y$ contains an uncountable G_δ-subset of $\mathbf{R}$. This immediately implies that X contains also a nonempty perfect subset of $\mathbf{R}$ (homeomorphic to the Cantor space), which again contradicts the fact that X is a Bernstein set in $\mathbf{R}$. Theorem 4 is thus proved.

A result much more general than Theorem 4 is presented in Exercises 7 and 8 of this chapter, and Exercise 9 yields a characterization of Bernstein subsets of Polish spaces in terms of topological measure theory.

All the constructions presented above were concerned with certain point sets either nonmeasurable in the Lebesgue sense or without the Baire property. The existence of such sets trivially implies the existence of real-valued functions either nonmeasurable in the Lebesgue sense or without the Baire property. We now wish to consider one direct construction of a Lebesgue nonmeasurable function acting from $\mathbf{R}$ into $\mathbf{R}$. An analogous construction can be carried out for the Baire property instead of Lebesgue measurability.

In our further considerations, we denote by the symbol λ_2 the standard two-dimensional Lebesgue measure on the Euclidean plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

Clearly, λ_2 is the completion of the product measure $\lambda \otimes \lambda$ where λ denotes, as usual, the standard Lebesgue measure on $\mathbf{R}$.

We recall that a subset X of $\mathbf{R}^2$ is λ_2-thick (or λ_2-massive) in $\mathbf{R}^2$ if, for every λ_2-measurable set $Z \subset \mathbf{R}^2$ with $\lambda_2(Z) > 0$, the relation $X \cap Z \neq \emptyset$ holds true.

In other words, X is λ_2-thick in $\mathbf{R}^2$ if and only if the equality

$$(\lambda_2)_*(\mathbf{R}^2 \setminus X) = 0$$

is satisfied, where the symbol $(\lambda_2)_*$ denotes the inner measure canonically associated with λ_2.

Let us observe that if a subset X of $\mathbf{R}^2$ is λ_2-measurable and λ_2-massive simultaneously, then it is of full λ_2-measure, i.e., $\lambda_2(\mathbf{R}^2 \setminus X) = 0$. Thus, if we a priori know that a set $X \subset \mathbf{R}^2$ is not of full λ_2-measure but is λ_2-thick, then we can immediately conclude that X is not λ_2-measurable.

The next statement (essentially due to Sierpiński) shows that there are functions acting from $\mathbf{R}$ into $\mathbf{R}$ whose graphs are λ_2-thick subsets of the plane $\mathbf{R}^2$.

Theorem 5. *There exists a function $f : \mathbf{R} \to \mathbf{R}$ whose graph is a λ_2-thick subset of $\mathbf{R}^2$. Consequently, the following two assertions are true:*

(1) the graph of f is not a λ_2-measurable subset of $\mathbf{R}^2$;

(2) f is not a λ-measurable function.

Proof. Let α be the least ordinal number of cardinality continuum. Consider the family $\{B_\xi \ : \ \xi < \alpha\}$ consisting of all Borel subsets of $\mathbf{R}^2$ having strictly positive λ_2-measure. We are going to construct, by transfinite recursion, a family of points $\{(x_\xi, y_\xi) \ : \ \xi < \alpha\} \subset \mathbf{R}^2$ satisfying these two conditions:

(a) if $\xi < \zeta < \alpha$, then $x_\xi \neq x_\zeta$;

(b) for each $\xi < \alpha$, the point (x_ξ, y_ξ) belongs to B_ξ.

Suppose that, for an ordinal $\beta < \alpha$, the partial family $\{(x_\xi, y_\xi) : \xi < \beta\}$ of points of $\mathbf{R}^2$ has already been defined. Let us take the set B_β. Applying the classical Fubini theorem to B_β, we see that the set

$$X_\beta = \{x \in \mathbf{R} \ : \ \lambda(B_\beta(x)) > 0\}$$

is λ-measurable and of strictly positive measure. Consequently, X_β is of cardinality $\mathbf{c}$, and there exists a point $x \in X_\beta$ distinct from all the points x_ξ $(\xi < \beta)$. We put $x_\beta = x$. Then we choose an arbitrary point y from the set $B_\beta(x_\beta)$ and put $y_\beta = y$. Proceeding in this manner, we will be able to construct the required family $\{(x_\xi, y_\xi) \ : \ \xi < \alpha\}$. Now, it follows from condition (a) that the set $F = \{(x_\xi, y_\xi) \ : \ \xi < \alpha\}$ can be regarded as the graph of some partial function acting from $\mathbf{R}$ into $\mathbf{R}$. We extend arbitrarily this partial function to a function acting from $\mathbf{R}$ into $\mathbf{R}$ and denote the latter function by f. Then condition (b) implies that the graph of f is λ_2-thick in $\mathbf{R}^2$.

Since there are continuum many pairwise disjoint translates of this graph in $\mathbf{R}^2$, we conclude that the graph is not of full λ_2-measure and hence it is not a λ_2-measurable subset of $\mathbf{R}^2$.

Finally, the function f is not λ-measurable. Indeed, otherwise the graph of f would be a λ_2-measure zero subset of $\mathbf{R}^2$, which is impossible. This completes the proof of the theorem.

Some generalizations of the above result are presented in Chapter 14.

If we deal with a class of subsets of $\mathbf{R}$ which are small in a certain sense, then, as a rule, it is not easy to establish the existence of a set belonging to this class and nonmeasurable in the Lebesgue sense (or lacking the Baire property).

More precisely, suppose that a σ-ideal $\mathcal{I}$ of subsets of $\mathbf{R}$ is given. Then the following natural question can be posed: does there exist at least one set $X \in \mathcal{I}$ nonmeasurable in the Lebesgue sense or without the Baire property? Obviously, the answer to this question depends on the structure of $\mathcal{I}$ and simple examples show that the answer can be negative.

However, let us consider the two classical σ-ideals:

$\mathcal{I}(\lambda)$ = the σ-ideal of all λ-measure zero subsets of $\mathbf{R}$;

$\mathcal{K}(\mathbf{R})$ = the σ-ideal of all first category subsets of $\mathbf{R}$.

These two σ-ideals are orthogonal, i.e., there exists a partition $\{A, B\}$ of $\mathbf{R}$ such that $A \in \mathcal{I}(\lambda)$ and $B \in \mathcal{K}(\mathbf{R})$.

The reader can verify this simple fact, which immediately implies the existence of a Lebesgue nonmeasurable set belonging to $\mathcal{K}(\mathbf{R})$ and the existence of a Lebesgue measure zero set without the Baire property. Indeed, let X be an arbitrary Bernstein subset of $\mathbf{R}$. Then, putting $X_0 = A \cap X$ and $X_1 = B \cap X$, it is easy to check that

(1) $X_0 \in \mathcal{I}(\lambda)$ and X_0 does not possess the Baire property;

(2) $X_1 \in \mathcal{K}(\mathbf{R})$ and X_1 is not measurable in the Lebesgue sense.

A more general result is presented in Exercise 16.

Returning to the question formulated above, we wish to consider it more thoroughly for the Marczewski σ-ideal in $\mathbf{R}$. In other words, it is natural to ask whether there exist Marczewski subsets of $\mathbf{R}$ nonmeasurable in the Lebesgue sense (or without the Baire property). This problem was originally raised by Marczewski (see [263]). The solution to it was independently obtained by Corazza [53] and Walsh [274]. We would like to present their result here.

First of all, we need one auxiliary proposition useful in many situations. The proof of this proposition is very similar to the argument utilized in the proof of Theorem 5.

Lemma 2. *Let $\{Z_j : j \in J\}$ be a family of subsets of the plane $\mathbf{R}^2$, satisfying the following two conditions:*

(1) $\mathrm{card}(J) \leq \mathbf{c}$*;*

(2) for each $j \in J$, the set of all $x \in \mathrm{pr}_1(Z_j)$ for which $\mathrm{card}(Z_j(x)) = \mathbf{c}$ *is of cardinality* $\mathbf{c}$.

Then there exist a set-valued mapping $F : J \to \mathcal{P}(\mathbf{R})$ and an injective family $\{x_j : j \in J\} \subset \mathbf{R}$ such that, for an arbitrary $j \in J$, the equalities $F(j) = Z_j(x_j)$ and $\mathrm{card}(F(j)) = \mathbf{c}$ *hold true.*

Proof. Obviously, we may assume without loss of generality that the set J is of cardinality $\mathbf{c}$. Also, we can identify J with the least ordinal number α such that $\mathrm{card}(\alpha) = \mathbf{c}$. Now, we are going to define a set-valued mapping F and a family $\{x_\xi : \xi < \alpha\}$ by the method of transfinite recursion.

Suppose that, for an ordinal $\beta < \alpha$, the partial families $\{F(\xi) : \xi < \beta\}$ and $\{x_\xi : \xi < \beta\}$ have already been constructed. Consider the set Z_β. According to our assumption, the set of all those $x \in \mathrm{pr}_1(Z_\beta)$ for which $\mathrm{card}(Z_\beta(x)) = \mathbf{c}$ is of cardinality continuum. Because

$$\mathrm{card}(\{x_\xi \ : \ \xi < \beta\}) < \mathbf{c},$$

there exists a point $x \in \mathbf{R}$ such that

$$(\forall \xi < \beta)(x \neq x_\xi), \qquad \mathrm{card}(Z_\beta(x)) = \mathbf{c}.$$

Therefore, we can put $F(\beta) = Z_\beta(x)$ and $x_\beta = x$.

Proceeding in this way, we will be able to define F and $\{x_\xi \ : \ \xi < \alpha\}$ with the required properties. Lemma 2 has thus been proved.

Starting with the previous lemma, it is not difficult to establish the following statement.

Theorem 6. *There exists a Marczewski subset of $\mathbf{R}^2$ nonmeasurable in the Lebesgue sense and lacking the Baire property.*

Proof. Let α be again the least ordinal number with $\mathrm{card}(\alpha) = \mathbf{c}$ and let $\{Z_\xi \ : \ \xi < \alpha\}$ denote the family of all Borel subsets of $\mathbf{R}^2$ with strictly positive λ_2-measure. Applying Lemma 2 to this family, we can find a set-valued mapping F and an injective family $\{x_\xi \ : \ \xi < \alpha\}$ of points of $\mathbf{R}$ with the corresponding properties. Let now $\{P_\xi \ : \ \xi < \alpha\}$ be the family of all nonempty perfect subsets of $\mathbf{R}^2$. For each $\beta < \alpha$, we put

$$y_\beta \in F(\beta) \setminus \cup\{P_\xi(x_\beta) \ : \ \xi \leq \beta, \ \mathrm{card}(P_\xi(x_\beta)) \leq \omega\}.$$

Notice that y_β is well-defined because of the equality $\mathrm{card}(F(\beta)) = \mathbf{c}$.

Let us check that $D_0 = \{(x_\xi, y_\xi) : \xi < \alpha\}$ is a Marczewski set nonmeasurable with respect to λ_2. Indeed, D_0 may be regarded as the graph of a certain partial function acting from $\mathbf{R}$ into $\mathbf{R}$. From the construction of D_0 we have that D_0 is also a λ_2-thick subset of $\mathbf{R}^2$. Consequently (cf. the proof of Theorem 5), we can assert that D_0 is nonmeasurable in the Lebesgue sense. It remains to show that D_0 is a Marczewski set. In order to do this, take an arbitrary nonempty perfect subset P of $\mathbf{R}^2$. We must verify that P contains a nonempty perfect set whose intersection with D_0 is empty. If there exists at least one point $x \in \mathbf{R}$ for which $\mathrm{card}(P(x)) = \mathbf{c}$, then there is nothing to prove. Suppose now that

$$(\forall x \in \mathbf{R})(\mathrm{card}(P(x)) \leq \omega).$$

For some $\beta < \alpha$, we have $P_\beta = P$. Then, taking into account the definition of y_ξ $(\xi < \alpha)$, we get

$$\mathrm{card}(D_0 \cap P_\beta) \leq \mathrm{card}(\beta) + \omega < \mathbf{c}.$$

The latter relation easily implies that there exists a nonempty perfect subset of P having no common points with D_0.

In a similar manner, starting with the family of all those Borel subsets of the plane $\mathbf{R}^2$ that do not belong to the σ-ideal $\mathcal{K}(\mathbf{R}^2)$, we can construct a Marczewski set $D_1 \subset \mathbf{R}^2$ thick in the category sense and coinciding with the graph of a partial function acting from $\mathbf{R}$ into $\mathbf{R}$. Then it is not hard to see that $D = D_0 \cup D_1$ is a Marczewski set in $\mathbf{R}^2$ nonmeasurable in the Lebesgue sense and without the Baire property. This finishes the proof of Theorem 6.

Now, it is not difficult to infer from Theorem 6 the existence of Marczewski subsets of $\mathbf{R}$ nonmeasurable in the Lebesgue sense (respectively, without the Baire property). For this purpose, consider a Borel isomorphism $\phi : \mathbf{R} \to \mathbf{R}^2$. It is a well-known fact that ϕ can be chosen in such a way that, simultaneously, ϕ will be an isomorphism between the measures λ and λ_2. Therefore, if X is a λ_2-nonmeasurable Marczewski subset of $\mathbf{R}^2$, then $\phi^{-1}(X)$ is a λ-nonmeasurable Marczewski subset of $\mathbf{R}$.

Analogously, a Borel isomorphism ϕ can be chosen in such a way that it will preserve the Baire category of sets (consequently, the Baire property of sets). So, if X is a Marczewski subset of $\mathbf{R}^2$ without the Baire property, then $\phi^{-1}(X)$ is a Marczewski subset of $\mathbf{R}$ without the Baire property.

A stronger result in this direction is presented in Exercise 18.

One more example of a Lebesgue nonmeasurable function acting from **R** into **R**, which also does not have the Baire property, can be obtained by using some results of Chapter 8.

Let us recall the theorem of Sierpiński and Zygmund proved in Chapter 8 and stating that there exists a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the following condition: for each subset X of **R** with $\mathrm{card}(X) = \mathbf{c}$, the restriction $f|X$ is not continuous on X.

Actually, this condition implies that f is not Lebesgue measurable and does not possess the Baire property.

Indeed, suppose for a moment that f has the Baire property. Then, according to a well-known theorem (see, e.g., [149], [202] or Exercise 13 from Chapter 0), we can find an everywhere dense G_δ-subset A of **R** such that $f|A$ is continuous. Obviously, $\mathrm{card}(A) = \mathbf{c}$, and we obtain a contradiction with the fact that f is a Sierpiński–Zygmund function.

Further, let us demonstrate that f is not measurable in the Lebesgue sense. We will prove a much more general result asserting that f is not measurable with respect to the completion of any nonzero σ-finite diffused Borel measure on **R**. Let μ be such a measure and let μ' denote the completion of μ. Because the analogue of Luzin's classical theorem holds true for μ', we can find a closed subset B of **R** with $\mu'(B) > 0$ such that the restricted function $f|B$ is continuous. Taking account of the diffusedness of μ' and of the inequality $\mu'(B) > 0$, we infer that B is uncountable and hence $\mathrm{card}(B) = \mathbf{c}$. This yields again a contradiction with the fact that f is a Sierpiński–Zygmund function.

Concluding this chapter, we wish to make several remarks about logical aspects of the question concerning the existence of a Lebesgue nonmeasurable subset of **R** (of a subset of **R** without the Baire property). Namely, in 1970, Solovay published his famous article [258] where he pointed out a model of **ZF** & **DC** in which all subsets of the real line are Lebesgue measurable and possess the Baire property. However, the existence of such a model was based on the assumption of the existence of an uncountable strongly inaccessible cardinal number and this seemed to be a weak side of the above-mentioned result. But, some time later, Shelah showed in his remarkable work [228] that a large cardinal appeared here not accidentally. More precisely, he established that

(1) there are models of **ZF** & **DC** in which all subsets of **R** possess the Baire property;

(2) the existence of a model of **ZF** & **DC** in which all subsets of **R** are Lebesgue measurable implies the existence of a large cardinal.

Solovay constructed also another model of **ZFC** set theory in which all projective subsets of $\mathbf{R}$ are Lebesgue measurable and possess the Baire property (see [258]). In this connection, it is reasonable to recall that in the Constructible Universe of Gödel $\mathbf{L}$ there are projective subsets of $\mathbf{R}$ (even belonging to the class $\mathcal{P}r_3(\mathbf{R})$) which are not Lebesgue measurable and do not have the Baire property (for more details, see, e.g., [18], [97], [198]).

From among many other results connected with the existence in $\mathbf{R}$ of sets nonmeasurable in the Lebesgue sense (respectively, of sets without the Baire property), we want to point out the following ones.

1. Kolmogorov showed in [139] that the existence in the theory **ZF** & **DC** of a universal operation of integration for all Lebesgue measurable functions acting from $[0, 1]$ into $\mathbf{R}$ implies the existence of a Lebesgue nonmeasurable function acting from $\mathbf{R}$ into $\mathbf{R}$. A similar result is true (in the same theory) for a universal operation of differentiation.

So that the two fundamental operations of mathematical analysis (i.e., integration and differentiation) lead directly to real-valued functions on $\mathbf{R}$ which are nonmeasurable in the Lebesgue sense. This result seems to be interesting and important from the point of view of foundations of real analysis. In Chapter 22, we shall present some statements concerning generalized derivatives which are closely related to the above-mentioned Kolmogorov result (in this context, see also Chapter 12).

2. Sierpiński proved that the existence of a nontrivial ultrafilter in the Boolean algebra of all subsets of ω implies (within **ZF** & **DC** theory) the existence of a subset of $\mathbf{R}$ nonmeasurable in the Lebesgue sense and without the Baire property.

The proof of this result can be found, e.g., in [248] (see also [43]).

3. Shelah and Raisonnier (see [228], [210]) established that the implication

$$\omega_1 \leq \mathbf{c} \;\Rightarrow\; \text{there exists a Lebesgue nonmeasurable set on } \mathbf{R}$$

holds true in **ZF** & **DC** theory.

The proof of this fact given in [210] is essentially based on some deep combinatorial properties of so-called rapid filters in ω (the notion of a rapid filter was first introduced by Mokobodzki [187]).

4. Pawlikowski showed in his short paper [204] that the Hahn–Banach theorem on extensions of partial linear continuous functionals implies (in **ZF** & **DC** theory) the famous Banach–Tarski paradox and, hence, implies the existence of a Lebesgue nonmeasurable subset of $\mathbf{R}$.

Various topics related to the Banach–Tarski paradox are discussed in the extensive monograph [273]. In particular, certain sections of this monograph are specially devoted to profound relationships between equidecomposability theory and the existence of Lebesgue nonmeasurable subsets of $\mathbf{R}$.

5. Roslanowski and Shelah have proved in their joint work [218] that there is a model of **ZFC** theory in which, for every function $f : \mathbf{R} \to \mathbf{R}$, there exists a set $X \subset \mathbf{R}$ nonmeasurable in the Lebesgue sense and such that the restriction $f|X$ is continuous.

It is useful to compare this result with the existence of a Sierpiński–Zygmund function (see Chapter 8).

Some other facts concerning the existence of nonmeasurable sets and of sets without the Baire property are presented in [1], [10], [40], [42], [47], [67], [83], [90], [108], [109], [115], [116], [127], [129], [152], [192], [202], [224], [237], [242], [256], [257], [270].

EXERCISES

1. Let $(G, \cdot)$ be an arbitrary topological group and let X be a subset of G such that $X \in \mathcal{B}a(G) \setminus \mathcal{K}(G)$.

Using the Banach result on first category open sets (see Exercise 30 from Chapter 0), show that the set $X \cdot X^{-1} = \{x \cdot y^{-1} : x \in X,\ y \in X\}$ is a neighborhood of the neutral element of G (this statement is sometimes called the Banach–Kuratowski–Pettis theorem; see, e.g., [107], [149]).

Deduce from this result that if A and B are any two subsets of G such that $\{A, B\} \subset \mathcal{B}a(G) \setminus \mathcal{K}(G)$, then the set $A \cdot B = \{a \cdot b : a \in A,\ b \in B\}$ has nonempty interior.

2. Let $(G, \cdot)$ be a σ-compact locally compact topological group with the neutral element e and let μ be the left invariant Haar measure on G. Denote by μ' the usual completion of μ and let X be an arbitrary μ'-measurable subset of G.

Starting with the fact that μ is a Radon measure, prove that

$$\lim_{g \to e} \mu'((g \cdot X) \cap X) = \mu'(X).$$

In particular, if $\mu'(X) > 0$, then there exists a neighborhood $U(e)$ of e such that

$$(\forall g \in U(e))(\mu'((g \cdot X) \cap X) > 0)$$

and, consequently,

$$(\forall g \in U(e))((g \cdot X) \cap X \neq \emptyset).$$

Conclude from this fact that if A and B are any two μ'-measurable subsets of G with $\mu'(A) > 0$ and $\mu'(B) > 0$, then the set $A \cdot B$ has nonempty interior.

3*. Prove that there exists a measure μ on the real line $\mathbf{R}$, satisfying the following five conditions:

(a) μ is a nonzero nonatomic complete σ-finite measure invariant under the group of all isometric transformations of $\mathbf{R}$;

(b) $\mathrm{dom}(\lambda) \subset \mathrm{dom}(\mu)$, where λ denotes the standard Lebesgue measure on $\mathbf{R}$;

(c) $(\forall Y \in \mathrm{dom}(\lambda))(\lambda(Y) = 0 \Rightarrow \mu(Y) = 0)$;

(d) $(\forall Y \in \mathrm{dom}(\lambda))(\lambda(Y) > 0 \Rightarrow \mu(Y) = +\infty)$;

(e) there exists a Vitali set X such that $X \in \mathrm{dom}(\mu)$.

Remark 2. Since μ is complete and σ-finite, we can consider a von Neumann topology $\mathcal{T}(\mu)$ associated with μ. Let $\mathbf{R}_\mu$ denote the set of all real numbers, equipped with $\mathcal{T}(\mu)$. Then the σ-ideal $\mathcal{K}(\mathbf{R}_\mu)$ and the σ-algebra $\mathcal{B}a(\mathbf{R}_\mu)$ are invariant under the group of all translations of $\mathbf{R}$ and the Vitali set X mentioned in (e) belongs to $\mathcal{B}a(\mathbf{R}_\mu)$, i.e., possesses the Baire property with respect to $\mathcal{T}(\mu)$.

4. Check that, for an arbitrary measure ν on $\mathbf{R}$ invariant under the group $(\mathbf{Q}, +)$ of all rational numbers and extending the measure λ, no Vitali subset of $\mathbf{R}$ is ν-measurable.

5· Work in **ZF** & **DC** theory and demonstrate that if there exists a totally imperfect subset of $\mathbf{R}$ of cardinality $\mathbf{c}$, then there exists a Lebesgue nonmeasurable subset of $\mathbf{R}$.

Prove also the analogous fact for the Baire property.

6. Let n be a natural number greater than or equal to 2, and let X be a totally imperfect subset of the n-dimensional Euclidean space $\mathbf{R}^n$.

Show that the set $\mathbf{R}^n \setminus X$ is connected (in the usual topological sense).

Infer from this fact that any Bernstein subset of $\mathbf{R}^n$ is connected.

7*. Let E be an infinite set and let $\{X_j : j \in J\}$ be a family of subsets of E such that

(a) $\mathrm{card}(J) \leq \mathrm{card}(E)$;

(b) $(\forall j \in J)(\mathrm{card}(X_j) = \mathrm{card}(E))$.

Prove, by applying the method of transfinite induction, that there exists a family $\{Y_j : j \in J\}$ of subsets of E, satisfying these two relations:

(c) $(\forall j \in J)(\forall j' \in J)(j \neq j' \Rightarrow Y_j \cap Y_{j'} = \emptyset)$;

(d) $(\forall j \in J)(\forall j' \in J)(\mathrm{card}(X_j \cap Y_{j'}) = \mathrm{card}(E))$.

In particular, conclude that there exists a disjoint family $\{Z_j : j \in J\}$ of subsets of E such that

$$Z_j \subset X_j, \quad \mathrm{card}(Z_j) = \mathrm{card}(X_j) \qquad (j \in J).$$

8. By starting with the result of the previous exercise, show that in every complete metric space E of cardinality continuum (hence, in every uncountable Polish topological space) there exists a Bernstein set.

Moreover, demonstrate that there exists a partition $\{Y_j : j \in J\}$ of E such that

(a) $\mathrm{card}(J) = \mathbf{c}$;

(b) for each $j \in J$, the set Y_j is a Bernstein subset of E.

Finally, show that if the space E has no isolated points, then all Bernstein subsets of E do not possess the Baire property.

9. Let E be an uncountable Polish space and let X be a subset of E.

Check that the following two assertions are equivalent:

(a) X is a Bernstein subset of E;

(b) for every nonzero σ-finite diffused Borel measure μ given on E, the set X is nonmeasurable with respect to the completion of μ.

Show also that, in general, these two assertions are not equivalent for a nonseparable complete metric space E.

10*. Consider the first uncountable ordinal number ω_1 equipped with its order topology, and put

$$\mathcal{I} = \{X \subset \omega_1 \ : \ (\exists F \subset \omega_1)(F \text{ is closed, } \mathrm{card}(F) = \omega_1, \ F \cap X = \emptyset)\}.$$

Verify that $\mathcal{I}$ is a σ-ideal of subsets of ω_1.

The elements of $\mathcal{I}$ are usually called nonstationary subsets of ω_1.

A set $Z \subset \omega_1$ is called a stationary subset of ω_1 if Z is not nonstationary.

Define $\mathcal{S} = \mathcal{I} \cup \mathcal{I}'$, where $\mathcal{I}'$ is the δ-filter dual to $\mathcal{I}$.

Observe that $\mathcal{S}$ is the σ-algebra generated by $\mathcal{I}$ and demonstrate that, for any set $X \subset \omega_1$, the following two properties are equivalent:

(a) the sets X and $\omega_1 \setminus X$ are stationary in ω_1;

(b) for every nonzero σ-finite diffused measure μ with $\mathrm{dom}(\mu) = \mathcal{S}$, the set X is not measurable with respect to the completion of μ.

Remark 3. Any set X with the above-mentioned properties can be considered as an analogue (for the topological space ω_1) of a Bernstein subset of $\mathbf{R}$. For more information about the σ-ideal $\mathcal{I}$ and stationary subsets of ω_1, see [18], [97], [103], [136], and [146].

The next exercise assumes that the reader is familiar with the notion of a complete Boolean algebra (for the definition, see, e.g., [52] or [153]).

11. Let $\mathcal{P}(\mathbf{R})$ denote the complete Boolean algebra of all subsets of the real line $\mathbf{R}$. Let $\mathcal{I}_0$ be the σ-ideal of all Lebesgue measure zero subsets of

$\mathbf{R}$ and let $\mathcal{I}_1$ be the σ-ideal of all first category subsets of $\mathbf{R}$. Consider the corresponding quotient algebras $\mathcal{P}(\mathbf{R})/\mathcal{I}_0$ and $\mathcal{P}(\mathbf{R})/\mathcal{I}_1$.

Prove that these Boolean algebras are not complete (apply Exercise 8 of this chapter).

12. Let E be an uncountable Polish space and let $\mathcal{I}$ be some σ-ideal of subsets of E. One says that $\mathcal{I}$ has a Borel base if, for each set $X \in \mathcal{I}$, there exists a Borel subset Y of E such that $X \subset Y$ and $Y \in \mathcal{I}$.

Supposing that all one-element subsets of E belong to $\mathcal{I}$ and that $\mathcal{I}$ possesses a Borel base, put

$\mathcal{S}(\mathcal{I})$ = the σ-algebra of subsets of E, generated by $\mathcal{B}(E) \cup \mathcal{I}$.

Let X be a Bernstein subset of E (the existence of Bernstein sets in E follows, for instance, from Exercise 8).

Show that $X \notin \mathcal{S}(\mathcal{I})$.

Conclude, in particular, that if μ is a nonzero σ-finite diffused Borel measure on E, then $X \notin \mathrm{dom}(\mu')$ where μ' stands for the completion of μ (cf. Exercise 9).

Formulate and prove the corresponding consequence for the Baire property.

13. Demonstrate that there exists a subset X of $\mathbf{R}$ which is simultaneously a Vitali set and a Bernstein set.

14. By applying the Kuratowski–Ulam theorem, which is a topological analogue of Fubini's classical theorem (see, e.g., [149] or [202]), establish for the Baire property a statement analogous to Theorem 5.

Namely, prove that there exists a function $g : \mathbf{R} \to \mathbf{R}$ such that its graph is thick in the sense of the Baire property, i.e., the graph of g intersects all those subsets of $\mathbf{R}^2$ which possess the Baire property and are not of first category in $\mathbf{R}^2$.

Deduce from this fact that the graph of g does not have the Baire property in $\mathbf{R}^2$ and g does not have the Baire property as a function acting from $\mathbf{R}$ into $\mathbf{R}$.

15*. Theorem 5 with the previous exercise show that there exist functions from $\mathbf{R}$ into $\mathbf{R}$ whose graphs are thick subsets of the plane $\mathbf{R}^2$ (in particular, those graphs are nonmeasurable in the Lebesgue sense or do not possess the Baire property).

On the other hand, demonstrate that there exists a measure μ on $\mathbf{R}^2$ satisfying the following three conditions:

(a) μ is an extension of the Lebesgue measure λ_2;

(b) μ is invariant under the group of all translations of $\mathbf{R}^2$ and under the central symmetry of $\mathbf{R}^2$ with respect to the origin $(0, 0)$;

(c) the graph of any function acting from $\mathbf{R}$ into $\mathbf{R}$ belongs to $\mathrm{dom}(\mu)$ and, for any such graph Γ, we have $\mu(\Gamma) = 0$.

Remark 4. We thus see that the graphs of all functions acting from $\mathbf{R}$ into $\mathbf{R}$ are small with respect to the above-mentioned measure μ. This is a common property of all functions, which act from $\mathbf{R}$ into $\mathbf{R}$. Another interesting common feature of all functions from $\mathbf{R}$ into $\mathbf{R}$ is described by Blumberg's theorem (see Chapter 8 of this book). Theorem 5 states that there exist functions acting from $\mathbf{R}$ into $\mathbf{R}$, whose graphs are nonmeasurable with respect to λ_2. At the same time, the graphs of such functions may be regarded as small subsets of $\mathbf{R}^2$ with respect to μ.

16. Let X be a Bernstein subset of $\mathbf{R}$, let Y be a λ-measurable set with $\lambda(Y) > 0$, and let Z be a subset of $\mathbf{R}$ having the Baire property but not belonging to $\mathcal{K}(\mathbf{R})$.

Check that the set $X \cap Y$ is not measurable in the Lebesgue sense and that the set $X \cap Z$ does not possess the Baire property.

17. Prove that there exists a Marczewski subset D of $\mathbf{R}^2$ such that

(a) D does not possess the Baire property;

(b) D is not measurable with respect to the completion of the product measure $\mu \otimes \nu$ where μ and ν are any two nonzero σ-finite diffused Borel measures on $\mathbf{R}$.

18. Work in **ZF** & **DC** theory and prove that there exists a Borel isomorphism $\psi : \mathbf{R} \to \mathbf{R}^2$ such that

(a) ψ preserves the category of sets (in particular, ψ preserves the Baire property);

(b) ψ is an isomorphism between the measures λ and λ_2.

By starting with this fact and applying Theorem 6, show that there exists a Marczewski subset of $\mathbf{R}$, nonmeasurable in the Lebesgue sense and without the Baire property.

19. Let K be an arbitrary compact subset of $\mathbf{R}^2$. Obviously, the set $\mathrm{pr}_1(K)$ is compact in $\mathbf{R}$.

Demonstrate that there exists a lower semicontinuous (hence, Borel) mapping $\phi : \mathrm{pr}_1(K) \to \mathbf{R}$ such that the graph of ϕ is contained in K.

In addition, give an example of a compact connected subset P of $\mathbf{R}^2$ for which there exists no continuous mapping $\psi : \mathrm{pr}_1(P) \to \mathbf{R}$ such that the graph of ψ is contained in P.

Remark 5. This simple result is a very special case of much more general statements about the existence of measurable selectors for set-valued mappings measurable in various senses. For example, suppose that a Borel subset B of the plane $\mathbf{R}^2$ is given satisfying the relation $\mathrm{pr}_1(B) = \mathbf{R}$. Can one assert that there exists a Borel function $h : \mathbf{R} \to \mathbf{R}$ whose graph is contained in B? Such a function h is usually called a Borel uniformization of B. Luzin and Novikov (see, e.g., [167]) showed that, in general, the answer to this question is negative, i.e., there exists a Borel subset B of $\mathbf{R}^2$ with $\mathrm{pr}_1(B) = \mathbf{R}$ for which there is no Borel uniformization.

On the other hand, suppose that we have an analytic subset A of $\mathbf{R}^2$ and consider its first projection $\mathrm{pr}_1(A)$ which is an analytic subset of $\mathbf{R}$. Then, according to the classical theorem of Luzin, Jankov, and von Neumann (see, for instance, [105]), there exists a function $g : \mathrm{pr}_1(A) \to \mathbf{R}$ satisfying the following relations:

(1) the graph of g is contained in A;

(2) g is measurable with respect to the σ-algebra generated by the family of all analytic subsets of $\mathbf{R}$.

In particular, one may assert that g has the Baire property in the restricted sense and is measurable with respect to the completion of any σ-finite Borel measure given on $\mathbf{R}$.

The above theorem has numerous applications in real analysis, general topology, and probability theory (some interesting applications of it are presented in [105]; cf. also [57], [154]).

20*. Let K be a compact subset of $\mathbf{R}^2$, all vertical sections of which are at most countable.

Prove that K can be covered by countably many graphs of partial Borel functions acting from $\mathbf{R}$ into $\mathbf{R}$ (for a more general statement, see [167] or [169]).

Applying this result, check that the graph of any function $f : \mathbf{R} \to \mathbf{R}$ of Sierpiński–Zygmund type is a Marczewski subset of the plane $\mathbf{R}^2$.

21*. Using the method of transfinite recursion, construct a Sierpiński–Zygmund function whose graph is a λ_2-thick subset of the plane $\mathbf{R}^2$.

Using the same method, construct a Sierpiński–Zygmund function whose graph is a thick subset of $\mathbf{R}^2$ in the category sense.

Infer from these results that there are λ_2-nonmeasurable Marczewski subsets of $\mathbf{R}^2$ (respectively, there are Marczewski subsets of $\mathbf{R}^2$ without the Baire property).

On the other hand, show that there exists a Sierpiński–Zygmund function whose graph is a λ_2-measure zero subset of $\mathbf{R}^2$.

Also, show that there exists a Sierpiński–Zygmund function whose graph is a first category subset of $\mathbf{R}^2$.

22. Prove that there exists a partition $\{X_i : i \in I\}$ of $\mathbf{R}$ such that

(a) $\mathrm{card}(I) = \mathbf{c}$ and $\mathrm{card}(X_i) = \mathbf{c}$ for all $i \in I$;
(b) every selector of $\{X_i : i \in I\}$ is a Bernstein set in $\mathbf{R}$.

23*. Let $(\mathbf{Q}, +)$ denote, as usual, the subgroup of all rational numbers in the additive group $(\mathbf{R}, +)$.

Demonstrate, within **ZF** & **DC** theory, that if there exists a linear ordering of the quotient set $\mathbf{R}/\mathbf{Q}$, then there exists a λ-nonmeasurable function from $\mathbf{R}$ into $\mathbf{R}$.

For this purpose, argue as follows. Denote $\{V_i : i \in I\} = (\mathbf{R}/\mathbf{Q}) \setminus \{\mathbf{Q}\}$ and suppose that $\preceq$ is a linear order on I. For each index $i \in I$, check that $-V_i = V_j$, where $j = j(i)$ is some (uniquely determined by i) index from I and $i \neq j$. Since the disjunction $i \prec j \ \vee \ j \prec i$ holds true, a function $f : \mathbf{R} \setminus \mathbf{Q} \to \mathbf{R}$ can be defined by putting

$f(x) = 1$ if $x \in V_i$ and $i \prec j$;

$f(x) = 0$ if $x \in V_i$ and $j \prec i$.

Further, verify that

(a) $f(x + q) = f(x)$ for any $x \in \mathbf{R} \setminus \mathbf{Q}$ and $q \in \mathbf{Q}$;

(b) $f(-x) = 1 - f(x)$ for any $x \in \mathbf{R} \setminus \mathbf{Q}$.

Conclude from (a) and (b) that the function f cannot be λ-measurable.

Deduce from the above result that

(*) it is impossible to prove, within **ZF** & **DC** theory, that the family of all countable subsets of $\mathbf{R}$ (even the family $\mathbf{R}/\mathbf{Q}$) admits a linear ordering;

(**) it is impossible to prove, within **ZF** & **DC** theory, that the cardinality of the family of all countable subsets of $\mathbf{R}$ is less than or equal to $\mathbf{c}$;

(***) it is impossible to prove, within **ZF** & **DC** theory, that the Baire class $Ba_2(\mathbf{R}, \mathbf{R})$ has cardinality less than or equal to $\mathbf{c}$.

Chapter 11
Hamel basis and Cauchy functional equation

In this chapter we would like to discuss some properties of Hamel bases of the real line $\mathbf{R}$ and highlight their remarkable role in various constructions of strange additive functions acting from $\mathbf{R}$ into itself.

The existence of such a basis was first established by Hamel in 1905 (see [87]). Later, it was also shown that the existence of a Hamel basis cannot be proved without the aid of uncountable forms of the Axiom of Choice (in particular, it is impossible to establish the existence of such a basis within **ZF** & **DC** theory).

The construction of a Hamel basis can be done by starting with one general theorem of the theory of vector spaces (over arbitrary fields). Let us recall that, according to this general theorem, for every vector space E, there exists a basis of E, i.e., a maximal (with respect to inclusion) linearly independent subset of E. This assertion follows almost immediately from the Zorn lemma or, equivalently, from the Axiom of Choice (**AC**). So, each element e of E can be represented as a linear combination of finitely many elements of the basis and such a representation of e is unique.

Let us now consider the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers. Then, applying the above-mentioned general theorem, we get that there are bases of $\mathbf{R}$ over $\mathbf{Q}$.

Any such basis is called a Hamel basis of $\mathbf{R}$.

Let $\{e_i : i \in I\}$ be a Hamel basis of $\mathbf{R}$. It is clear that $\mathrm{card}(I) = \mathbf{c}$. Every element x of $\mathbf{R}$ can be uniquely represented in the form

$$x = \sum_{i \in I} q_i(x) \cdot e_i,$$

where $\{q_i(x) : i \in I\}$ is some indexed family of rational numbers, such that

$$\mathrm{card}(\{i \in I \ : \ q_i(x) \neq 0\}) < \omega.$$

For each index $i \in I$, the rational number $q_i(x)$ is called the coordinate of x corresponding to this index.

From the purely group-theoretical point of view, this representation simply means that the additive group $(\mathbf{R}, +)$ can be regarded as the direct sum of a family of groups $\{Q_i : i \in I\}$, satisfying the following conditions:

(1) $\mathrm{card}(I) = \mathbf{c}$;

(2) for each index $i \in I$, the group Q_i is isomorphic to the additive group $(\mathbf{Q}, +)$ of all rational numbers.

In other words, we may write $\mathbf{R} = \sum_{i \in I} Q_i$.

Clearly, an analogous representation will be true for the n-dimensional Euclidean space $\mathbf{R}^n$, where $n \geq 1$, for any infinite-dimensional separable Hilbert space and, more generally, for an arbitrary vector space over $\mathbf{Q}$ having cardinality $\mathbf{c}$.

Thus, from the group-theoretical point of view, all these spaces are isomorphic (i.e., all of them are isomorphic to the group $\sum_{i \in I} Q_i$).

The above observation seems to be quite trivial. However, by using some additional argument, it yields highly nontrivial consequences.

We wish to consider here one of such consequences.

Recall that the existence of Lebesgue nonmeasurable subsets of $\mathbf{R}$ was thoroughly discussed in the previous chapter of this book. More precisely, we discussed therein the two classical constructions of Lebesgue nonmeasurable sets, due to Vitali and Bernstein, respectively. The first of them was based on certain algebraic properties of the Lebesgue measure λ, namely, on the invariance of this measure under the group of all translations of $\mathbf{R}$, and the second one was based on some topological properties of λ (actually, we essentially utilized the fact that λ is a Radon measure).

Also, it was mentioned in the previous chapter that Vitali's construction cannot be generalized to the class of all nonzero σ-finite measures given on $\mathbf{R}$ and quasi-invariant with respect to the group of all translations of $\mathbf{R}$ (we recall that a measure μ given on $\mathbf{R}$ is quasi-invariant under the group of all translations of $\mathbf{R}$ if $\mathrm{dom}(\mu)$ and the σ-ideal generated by the family of all μ-measure zero sets are invariant under this group). In other words, the argument used in Vitali's construction does not work for nonzero σ-finite translation quasi-invariant measures on $\mathbf{R}$.

Let μ be a nonzero σ-finite measure on $\mathbf{R}$ quasi-invariant with respect to the group of all translations of $\mathbf{R}$.

For this μ, the following two closely related problems arise:

1. Does there exist a subset of $\mathbf{R}$ nonmeasurable with respect to μ?

2. Does there exist a subgroup of $\mathbf{R}$ nonmeasurable with respect to μ?

Our purpose is to establish that the answer to the second question is positive (consequently, the answer to the first question turns out to be positive, too). In the process of establishing this fact we shall substantially exploit the existence of a Hamel basis for **R** (hence an appropriate uncountable form of the Axiom of Choice will be assumed automatically).

In our further consideration, we need several auxiliary statements on commutative groups. These statements are not difficult to prove. We shall see that they directly lead us to the required result.

Lemma 1. *Let $(G, +)$ be a commutative group, G_0 be a subgroup of G, and let μ be a σ-finite G-quasi-invariant measure on G. Assume also that*

(1) the quotient group G/G_0 is uncountable;

(2) $\mu^*(G_0) > 0$ where μ^* denotes, as usual, the outer measure associated with the initial measure μ.

Then G_0 is nonmeasurable with respect to μ.

Proof. Suppose for a moment that $G_0 \in \mathrm{dom}(\mu)$. Then, according to condition (2), we must have the inequality $\mu(G_0) > 0$. On the other hand, according to condition (1), there is an uncountable family of pairwise disjoint G-translates of G_0 in G. Taking account of the σ-finiteness and G-quasi-invariance of our measure μ, we infer that the equality $\mu(G_0) = 0$ must be true, which contradicts the above-written inequality. The contradiction obtained finishes the proof of Lemma 1.

Exercise 2 of this chapter presents a slightly more general result than the preceding lemma.

Lemma 2. *Let $(G, +)$ be a commutative group and let $\{G_n : n \in \omega\}$ be a countable family of subgroups of G. Assume that the following two conditions are satisfied:*

(1) for each natural number n, the quotient group G/G_n is uncountable;

(2) $\cup\{G_n : n \in \omega\} = G$.

Further, let μ be a nonzero σ-finite G-quasi-invariant measure on G. Then there exists at least one $n \in \omega$ such that the corresponding group G_n is nonmeasurable with respect to μ.

Proof. Because μ is not identically equal to zero and the given family of subgroups $\{G_n : n \in \omega\}$ is a countable covering of G, there exists a natural number n such that $\mu^*(G_n) > 0$. Now, applying Lemma 1 to the subgroup G_n, we see that this subgroup is necessarily nonmeasurable with respect to μ. Lemma 2 has thus been proved.

Lemma 3. *Let $(G, +)$ be a commutative group, H be a proper subgroup of G, and let I be an infinite set of indices. Consider two direct sums of groups $\sum_{i \in I} G_i$ and $\sum_{i \in I} H_i$ where, for each index $i \in I$, the group G_i*

coincides with G and the group H_i coincides with H. Then we have

$$\mathrm{card}((\sum_{i\in I} G_i)/(\sum_{i\in I} H_i)) \geq \mathrm{card}(I).$$

In particular, if the set I is uncountable, then the quotient group

$$(\sum_{i\in I} G_i)/(\sum_{i\in I} H_i)$$

is uncountable, too.

Proof. Since H is a proper subgroup of G, there exists at least one element $g \in G \setminus H$. Denote by the symbol D the family of all those elements $\{g_i : i \in I\} \in \sum_{i\in I} G_i$ which satisfy $(\forall i \in I)(g_i \neq 0 \Rightarrow g_i = g)$. Obviously, we have $\mathrm{card}(D) = \mathrm{card}(I)$. Also, it is not hard to verify that if d' and d'' are any two distinct elements from D, then $d' - d'' \notin \sum_{i\in I} H_i$. Consequently, we obtain the required inequality

$$\mathrm{card}((\sum_{i\in I} G_i)/(\sum_{i\in I} H_i)) \geq \mathrm{card}(I),$$

and the proof of Lemma 3 is completed.

Lemma 4. *The additive group $(\mathbf{R}, +)$ can be represented in the form*

$$\mathbf{R} = \cup\{G_n \ : \ n \in \omega\},$$

where $\{G_n : n \in \omega\}$ is a countable family of subgroups of $\mathbf{R}$ and, for each $n \in \omega$, the quotient group $\mathbf{R}/G_n$ is uncountable.

Proof. First, let us consider the additive group $(\mathbf{Q}, +)$ and show that this group can be represented as the union of a countable increasing (with respect to inclusion) family of its proper subgroups. Indeed, for any natural number n, define $Q^{(n)} = \{k/n! \ : \ k \in \mathbf{Z}\}$, where $\mathbf{Z}$ stands for the set of all integers. Evidently, $Q^{(n)}$ is a proper subgroup of $\mathbf{Q}$. Moreover, we may write

$$(\forall n)(\forall m)(n \leq m < \omega \Rightarrow Q^{(n)} \subset Q^{(m)}),$$

i.e., the family of groups $\{Q^{(n)} \ : \ n < \omega\}$ is increasing with respect to inclusion. Also, it is clear that $\mathbf{Q} = \cup\{Q^{(n)} \ : \ n < \omega\}$.

As demonstrated earlier (with the aid of a Hamel basis), we have a representation of $\mathbf{R}$ in the form of a direct sum $\mathbf{R} = \sum_{i\in I} Q_i$, where I is a set of indices with $\mathrm{card}(I) = \mathbf{c}$ and, for each index $i \in I$, the group Q_i

coincides with $\mathbf{Q}$. Now, for any natural number n, we put $G_n = \sum_{i \in I} Q_i^{(n)}$. It can easily be shown, by applying Lemma 3, that the family $\{G_n : n \in \omega\}$ of subgroups of $(\mathbf{R}, +)$ is the required one, and this finishes the proof of Lemma 4.

The preceding lemmas immediately give us the following generalization of Vitali's classical theorem.

Theorem 1. *For any nonzero σ-finite measure μ on $\mathbf{R}$ which is quasi-invariant under the group of all translations of $\mathbf{R}$, there exists a subgroup of $(\mathbf{R}, +)$ nonmeasurable with respect to μ.*

Exercise 3 of this chapter yields a more general result.

Let us return to Hamel bases of the real line $\mathbf{R}$.

These bases were found not accidentally but as a tool for solving a concrete question in mathematical analysis. Now, we are going to formulate this question and discuss its close relationships with Hamel bases of $\mathbf{R}$.

Consider the class of all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying the following functional equation:

$$f(x + y) = f(x) + f(y) \qquad (x \in \mathbf{R},\ y \in \mathbf{R}),$$

which is usually called Cauchy's functional equation. Notice that this equation simply says that f is a homomorphism acting from the additive group $(\mathbf{R}, +)$ into itself. Also, it is easy to see that any homomorphism f from the group $(\mathbf{R}, +)$ into itself satisfies the relation $f(qx) = q \cdot f(x)$ for each $x \in \mathbf{R}$ and for each $q \in \mathbf{Q}$. In other words, f can be regarded as a linear mapping when $(\mathbf{R}, +)$ is treated as a vector space over the field $\mathbf{Q}$.

The problem is to find all solutions of Cauchy's functional equation.

It is clear that there are very natural solutions of this equation. Namely, every function $f : \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = a \cdot x \qquad (x \in \mathbf{R}),$$

where a is an arbitrary real number, is a solution of this equation.

In the following considerations, we shall call such solutions trivial ones.

Let us stress that any continuous solution of the Cauchy functional equation is trivial. Moreover, a much stronger result will be presented below stating that any Lebesgue measurable (or having the Baire property) solution of the Cauchy functional equation is necessarily trivial (see Theorem 4 of this chapter).

Hamel bases allow us to construct nontrivial solutions of this equation. Namely, we have the following statement due to Hamel (see [87]).

Theorem 2. *There are nontrivial solutions of the Cauchy functional equation.*

Proof. Let $\{e_i : i \in I\}$ be a Hamel basis of $\mathbf{R}$. As was noticed earlier, each $x \in \mathbf{R}$ can be uniquely represented in the form $x = \sum_{i \in I} q_i(x) \cdot e_i$. Fix an index $i_0 \in I$ and define a function $\phi : \mathbf{R} \to \mathbf{R}$ by the following formula:

$$\phi(x) = q_{i_0}(x) \qquad (x \in \mathbf{R}).$$

Since the function ϕ is additive, it is clear that ϕ satisfies Cauchy's functional equation. Moreover, the range of this function is contained in $\mathbf{Q}$. Also, this function is not constant because of $\phi(0) = 0$ and $\phi(e_{i_0}) = 1$. So ϕ is not a continuous function, and we shall see later that ϕ is not even Lebesgue measurable and does not possess the Baire property. Theorem 2 has thus been proved.

The next statement yields a characterization of nontrivial solutions of the Cauchy functional equation in terms of their graphs.

Theorem 3. *Let f be a solution of Cauchy's functional equation. Then the following two assertions are equivalent:*

(1) the graph of f is dense everywhere in the plane $\mathbf{R}^2$;

(2) f is a nontrivial solution of Cauchy's functional equation.

Proof. Obviously, if f is a trivial solution of Cauchy's functional equation, then the graph of f, being a straight line, is nowhere dense in $\mathbf{R}^2$. Consequently, if a solution of Cauchy's functional equation has the graph everywhere dense in $\mathbf{R}^2$, then this solution must be nontrivial. We thus see that the implication (1) $\Rightarrow$ (2) is valid.

Let us establish the converse implication (2) $\Rightarrow$ (1). Suppose to the contrary that (2) is satisfied but the graph of f is not dense everywhere in $\mathbf{R}^2$. Then there is a nonempty open rectangle $]a, b[\times]c, d[\ \subset \mathbf{R} \times \mathbf{R} = \mathbf{R}^2$ such that $G_f \cap (]a, b[\times]c, d[) = \emptyset$, where G_f denotes the graph of f. Let us show that at least one of the following two relations holds:

(i) for all $x \in\ ]a, b[$, we have $f(x) \geq c$;

(ii) for all $x \in\ ]a, b[$, we have $f(x) \leq d$.

Indeed, assuming for a moment that there are two points x_1 and x_2 from $]a, b[$ satisfying the inequalities $f(x_1) < c$ and $f(x_2) > d$, we can readily deduce that, for some rational numbers $q_1 > 0$ and $q_2 > 0$ with $q_1 + q_2 = 1$, the relation

$$f(q_1 x_1 + q_2 x_2) = q_1 f(x_1) + q_2 f(x_2) \in\ ]c, d[$$

must be fulfilled, i.e.,

$$(q_1x_1 + q_2x_2, f(q_1x_1 + q_2x_2)) \in \ ]a, b[\times]c, d[,$$

which is impossible. The contradiction obtained yields that at least one of relations (i) and (ii) holds true. We may suppose, without loss of generality, that relation (ii) holds. Now, it is easy to see that there exist a real number $\delta > 0$ and $h \in \mathbf{R}$, such that $]-\delta, \delta[\ + \ h \subset \ ]a, b[$. Taking relation (ii) into account and applying the additivity of our function f, we deduce that f is bounded from above on $]-\delta, \delta[$. Since $f(-x) = -f(x)$ for all $x \in \mathbf{R}$, we also infer that f is bounded on the same interval and, consequently, is continuous at the point $0 \in \mathbf{R}$. Finally, using the additivity of f again, we conclude that f is continuous at all points of $\mathbf{R}$, which contradicts assumption (2). This contradiction establishes the converse implication (2) $\Rightarrow$ (1). Theorem 3 has thus been proved.

Suppose now that a function $f : \mathbf{R} \to \mathbf{R}$ is given satisfying Cauchy's functional equation. Then, as mentioned above, for every $x \in \mathbf{R}$ and for every $q \in \mathbf{Q}$, we have the equality $f(q \cdot x) = q \cdot f(x)$. It immediately follows from this fact that if f is a continuous function at least at one point of $\mathbf{R}$, then $f(x) = f(1) \cdot x$ for each $x \in \mathbf{R}$. This simple result was first established by Cauchy. It is also easy to prove a slightly more general result stating that if f is a solution of Cauchy's functional equation and, in addition, has an upper (or lower) bound on some nonempty open interval, then f is a trivial solution of this equation (cf. the proof of Theorem 3).

Thus, we see that nontrivial solutions of Cauchy's functional equation are very bad from the topological point of view: they are discontinuous at each element of their domain. The next result (due to Frechet) shows that nontrivial solutions of Cauchy's functional equation are also bad from the point of view of the Lebesgue measurability and Baire property.

Theorem 4. *All nontrivial solutions of Cauchy's functional equation are Lebesgue nonmeasurable and do not possess the Baire property.*

Proof. There are many proofs of this remarkable theorem. We shall present here a very simple argument based on the Steinhaus property for the Lebesgue measure (respectively, for the Baire property). Suppose that a function $f : \mathbf{R} \to \mathbf{R}$ is given satisfying Cauchy's functional equation, and suppose to the contrary that this f is Lebesgue measurable (respectively, possesses the Baire property). Consider the sets of the form $f^{-1}([-n, n])$, where $n \in \omega$. All these sets are Lebesgue measurable (respectively, have the Baire property) and $\cup\{f^{-1}([-n, n]) : n \in \omega\} = \mathbf{R}$. Hence there exists a natural number n such that $\lambda(f^{-1}([-n, n])) > 0$ or, respectively,

$f^{-1}([-n,n]) \in \mathcal{B}a(\mathbf{R}) \setminus \mathcal{K}(\mathbf{R})$. Now, the Steinhaus property for the Lebesgue measure (respectively, its analogue for the Baire property) implies that the set $f^{-1}([-n,n]) - f^{-1}([-n,n])$ is a neighborhood of the point $0 \in \mathbf{R}$. Let now V be any open interval in $\mathbf{R}$ such that

$$0 \in V \subset f^{-1}([-n,n]) - f^{-1}([-n,n]).$$

Then we obviously get $f(V) \subset [-2n, 2n]$, so the function f is bounded on V. From this fact it immediately follows that f is continuous at point 0. Hence f is a trivial solution of Cauchy's functional equation, and the proof of Theorem 4 is completed.

As mentioned earlier, there are also many other proofs of the preceding theorem which are based on essentially different ideas and methods. For example, one purely analytic proof of this theorem was suggested by Orlicz (see [200]). However, the argument presented above seems to be more natural and can be applied in situations where a given Lebesgue measurable function (or a function having the Baire property) is not necessarily additive (see, for instance, the proof of Theorem 5 below).

Considering Theorems 2 and 4, we can conclude that the existence of a Hamel basis in $\mathbf{R}$ implies the existence of a subset of $\mathbf{R}$ which is not Lebesgue measurable and does not have the Baire property. More exactly, we see that the proof of this implication can be done within **ZF** & **DC** theory.

Let us briefly discuss the weakest form of so-called Jensen's inequality, which plays a basic role in the theory of real-valued convex functions and is closely connected with Cauchy's functional equation.

Let $]a,b[$ be an open subinterval of $\mathbf{R}$ (here we do not exclude the cases $a = -\infty$ or $b = +\infty$).

We recall that a function $f : \ ]a,b[\ \to \mathbf{R}$ satisfies Jensen's inequality (or is midpoint convex) if $f((x+y)/2) \leq (f(x)+f(y))/2$ for all $x \in \ ]a,b[$ and $y \in \ ]a,b[$.

The standard form of Jensen's inequality (often used in classical mathematical analysis) is

$$f(q_1 x + q_2 y) \leq q_1 f(x) + q_2 f(y)$$

for all $x \in \ ]a,b[$ and $y \in \ ]a,b[$ and for all positive real numbers q_1 and q_2 whose sum is equal to 1.

It is widely known that the above form of Jensen's inequality describes the class of all convex functions acting from $]a,b[$ into $\mathbf{R}$. This class is very important from the point of view of applications. Also, functions belonging to this class have nice properties. For instance, they are continuous and possess a derivative almost everywhere. Moreover, a well-known theorem from mathematical analysis states that any convex function possesses a derivative at all points of $]a, b[$ except countably many of them, and possesses also a second derivative at almost all points of $]a, b[$. Indeed, the derivative of a convex function is monotone, so the desired result follows at once from the Lebesgue theorem on the differentiability almost everywhere of any monotone function (see Theorem 2 of Chapter 4).

At the same time, the weakest form of Jensen's inequality written above does not restrict essentially the class of admissible functions. Indeed, if f is any solution of Cauchy's functional equation, then f obviously satisfies the weakest form of Jensen's inequality. But we know that f may be a nontrivial solution of Cauchy's functional equation and, in this case, f is Lebesgue nonmeasurable (and does not possess the Baire property).

In this connection, the following result due to Sierpiński [239] and generalizing Theorem 4 is of some interest.

Theorem 5. *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function (respectively, a function with the Baire property) satisfying the inequality*

$$f((x+y)/2) \leq (f(x)+f(y))/2 \qquad (x \in \mathbf{R},\ y \in \mathbf{R}).$$

Then f is continuous and, consequently, f is a convex function in the usual sense.

Proof. The argument is analogous to that used in the proof of Theorem 4. However, some delicate moments occur here.

For any natural number n, consider the set $X_n = f^{-1}(]-\infty, n])$. According to our assumption, all these sets are measurable in the Lebesgue sense (respectively, possess the Baire property). Because

$$\mathbf{R} = \cup\{X_n \ :\ n \in \omega\},$$

at least one of these sets is not of Lebesgue measure zero (respectively, is not of first category). Let X_m be such a set. Then the Steinhaus property implies that the vector sum $Z_m = X_m + X_m$ has nonempty interior and hence contains some nondegenerate open interval Δ. Now, the inequality

$$f((x+y)/2) \leq (f(x)+f(y))/2 \qquad (x \in X_m,\ y \in X_m)$$

immediately yields that $f(t) \leq m$ for all $t \in\ \Delta/2$, i.e., our function f is bounded from above on the interval $\Delta/2$. Further, the inequality

$$f(t+r) \leq (f(2t)+f(2r))/2 \qquad (t \in \Delta/4,\ r \in \mathbf{R})$$

shows that f is bounded from above on the interval $\Delta/4 + r$ where $r \in \mathbf{R}$. Consequently, f is bounded from above on each subinterval of $\mathbf{R}$ with the compact closure and, actually, f turns out to be locally bounded on $\mathbf{R}$.

Let now t be an arbitrary point of $\mathbf{R}$ and let h be a nonzero real number such that $|h| < 1$. Obviously, we may write

$$f(t + h) - f(t) \leq (f(t + 2h) - f(t))/2.$$

From this inequality, applying an easy induction on k, we get

$$f(t + h) - f(t) \leq (f(t + 2^k h) - f(t))/2^k.$$

But it is clear that, for some $k \in \mathbf{N}$, we have $1/2^{k+1} \leq |h| < 1/2^k$. If we denote

$$L_0 = L_0(t, f) = \sup_{x \in [t-1,t+1]} f(x) < +\infty,$$

then the preceding inequalities give us

$$f(t + h) - f(t) \leq (L_0 - f(t)) \cdot 2|h| = L_1 \cdot |h|,$$

where $L_1 = L_1(t, f) \geq 0$ is some constant depending only on f and t.

Using an analogous argument, we easily come to the relation

$$f(t) - f(t + h) \leq L_1 \cdot |h|.$$

Actually, it suffices to apply the special case of Jensen's inequality

$$f(t) - f(t + h) \leq f(t - h) - f(t)$$

and to keep in mind that

$$f(t - h) - f(t) \leq L_1 \cdot | - h| = L_1 \cdot |h|.$$

Finally, we obtain the relation $|f(t + h) - f(t)| \leq L_1 \cdot |h|$, which trivially implies that the function f is continuous at t. Hence f is continuous at all points of $\mathbf{R}$ (because t was taken arbitrarily from $\mathbf{R}$). In other words, f is a convex function in the usual sense. Theorem 5 has thus been proved.

A detailed account of Cauchy's functional equation and of the weakest form of Jensen's inequality is presented in the extensive monograph by Kuczma [145].

There are also many textbooks and monographs devoted to the theory of convex sets and convex functions (this theory is usually called convex analysis). A few additional facts about convex functions, interesting from the purely analytical point of view, are presented in Exercises 12, 13, and 14 of this chapter.

Let f be a function acting from $\mathbf{R}$ into $\mathbf{R}$ and let x be a point of $\mathbf{R}$.

Recall that f is symmetric with respect to x if $f(x-t) = f(x+t)$ for all $t \in \mathbf{R}$.

In other words, f is symmetric with respect to x if its graph is a symmetric subset of $\mathbf{R}^2$ with respect to the straight line $\{x\} \times \mathbf{R}$.

We shall say that a function $f : \mathbf{R} \to \mathbf{R}$ is almost symmetric with respect to $x \in \mathbf{R}$ if $\mathrm{card}(\{t \in \mathbf{R} \ : \ f(x-t) \neq f(x+t)\}) < \mathbf{c}$.

It is easy to see that if f is symmetric with respect to each $x \in \mathbf{R}$, then f is constant.

At the same time, by using a special Hamel basis of $\mathbf{R}$, it can be proved that there exist almost symmetric functions with respect to all $x \in \mathbf{R}$ which are not constant. This result is due to Sierpiński (see [242]). More precisely, Sierpiński established that the following statement is true.

Theorem 6. *There exists a Lebesgue nonmeasurable (and lacking the Baire property) function $f : \mathbf{R} \to \mathbf{R}$ which is almost symmetric with respect to all points of $\mathbf{R}$. In particular, such a function is not equivalent to a constant function, i.e., there exists no set $Z \subset \mathbf{R}$ of Lebesgue measure zero (of first category) such that the restriction of f to $\mathbf{R} \setminus Z$ is constant.*

Proof. Let α denote the least ordinal number for which $\mathrm{card}(\alpha) = \mathbf{c}$. Let $\{P_\xi : \xi < \alpha\}$ be an enumeration of all nonempty perfect subsets of $\mathbf{R}$. We may assume without loss of generality that each of the two partial families

$$\{P_\xi : \xi < \alpha, \ \xi \text{ is an even ordinal}\}, \quad \{P_\xi : \xi < \alpha, \ \xi \text{ is an odd ordinal}\}$$

consists of all nonempty perfect subsets of $\mathbf{R}$, too.

Now, by applying the method of transfinite recursion, it is not hard to define a family $\{e'_\xi \ : \ \xi < \alpha\}$ of elements of $\mathbf{R}$ such that

(a) $e'_\xi \in P_\xi$ for each $\xi < \alpha$;

(b) $\{e'_\xi \ : \ \xi < \alpha\}$ is linearly independent over $\mathbf{Q}$.

This construction is very similar to the classical Bernstein construction (see Chapter 10), so we leave the corresponding details to the reader.

Further, we can extend the family $\{e'_\xi : \xi < \alpha\}$ to some Hamel basis of $\mathbf{R}$. This Hamel basis will be denoted by the symbol $\{e_\xi : \xi < \alpha\}$. Obviously, we may also suppose that each of the sets

$$E_0 = \{e_\xi : \xi < \alpha, \ \xi \text{ is an even ordinal}\},$$

$$E_1 = \{e_\xi : \xi < \alpha, \ \xi \text{ is an odd ordinal}\}$$

is a Bernstein subset of $\mathbf{R}$ (hence, these two sets are nonmeasurable in the Lebesgue sense and do not possess the Baire property).

Now, for any real number r, we have a unique representation of r in the form

$$r = q_1 e_{\xi_1} + q_2 e_{\xi_2} + ... + q_n e_{\xi_n},$$

where $n = n(r)$ is a natural number, all $q_1, \ q_2, ..., \ q_n$ are nonzero rational numbers, and $\xi_1 < \xi_2 < ... < \xi_n < \alpha$. We define a mapping

$$\phi \ : \ \mathbf{R} \setminus \{0\} \to E_0 \cup E_1$$

by the formula

$$\phi(r) = e_{\xi_n} \qquad (r \in \mathbf{R} \setminus \{0\}).$$

Further, we put $X = \{r \in \mathbf{R} : \phi(r) \in E_0\}$ and denote by f the characteristic function of X.

Clearly, f is Lebesgue nonmeasurable and does not possess the Baire property. Fix a point $x \in \mathbf{R}$ and let t be an arbitrary point of $\mathbf{R} \setminus \{0\}$. If $x \neq 0$, then we can write $\phi(x) = e_\xi$ and $\phi(t) = e_\zeta$ for some ordinal numbers $\xi < \alpha$ and $\zeta < \alpha$. If $\xi < \zeta$, then we have the relation

$$x - t \in X \Leftrightarrow x + t \in X.$$

This relation is also true for $x = 0$. Keeping in mind the equality

$$\mathrm{card}(\mathrm{lin}(\{e_\zeta \ : \ \zeta \leq \xi\})) = \mathrm{card}(\xi) + \omega,$$

where $\mathrm{lin}(\{e_\zeta \ : \ \zeta \leq \xi\})$ denotes the vector space (over $\mathbf{Q}$) generated by $\{e_\zeta \ : \ \zeta \leq \xi\}$, we obtain that

$$\mathrm{card}(\{t \in \mathbf{R} \ : \ f(x - t) \neq f(x + t)\}) < \mathbf{c},$$

i.e., our function f turns out to be almost symmetric with respect to x.

Since x was taken arbitrarily, we conclude that f is an almost symmetric function with respect to all points of $\mathbf{R}$. Theorem 6 has thus been proved.

Some additional information around Theorem 6 can be found in [74].

Hamel bases have many other interesting and important applications, and not only in analysis. One of the most beautiful applications may be found in geometry of Euclidean space, more precisely, in the theory of polyhedra lying in this space. We mean here that part of this theory, which is connected with Hilbert's third problem about the nonequivalence (by a finite decomposition) of a three-dimensional unit cube and a regular three-dimensional simplex of volume 1. Hilbert's third problem will be briefly discussed in Exercise 17 of this chapter. It is reasonable to mention, in this connection, that Hamel was one of Hilbert's many students and worked in the geometry of Euclidean space, as well.

EXERCISES

1. Let E be a vector space (over an arbitrary field).

Show that any two bases of E have the same cardinality (their common cardinality is called the algebraic (or linear) dimension of the space E).

Remark 1. Notice that the result presented in Exercise 1 is a very special case of the general theorem on the cardinality of any system of free generators of a free universal algebra (in this connection, see, e.g., [52]).

2. Let $(G, \cdot)$ be an arbitrary group and let X be a subset of G. Suppose that G is equipped with a σ-finite left (respectively, right) G-quasi-invariant measure μ and suppose also that these two relations are fulfilled:

(a) there exists an uncountable family of pairwise disjoint left (respectively, right) G-translates of X in G;

(b) $\mu^*(X) > 0$.

Show that the set X is nonmeasurable with respect to μ.

3. Let E be an uncountable vector space over the field $\mathbf{Q}$.

Prove that there exists a countable family $\{E_n : n \in \omega\}$ of vector subspaces of E, satisfying the following conditions:

(a) $\cup\{E_n : n \in \omega\} = E$;

(b) for each $n \in \omega$, the inequality $\mathrm{card}(E/E_n) > \omega$ holds true.

Infer from (a) and (b) that, for any nonzero σ-finite measure μ on E quasi-invariant under the group of all translations of E, there exists at least one $n \in \omega$ for which E_n is nonmeasurable with respect to μ.

Remark 2. Theorem 1 and Exercise 3 above can be significantly generalized. More precisely, it can be proved that, for every uncountable commutative group $(G, +)$ and for any nonzero σ-finite G-quasi-invariant measure ν on G, there exists a subgroup of G nonmeasurable with respect to ν. Notice that, in order to obtain this result, one needs some deep theorems concerning the algebraic structure of infinite commutative groups (for more details, see [122] and [129]). Notice also that similar statements are valid (for uncountable commutative groups again) in terms of the Baire property (see [122]). On the other hand, there are examples of uncountable noncommutative groups for which analogous results fail to be true.

4. Show that there exists a nontrivial automorphism of the additive group $(\mathbf{R}, +)$ onto itself and, more generally, demonstrate that the number of all such automorphisms is equal to $2^{\mathbf{c}}$.

5*. Let $(E, ||\cdot||)$ be an arbitrary normed vector space over the field of all real numbers (or over the field of all complex numbers) and let E have an infinite algebraic dimension.

Prove that there exists a linear functional defined on E and discontinuous at every point of E.

Remark 3. It is useful to compare the above result with the well-known fact that every linear functional defined on a finite-dimensional normed vector space is continuous.

6. Let $\{e_i : i \in I\}$ be a Hamel basis of $\mathbf{R}$. Fix an index $i_0 \in I$ and put

$\Gamma =$ the vector space (over $\mathbf{Q}$) generated by $\{e_i \ : \ i \in I \setminus \{i_0\}\}$.

In other words, Γ is a hyperplane in $\mathbf{R}$ regarded again as a vector space over $\mathbf{Q}$.

Work in **ZF** & **DC** theory and show that

(a) $\mathbf{R}$ can be covered by a countable family of translates of Γ;

(b) Γ does not have the Steinhaus property, i.e., for each real $\varepsilon > 0$, there exists a number $q \in \ ]-\varepsilon, \varepsilon[$ such that $(q + \Gamma) \cap \Gamma = \emptyset$.

Deduce from relations (a) and (b) that Γ is not measurable in the Lebesgue sense and does not possess the Baire property.

Conclude from this result (in the same **ZF** & **DC** theory) that the existence of a Hamel basis in $\mathbf{R}$ implies the existence of a subset of $\mathbf{R}$ which is nonmeasurable in the Lebesgue sense and does not possess the Baire property.

7. By using the theorem on the existence of a Hamel basis of $\mathbf{R}$, describe all solutions of Cauchy's functional equation.

8. Describe all continuous functions $f : \mathbf{R} \to \mathbf{R}$ that satisfy the following functional equation:

$$f(x + y) = f(x) \cdot f(y) \qquad (x \in \mathbf{R}, \ y \in \mathbf{R}).$$

Show that there are discontinuous (and Lebesgue nonmeasurable) solutions of this functional equation and describe all of them.

Also, consider the functional equation

$$f(x + y) = f(x) + f(y) + r \qquad (x \in \mathbf{R}, \ y \in \mathbf{R}),$$

where r is some real constant, and describe all solutions of this equation.

Remark 4. One can prove that there are Lebesgue measurable and Lebesgue nonmeasurable Hamel bases (see Exercise 9 below). This fact underlines an essential difference between Hamel bases and Vitali sets from the point of view of Lebesgue measurability (the analogous fact may be observed for the Baire property).

9*. Demonstrate that there are two sets $A \subset \mathbf{R}$ and $B \subset \mathbf{R}$, both of Lebesgue measure zero and of first category, such that

$$A + B = \{a + b : a \in A, \ b \in B\} = \mathbf{R}.$$

In particular, deduce the equality $(A \cup B) + (A \cup B) = \mathbf{R}$, where the set $A \cup B$ is also of Lebesgue measure zero and of first category (cf. Exercise 19 from Chapter 1).

Starting with this fact and applying the method of transfinite induction, show that there exists a Hamel basis H of $\mathbf{R}$ contained in the set $A \cup B$. Consequently, H is of first category and of Lebesgue measure zero.

In addition, prove that if μ is an arbitrary σ-finite measure on $\mathbf{R}$ invariant (or, more generally, quasi-invariant) under the group of all translations of $\mathbf{R}$ and $\{e_i : i \in I\}$ is an arbitrary Hamel basis in $\mathbf{R}$, then the implication

$$\{e_i : i \in I\} \in \mathrm{dom}(\mu) \Rightarrow \mu(\{e_i : i \in I\}) = 0$$

holds true.

10. By using the method of transfinite induction, demonstrate that there exists a Hamel basis of $\mathbf{R}$ which simultaneously is a Bernstein subset of $\mathbf{R}$ (cf. the proof of Theorem 6).

Conclude that such a Hamel basis is λ-nonmeasurable and does not possess the Baire property (where λ denotes, as usual, the standard Lebesgue measure on $\mathbf{R}$)

Remark 5. In connection with the results presented in the two preceding exercises, the following question arises naturally: is any Hamel basis of $\mathbf{R}$ totally imperfect in $\mathbf{R}$? It turns out that the answer to this question is negative. Namely, it can be shown that there are nonempty perfect subsets of $\mathbf{R}$ linearly independent over the field $\mathbf{Q}$ (see, for instance, [192] or [273] where a much stronger statement is discussed with its applications). Let P be such a subset. Then, by using the Zorn lemma, one can readily prove that there exists a Hamel basis of $\mathbf{R}$ containing P. Obviously, this Hamel basis is not totally imperfect.

Remark 6. The following question also seems to be natural: if μ is a nonzero σ-finite measure on $\mathbf{R}$, invariant under the group of all translations of **R**, does there exist a Hamel basis in **R** nonmeasurable with respect to μ? As pointed out above (see Exercise 10), the answer is positive for the standard Lebesgue measure λ on **R**. However, in the general case, this question is undecidable within **ZFC** set theory. The next exercise presents a more precise result concerning the question.

11*. Prove that the following two statements are equivalent:

(a) the Continuum Hypothesis ($\mathbf{c} = \omega_1$);

(b) for every nonzero σ-finite measure μ given on **R** and quasi-invariant under the group of all translations of **R**, there exists a Hamel basis in **R** nonmeasurable with respect to μ.

In connection with the equivalence of (a) and (b), see also [65] and [247].

12*. Let f and g be two functions acting from **R** into **R**. Suppose that, for all points $x \in \mathbf{R}$ and $y \in \mathbf{R}$ and for each $q \in [0, 1]$, the inequality

$$f(qx + (1 - q)y) \leq qg(x) + (1 - q)g(y)$$

is satisfied. Consider the set $\Gamma^*(g) = \{(x, t) \in \mathbf{R}^2 : g(x) \leq t\}$ and let $\mathrm{conv}(\Gamma^*(g))$ denote the convex hull of $\Gamma^*(g)$ (recall that, according to the standard definition, the convex hull $\mathrm{conv}(A)$ of a set A lying in a vector space E over **R** is the smallest convex set in E containing A; actually, $\mathrm{conv}(A)$ coincides with the intersection of all those convex subsets of E which contain A).

Check that $\mathrm{conv}(\Gamma^*(g))$ is identical with the union of the family of all those triangles whose vertices belong to the set $\Gamma^*(g)$.

Show also that the boundary of $\mathrm{conv}(\Gamma^*(g))$ can be regarded as the graph of some convex function $\phi : \mathbf{R} \to \mathbf{R}$ for which the relation

$$f(x) \leq \phi(x) \leq g(x) \qquad (x \in \mathbf{R})$$

is fulfilled (in other words, the convex function ϕ separates the given functions f and g).

13. Let f be a function acting from **R** into **R** and let ε be a strictly positive real number.

This f is called an ε-convex function if, for any points $x \in \mathbf{R}$ and $y \in \mathbf{R}$ and for each $q \in [0, 1]$, the inequality

$$f(qx + (1 - q)y) \leq qf(x) + (1 - q)f(y) + \varepsilon$$

holds true.

By applying the result of the previous exercise, demonstrate that if f is an ε-convex function, then there exists a convex function $\phi : \mathbf{R} \to \mathbf{R}$ such that

$$f(x) \le \phi(x) \le f(x) + \varepsilon \qquad (x \in \mathbf{R}).$$

Furthermore, define a function $\psi : \mathbf{R} \to \mathbf{R}$ by the formula

$$\psi(x) = \phi(x) - \varepsilon/2 \qquad (x \in \mathbf{R})$$

and show that $|\psi(x) - f(x)| \le \varepsilon/2$ for all $x \in \mathbf{R}$.

In other words, for every ε-convex function f, there exists a convex function ψ such that the norm $||\psi - f||$ is less than or equal to $\varepsilon/2$.

Remark 7. The above result is due to Hyers and Ulam (see [93]; some related statements and problems are also discussed in [271]).

Remark 8. Let f be a function acting from $\mathbf{R}$ into $\mathbf{R}$ and suppose that, for a real $\varepsilon > 0$, we have the relation

$$f((x+y)/2) \le (f(x) + f(y))/2 + \varepsilon \qquad (x \in \mathbf{R},\ y \in \mathbf{R}).$$

In general, we cannot assert that there exists a convex function $\psi : \mathbf{R} \to \mathbf{R}$ satisfying the inequalities $\psi(x) - \varepsilon \le f(x) \le \psi(x) + \varepsilon$ for each $x \in \mathbf{R}$. Indeed, if f is a nontrivial solution of the Cauchy functional equation, then f is a Lebesgue nonmeasurable function and, obviously, it cannot be uniformly approximated by convex functions (which are measurable in the Lebesgue sense).

14*. Let $\{f_n : n < \omega\}$ be a sequence of real-valued convex functions on $\mathbf{R}$ and suppose that $f(x) = \lim_{n \to +\infty} f_n(x)$ for each $x \in \mathbf{R}$.

Show that $f : \mathbf{R} \to \mathbf{R}$ is a convex function, too.

In addition, assuming that all functions f_n $(n < \omega)$ and the function f are differentiable, show that $f'(x) = \lim_{n \to +\infty} f'_n(x)$ for all $x \in \mathbf{R}$.

Compare this result with Exercise 19 from Chapter 2.

15*. Let H be an arbitrary Hamel basis in $\mathbf{R}$.

By starting with the fact that any analytic subset of $\mathbf{R}$ is Lebesgue measurable and, applying the Steinhaus property for Lebesgue measurable sets with positive measure, prove that H is not an analytic subset of $\mathbf{R}$.

Remark 9. The previous exercise provides us with some information about the descriptive structure of a Hamel basis of the real line. The result formulated above is due to Sierpiński. In connection with it, let us remark that there are uncountable linearly independent (over $\mathbf{Q}$) subsets of

R having a good descriptive structure. For instance, as mentioned earlier, there are nonempty perfect subsets of **R** linearly independent over **Q**. In this context, it should also be pointed out that there are models of **ZFC** theory (e.g., the Constructible Universe **L** of Gödel) in which some Hamel basis is a co-analytic set, i.e., is the complement of an analytic subset of **R** (see [185], [186]).

16. Let f be a function acting from **R** into **R** and let $x \in \mathbf{R}$.

This f is called λ-almost symmetric with respect to x if f is λ-measurable and

$$\lambda(\{t \in \mathbf{R} \ : \ f(x-t) \neq f(x+t)\}) = 0.$$

Show that if f is λ-almost symmetric with respect to all points belonging to some everywhere dense subset of **R**, then f is λ-equivalent to a constant function.

Formulate and prove an analogous result in terms of the Baire property.

17*. Recall that a (convex) polyhedron in the space $\mathbf{R}^3$ is an arbitrary convex subset of this space which can be represented as the union of a nonempty finite family of closed three-dimensional simplices.

For two polyhedra $X \subset \mathbf{R}^3$ and $Y \subset \mathbf{R}^3$, we say that they are equivalent by a finite decomposition if there exist finite families $\{X_i \ : \ i \in I\}$ and $\{Y_i \ : \ i \in I\}$ of polyhedra such that

(a) $X = \cup_{i \in I} X_i$ and $Y = \cup_{i \in I} Y_i$;

(b) for any two distinct indices $i \in I$ and $i' \in I$, we have

$$\mathrm{int}(X_i) \cap \mathrm{int}(X_{i'}) = \mathrm{int}(Y_i) \cap \mathrm{int}(Y_{i'}) = \emptyset,$$

where $\mathrm{int}(\cdot)$ stands, as usual, for the operation of taking the interior of a set lying in $\mathbf{R}^3$;

(c) for each index $i \in I$, the polyhedra X_i and Y_i are congruent with respect to the group of all motions (i.e., isometric transformations) of $\mathbf{R}^3$.

Denote by the symbol $\mathcal{P}_3$ the class of all polyhedra in $\mathbf{R}^3$.

Consider any solution $f : \mathbf{R} \to \mathbf{R}$ of the Cauchy functional equation, such that $f(\pi) = 0$. Starting with this f, one may define a functional $\Phi_f : \mathcal{P}_3 \to \mathbf{R}$ by putting

$$\Phi_f(X) = \sum_{j \in J} f(\alpha_j) \cdot |b_j| \qquad (X \in \mathcal{P}_3),$$

where $\{b_j : j \in J\}$ is an injective family of all edges of the polyhedron X and α_j is the value of the dihedral angle of X corresponding to the edge b_j

(i.e., α_j is the value of the dihedral angle which is formed by two faces of X meeting at the edge b_j) and, finally, $|b_j|$ denotes the length of the edge b_j.

Verify that the functional Φ_f is invariant under the group of all motions of the space $\mathbf{R}^3$ and has equal values on any two polyhedra which are equivalent by a finite decomposition.

This Φ_f is called the Dehn functional on $\mathcal{P}_3$ associated with a solution f of the Cauchy functional equation.

Let α be the value of the dihedral angle corresponding to an edge of a regular three-dimensional simplex (tetrahedron).

Show that the two-element set $\{\alpha, \pi\} \subset \mathbf{R}$ is linearly independent over the field $\mathbf{Q}$. Therefore, by applying the Zorn lemma, this set can be extended to a Hamel basis of $\mathbf{R}$. Conclude from this fact that there exists a solution $g : \mathbf{R} \to \mathbf{R}$ of the Cauchy functional equation such that $g(\alpha) = 1$ and $g(\pi) = 0$.

Check that the functional Φ_g assigns a strictly positive value to a regular three-dimensional simplex of volume 1 and, simultaneously, Φ_g assigns value 0 to the closed unit cube in $\mathbf{R}^3$. Hence these two polyhedra, being of the same volume, are not equivalent by a finite decomposition.

This yields a solution of Hilbert's third problem, first obtained by his disciple Dehn (see [56]).

Remark 10. The result presented in the above exercise explains why, in the standard course of elementary geometry, we need to use some infinite procedures or limit processes for calculation of the volume of a three-dimensional simplex in the Euclidean space $\mathbf{R}^3$.

Notice that for the plane $\mathbf{R}^2$ the situation is more nice. Namely, it is well known that any two polygons in $\mathbf{R}^2$ with equal areas are equivalent by a finite decomposition (see, e.g., [24] where a much stronger statement due to Hadwiger and Glur is formulated and proved).

After Hilbert's third problem was solved, another problem arose naturally concerning necessary and sufficient conditions for equivalence by a finite decomposition of two given polyhedra. This essentially more difficult problem was finally solved by Sydler, who proved that two given polyhedra X and Y in the space $\mathbf{R}^3$ are equivalent by a finite decomposition if and only if they have the same volume and, for every Dehn functional Φ_f, the equality $\Phi_f(X) = \Phi_f(Y)$ is fulfilled (see the original paper by Sydler [262]; cf. also [98] where a much simpler argument is given). This topic and a number of related questions are thoroughly considered in the excellent small textbook by Boltjanskii [24].

18. Let G be an everywhere dense subgroup of the space $\mathbf{R}^n$ ($n \geq 1$).

Starting with the Vitali construction of a Lebesgue nonmeasurable set, show that any two sets $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^n$ with nonempty interiors are countably G-equidecomposable, which means that there exist countable partitions $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ of X and Y, respectively, and a family $\{g_i : i \in I\}$ of elements from G such that $(\forall i \in I)(Y_i = g_i + X_i)$.

Infer from this fact that there is no G-invariant measure extending the Lebesgue measure λ_n and defined on the family of all subsets of $\mathbf{R}^n$.

19*. Let E be a real locally convex topological vector space of second category, U be an open convex subset of E, and let $f : U \to \mathbf{R}$ be a midpoint convex function possessing the Baire property.

By using an argument similar to the proof of Theorem 5, demonstrate that f is continuous and hence is a convex function on U.

For this purpose, first check that the family of all nonempty convex open sets $V \subset U$ such that $f(V)$ is bounded from above in $\mathbf{R}$ and $f|V$ is continuous forms a π-base for the topological space U (see Exercise 1 from Chapter 18).

20. Prove that there exists a solution f of Cauchy's functional equation, which is surjective and is not injective (consequently, f is a nontrivial solution of Cauchy's functional equation).

Verify that any such f satisfies the equality $f(\triangle) = \mathbf{R}$ for every nondegenerate subinterval $\triangle$ of $\mathbf{R}$. In particular, f has the Darboux property.

21*. Let $g : \mathbf{R} \to \mathbf{R}$ be defined by the formula $g(x) = \exp(x^2)$.

Demonstrate that

(a) g cannot be represented as the limit of a uniformly convergent sequence of finite sums of periodic functions on $\mathbf{R}$;

(b) $g = \lim_{n\to+\infty} g_n$ for some sequence $\{g_n : n \in \mathbf{N}\}$ of functions on $\mathbf{R}$ converging uniformly on any bounded subinterval of $\mathbf{R}$, and every g_n is a finite sum of periodic functions each of which has a period belonging to the segment $[1, 2]$.

For establishing (b), take an appropriate Hamel basis H of $\mathbf{R}$ and show, with the aid of this H, that any real-valued polynomial on $\mathbf{R}$ can be expressed as a finite sum of periodic functions on $\mathbf{R}$ with periods belonging to $[1, 2]$.

22*. Prove that there exists a translation quasi-invariant measure μ on $\mathbf{R}$ extending the Lebesgue measure λ and such that some additive functional $f : \mathbf{R} \to \mathbf{R}$ is μ-measurable and is not λ-measurable.

Chapter 12
Summation methods and Lebesgue nonmeasurable functions

From a naive point of view, all mathematics can be considered as an intriguing interplay between finite and infinite collections of objects or, if one prefers, between various discrete and continuous structures. During the long-term history of human civilization, a lot of thrilling contrasts were discovered concerning discrete and continuous entities, which afterwards have been realized in well-known paradoxes or antinomies. One might say that the progress of mathematical sciences was (and still remains) heavily dependent on successful decisions or reasonable explanations of paradoxes of such a kind.

The emergence and further extensive development of classical mathematical analysis is a very good illustration of deep connections between finite and infinite mathematical objects. In any lecture course of mathematical analysis infinite sequences of real numbers (reals) and the limits of infinite sequences of reals are thoroughly considered.

Recall that an infinite real sequence is a function whose domain coincides with the set $\mathbf{N}$ of all natural numbers (or with some of its infinite subsets) and whose range is contained in the real line $\mathbf{R}$. Such a sequence is usually denoted by $\{a_n : n \in \mathbf{N}\}$ or simply by $a_0, a_1, a_2, ..., a_n, ...$. The terms a_n of this sequence are often produced by a certain rule or by applying a concrete algorithm to each natural index n. In particular, quite often a_n are defined recursively. However, in many cases restrictions of this kind turn out to be somewhat artificial and, in fact, are not needed when dealing with the general properties of infinite sequences.

As known, among all real sequences the most important for applications are those which are convergent. They are defined as follows.

A real number a is a limit of a sequence $\{a_n : n \in \mathbf{N}\}$ if, for any real $\varepsilon > 0$, there exists a natural index $n_0 = n_0(\varepsilon)$ such that $|a_n - a| < \varepsilon$ whenever $n > n_0$. In this case it is usually said that the above-mentioned sequence converges (or tends) to a as n tends to infinity, and it can readily be shown that a is uniquely determined. The commonly used abbreviation is

$$a = \lim_{n \to +\infty} a_n \quad \text{or} \quad a = \lim\{a_n : n \in \mathbf{N}\}.$$

Cauchy's classical criterion states that $\{a_n : n \in \mathbf{N}\}$ is convergent if and only if, for any real $\varepsilon > 0$, there exists a natural index $n_0 = n_0(\varepsilon)$ such that $|a_n - a_m| < \varepsilon$ whenever $n > n_0$ and $m > n_0$.

This fundamental criterion reflects the so-called completeness property of the real line $\mathbf{R}$ and yields a necessary and sufficient condition for the existence of $\lim_{n\to+\infty} a_n$.

There are many other sufficient conditions for the convergence of a real sequence, which are more convenient in practice and admit simpler formulations. For instance, if a sequence of reals is monotone and bounded, then it necessarily has a limit (actually, this is another form of the completeness of $\mathbf{R}$).

If two sequences $\{a_n : n \in \mathbf{N}\}$ and $\{b_n : n \in \mathbf{N}\}$ converge to a and b, respectively, then, for any reals r and t, the sequence $\{ra_n + tb_n : n \in \mathbf{N}\}$ converges to $ra + tb$. Briefly speaking, the operation lim on the vector space of all convergent real sequences turns out to be a linear functional (cf. Appendix 2).

Suppose that, for given sequences $\{a_n : n \in \mathbf{N}\}$ and $\{b_n : n \in \mathbf{N}\}$, two natural numbers k and m can be found such that $a_{k+i} = b_{m+i}$ whenever $i \in \mathbf{N}$. Then $\{a_n : n \in \mathbf{N}\}$ converges if and only if $\{b_n : n \in \mathbf{N}\}$ converges and, in this case, the equality

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$$

trivially holds true. In other words, the convergence of a sequence of reals depends only on the behavior of its tail.

Let us recall several widely known examples from the university course of Calculus.

Example 1. For any $\alpha \in \mathbf{R}$, the sequence $1^{\alpha/1}, 2^{\alpha/2}, 3^{\alpha/3}, \ldots, n^{\alpha/n}, \ldots$ converges and its limit is equal to 1 (which is easy to prove).

Example 2. The sequence $\{b_n : n \in \mathbf{N}\}$, where

$$b_n = 1/0! + 1/1! + 1/2! + ... + 1/n!,$$

converges and its limit is Napier's constant $\mathrm{e} = 2,71828...$.

Example 3. The sequence $\{c_n : n \in \mathbf{N}\}$, where c_0 is a real number and

$$c_n = 1 + 1/2 + ... + 1/n - \ln(n) \qquad (n > 0),$$

converges and its limit is Euler's constant $\mathrm{C} = 0,57721...$.

According to the standard definition, if there exists no limit of a sequence $\{a_n : n \in \mathbf{N}\}$, then this sequence is called divergent.

In some sense most real sequences turn out to be divergent. This important circumstance will be explained later, in more detail.

Example 4. Any unbounded sequence is divergent; in particular, the unbounded sequence $\{a_n : n \in \mathbf{N}\}$, where a_0 is again an arbitrary real number and

$$a_n = 1 + 1/2 + ... + 1/n \qquad (n > 0),$$

is divergent (cf. Example 3).

Example 5. The sequence $\{b_n : n \in \mathbf{N}\}$, where $b_n = 1$ for all even natural indices n and $b_n = -1$ for all odd natural indices n, is divergent.

Example 6. The sequence $\{c_n : n \in \mathbf{N}\}$, where c_0 is some real number and

$$c_n = (n!)^{1/n} \qquad (n > 0),$$

is unbounded, so is divergent.

Remark 1. As is well known, in classical mathematical analysis it is convenient sometimes to treat certain divergent sequences of reals as converging (tending) to $+\infty$ or to $-\infty$.

More precisely, a sequence $\{a_n : n \in \mathbf{N}\}$ tends to $+\infty$ if, for every real $r > 0$, there exists a natural index $n_0 = n_0(r)$ such that $a_n > r$ whenever $n > n_0$.

Analogously, a sequence $\{b_n : n \in \mathbf{N}\}$ tends to $-\infty$ if, for every real $r < 0$, there exists a natural index $n_0 = n_0(r)$ such that $b_n < r$ whenever $n > n_0$.

If a sequence $\{a_n : n \in \mathbf{N}\}$ of reals is given, one can associate to it another sequence $\{s_n : n \in \mathbf{N}\}$ of reals, where $s_n = \sum\{a_k : k \leq n\}$ denotes the sum of the first $n+1$ terms of $\{a_n : n \in \mathbf{N}\}$. Observe that the initial sequence $\{a_n : n \in \mathbf{N}\}$ is completely determined by $\{s_n : n \in \mathbf{N}\}$, because of the equality $a_n = s_n - s_{n-1}$ which is valid for all $n \in \mathbf{N}$.

The pair $(\{a_n : n \in \mathbf{N}\}, \{s_n : n \in \mathbf{N}\})$ is called an infinite series or, simply, series (see, for instance, [59]). In practice, the symbolic notation

$$a_0 + a_1 + a_2 + ... + a_n + ...$$

is systematically utilized instead of writing the above-mentioned pair.

A series $a_0 + a_1 + a_2 + ... + a_n + ...$ is convergent if the corresponding sequence $\{s_n : n \in \mathbf{N}\}$ has a finite limit s which is called the sum of this series. Otherwise, the series is called divergent.

Cauchy's criterion, for establishing whether $a_0 + a_1 + a_2 + ... + a_n + ...$ is convergent, reads as follows:

A series $a_0 + a_1 + a_2 + ... + a_n + ...$ has a sum if and only if, for any real $\varepsilon > 0$, there exists a natural index $n_0 = n_0(\varepsilon)$ such that

$$|a_n + a_{n+1} + ... + a_{n+m}| < \varepsilon$$

whenever $n > n_0$ and $m \in \mathbf{N}$.

In particular, the convergence of $a_0 + a_1 + a_2 + ... + a_n + ...$ trivially implies the equality $\lim_{n\to+\infty} a_n = 0$.

In practice various other much simpler tests are utilized which give sufficient conditions for the convergence of a series. Among them there are the D'Alembert test, n-th root test, Leibniz test, Raabe test, Kummer test, Bertrand test, Gauss test, integral test, Abel test, Dirichlet test, Ermakov test, and so on (see, e.g., [4], [72], [137], [219], [277]).

Example 7. The series

$$1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 + ...$$

converges, because all of its terms $1/n^2$, where $n > 1$, are strictly less than

$$1/(n(n-1)) \qquad (= 1/(n-1) - 1/n)$$

and the series

$$1 + 1/2 + 1/6 + 1/12 + ... + 1/(n(n-1)) + \ldots$$

converges to the sum equal to 2 (here the so-called comparison test does work). A more delicate calculation (first made by Euler) shows that

$$1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 + ... = \pi^2/6.$$

In this connection, see, e.g., [72], [277] or other textbooks of mathematical analysis.

There are standard operations over series, which are useful in various topics of mathematical analysis (for instance, the product of two series). Also, many standard transformations of a series are known, the main purpose of which is to make the behavior of the series much better in the sense of its convergence. Among such transformations, Abel's transformation should be mentioned especially, which can be treated as a discrete version of integration by parts. In addition, Euler's transformation for so-called alternating series, Kummer's transformation, Markov's transformation, and others are frequently useful for improvement of convergence (in particular, for acceleration of convergence).

Recall that there are special types of series: geometric series, harmonic series, hyper-geometric series, alternating series, etc.

In many cases a series of the form

$$a_0(x) + a_1(x) + a_2(x) + ... + a_n(x) + ...$$

is under consideration, where all terms a_n $(n \in \mathbf{N})$ are real-valued functions of a real variable x (a function series). Here, for each admissible value r of a variable x, one has the usual series

$$a_0(r) + a_1(r) + a_2(r) + ... + a_n(r) + ...$$

There are also more concrete types of function series, e.g., power series, Taylor series, Laurent series, Dirichlet series, trigonometric series (in particular, Fourier series), binomial series, and so on (see [4], [72], [277], [285]).

Let us remind further well-known examples of series.

Example 8. The standard geometric series

$$1 + q^1 + q^2 + ... + q^n + ...,$$

where a real number q satisfies the inequality $|q| < 1$. This series is trivially convergent and its sum is equal to $1/(1-q)$.

Example 9. Grandi's classical series

$$1 - 1 + 1 - 1 + 1 - 1 + ... + (-1)^n + ...$$

Obviously, this series is divergent.

Example 10. The series

$$0! - 1! + 2! - 3! + ... + (-1)^n n! + ...$$

is trivially divergent, because the absolute values of its terms tend to $+\infty$.

Example 11. The Dirichlet series

$$f(x) = a_1/1^x + a_2/2^x + a_3/3^x + ... + a_n/n^x + \ \ldots,$$

where $x \in \mathbf{R}$ and $\{a_n : n \in \mathbf{N} \setminus \{0\}\}$ is a sequence of reals. In particular, the generalized harmonic series (a part of Riemann's zeta function)

$$\zeta(x) = 1/1^x + 1/2^x + 1/3^x + ... + 1/n^x + ...$$

and the generalized alternating harmonic series (a part of Dirichlet's eta function)

$$\eta(x) = 1/1^x - 1/2^x + 1/3^x - ... + (-1)^{n-1}/n^x + ...$$

are obtained in this way. It makes sense to recall that the generalized harmonic series converges for all real numbers $x > 1$, while the generalized alternating harmonic series converges for all real numbers $x > 0$.

Example 12. A series $a_0 + a_1 + a_2 + ... + a_n + ...$ is called absolutely convergent if the series $|a_0| + |a_1| + |a_2| + ... + |a_n| + ...$ converges. Any absolutely convergent series converges. However, the converse assertion is not true in general. For instance, the alternating harmonic series

$$1 - 1/2 + 1/3 - 1/4 + ... + (-1)^{n-1}(1/n) + ...$$

converges and its sum is equal to ln(2), but the same series is not absolutely convergent (see Example 4).

It is usually said that a convergent series $a_0 + a_1 + a_2 + ... + a_n + ...$ is conditionally convergent if the associated series $|a_0|+|a_1|+|a_2|+...+|a_n|+...$ diverges.

If a series converges conditionally, then for any real r there exists a permutation (rearrangement) of the terms of this series such that the obtained new series is convergent to r. This is Riemann's classical theorem, which also admits a nontrivial generalization to a series of vectors in a finite-dimensional Euclidean space.

Abel wrote in one of his essays that divergent series are the devil's invention, because they lead to strange and paradoxical conclusions. Nevertheless, Euler systematically was dealing with series of such a kind and manipulated with them quite successfully. In his works he was trying to show that even in the case of a divergent series, it is possible to speak of meaningful value of the sum of the series. After many clever manipulations over divergent series, Euler claimed that any of such series must have a certain sum (in the generalized sense). This position was under strong criticism of D'Alembert, Abel, and other mathematicians of that time. Only much later a rational fragment of Euler's idea was transformed into a rich theory of summability (summation) of divergent series. A lot of summability methods were introduced and studied. Among them the following should be mentioned especially: Abel's summation method, Borel's summation method, Cesáro's summation method, Euler's summation method, Hölder's summation method, Kummer's summation method, Lambert's summation method, Lindelöf's summation method, Poisson's summation method, Voronoy's summation method, and many others (see [25], [72], [88], [137], [250], [277]).

The simplest method of summation is named after Cesáro. It looks as follows. If a series $a_0 + a_1 + ... + a_n + ...$ is given, then sometimes it may happen that the corresponding sequence of arithmetic means of its partial sums

$$s_0, \quad (s_0 + s_1)/2, \quad (s_0 + s_1 + s_2)/3, \ \ldots, \quad (s_0 + s_1 + \ + s_n)/(n+1), \ \ldots$$

converges to a certain real number r, and in this case r is called the Cesáro sum of the series. It should be stressed that the introduced notion of Cesáro's sum is compatible with the usual notion of sum of a convergent series. Indeed, one of Cauchy's theorems states that if a sequence of reals has a limit, then the corresponding sequence of arithmetic means of these reals also converges and has the same limit. Simple counterexamples show that the converse assertion is not true, in general.

Notice, by the way, that the Cesáro summability method inspired Tauberian type results in the general theory of series. Speaking more precisely, those results allow one to establish the convergence of a given series $a_0 + a_1 + ... + a_n + ...$ if it is a priori known that $a_0 + a_1 + ... + a_n + ...$ is summable by some method and satisfies minimal additional conditions (for more details, see [25], [72], [88], [137], [277]).

A widely known summation method is based on fruitful ideas of Abel, so is called the Abel summation method (or, sometimes, the Abel–Poisson summation method). Again, suppose that a series $a_0 + a_1 + ... + a_n + ...$ is given and, for a variable x ranging over the open interval $]0, 1[$, consider the associated power series

$$a_0 + a_1 x^1 + a_2 x^2 + ... + a_n x^n + ...$$

It may happen that this series converges for each x from $]0, 1[$, so determines a concrete function $f(x)$ on the above-mentioned interval. Moreover, in some cases it may happen that there exists a limit of $f(x)$ as x tends to 1. Then this limit is called the Abel sum of the given series. Similarly to Cesáro's summation method, the introduced notion is compatible with the usual notion of the sum of a convergent series. In other words, if a series $a_0 + a_1 + ... + a_n + ...$ converges to s, then, by virtue of Abel's classical theorem (see, e.g., [25], [72], [137]), the Abel sum of $a_0 + a_1 + ... + a_n + ...$ is also equal to s. In general, the converse assertion does not hold.

Example 13. Grandi's series

$$1 - 1 + 1 - 1 + 1 - 1 + ... + (-1)^n + ...$$

is summable by Abel's method to 1/2. The same series is also summable by Cesáro's method and its Cesáro's sum is again equal to 1/2. The latter circumstance is not occasional, because Abel's method is substantially stronger than Cesáro's. In other words, if a series is summable in the sense of Cesáro, then it is also summable in the sense of Abel, and both generalized sums of the series obtained by these two summation methods are equal to each other.

There are many examples showing the existence of series summable by Abel's method and not summable by Cesáro's method.

Example 14. Consider the series

$$1 - 2 + 3 - 4 + ... + (-1)^n(n + 1) + ...$$

which trivially is divergent and, moreover, is not summable in the Cesáro sense. However, Abel's summation method can be successfully applied to this series. Indeed, if $0 < x < 1$, then the series

$$1 - 2x^1 + 3x^2 - 4x^3 + ...$$

converges and its ordinary sum is equal to $1/(x + 1)^2$. Therefore, by putting $x = 1$, Abel's method yields the equality

$$1 - 2 + 3 - 4 + ... + (-1)^n(n + 1) + ... = 1/4.$$

It should also be noticed that Abel's method is very helpful when dealing with the products of infinite series. If one is given two series

$$a_0 + a_1 + ... + a_n + ..., \qquad b_0 + b_1 + ... + b_n + ...,$$

then they form the product series $c_0 + c_1 + ... + c_n + ...$ whose terms are defined by the equalities

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_{n-2} b_2 + a_{n-1} b_1 + a_n b_0 \qquad (n \in \mathbf{N}).$$

It is well known that the product series of two conditionally convergent series is not, in general, convergent. However, if $a_0 + a_1 + ... + a_n + ...$ and $b_0 + b_1 + ... + b_n + ...$ are summable in the sense of Abel, then their product series $c_0 + c_1 + ... + c_n + ...$ is also summable in the same sense, and Abel's sum of the product series is equal to the usual product of Abel's sums of the given series $a_0 + a_1 + ... + a_n + ...$ and $b_0 + b_1 + ... + b_n + ...$.

Example 15. In particular, using the fact mentioned above, one immediately obtains the equality

$$(1 - 1 + 1 - 1 + ...)(1 - 1 + 1 - 1 + ...) = 1 - 2 + 3 - 4 + ... \; .$$

Example 16. Several important summation methods were suggested by Borel. A special version of his method may be described as follows. If one is given a series $a_0 + a_1 + a_2 + ... + a_n + ...$, then the function

$$f(x) = e^{-x}(\sum\{s_n x^n/n! : n \in \mathbf{N}\})$$

can be considered on the set of all those real numbers x, for which this function is well-defined (observe that if $\{s_n : n \in \mathbf{N}\}$ is bounded, then $f(x)$ is defined on the whole of $\mathbf{R}$). It may happen that $f(x)$ has a finite limit r as x tends to $+\infty$. In this case, r is called Borel's sum of the series. It is not difficult to prove that if a series $a_0 + a_1 + a_2 + ... + a_n + ...$ is convergent, then it is Borel summable, too, and its usual sum coincides with its Borel sum.

Example 17. Borel also introduced another slightly more complicated integral summation method which turned out to be very strong from the point of view of summability. For instance, the series

$$0! - 1! + 2! - 3! + ... + (-1)^n n! + ...$$

is not summable by Abel's method but is summable by Borel's integral summation method and the corresponding Borel sum of this series is equal to

$$\int_0^{+\infty} (e^{-t}/(1+t))dt.$$

For further details, see, e.g., [25], [88], [137], [277].

Example 18. Consider partial Riemann's zeta function $\zeta(x)$ and partial Dirichlet's eta function $\eta(x)$ for all those real numbers x which are strictly greater than 1 (see Example 11). In this case, the convergence of the corresponding two series

$$1/1^x + 1/2^x + 1/3^x + ... + 1/n^x + \ldots,$$

$$1/1^x - 1/2^x + 1/3^x - ... + (-1)^{n-1}/n^x + \ldots$$

is readily obtained, which also yields the following equality:

$$\eta(x) = (1 - 2^{1-x})\zeta(x).$$

Further, one can consider the series $1/1^x - 1/2^x + 1/3^x - ... + (-1)^{n-1}/n^x + ...$ corresponding to $\eta(x)$ not only for real numbers x but also for complex numbers z. So, replacing x by z, one obtains the series

$$1/1^z - 1/2^z + 1/3^z - ... + (-1)^{n-1}/n^z + ...$$

which converges to a complex value whenever the real part of z is strictly greater than 1. It was proved that $\eta(x)$ admits an analytic continuation defined on the complex plane $\mathbf{C}$. Then the above equality takes the form

$$\eta(z) = (1 - 2^{1-z})\zeta(z),$$

where $\zeta(z)$ is an analytic function on the open set $\mathbf{C} \setminus \{1\}$, extending $\zeta(x)$. Taking into account that $\eta(-1) = 1/4$ (cf. Example 14), one finally comes to the remarkable relation

$$1 + 2 + 3 + ... + n + ... = \zeta(-1) = -1/12,$$

which has nontrivial applications in certain questions and topics of contemporary theoretical physics.

In this chapter we are not going to touch upon numerous other summability methods and once again refer the reader to [25], [72], [88], [137], [277]. Instead of more or less detailed consideration of such methods, we would like to briefly discuss several set-theoretical aspects of summation theory.

A general scheme of summability methods can be described as follows. Suppose that a sequence $\{g_n : n \in \mathbf{N}\}$ of real-valued functions is given on some subset D of $\mathbf{R}$ such that there is an accumulation point d of D. Here we do not exclude the case $d = +\infty$ (or the case $d = -\infty$), which simply means that D is not bounded from above (from below). Suppose also that, for each $n \in \mathbf{N}$, there exists a limit of $g_n(t)$ as $t \in D$ tends to d, and this limit is equal to 1. Consider an arbitrary series $a_0 + a_1 + ... + a_n + ...$ and associate to it the following function series:

$$g(t) = a_0 g_0(t) + a_1 g_1(t) + ... + a_n g_n(t) + ... \qquad (t \in D).$$

It may happen that, for some neighborhood $U(d)$ of d, the function g is well-defined on $U(d) \cap D$ and that there exists a finite limit

$$s = s(a_0 + a_1 + ... + a_n + ...)$$

of $g(t)$ as $t \in U(d) \cap D$ tends to d.

In this case, the series $a_0 + a_1 + ... + a_n + ...$ is called summable by the described method and its generalized sum is taken to be equal to s.

Example 19. Assume that a sequence of functions $\{g_n : n \in \mathbf{N}\}$ mentioned above satisfies these two conditions:

(1) all functions from $\{g_n : n \in \mathbf{N}\}$ are uniformly bounded, i.e., there exists a positive real constant r such that $|g_n(t)| < r$ for all $n \in \mathbf{N}$ and all $t \in D$;

(2) for each $t \in D$, the sequence $\{g_n(t) : n \in \mathbf{N}\}$ is monotone.

In this case one can assert that if a given series $a_0 + a_1 + ... + a_n + ...$ converges to s, then the same series is also summable to s by the just described method.

Many classical summation methods correspond to appropriate choices of a sequence $\{g_n : n \in \mathbf{N}\}$ satisfying conditions (1) and (2) of Example 19.

Indeed, for Cesáro's summation method, the functions g_n $(n \in \mathbf{N})$ can be defined on the set $\mathbf{N}$ of all natural numbers as follows:

$g_n(m) = (m + 1 - n)/(m + 1)$ if $n \leq m$, and $g_n(m) = 0$ if $n > m$.

For Abel's summation method, the functions g_n $(n \in \mathbf{N})$ can be defined on the interval $]0,1[$ as follows:

$$g_n(t) = t^n \qquad (0 < t < 1).$$

Further, taking the positive ray $]0, +\infty[$ of $\mathbf{R}$ as D and defining the functions g_n $(n \in \mathbf{N})$ on it by the formula

$$g_n(t) = 1 - \frac{\sum\{t^k/k! : k < n\}}{e^t} \qquad (t > 0),$$

one gets a certain version of Borel's summation method.

There are also other general schemes of summability methods (see, e.g., [25], [72], [88], [277], [285]). Notice also that two different summability methods can be incompatible (inconsistent) in the sense that, for some concrete divergent series, they yield respectively two distinct generalized sums.

Example 20. Quite often a summability method is formulated in terms of a real-valued matrix T with countably many rows and columns. In this case, we have the so-called T-summability method. Such a matrix transforms any given infinite sequence of reals into another one that may have better properties from the convergence viewpoint. The celebrated Silverman–Toeplitz theorem establishes necessary and sufficient conditions under which the class of all convergent sequences is transformed by T into itself (for more details, see [25], [72], [88], [250], [285]). Among many interesting and important statements about matrix summability methods, we only wish to mention the result of [165]. For a given matrix T and a series $a_0 + a_1 + ... + a_n + ...$, consider all those rearrangements of the terms of $a_0 + a_1 + ... + a_n + ...$ which yield T-summable series, and denote by the symbol

$$S(a_0 + a_1 + ... + a_n + ..., T)$$

the set of all T-sums obtained in this manner. It turns out that, changing $a_0 + a_1 + ... + a_n + ...$ and T in all possible ways, the family of the above-mentioned sets $S(a_0 + a_1 + ... + a_n + ..., T)$ yields the family of all analytic (i.e., Suslin) subsets of $\mathbf{R}$. Recall that this family is much wider than the family of all Borel subsets of $\mathbf{R}$ (the theory of Borel and analytic sets in Polish topological spaces is presented in detail in the two classical monographs [149] and [167]; see also [105]).

The previous example vividly shows that various summability methods naturally lead to a large family of point sets in $\mathbf{R}$ and are connected with more or less delicate facts of classical descriptive set theory. Actually, connections of this sort are much deeper and even touch upon the problem of the Lebesgue measurability of all subsets of $\mathbf{R}$ (cf. Chapter 10).

From the general position, any summability method can be described as a functional f defined on some vector subspace $L = \mathrm{dom}(f)$ of the space $\mathbf{R}^{\mathbf{N}}$ of all infinite real sequences. The elements of L are of the form $\{s_n : n \in \mathbf{N}\}$, where s_n ($n \in \mathbf{N}$) are the partial finite sums of a series $a_0 + a_1 + ... + a_n + ...$.

As a rule, the following two natural conditions are fulfilled for a summability method: linearity and regularity. The first condition means that f is a linear functional on L and the second condition means that f extends the standard linear functional lim defined on the space of all convergent sequences in $\mathbf{R}^{\mathbf{N}}$.

Sometimes, the third condition is imposed, which is an analogue of translation invariance. This translativity condition requires that if s is the f-sum of a series $a_0 + a_1 + ... + a_n + ...$ and r is the f-sum of the series $a_1 + a_2 + ... + a_n + ...$, then the equality $s = a_0 + r$ must be valid.

Example 21. No summability method satisfying the translativity condition can be applied to the series $1 + 1 + 1 + ... + 1 + ...$, i.e., no such method can assign a finite sum s to this series. Indeed, otherwise we would have

$$s = 1 + (1 + 1 + 1 + ... + 1 + ...) = 1 + s,$$

which implies the equality $0 = 1$, so yields a contradiction.

Now, if a sequence $\{t_n : n \in \mathbf{N}\}$ does not belong to the space $L \subset \mathbf{R}^{\mathbf{N}}$ and r is an arbitrary real number, then the linear functional f can be extended to a linear functional g defined on a larger vector subspace K of $\mathbf{R}^{\mathbf{N}}$ such that

$$\{t_n : n \in \mathbf{N}\} \in K, \qquad g(\{t_n : n \in \mathbf{N}\}) = r.$$

Moreover, this fact admits a constructive proof which does not rely on the Axiom of Choice. So Euler's idea may be justified in the following manner: for any concrete divergent series, there exists a linear regular summability method with the aid of which it is possible to assign a (finite) sum to this series and, moreover, the sum can be an arbitrary real number. Consequently, the series

$$1 + 2 + 3 + ... + n + ...$$

can be made summable to an arbitrary finite value (cf. Example 18).

However, there exists no linear summability method satisfying the translativity condition, which assigns a finite sum to $1+2+3+...+n+...$. Indeed, suppose for a moment that $1 + 2 + 3 + ... + n + ... = s$, where s is a real number. Then we must have

$$s - 1 = 2 + 3 + ... + n + ...,$$

$$-1 = (s-1) - s = (2-1) + (3-2) + ... + (n+1-n) + ... =$$
$$1 + 1 + 1 + ... + 1 + ... ,$$

which is impossible in view of Example 21.

Remark 2. If one wishes to assign certain generalized sums to all divergent series simultaneously, then one must appeal to strong forms of the Axiom of Choice (more precisely, to uncountable forms of this axiom). Indeed, a standard application of the Kuratowski–Zorn lemma (which is a logical equivalent of the Axiom of Choice), shows that there exists a linear regular summation method whose domain is the entire space $\mathbf{R}^{\mathbf{N}}$. However, this result is useless in practice, because of its extremely nonconstructive character.

On the other hand, it can be deduced from the fundamental results of Solovay [258] and Shelah [228] that, within the framework of **ZF** & **DC** theory, it is logically consistent to assume that the domain of any linear regular summability method is a first category subspace of $\mathbf{R}^{\mathbf{N}}$. Since $\mathbf{R}^{\mathbf{N}}$ is a Polish topological space, $\mathbf{R}^{\mathbf{N}}$ is of second category on itself. Consequently, one may assume that the domain of any effectively (constructively) defined linear regular summability method is a very small part of $\mathbf{R}^{\mathbf{N}}$. Moreover, within the same weak fragment of set theory, one may suppose that the class of all linear regular summation methods, equipped with its natural partial ordering, does not possess maximal elements.

In this context, it should also be mentioned that, by using uncountable forms of the Axiom of Choice, some remarkable linear functionals were constructed on various vector subspaces of $\mathbf{R}^{\mathbf{N}}$. Among them the best known is Banach's functional Lim which is defined on the space of all bounded real sequences, is linear, extends the standard functional lim, and satisfies the translativity condition (see, e.g., [15], [273], and Appendix 2 of the present book). However, the existence of functionals of such a kind always implies (in **ZF** & **DC** theory) the existence of ultimately pathological subsets of **R**, namely, sets nonmeasurable in the Lebesgue sense or sets not having the Baire property (see again Appendix 2; a lot of interesting and deep analogies between the Lebesgue measurability and Baire property of point sets are thoroughly discussed in a small but very informative textbook by Oxtoby [202]).

Example 22. Consider the following function series:

$$\cos(t) + \cos(2t) + \cos(4t) + ... + \cos(2^n t) + ... ,$$

where a variable t ranges over **R**. Assuming that, for each t from some Lebesgue measurable subset of **R** with strictly positive measure, this series is summable to the finite value $\phi(t)$ by a certain linear regular summation method satisfying the translativity condition, one necessarily comes to the conclusion that the obtained function ϕ is nonmeasurable in the Lebesgue sense (Kolmogorov's theorem [139]). For more information about this and similar results, see [285] or the extensive article [171] (cf. also Chapter 22).

Some aspects of summability methods, treated from the point of view of modern functional analysis and general theory of linear operators, are presented, e.g., in [25] and [61].

EXERCISES

1. Show that the generalized harmonic series converges for all reals $x > 1$ and the generalized alternating harmonic series converges for all reals $x > 0$.

2. Give an example of a divergent sequence of reals such that the corresponding sequence of arithmetic means of these reals converges to a finite limit.

On the other hand, prove that if a sequence of reals converges to a finite limit, then the corresponding sequence of arithmetic means of these reals also converges to the same limit.

3. Verify that the series

$$1 - 2 + 3 - 4 + ... + (-1)^n(n+1) + ...$$

is not summable by Cesáro's summation method.

4. Present an example of two conditionally convergent series whose product series is divergent.

5. Give a proof of the assertion formulated in Example 16.

6*. Give a proof of the assertion formulated in Example 19.

7*. Let I be a countable set of indices. Equip $\mathbf{R}^I$ with the product topology and canonical vector structure. For each index $i \in I$, denote by pr_i the canonical projection of $\mathbf{R}^I$ onto $\mathbf{R}$ corresponding to i.

Work in **ZF** & **DC** theory and show that, for a linear functional $f : \mathbf{R}^I \to \mathbf{R}$, the following three assertions are equivalent:

(a) f has the Baire property;

(b) f is continuous;

(c) $f = t_1\mathrm{pr}_{i_1} + t_2\mathrm{pr}_{i_2} + ... + t_k\mathrm{pr}_{i_k}$, where k is some natural number and

$$\{t_1, t_2, ..., t_k\} \subset \mathbf{R}, \quad \{i_1, i_2, ..., i_k\} \subset I.$$

8*. Let $\{X_j : j \in J\}$ be a family of subsets of $\mathbf{R}$ such that $\mathrm{card}(J) = \mathbf{c}$ and $\mathrm{card}(X_j) = \mathbf{c}$ for any index $j \in J$.

Using the method of transfinite induction, demonstrate that there exists a disjoint family $\{Y_j : j \in J\}$ satisfying the following two conditions:

(a) $Y_j \subset X_j$ and $\mathrm{card}(Y_j) = \mathbf{c}$ for each $j \in J$;

(b) the set $\cup\{Y_j : j \in J\}$ is linearly independent over the field $\mathbf{Q}$ of all rational numbers.

9. Let λ denote, as usual, the standard Lebesgue measure on $\mathbf{R}$.

Show that there exists an additive function $\psi : \mathbf{R} \to \mathbf{R}$ such that, for every λ-measurable set X with $\lambda(X) > 0$, the equality $\psi(X) = \mathbf{R}$ holds true.

For this purpose, apply the result of Exercise 8.

Check that this ψ satisfies the following two relations:

(a) ψ is not λ-measurable;

(b) for an arbitrary λ-nonmeasurable set $Z \subset \mathbf{R}$, the set $\psi^{-1}(Z)$ is also λ-nonmeasurable.

10. Formulate and prove the analogue of Exercise 9 in terms of the Baire property.

Chapter 13
Luzin sets, Sierpiński sets, and their applications

This chapter is devoted to the so-called Luzin subsets of the real line **R** and to the Sierpiński subsets of **R**. These sets are useful in various questions of real analysis and measure theory. Also, they have a number of applications in modern set theory (in particular, in constructing some special models of **ZFC** theory) and in certain topics of general topology.

First of all, we wish to emphasize the fact that the existence of Luzin and Sierpiński subsets of **R** cannot be established within **ZFC** theory, so if we want to deal with such subsets, then we need additional set-theoretical axioms (see Theorems 1 and 4 below).

We begin our consideration with some properties of Luzin sets. These sets were constructed by Luzin, in 1914, under the assumption of the Continuum Hypothesis (**CH**). The same sets were constructed by Mahlo one year before Luzin. However, in the mathematical literature the notion of a Luzin set is usually utilized, instead of Mahlo's set, probably because Luzin investigated these sets more deeply and also showed their applications to the theory of real-valued functions and classical measure theory (see, for example, [145], [149], [166], [192], [202]).

We now give the precise definition of Luzin sets.

Let X be a subset of **R**. We say that X is a Luzin set if X is uncountable and, for every first category subset Y of **R**, the intersection $X \cap Y$ is at most countable.

It is obvious that the family of all Luzin subsets of **R** generates the σ-ideal on **R** invariant under the group of all those transformations of **R** which preserve the σ-ideal of all first category subsets of **R**. Consequently, the above-mentioned family is invariant with respect to the group of all homeomorphisms of **R** and, in particular, with respect to the group of all translations of **R**.

We would like to underline once again that it is impossible to prove in

ZFC theory the existence of a Luzin set. Namely, assume that the relation

$$(\text{Martin's Axiom}) \,\&\, (2^{\omega} > \omega_1)$$

holds and take an arbitrary set $X \subset \mathbf{R}$. If X is at most countable, then it is not a Luzin set. Suppose now that X is uncountable. As known (see, e.g., [18], [146]), Martin's Axiom implies that the union of an arbitrary family $\{Y_i : i \in I\}$ of first category (respectively, of Lebesgue measure zero) subsets of $\mathbf{R}$, where $\mathrm{card}(I) < \mathbf{c}$, is also of first category (respectively, of Lebesgue measure zero). In particular, under Martin's Axiom, each subset of $\mathbf{R}$ with cardinality strictly less than $\mathbf{c}$ is of first category and of Lebesgue measure zero. Let now Y be any subset of X of cardinality ω_1. Then, taking into account the inequality $\omega_1 < \mathbf{c}$, we readily infer that Y is a first category subset of $\mathbf{R}$ and $\mathrm{card}(X \cap Y) = \mathrm{card}(Y) = \omega_1 > \omega$, so we obtain again that X is not a Luzin set.

Fortunately, if one assumes the Continuum Hypothesis, then one can prove that Luzin sets exist on $\mathbf{R}$ (we recall once more that this classical result is due to Luzin and Mahlo).

Theorem 1. *If the Continuum Hypothesis (***CH***) holds, then there are Luzin subsets of* $\mathbf{R}$.

Proof. As we know, **CH** means the equality $\mathbf{c} = \omega_1$, which implies, in particular, that the family of all Borel subsets of $\mathbf{R}$ has cardinality ω_1. Let $\{X_\xi : \xi < \omega_1\}$ denote the family of all first category Borel subsets of $\mathbf{R}$. We define, by the method of transfinite recursion, a family $\{x_\xi : \xi < \omega_1\}$ of points from $\mathbf{R}$. Suppose that $\xi < \omega_1$ and that the partial family $\{x_\zeta : \zeta < \xi\}$ has already been constructed. Let us consider the set

$$Z_\xi = (\cup\{X_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\}.$$

It is clear that Z_ξ is a first category subset of $\mathbf{R}$. Hence, by the classical Baire theorem applied to $\mathbf{R}$, there exists a point $x \in \mathbf{R} \setminus Z_\xi$. Thus we may put $x_\xi = x$. This ends the construction of the required family $\{x_\xi : \xi < \omega_1\}$.

Further, we define $X = \{x_\xi \ : \ \xi < \omega_1\}$. Observe that if $\zeta < \xi < \omega_1$, then $x_\zeta \neq x_\xi$. Therefore, we immediately get $\mathrm{card}(X) = \omega_1 = \mathbf{c}$.

Suppose now that Z is an arbitrary first category subset of $\mathbf{R}$. Then there is an ordinal $\xi < \omega_1$ such that $Z \subset X_\xi$ (because the family $\{X_\xi : \xi < \omega_1\}$ forms a base of the σ-ideal of all first category subsets of $\mathbf{R}$). Obviously, we have the relations

$$X \cap Z \subset X \cap X_\xi \subset \{x_\zeta : \zeta \le \xi\}, \quad \mathrm{card}(X \cap Z) \le \omega,$$

which enable us to conclude that X is a Luzin set. This finishes the proof of Theorem 1.

Notice that if X is a Luzin set on $\mathbf{R}$, then the set $X \cup \mathbf{Q}$ is also a Luzin set (where $\mathbf{Q}$ denotes, as usual, the set of all rational numbers). Hence, **CH** implies that there are Luzin sets everywhere dense in $\mathbf{R}$. We can easily get a much stronger result. Namely, a slight modification of the proof presented above gives us a Luzin set X such that, for any set $Y \subset \mathbf{R}$ with the Baire property, we have the relation

$$(\mathrm{card}(Y \cap X) \leq \omega) \Leftrightarrow (Y \text{ is of first category}).$$

In this connection, see Exercise 1 of the present chapter.

Remark 1. It makes sense to point out here that the existence of Luzin subsets of $\mathbf{R}$ is also possible in some cases when **CH** does not hold. Namely, there are certain Cohen-type models of set theory in which the negation of **CH** is true and there exist Luzin sets of cardinality $\mathbf{c}$ (for more details, see, e.g., [146], [147]). Notice, in addition, that in those models we also have a subset of $\mathbf{R}$ of cardinality $\omega_1 < \mathbf{c}$ which does not possess the Baire property (cf. Exercise 2).

Luzin sets possess a number of specific features important from the point of view of their applications in real analysis and topology. The following statement was proved by Luzin.

Theorem 2. *Suppose that X is a Luzin set on $\mathbf{R}$. Then X does not possess the Baire property in $\mathbf{R}$ (and, moreover, any uncountable subset of X, being also a Luzin set, does not possess the Baire property). Furthermore, in the space $X \cup \mathbf{Q}$ equipped with the topology induced by the standard topology of $\mathbf{R}$, every first category set is at most countable and, conversely, every at most countable subset of the space $X \cup \mathbf{Q}$ is of first category in $X \cup \mathbf{Q}$.*

Proof. Let X be an arbitrary Luzin set on $\mathbf{R}$. Suppose, for a moment, that X has the Baire property. Because X is uncountable and, for every first category set $Y \subset \mathbf{R}$, we have $\mathrm{card}(X \cap Y) \leq \omega$, we infer that X is not a first category subset of $\mathbf{R}$. But then X contains some uncountable G_δ-subset Z (let us stress in this place that Z is a G_δ-set in $\mathbf{R}$). Let Y be any subset of Z homeomorphic to the classical Cantor discontinuum. Then Y is nowhere dense in $\mathbf{R}$ and $Y \subset X$. We also may write

$$\mathrm{card}(X \cap Y) = \mathrm{card}(Y) = \mathbf{c} \geq \omega_1,$$

which contradicts the definition of a Luzin set. The contradiction obtained yields us that X does not have the Baire property.

The second part of this theorem follows from the fact that $X \cup \mathbf{Q}$ is a Luzin set and, in addition, $X \cup \mathbf{Q}$ is everywhere dense in $\mathbf{R}$. The proof of Theorem 2 is thus completed.

The result established above shows us that if the Continuum Hypothesis (**CH**) holds, then there exists a subspace X of $\mathbf{R}$ everywhere dense in $\mathbf{R}$ and such that card$(X) = \mathbf{c}$ and $\mathcal{K}(X) = [X]^{\leq\omega}$, where $\mathcal{K}(X)$ denotes, as usual, the σ-ideal of all first category subsets of the space X and $[X]^{\leq\omega}$ denotes the family of all (at most) countable subsets of X.

The latter equality also implies that, for the same X, we have the relation $\mathcal{B}a(X) = \mathcal{B}(X)$, where $\mathcal{B}a(X)$ denotes the class of all subsets of X with the Baire property and $\mathcal{B}(X)$ denotes the Borel σ-algebra of X.

Taking this result into account, it is reasonable to introduce the following definition.

A Hausdorff topological space E without isolated points is called a Luzin space if the equality $\mathcal{K}(E) = [E]^{\leq\omega}$ is valid.

Hence we see that, under **CH**, there exists an everywhere dense Luzin set on the real line $\mathbf{R}$, which can be regarded as an example of a topological Luzin space E with card$(E) = \mathbf{c}$.

In our further considerations, we need one simple auxiliary statement concerning first category supports of σ-finite diffused Borel measures given on separable metric spaces.

Recall that a measure μ defined on a σ-algebra of subsets of a ground set E is diffused (continuous) if, for each $e \in E$, we have $\{e\} \in \mathrm{dom}(\mu)$ and $\mu(\{e\}) = 0$.

Lemma 1. *Let E be an arbitrary separable metric space and let μ be a σ-finite diffused Borel measure on E. Then there exists a subset Z of E such that*

(1) $Z \in \mathcal{K}(E)$ and Z is an F_σ-subset of E;

(2) $\mu(E \setminus Z) = 0$.

In other words, Z is a first category support of μ.

Proof. Without loss of generality, we may assume that μ is a probability measure, i.e., the equality $\mu(E) = 1$ holds true. Denote by $\{e_n : n < \omega\}$ a countable everywhere dense subset of the given space E. Fix a natural number k. Since our μ is a diffused measure, we can find, for each point e_n, an open neighborhood $V_k(e_n)$ such that $\mu(V_k(e_n)) < 1/2^{k+n}$. Let us put

$$V_k = \cup\{V_k(e_n) \; : \; n < \omega\}.$$

Then V_k is an everywhere dense open subset of E and $\mu(V_k) \leq 1/2^{k-1}$. Now, putting $Y = \cap\{V_k : k < \omega\}$, we obtain a G_δ-subset Y of E of μ-measure zero. According to the definition of Y, the set $Z = E \setminus Y$ is a first category F_σ-subset of E such that $\mu(E \setminus Z) = \mu(Y) = 0$. Thus, Z is the required support of our measure μ, and the proof is finished (a slightly more general result than Lemma 1 is formulated in Exercise 3).

The next statement, also essentially due to Luzin, highlights an interesting connection between Luzin sets on $\mathbf{R}$ and topological measure theory.

Theorem 3. *Let X be an arbitrary Luzin subset of $\mathbf{R}$. The following two assertions are valid:*

(1) if μ is any σ-finite diffused Borel measure on $\mathbf{R}$, then $\mu^*(X) = 0$, where μ^* denotes the outer measure canonically associated with μ;

(2) if μ is any σ-finite diffused Borel measure on the topological space X, then μ is identically equal to zero.

Proof. It is almost obvious that assertions (1) and (2) are equivalent. Hence it is sufficient to prove only the second assertion.

Without loss of generality, we may assume that X is an everywhere dense subset of $\mathbf{R}$. Let μ be an arbitrary σ-finite diffused Borel measure given on the topological space X. Because X is a separable metric space, we may apply to X the preceding lemma on first category supports of σ-finite diffused Borel measures. So, according to Lemma 1, there exists a first category subset of X on which our μ is concentrated. But we know that each first category subset of X is at most countable (see Theorem 2). Since μ is diffused, we conclude that μ must be identically equal to zero, and the theorem has thus been proved.

Theorems 2 and 3 show us that, on the one hand, from the topological point of view Luzin sets are extremely pathological (because any uncountable subset of a Luzin set does not have the Baire property in $\mathbf{R}$) but, on the other hand, from the point of view of topological measure theory, Luzin sets are very small (because they have measure zero with respect to the completion of any σ-finite diffused Borel measure on $\mathbf{R}$).

Let E be a topological space such that all one-element sets $\{e\}$, where $e \in E$, are Borel in E. We shall say that E is a universal measure zero space (or an absolute null space) if every σ-finite diffused Borel measure on E is identically equal to zero.

It immediately follows from Theorem 3 that, under **CH**, there exist universal measure zero subspaces of $\mathbf{R}$ having the cardinality of the continuum (namely, any Luzin subset of $\mathbf{R}$ is such a space). In particular, **CH** implies that **c** is not a real-valued measurable cardinal (for the definition of real-valued measurable cardinals and their properties, see [18], [97], [202], [232], [259], and [270]).

As mentioned above, any Luzin subset of **R** is universal measure zero. But the existence of Luzin sets cannot be proved within **ZFC** theory. On the other hand, it is known that the existence of uncountable universal measure zero subsets of **R** can be established within **ZFC** (see, for instance, [84], [149], [169], [184], [207], [283], and Chapter 23). One of the earliest examples of an uncountable universal measure zero subset of **R** was constructed by Luzin, who applied, in his construction, some specific methods of the theory of analytic sets. In fact, the construction of Luzin yields a universal measure zero set $Y \subset \mathbf{R}$ with $\mathrm{card}(Y) = \omega_1$. In particular, one may conclude that it is impossible to establish within **ZFC** theory that Y is a Luzin set in the sense of the definition presented in this chapter. In addition, let us remark that Luzin's result concerning the cardinality of universal measure zero subsets of **R** is quite precise. Namely, as shown by Laver (cf. [157], [184]), in a certain model of set theory each universal measure zero subspace of **R** has cardinality less than or equal to ω_1, and the inequality $\omega_1 < \mathbf{c}$ holds in the same model.

As we see, Luzin sets and universal measure zero sets on **R** may be regarded as small subsets of **R**. There are also many other notions of small subsets of **R**. One of such notions was introduced by Borel in 1919.

Let X be a subset of **R**. We say that X is small in the Borel sense (or is a strong measure zero set) if, for any sequence $\{\varepsilon_n : n < \omega\}$ of strictly positive real numbers, there exists a countable covering $\{\,]a_n, b_n[\ : n < \omega\}$ of X by open intervals, such that $(\forall n < \omega)(b_n - a_n < \varepsilon_n)$.

It immediately follows from this definition that the family of all subsets of **R** small in the Borel sense forms a σ-ideal in the Boolean algebra of all subsets of **R**.

It turns out that any Luzin set is small in the Borel sense and that any subset of **R** small in the Borel sense is universal measure zero (see Exercises 8 and 9 for this chapter). So, assuming **CH** and taking into account the result of Exercise 8, we come to the existence of uncountable subsets of **R** small in the Borel sense.

The Borel Conjecture is the following set-theoretical statement:

Every set on **R** small in the Borel sense is at most countable.

Thus, under **CH**, the Borel Conjecture is false. However, Laver demonstrated that there are some models of set theory in which this conjecture holds true (for more detailed information, see [157], [184] and the corresponding references therein).

Exercises 10 and 11 present some additional information about Luzin subsets of **R**.

Dual (in a certain sense) objects to Luzin sets are the so-called Sierpiński sets, which were constructed by Sierpiński in 1924, also under the assumption of the Continuum Hypothesis.

Let us introduce the notion of Sierpiński sets and consider some properties of these sets.

Let X be a subset of **R**. We say that X is a Sierpiński set if X is uncountable and, for each Lebesgue measure zero subset Y of **R**, the intersection $X \cap Y$ is at most countable.

Many facts concerning Sierpiński sets are similar to the corresponding facts concerning Luzin sets. For example, we have

(a) the family of all Sierpiński sets generates the σ-ideal of subsets of **R** invariant under the group of all those transformations of **R** which preserve the σ-ideal of all Lebesgue measure zero subsets of **R** (in particular, this family is invariant with respect to the group of all translations of **R**);

(b) the assumption

$$(\text{Martin}'\text{s Axiom}) \ \& \ (2^{\omega} > \omega_1)$$

implies that there are no Sierpiński sets on **R**.

Analogously, we have the following theorem due to Sierpiński.

Theorem 4. *Assume the Continuum Hypothesis (***CH***). Then there exist Sierpiński subsets of* **R**.

Proof. The argument is very similar to the proof of Theorem 1. The only change is the replacement of the family $\{X_\xi : \xi < \omega_1\}$ of all first category Borel subsets of **R** by the family $\{Y_\xi : \xi < \omega_1\}$ of all λ-measure zero Borel subsets of **R** (where λ denotes, as usual, the standard Lebesgue measure on **R**). Also, instead of the classical Baire theorem, here we should refer to the trivial fact that λ is not identically equal to zero. The corresponding details are left to the reader.

In connection with Theorem 4, we wish to notice that the existence of Sierpiński subsets of **R** is possible in some situations when the Continuum Hypothesis does not hold. More precisely, there are models of **ZFC** theory in which the negation of **CH** is valid and there exist Sierpiński sets of cardinality **c** (see, e.g., [146] and [147]). Evidently, in such models we also have Lebesgue nonmeasurable subsets of **R** whose cardinality is equal to ω_1 and $\omega_1 < \mathbf{c}$.

Remark 2. There is a quite general result due to Sierpiński and Erdös, which states that under certain additional set-theoretical hypotheses (in particular, under **CH** or Martin's Axiom), the σ-ideal of all first category subsets of $\mathbf{R}$ is isomorphic to the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$. An isomorphism between these two classical σ-ideals is purely set-theoretical and does not have nice descriptive properties. However, the existence of such an isomorphism allows one to obtain automatically many theorems for the Lebesgue measure (the Baire category) starting with the corresponding theorems for the Baire category (the Lebesgue measure). In particular, in this way one can easily deduce Theorem 4 from Theorem 1 (and, conversely, Theorem 1 from Theorem 4). A detailed proof of the Sierpiński–Erdös result mentioned above (the so-called Sierpiński–Erdös Duality Principle) is given in the well-known textbooks [192] and [202] where numerous applications of this principle are presented as well. Some general version of the Duality Principle is formulated and proved in [43].

Let us point out another similarity between Luzin and Sierpiński sets.

Theorem 5. *Every Sierpiński set is a first category subset of* $\mathbf{R}$. *No uncountable subset of a Sierpiński set is Lebesgue measurable.*

Proof. Let X be a Sierpiński set. Let $\mathcal{I}(\lambda)$ denote, as usual, the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$. As we know, the σ-ideals $\mathcal{K}(\mathbf{R})$ and $\mathcal{I}(\lambda)$ are orthogonal, i.e., there exists a partition $\{A, B\}$ of $\mathbf{R}$ such that $A \in \mathcal{K}(\mathbf{R})$ and $B \in \mathcal{I}(\lambda)$. So we have the inequality $\mathrm{card}(X \cap B) \leq \omega$ and the inclusion $X \subset A \cup (X \cap B)$, from which we immediately obtain that $X \in \mathcal{K}(\mathbf{R})$.

Let now Y be an arbitrary uncountable subset of X (so Y is a Sierpiński set, too). Because $X \cap Y$ is uncountable, we observe that $Y \notin \mathcal{I}(\lambda)$. Suppose, to the contrary, that Y is Lebesgue measurable. Then $\lambda(Y) > 0$ and we can find an uncountable set $Z \subset Y$ of Lebesgue measure zero. But then the set $X \cap Z$ is uncountable, so we get a contradiction with the definition of the Sierpiński set X. The contradiction obtained finishes the proof of Theorem 5.

If we replace the Continuum Hypothesis by Martin's Axiom (which is a much weaker assertion than **CH**), then we can prove the existence of some analogues of Luzin and Sierpiński sets.

Namely, if Martin's Axiom (**MA**) holds, then there exists a set $X \subset \mathbf{R}$ such that

(1) $\mathrm{card}(X) = \mathbf{c}$;

(2) for each set $A \in \mathcal{K}(\mathbf{R})$, we have $\mathrm{card}(A \cap X) < \mathbf{c}$.

Any set X with the above two properties is usually called a generalized Luzin subset of the real line $\mathbf{R}$.

Similarly, if Martin's Axiom (**MA**) holds, then there exists a set $Y \subset \mathbf{R}$ such that

(1′) $\mathrm{card}(Y) = \mathbf{c}$;

(2′) for each set $B \in \mathcal{I}(\lambda)$, we have $\mathrm{card}(B \cap Y) < \mathbf{c}$.

Any set Y with the above two properties is usually called a generalized Sierpiński subset of the real line $\mathbf{R}$.

Observe that, for the existence of generalized Luzin sets or generalized Sierpiński sets, we do not need the full power of Martin's Axiom. In fact, the existence of generalized Luzin and Sierpiński sets is implied by some additional set-theoretical assumptions that are essentially weaker than Martin's Axiom. Exercise 19 for this chapter presents the corresponding result for an abstract σ-ideal of subsets of a given uncountable ground set.

We wish to give here one application of a generalized Luzin set to the construction of a function which is extremely bad from the point of view of measure theory.

Let E be a set (in particular, a topological space) and let f be a function acting from E into the real line $\mathbf{R}$. We shall say that f is absolutely nonmeasurable if, for any nonzero σ-finite diffused measure μ on E, this f is nonmeasurable with respect to μ.

Let us underline that, in this definition, the domain of μ is not a fixed σ-algebra of subsets of E (actually, $\mathrm{dom}(\mu)$ may be an arbitrary σ-algebra of subsets of E, containing all singletons in E).

Theorem 6. *Suppose that Martin's Axiom (***MA***) is valid. Then there exists an injective absolutely nonmeasurable function $f : \mathbf{R} \to \mathbf{R}$.*

Proof. We know that Martin's Axiom implies the existence of a generalized Luzin subset of $\mathbf{R}$. Let X be such a subset. Because we have

$$\mathrm{card}(X) = \mathbf{c} = \mathrm{card}(\mathbf{R}),$$

there exists a bijection $f : \mathbf{R} \to X$. Obviously, we can consider f as an injection acting from $\mathbf{R}$ into itself. Let us verify that f is the required function. Suppose, for a moment, that our f is not absolutely nonmeasurable. Then there exists a nonzero σ-finite diffused measure μ on $\mathbf{R}$ such that f is μ-measurable, i.e., for any Borel set $B \subset \mathbf{R}$, the relation $f^{-1}(B) \in \mathrm{dom}(\mu)$ is satisfied. So, for any Borel subset B' of X, we have $f^{-1}(B') \in \mathrm{dom}(\mu)$. Clearly, without loss of generality, we may assume that μ is a probability measure. Now, for each Borel subset B' of X, we put $\nu(B') = \mu(f^{-1}(B'))$. In this way we come to a Borel diffused probability measure ν on X, which is impossible since X is a universal measure zero space (see Exercise 21 of the present chapter). The contradiction obtained finishes the proof of Theorem 6.

Remark 3. The preceding theorem was formulated and proved under the assumption that Martin's Axiom is valid. In this connection, it is reasonable to mention that one cannot establish the existence of an absolutely nonmeasurable function on $\mathbf{R}$ within **ZFC** set theory. Indeed, if the cardinality of the continuum is real-valued measurable, then such functions do not exist. At the same time, one can easily demonstrate in **ZFC** theory that there is an absolutely nonmeasurable function $f : \omega_1 \to \mathbf{R}$. In order to demonstrate this, it suffices to pick a universal measure zero subspace X of $\mathbf{R}$ with $\mathrm{card}(X) = \omega_1$ and then to take as f any bijection acting from ω_1 onto X. Some additive version of Theorem 6 will be discussed in Chapter 14 in connection with invariant extensions of the standard Borel measure on $\mathbf{R}$.

In our further considerations, we shall meet many other applications of Luzin sets and Sierpiński sets (respectively, of generalized Luzin sets and generalized Sierpiński sets). But now we shall use once more Martin's Axiom for giving a construction of a generalized Sierpiński set with the Baire property in the restricted sense.

Theorem 7. *Suppose that Martin's Axiom (*$\mathbf{MA}$*) holds. Then there exists a set* $X \subset \mathbf{R}$ *such that*

(1) for every nonempty perfect set $P \subset \mathbf{R}$*, the intersection* $X \cap P$ *is a first category set in* P*;*

(2) for each Lebesgue measurable set $Y \subset \mathbf{R}$ *with* $\lambda(Y) > 0$*, the intersection* $X \cap Y$ *is nonempty;*

(3) X *is a generalized Sierpiński subset of* $\mathbf{R}$*.*

Proof. Let α denote the smallest ordinal number whose cardinality is equal to $\mathbf{c}$. Actually, we may identify α with $\mathbf{c}$ (see Chapter 0).

Let $\{Z_\xi : \xi < \alpha\}$ denote the family of all Borel subsets of $\mathbf{R}$ having strictly positive Lebesgue measure, i.e., $\{Z_\xi : \xi < \alpha\} = \mathcal{B}(\mathbf{R}) \setminus \mathcal{I}(\lambda)$, and let $\{T_\xi : \xi < \alpha\}$ denote the family of all Borel subsets of $\mathbf{R}$ having Lebesgue measure zero, i.e., $\{T_\xi : \xi < \alpha\} = \mathcal{B}(\mathbf{R}) \cap \mathcal{I}(\lambda)$. For each ordinal $\xi < \alpha$, we fix a partition $\{Z^0_\xi, Z^1_\xi\}$ of Z_ξ such that Z^0_ξ is a Lebesgue measure zero set and Z^1_ξ is a first category set in Z_ξ. Notice that the existence of such a partition follows directly from Lemma 1. Now, we define an injective α-sequence $\{x_\xi : \xi < \alpha\}$ of real numbers.

Suppose that $\xi < \alpha$ and that the partial sequence $\{x_\zeta : \zeta < \xi\}$ has already been determined. Let us consider the set

$$D_\xi = (\bigcup_{\zeta \le \xi} Z^0_\zeta) \cup \{x_\zeta : \zeta < \xi\} \cup (\bigcup_{\zeta \le \xi} T_\zeta).$$

Martin's Axiom implies that the set D_ξ is of Lebesgue measure zero. Hence we have the relation $Z_\xi \setminus D_\xi \neq \emptyset$. So we can choose a point x_ξ from the above-mentioned nonempty difference of sets.

Proceeding in this manner, we are able to define the whole α-sequence $\{x_\xi : \xi < \alpha\}$ of points of $\mathbf{R}$. Afterwards, we put $X = \{x_\xi : \xi < \alpha\}$ and we are going to show that the set X is as required.

Let P be any nonempty perfect subset of $\mathbf{R}$. If its Lebesgue measure is equal to zero, then, for some ordinal $\xi < \alpha$, we may write $P = T_\xi$. Consequently, from the method of construction of the set X, we immediately obtain the inequality $\mathrm{card}(P \cap X) < \mathbf{c}$. Applying Martin's Axiom again, we see that the intersection $P \cap X$ is a first category set in P.

Suppose now that $\lambda(P) > 0$. Then, for some ordinal $\xi < \alpha$, we can write $P = Z_\xi$. Therefore, $P \cap X \subset Z^1_\xi \cup \{x_\zeta : \zeta < \xi\}$. Taking account of the fact that the set P does not have isolated points, we obtain from the last inclusion that $P \cap X$ is a first category set in P. Hence condition (1) is satisfied for our set X. Furthermore, since $x_\xi \in Z_\xi$ for each ordinal $\xi < \alpha$, we conclude that condition (2) holds for the set X, too. The validity of condition (3) follows directly from our construction. Theorem 7 has thus been proved.

Remark 4. This theorem enables us to conclude that Martin's Axiom implies the existence of a generalized Sierpiński subset of $\mathbf{R}$ which is thick (with respect to the Lebesgue measure λ) and simultaneously has the Baire property in the restricted sense (cf. Exercise 12 from Chapter 0).

We want to finish this chapter with one interesting fact concerning universal measure zero sets and Lebesgue measure zero sets. The following statement (due to Marczewski) yields a characterization of universal measure zero subsets of $\mathbf{R}$ in terms of Lebesgue measure zero sets.

Theorem 8. *Let X be a subset of $\mathbf{R}$. Then these two assertions are equivalent:*

(1) X is a universal measure zero space;

(2) each homeomorphic image of X lying in $\mathbf{R}$ has Lebesgue measure zero.

Proof. Suppose first that X satisfies relation (1). Let Y be a subset of $\mathbf{R}$ homeomorphic to X. Fix any homeomorphism $f : X \to Y$. If $\lambda^*(Y) > 0$, then obviously there exists a nonzero σ-finite diffused Borel measure μ on Y. Putting

$$\nu(Z) = \mu(f(Z)) \qquad (Z \in \mathcal{B}(X)),$$

we obtain a nonzero σ-finite diffused Borel measure ν on X. But this is impossible because X is a universal measure zero space. Therefore, the equality $\lambda(Y) = 0$ must be valid. The implication (1) $\Rightarrow$ (2) has thus been proved.

Let now X satisfy relation (2). We are going to demonstrate that relation (1) holds for X, too. Of course, without loss of generality, we may assume that our X is a subset of the unit segment $[0,1]$. Suppose for a moment that X is not a universal measure zero space. Then there exists a Borel diffused probability measure μ on $[0,1]$ such that $\mu^*(X) > 0$. We may also assume that μ does not vanish on any nonempty open subinterval of $[0,1]$ (replacing, if necessary, μ by $(\mu + \lambda)/2$). Now, define a function $f : [0,1] \to [0,1]$ by the formula

$$f(x) = \mu([0,x]) \qquad (x \in [0,1]).$$

Evidently, f is an increasing homeomorphism from $[0,1]$ onto itself. Let us put

$$\nu(Z) = \mu(f^{-1}(Z)) \qquad (Z \in \mathcal{B}([0,1])).$$

In this way we get a Borel probability measure ν on $[0,1]$ such that

$$\nu^*(f(X)) = \mu^*(X) > 0.$$

Furthermore, it turns out that ν coincides with the standard Borel measure on $[0,1]$. Indeed, for each interval $[a,b] \subset [0,1]$, we may write

$$f^{-1}([a,b]) = \{t \in [0,1] \ : \ \mu([0,t]) \in [a,b]\} = [c,d],$$

where $\mu([0,c]) = a$ and $\mu([0,d]) = b$. Then we have

$$\nu([a,b]) = \mu(f^{-1}([a,b])) = \mu([c,d]) = \mu([0,d]) - \mu([0,c]) = b - a.$$

Consequently, the measures ν and λ are identical on the family of all subintervals of $[0,1]$ and hence these two measures coincide on the whole Borel σ-algebra of $[0,1]$. Thus $\lambda^*(f(X)) = \nu^*(f(X)) > 0$, which contradicts relation (2). The contradiction obtained establishes the implication (2) $\Rightarrow$ (1) and completes the proof of Theorem 8.

EXERCISES

1. By assuming the Continuum Hypothesis (**CH**) and applying the method of transfinite induction, show that there exists a Luzin subset X of

$\mathbf{R}$ such that, for each set $Y \subset \mathbf{R}$ possessing the Baire property, the following two relations are equivalent:

(a) $\mathrm{card}(Y \cap X) \leq \omega$;

(b) Y is a first category subset of $\mathbf{R}$.

Also, check whether the assumption that all Y possess the Baire property is essential for the validity of (a) $\Leftrightarrow$ (b).

2. Let X be a Luzin set on $\mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$ and let κ be a cardinal number satisfying the inequalities $\omega_1 \leq \kappa < \mathbf{c}$.

Demonstrate that no subset Y of X with $\mathrm{card}(Y) = \kappa$ possesses the Baire property in $\mathbf{R}$.

3. Let E be a topological space satisfying the following two conditions:

(a) there exists a countable subset D of E everywhere dense in E;

(b) each one-element set $\{e\}$, where $e \in D$, is a G_δ-subset of E.

Let μ be a σ-finite Borel measure on E such that $(\forall e \in D)(\mu(\{e\}) = 0)$.

Prove that there exists a first category F_σ-subset Z of E for which the equality $\mu(E \setminus Z) = 0$ holds true.

4. Verify that

(a) any subspace of a universal measure zero space is also a universal measure zero space;

(b) the class of all universal measure zero spaces is closed under finite Cartesian products, but is not closed under countable Cartesian products;

(c) if I is a set of indices whose cardinality is not real-valued measurable, and $\{E_i : i \in I\}$ is a family of universal measure zero spaces, then the topological sum of $\{E_i : i \in I\}$ is a universal measure zero space, too.

5. Assume that E is a topological space such that all singletons in E are Borel subsets of E.

Prove that the following two assertions are equivalent:

(a) E is a universal measure zero space;

(b) for any topological space E' satisfying the same assumption and containing E as a subspace, and for any σ-finite diffused Borel measure μ on E', the equality $\mu^*(E) = 0$ holds true, where μ^* denotes, as usual, the outer measure associated with μ.

Deduce from this result that if $\{E_n : n \in \omega\}$ is a countable family of universal measure zero subspaces of a topological space E', then the space $\cup\{E_n : n \in \omega\}$ is universal measure zero, too. In other words, if a topological space E' is not universal measure zero, then the family of all universal measure zero subspaces of E' forms a σ-ideal in the Boolean algebra of all subsets of E'.

6. Let E_1 and E_2 be two topological spaces satisfying the assumption of Exercise 5, and let $f : E_1 \to E_2$ be an injective Borel mapping.

Check that if the space E_2 is universal measure zero, then the space E_1 is universal measure zero, too.

Show also that the condition of injectivity of f is essential here. Moreover, show that a bijective continuous image of a universal measure zero space is not, in general, universal measure zero.

7. Verify that any universal measure zero subspace X of $\mathbf{R}$ is a Marczewski subset of $\mathbf{R}$ (see Chapter 10 of this book), i.e., for each nonempty perfect set $P \subset \mathbf{R}$, there exists a nonempty perfect set $P' \subset \mathbf{R}$ such that $P' \subset P$ and $P' \cap X = \emptyset$.

8. Demonstrate that any Luzin subset of $\mathbf{R}$ is small in the Borel sense.

9. Let $[a, b]$ be an arbitrary compact subinterval of $\mathbf{R}$ and let μ be a finite diffused Borel measure on $[a, b]$.

Show that, for each real $\varepsilon > 0$, there exists a real $\delta > 0$ such that, for any subinterval $[t', t'']$ of $[a, b]$ with $t'' - t' < \delta$, one has $\mu([t', t'']) < \varepsilon$.

Deduce from this fact that every subset of $\mathbf{R}$ small in the Borel sense is universal measure zero (notice that the converse assertion is not true; in this connection, see Exercise 25 below).

10. Let X be a Luzin subset of the real line $\mathbf{R}$ and let μ be an arbitrary σ-finite diffused Borel measure on $\mathbf{R}$. Suppose also that $f : X \to \mathbf{R}$ is a mapping which has the Baire property.

Prove that $\mu^*(f(X)) = 0$, where μ^* denotes, as usual, the outer measure associated with μ. Hence, $f(X)$ is a universal measure zero subset of $\mathbf{R}$.

11. Suppose that the Continuum Hypothesis holds, and let X be an uncountable everywhere dense subspace of $\mathbf{R}$ such that $\mathcal{B}a(X) = \mathcal{B}(X)$.

Check that X is a Luzin subset of $\mathbf{R}$.

12. Give a detailed proof of Theorem 4.

13. Let $\mathcal{T} = \mathcal{T}_d$ denote the density topology on $\mathbf{R}$.

Show that a set $Z \subset \mathbf{R}$ is a Sierpiński set in $\mathbf{R}$ if and only if Z is a Luzin set in the space $(\mathbf{R}, \mathcal{T})$ (the latter means that Z is uncountable and, for every first category set Y in $(\mathbf{R}, \mathcal{T})$, the intersection $Z \cap Y$ is at most countable).

14. Let X be a Sierpiński subset of $\mathbf{R}$ considered as a topological space with the induced topology.

Prove that any Borel subset of X is simultaneously an F_σ-set and a G_δ-set in X (in particular, each countable subset of X is a G_δ-set in X).

15. Let X be a Sierpiński subset of $\mathbf{R}$.

Demonstrate that, for any nonempty perfect set $P \subset \mathbf{R}$, the set $X \cap P$ is of first category in P (this result strengthens the corresponding part of Theorem 5).

16. Let X be a Sierpiński subset of $\mathbf{R}$. Equip X with the topology induced by the density topology of $\mathbf{R}$.

Check that the topological space X is nonseparable and hereditarily Lindelöf (the latter means that each subspace of X is Lindelöf, i.e., any open covering of a subspace contains a countable subcovering).

17*. Assume that **CH** holds and let X be a Sierpiński set on $\mathbf{R}$. Equip X with the topology induced by the Euclidean topology of $\mathbf{R}$.

Prove that $\mathcal{A}(X) = \mathcal{B}(X)$, where $\mathcal{A}(X)$ denotes the class of all analytic subsets of X and $\mathcal{B}(X)$ denotes the class of all Borel subsets of X.

18*. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two orthogonal σ-ideals of subsets of $\mathbf{R}$, each of which is invariant with respect to the group of all translations of $\mathbf{R}$. Let A_1 and A_2 be two subsets of $\mathbf{R}$ satisfying the relations $A_1 \notin \mathcal{J}_1$ and $A_2 \notin \mathcal{J}_2$.

Demonstrate that

(i) there exists a set $B_1 \in \mathcal{J}_1$ for which one has

$$B_1 - A_2 = \cup\{B_1 - a \ : \ a \in A_2\} = \mathbf{R};$$

(ii) there exists a set $B_2 \in \mathcal{J}_2$ for which one has

$$B_2 - A_1 = \cup\{B_2 - a \ : \ a \in A_1\} = \mathbf{R}.$$

Further, put

$\mathcal{J}_1$ = the σ-ideal of all first category subsets of $\mathbf{R}$;

$\mathcal{J}_2$ = the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$.

Deduce from relations (i) and (ii) that if X is a Luzin set on $\mathbf{R}$ and Y is a Sierpiński set on $\mathbf{R}$, then the equalities $\mathrm{card}(X) = \mathrm{card}(Y) = \omega_1$ are fulfilled, and conclude that the simultaneous existence in $\mathbf{R}$ of Luzin and Sierpiński sets immediately implies that the cardinality of these sets is as minimal as possible (i.e., is equal to the least uncountable cardinal).

Remark 5. The above result was first obtained by Rothberger (see [220]).

19. Let E be a set with $\mathrm{card}(E) \geq \omega_1$ and let $\mathcal{J}$ be a σ-ideal of subsets of E, containing in itself all one-element subsets of E. Denote

$\mathrm{cov}(\mathcal{J})$ = the smallest cardinality of a covering of E by sets belonging to $\mathcal{J}$;

$\mathrm{cof}(\mathcal{J})$ = the smallest cardinality of a base of $\mathcal{J}$.

Prove that if the equalities $\mathrm{cov}(\mathcal{J}) = \mathrm{cof}(\mathcal{J}) = \mathrm{card}(E)$ are fulfilled, then there exists a subset D of E such that $\mathrm{card}(D) = \mathrm{card}(E)$ and, for any set $Z \in \mathcal{J}$, one has $\mathrm{card}(Z \cap D) < \mathrm{card}(E)$.

In particular, if a ground set E coincides with the real line $\mathbf{R}$ and $\mathcal{J}$ is the σ-ideal of all first category subsets of $\mathbf{R}$ (respectively, the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$), then one obtains, under Martin's Axiom, the existence of a generalized Luzin subset of $\mathbf{R}$ (respectively, the existence of a generalized Sierpiński subset of $\mathbf{R}$).

20*. Assume that the Continuum Hypothesis (**CH**) holds.

Demonstrate that there exists a set $X \subset \mathbf{R}$ satisfying the following two conditions:

(a) X is a vector space over the field $\mathbf{Q}$;

(b) X is an everywhere dense Luzin subset of $\mathbf{R}$.

Show also that there exists a set $Y \subset \mathbf{R}$ satisfying the following two conditions:

(a$'$) Y is a vector space over the field $\mathbf{Q}$;

(b$'$) Y is an everywhere dense Sierpiński subset of $\mathbf{R}$.

Moreover, by assuming Martin's Axiom, formulate and prove analogous results for generalized Luzin sets and for generalized Sierpiński sets.

In addition, infer from these results, by assuming Martin's Axiom again, that there exist an isomorphism f of the additive group $(\mathbf{R}, +)$ onto itself and a generalized Luzin set Z in $\mathbf{R}$ such that $f(Z)$ is a generalized Sierpiński set in $\mathbf{R}$.

21*. Suppose that Martin's Axiom (**MA**) holds.

Prove that any generalized Luzin subset of $\mathbf{R}$ is universal measure zero.

In addition, by using a generalized Luzin set on $\mathbf{R}$, show that there exists a σ-algebra $\mathcal{S}$ of subsets of $\mathbf{R}$, such that

(a) for each point $x \in \mathbf{R}$, we have $\{x\} \in \mathcal{S}$;

(b) $\mathcal{S}$ is a countably generated σ-algebra, i.e., there exists a countable subfamily of $\mathcal{S}$, which generates $\mathcal{S}$;

(c) there is no nonzero σ-finite diffused measure whose domain coincides with $\mathcal{S}$.

Conclude that Martin's Axiom implies that the cardinal $\mathbf{c}$ is not real-valued measurable (i.e., there exists no nonzero σ-finite diffused measure defined on the family of all subsets of $\mathbf{R}$).

Remark 6. A result similar to the one presented in Exercise 21 can be proved within **ZFC** theory if we replace $\mathbf{R}$ by a certain uncountable subspace E of $\mathbf{R}$. Namely, it suffices to take as E a universal measure zero subset of $\mathbf{R}$ with cardinality equal to ω_1. In particular, we immediately obtain from this result that ω_1 is not real-valued measurable (cf. [270]).

22. Assume Martin's Axiom (**MA**).

By applying a generalized Luzin set, prove that there exist two σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ of subsets of $\mathbf{R}$, satisfying the following five conditions:

(a) $\mathcal{B}(\mathbf{R}) \subset \mathcal{S}_1 \cap \mathcal{S}_2$;

(b) both $\mathcal{S}_1$ and $\mathcal{S}_2$ are countably generated σ-algebras;

(c) there exists a measure μ_1 on $\mathcal{S}_1$ extending the standard Borel measure on $\mathbf{R}$;

(d) there exists a measure μ_2 on $\mathcal{S}_2$ extending the standard Borel measure on $\mathbf{R}$;

(e) there is no nonzero σ-finite diffused measure defined on the σ-algebra of sets, generated by $\mathcal{S}_1 \cup \mathcal{S}_2$.

Remark 7. The result of Exercise 22 will be significantly strengthened in Chapter 14.

23*. Check that if A is an arbitrary first category subset of $\mathbf{R}$, then the equality $(\mathbf{R} \setminus A) + (\mathbf{R} \setminus A) = \mathbf{R}$ holds true.

By starting with this fact, assuming the Continuum Hypothesis and using the method of transfinite recursion, construct a Luzin set X in $\mathbf{R}$ such that $X + X = \mathbf{R}$.

Formulate and prove the analogous result (under Martin's Axiom) for a generalized Luzin set.

24. Let Z be a subset of the Euclidean plane $\mathbf{R}^2$.

This Z is called strong measure zero if, for any sequence $\{\varepsilon_n : n \in \omega\}$ of strictly positive reals, there exists a countable covering $\{V_n : n \in \omega\}$ of Z by squares, such that $(\forall n \in \omega)(\mathrm{diam}(V_n) < \varepsilon_n)$.

Let l be a straight line in $\mathbf{R}^2$ not parallel to the line $\mathbf{R} \times \{0\}$. Consider the projection $\mathrm{pr}_{(\mathbf{R},l)} : \mathbf{R}^2 \to \mathbf{R}$ of $\mathbf{R}^2$ onto $\mathbf{R}$, canonically associated with the direction l. According to the definition of $\mathrm{pr}_{(\mathbf{R},l)}$, for each point (x, y) of $\mathbf{R}^2$, one has $\mathrm{pr}_{(\mathbf{R},l)}(x, y) = (x', 0)$, where the vector $(x - x', y)$ is parallel to the line l.

Show that if Z is a strong measure zero subset of $\mathbf{R}^2$, then $\mathrm{pr}_{(\mathbf{R},l)}(Z)$ is a strong measure zero subset of $\mathbf{R}$.

In addition, introduce the notion of a Luzin subset (respectively, of a generalized Luzin subset) of the plane $\mathbf{R}^2$ and prove that, under the Continuum Hypothesis (respectively, under Martin's Axiom), there exist Luzin subsets (respectively, generalized Luzin subsets) of $\mathbf{R}^2$.

Finally, verify that any Luzin set in $\mathbf{R}^2$ is strong measure zero (and hence universal measure zero).

25*. Let X be the Luzin set of Exercise 23 and let $\phi : \mathbf{R}^2 \to \mathbf{R}$ be the mapping defined by

$$\phi((x,y)) = x + y \qquad ((x,y) \in \mathbf{R}^2).$$

Observe that ϕ coincides with the projection $\mathrm{pr}_{(\mathbf{R},l)}$ of $\mathbf{R}^2$ onto $\mathbf{R}$, where $l = \{(x,y) \in \mathbf{R}^2 \ : \ x + y = 1\}$.

Check also that $\phi(X \times X) = \mathbf{R}$ and infer from this circumstance that $X \times X$ is not a strong measure zero subset of $\mathbf{R}^2$ (cf. the result of Exercise 24). Conclude from the above-mentioned fact that the Cartesian product of two strong measure zero subsets of $\mathbf{R}$ is not, in general, a strong measure zero subset of $\mathbf{R}^2$.

Compare the above result with the fact that the class of all universal measure zero spaces is closed under finite Cartesian products (see Exercise 4 of the present chapter).

Remark 8. The results given in Exercises 23, 24, and 25 are essentially due to Sierpiński (cf. [241]).

26. Let X be an uncountable topological space such that all one-element subsets of X are Borel in X.

This X is called a Sierpiński space if X contains no universal measure zero subspace with cardinality equal to $\mathrm{card}(X)$.

Demonstrate that

(a) any generalized Sierpiński subset of $\mathbf{R}$ is a Sierpiński space;

(b) if X is a Sierpiński space of cardinality ω_1, then $\mathcal{A}(X) = \mathcal{B}(X)$, where $\mathcal{A}(X)$ denotes the class of all analytic subsets of X (i.e., the class of all those sets which can be obtained by applying the (A)-operation to various (A)-systems consisting of Borel subsets of X) and $\mathcal{B}(X)$ denotes, as usual, the class of all Borel subsets of X;

(c) if X_1 and X_2 are two Sierpiński spaces and X is their topological sum, then X is a Sierpiński space, too;

(d) if X is any Sierpiński space, Y is any topological space such that $\mathrm{card}(Y) = \mathrm{card}(X)$, all one-element subsets of Y are Borel in Y, and there exists a Borel surjection from X onto Y, then Y is a Sierpiński space, too; consequently, if X is a Sierpiński subset of $\mathbf{R}$ and $f : X \to \mathbf{R}$ is a Borel mapping such that $\mathrm{card}(f(X)) = \mathrm{card}(X)$, then $f(X)$ is a Sierpiński subspace of $\mathbf{R}$.

27. By assuming Martin's Axiom and applying the method of transfinite recursion, construct a generalized Sierpiński subset X of $\mathbf{R}$ satisfying the equality $X + X = \mathbf{R}$.

Infer from this equality that there exists a continuous surjection from the product space $X \times X$ onto $\mathbf{R}$.

Further, by starting with this property of X and keeping in mind assertion (d) of Exercise 26, conclude that the topological product $X \times X$ is not a Sierpiński space.

So, the topological product of two Sierpiński spaces is not, in general, a Sierpiński space.

28. Let H be a Hilbert space (over the field $\mathbf{R}$) whose Hilbert dimension is equal to $\mathbf{c}$ (in particular, the cardinality of H equals $\mathbf{c}$, too).

Assuming that $\mathbf{c}$ is not a real-valued measurable cardinal, demonstrate that there exists a subset X of H satisfying the following three conditions:

(a) $\mathrm{card}(X) = \mathbf{c}$;

(b) X is everywhere dense in H (in particular, X is nonseparable);

(c) X is a universal measure zero subspace of H.

Suppose now that $\mathbf{c}$ is not cofinal with ω_1, i.e., $\mathbf{c}$ cannot be represented in the form $\mathbf{c} = \sum_{\xi<\omega_1} \kappa_\xi$, where all cardinal numbers κ_ξ $(\xi < \omega_1)$ are strictly less than $\mathbf{c}$.

By starting with the fact that there exists an ω_1-sequence of nowhere dense subsets of H which cover H, show that there is no generalized Luzin subset of H; in other words, show that there is no subset Y of H satisfying these two relations:

(d) $\mathrm{card}(Y) = \mathbf{c}$;

(e) for each first category set $Z \subset H$, the inequality $\mathrm{card}(Z \cap Y) < \mathbf{c}$ is fulfilled.

29*. Consider the Hilbert cube $[0,1]^\omega$. A deep theorem due to Hurewicz states that this cube cannot be covered by countably many zero-dimensional subspaces (see, for instance, [149]). Also, it is well known that

(a) any zero-dimensional subset of $[0,1]^\omega$ can be included in some zero-dimensional G_δ-subspace of $[0,1]^\omega$;

(b) any finite-dimensional subset of $[0,1]^\omega$ is contained in a finite union of zero-dimensional subspaces of $[0,1]^\omega$.

For more details, see again [149] (notice that (a) is a direct consequence of the Lavrentieff theorem on extensions of homeomorphisms).

Starting with these facts and assuming the Continuum Hypothesis, construct an uncountable Luzin type set $X \subset [0,1]^\omega$ for the σ-ideal generated by the family of all zero-dimensional G_δ-sets in $[0,1]^\omega$.

Check that no uncountable subspace of X is finite-dimensional.

Now, fix a Peano type mapping $\phi : C \to [0,1]^\omega$, where C denotes the classical Cantor discontinuum on the segment $[0,1]$, and define two functions

$f_1 : [0,1]^\omega \to C$ and $f_2 : [0,1]^\omega \to C$ by the formulas

$$f_1(x) = \min(\phi^{-1}(x)), \quad f_2(x) = \max(\phi^{-1}(x)) \qquad (x \in [0,1]^\omega).$$

Both these functions are semicontinuous (see Exercise 4 from Chapter 3).

Show that, for any uncountable set $Y \subset X$, the restrictions $f_1|Y$ and $f_2|Y$ are not continuous; deduce from this circumstance that $f_1|Y$ and $f_2|Y$ are not countably continuous.

30*. Assume Martin's Axiom (**MA**).

Starting with the existence of a generalized Luzin set in $\mathbf{R}$ and using the topological invariance of the Baire property in the restricted sense (see Chapter 0), demonstrate that there exist a function f acting from $\mathbf{R}$ into $[0,1]$ and a set $X \subset \mathbf{R}$ having the following properties:

(a) f is upper semicontinuous;

(b) $\mathrm{card}(X) = \mathbf{c}$;

(c) for any set $Y \subset X$ with $\mathrm{card}(Y) = \mathbf{c}$, the restriction $f|Y$ is not continuous.

Infer from these three properties that if $f = \cup\{f_i : i \in I\}$, where $\mathrm{card}(I) < \mathbf{c}$, then at least one partial function f_i is not continuous (in particular, f is not countably continuous).

Remark 9. The result of Exercise 30 is essentially due to Sierpiński (see [243]).

31*. Assuming **CH**, establish that there exists a family $\{X_i : i \in I\}$ of Luzin sets in $\mathbf{R}$ such that

(a) $\mathrm{card}(I) > \mathbf{c}$;

(b) for any two distinct indices $i \in I$ and $j \in I$, there is no Borel isomorphism of X_i onto a subset of X_j.

Formulate and prove the analogous result for Sierpiński sets in $\mathbf{R}$.

Remark 10. Additional information about Luzin sets and Sierpiński sets (also, about other small subsets of the real line) can be found in [43], [84], [145], [149], [184], [192], [202], [207], and [283].

Chapter 14
Absolutely nonmeasurable additive functions

It was shown in Chapter 13 that, assuming Martin's Axiom (**MA**), there exists an injective absolutely nonmeasurable function $f : \mathbf{R} \to \mathbf{R}$. In other words, it was demonstrated therein that some functions f acting from $\mathbf{R}$ into $\mathbf{R}$ are extremely bad from the measure-theoretical point of view, i.e., those f are nonmeasurable with respect to any nonzero σ-finite diffused measure defined on a σ-algebra of subsets of $\mathbf{R}$. In the same chapter it was also pointed out that the existence of absolutely nonmeasurable functions acting from $\mathbf{R}$ into $\mathbf{R}$ cannot be proved within **ZFC** set theory, so necessarily needs additional set-theoretical axioms.

Here we wish to develop this topic and to establish some interesting connections between absolutely nonmeasurable functions and the measure extension problem (the latter was raised by Banach many years ago).

First, it is natural to ask whether there exist (under **MA**) absolutely nonmeasurable functions $f : \mathbf{R} \to \mathbf{R}$ having important additional (e.g., algebraic or topological) properties. For instance, one can ask whether there exists an absolutely nonmeasurable homomorphism of the additive group $(\mathbf{R}, +)$ into itself.

In fact, Exercise 20 from Chapter 13 gives a positive answer to this question (assuming Martin's Axiom). More precisely, if we suppose that there exists a generalized Luzin subset X of $\mathbf{R}$ being simultaneously a vector space over the field $\mathbf{Q}$ of all rational numbers, then the required absolutely nonmeasurable homomorphism from $\mathbf{R}$ into $\mathbf{R}$ can be constructed without any difficulty. Namely, let us treat $\mathbf{R}$ as a vector space over $\mathbf{Q}$. Because we have $\mathrm{card}(\mathbf{R}) = \mathrm{card}(X) = \mathbf{c}$, the vector spaces $\mathbf{R}$ and X are isomorphic, so we may take any isomorphism $h : \mathbf{R} \to X$ between these two spaces. Then we may consider h as an injective homomorphism from $\mathbf{R}$ into $\mathbf{R}$. A simple argument presented in the same chapter (see the proof of Theorem 6 therein) yields that h is an absolutely nonmeasurable function.

Remark 1. We thus conclude that the existence of absolutely non-measurable solutions of Cauchy's functional equation can be proved under

Martin's Axiom. In this context, it is reasonable to recall that any nontrivial solution of Cauchy's functional equation is necessarily nonmeasurable with respect to the classical Lebesgue measure λ on $\mathbf{R}$ (see Theorem 4 from Chapter 11).

In the present chapter, our main goal is to obtain a certain generalization of the result of Pelc and Prikry [205] concerning the above-mentioned measure extension problem (cf. Exercise 22 from Chapter 13). The existence of absolutely nonmeasurable additive functions acting from $\mathbf{R}$ into $\mathbf{R}$ will be a starting point for our further considerations. Some other questions closely connected with measurability properties of additive functions will be discussed, too.

First, we would like to recall several notions and facts.

Let E be a nonempty set and let μ be a measure defined on a σ-algebra of subsets of E. We recall that μ is diffused (or continuous) if all singletons in E are of μ-measure zero.

Let us emphasize that the measures considered below in this chapter are always assumed to be diffused.

Let $\mathcal{M}$ be a class of measures given on various σ-algebras of subsets of E and let $f : E \to \mathbf{R}$ be a function. We shall say that f is absolutely nonmeasurable with respect to $\mathcal{M}$ if there exists no measure $\mu \in \mathcal{M}$ such that f is measurable with respect to μ.

We shall say that f is absolutely nonmeasurable if f is absolutely nonmeasurable with respect to the class of all nonzero σ-finite diffused measures on E.

Remark 2. In the definition above, the real line $\mathbf{R}$ can be replaced by any uncountable Polish topological space (or, more generally, by any uncountable Borel subset of a Polish space). Indeed, since an arbitrary uncountable Borel subset B of a Polish space is Borel isomorphic to $\mathbf{R}$ (see, e.g., [105], [149]), we have a one-to-one correspondence between all functions $f : E \to \mathbf{R}$ that are absolutely nonmeasurable with respect to a class $\mathcal{M}$, and all functions $g : E \to B$ that are absolutely nonmeasurable with respect to the same $\mathcal{M}$.

Example 1. Let $E = \mathbf{R}$. Take as $\mathcal{M}$ the class of all those measures on $\mathbf{R}$ which extend λ and are invariant under the group of all translations of $\mathbf{R}$. Let X be a Vitali set in $\mathbf{R}$ and denote by $f = f_X$ its characteristic function. In view of the Vitali theorem (see also Exercise 4 from Chapter 10), we can assert that f is absolutely nonmeasurable with respect to $\mathcal{M}$.

Example 2. Let E be an uncountable Polish topological space and let $\mathcal{M}$ denote the class of the completions of all nonzero σ-finite diffused Borel measures on E. As we know, there exists a Bernstein subset X of E (see Chapter 10). Let $f = f_X$ be the characteristic function of X. Then we can assert that f is absolutely nonmeasurable with respect to $\mathcal{M}$ (cf. Exercise 9 from Chapter 10).

Let λ_0 stand for the restriction of λ to the Borel σ-algebra $\mathcal{B}(\mathbf{R})$ (in other words, λ_0 is the standard Borel measure on $\mathbf{R}$).

Pelc and Prikry proved in their work [205] the following statement.

If the Continuum Hypothesis holds, then there exist two σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ of subsets of $\mathbf{R}$ such that

(1) the Borel σ-algebra of $\mathbf{R}$ is contained in $\mathcal{S}_1 \cap \mathcal{S}_2$;

(2) both σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ are countably generated;

(3) both σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ are invariant under the group Γ of all isometric transformations of $\mathbf{R}$;

(4) there exists a Γ-invariant measure μ_1 on $\mathcal{S}_1$ extending λ_0;

(5) there exists a Γ-invariant measure μ_2 on $\mathcal{S}_2$ extending λ_0;

(6) there is no nonzero σ-finite diffused measure defined on the σ-algebra generated by $\mathcal{S}_1 \cup \mathcal{S}_2$.

The proof of this statement given in [205] essentially utilizes the methods developed in [83] and [100]. Also, in [205] the question is posed whether the statement remains valid assuming Martin's Axiom (**MA**) instead of the Continuum Hypothesis (**CH**).

By applying the method of Kodaira and Kakutani [138] and by using absolutely nonmeasurable homomorphisms from $(\mathbf{R}, +)$ into the one-dimensional unit torus, it will be demonstrated below that the question of Pelc and Prikry is solvable positively (Theorem 2 of this chapter). Also, the reader will be able to see that our approach essentially differs from the one presented in [205] and leads to a much stronger result in terms of absolutely nonmeasurable additive functions.

Let $\mathbf{T}$ be the one-dimensional unit torus in the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Actually, this torus is defined by $\mathbf{T} = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ and therefore coincides with the unit circle in $\mathbf{R}^2$.

We will consider $\mathbf{T}$ as a commutative compact topological group with respect to the natural group operation and topology (the latter is induced by the standard Euclidean topology on $\mathbf{R}^2$). The group operation in $\mathbf{T}$ will be denoted by $+$ and, accordingly, the neutral element in $\mathbf{T}$ will be denoted by 0. The pair $(\mathbf{T}, +)$ is often called the circle group. Observe also that $(\mathbf{T}, +)$ may be identified with $(\mathbf{S}_1, \cdot)$, where $\mathbf{S}_1 = \mathbf{T}$ and $\cdot$ is the restriction to $\mathbf{S}_1$ of the standard multiplication operation for complex numbers.

Being a compact topological group, $(\mathbf{T}, +)$ is equipped with the Haar probability measure which will be denoted by ν. In fact, the completion of ν coincides with the standard Lebesgue measure on $\mathbf{T}$ taken with coefficient $1/(2\pi)$. Further, the group $(\mathbf{T}, +)$ is divisible, i.e., for each $t \in \mathbf{T}$ and for any natural number $n > 0$, there exists $z \in \mathbf{T}$ such that

$$t = nz = z + z + ... + z,$$

where in the sum of the right-hand side of this relation z is taken n times.

Some nontrivial subgroups of $(\mathbf{T}, +)$ are divisible, too. In particular, consider the subgroup G of $\mathbf{T}$ consisting of all those elements from $\mathbf{T}$ which have finite order; in other words, put

$$G = \{t \in \mathbf{T} : (\exists n < \omega)(n \neq 0 \ \&\ nt = 0)\}.$$

It can easily be verified that G is infinite, countable, and divisible.

By virtue of Exercise 3 of this chapter, the group $(\mathbf{T}, +)$ is representable in the form

$$\mathbf{T} = G + H \qquad (G \cap H = \{0\}),$$

where G is again the countable group of all those elements in $\mathbf{T}$ which have finite order, and H is a complemented subgroup of $\mathbf{T}$.

It can readily be checked that the following three relations are satisfied:

(a) H is a vector space (over $\mathbf{Q}$) isomorphic to $\mathbf{R}$;

(b) H is a ν-thick subset of $\mathbf{T}$, i.e., $\nu^*(H) = 1$;

(c) H is a second category subset of $\mathbf{T}$; moreover, H is thick in the sense of category (i.e., H intersects each second category subset of $\mathbf{T}$ possessing the Baire property).

Taking the above-mentioned facts into account, we come to the following auxiliary proposition.

Lemma 1. *Under Martin's Axiom (*$\mathbf{MA}$*), there exists a set* L *in* $\mathbf{T}$ *such that*

(1) $L \subset H$*;*

(2) L *is a generalized Luzin subset of* $\mathbf{T}$*;*

(3) L *is a vector space over* $\mathbf{Q}$*.*

Proof. Indeed, by using the standard argument (see the proof of Theorem 1 and Exercise 20 from Chapter 13), a generalized Luzin set $L \subset \mathbf{T}$ can be constructed in such a manner that all points of L would be in H and L would be a vector space over $\mathbf{Q}$. We omit the details of this construction (which are not difficult). Let us only notice that, because H is a thick subspace of $\mathbf{T}$ in the sense of category, L is a generalized Luzin subset of H.

The next auxiliary statement plays the key role for our further considerations.

Lemma 2. *Assuming Martin's Axiom (* **MA** *), there exist two functions $\phi : \mathbf{R} \to H$ and $\psi : \mathbf{R} \to H$ which satisfy the following four conditions:*

(1) ϕ and ψ are homomorphisms of vector spaces (over $\mathbf{Q}$);

(2) the graph of ϕ is $(\lambda \otimes \nu)$-thick in the product space $\mathbf{R} \times \mathbf{T}$;

(3) the graph of ψ is $(\lambda \otimes \nu)$-thick in the product space $\mathbf{R} \times \mathbf{T}$;

(4) $\phi+\psi$ is an isomorphism between $\mathbf{R}$ and L, where L is the generalized Luzin set of Lemma 1.

Proof. Let α denote the least ordinal number of cardinality $\mathbf{c}$ and let $\{Z_\xi : \xi < \alpha\}$ be the family of all Borel subsets of $\mathbf{R} \times \mathbf{T}$ with strictly positive $(\lambda \otimes \nu)$-measure. Moreover, fix some partition $\{\Xi_1, \Xi_2, \Xi_3\}$ of $[0, \alpha[$ into three sets, each of cardinality $\mathbf{c}$, and assume that every Borel subset of $\mathbf{R} \times \mathbf{T}$ whose $(\lambda \otimes \nu)$-measure is strictly positive belongs to both partial families $\{Z_\xi : \xi \in \Xi_1\}$ and $\{Z_\xi : \xi \in \Xi_2\}$. Let L be as in Lemma 1. Pick a Hamel basis $\{l_\xi : \xi < \alpha\}$ of L. By using the method of transfinite recursion, it is not hard to construct three α-sequences

$$\{x_\xi : \xi < \alpha\}, \quad \{y_\xi : \xi < \alpha\}, \quad \{y'_\xi : \xi < \alpha\},$$

which satisfy these four relations:

(a) $\{x_\xi : \xi < \alpha\}$ is a Hamel basis of $\mathbf{R}$;

(b) $(x_\xi, y_\xi) \in Z_\xi$ for any ordinal number $\xi \in \Xi_1$;

(c) $(x_\xi, y'_\xi) \in Z_\xi$ for any ordinal number $\xi \in \Xi_2$;

(d) $\{y_\xi, y'_\xi\} \subset H$ and $y_\xi + y'_\xi = l_\xi$ for each ordinal $\xi < \alpha$.

Now, we define

$$\phi(x_\xi) = y_\xi, \quad \psi(x_\xi) = y'_\xi \qquad (\xi < \alpha).$$

In view of the linear independence of $\{x_\xi : \xi < \alpha\}$, both ϕ and ψ can uniquely be extended to homomorphisms $\phi : \mathbf{R} \to H$ and $\psi : \mathbf{R} \to H$, which also determine the homomorphism $\phi+\psi : \mathbf{R} \to H$. Since the relation $(\phi + \psi)(x_\xi) = l_\xi$ holds for all $\xi < \alpha$ and $\{l_\xi : \xi < \alpha\}$ is a Hamel basis of L, we infer that $\phi + \psi$ is an isomorphism between $\mathbf{R}$ and L. Finally, by virtue of relations (b) and (c), it is clear that the graphs of ϕ and ψ are $(\lambda \otimes \nu)$-thick in the product space $\mathbf{R} \times \mathbf{T}$. This obviously finishes the proof of Lemma 2.

Let $f : \mathbf{R} \to \mathbf{T}$ be an arbitrary group homomorphism whose graph is $(\lambda \otimes \nu)$-thick in $\mathbf{R} \times \mathbf{T}$. Then f can be made measurable with respect to an appropriate invariant extension of λ_0 (see, for instance, the well-known article by Kodaira and Kakutani [138]). Indeed, for each set Z belonging to $\mathrm{dom}(\lambda_0 \otimes \nu)$, denote $Z' = \{x \in \mathbf{R} : (x, f(x)) \in Z\}$ and consider the family of sets $\mathcal{S}' = \{Z' : Z \in \mathrm{dom}(\lambda_0 \otimes \nu)\}$. It is not difficult to verify that $\mathcal{S}'$ is a σ-algebra of subsets of $\mathbf{R}$, containing $\mathrm{dom}(\lambda_0)$ and invariant under the group Γ of all isometric transformations of $\mathbf{R}$. Also, we can define a functional μ on $\mathcal{S}'$ by putting

$$\mu(Z') = (\lambda_0 \otimes \nu)(Z) \qquad (Z' \in \mathcal{S}').$$

It turns out that μ is a measure on $\mathcal{S}'$ extending λ_0 and invariant under Γ (cf. [118], [125], [138]). An easy verification of this fact is left to the reader. Besides, the definition of μ directly implies that the original homomorphism f becomes measurable with respect to μ.

Theorem 1. *Assuming Martin's Axiom (**MA**), there exist two group homomorphisms $f_1 : \mathbf{R} \to \mathbf{T}$ and $f_2 : \mathbf{R} \to \mathbf{T}$ and two measures μ_1 and μ_2 on $\mathbf{R}$, such that*

(1) μ_1 extends λ_0 and is invariant under Γ;

(2) μ_2 extends λ_0 and is invariant under Γ;

(3) f_1 is measurable with respect to μ_1;

(4) f_2 is measurable with respect to μ_2;

(5) the homomorphism $f_1 + f_2$ is absolutely nonmeasurable.

Proof. It suffices to put $f_1 = \phi$ and $f_2 = \psi$, where ϕ and ψ are as in Lemma 2. Because the graphs of ϕ and ψ are $(\lambda \otimes \nu)$-thick subsets of $\mathbf{R} \times \mathbf{T}$, they determine the corresponding Γ-invariant extensions μ_1 and μ_2 of the Borel measure λ_0. At the same time, the group homomorphism $f_1 + f_2$ is injective and its range is a generalized Luzin subset of $\mathbf{T}$. This yields at once that $f_1 + f_2$ is an absolutely nonmeasurable function (cf. Theorem 6 from Chapter 13), and so completes the proof of the statement.

Theorem 2. *Assume again Martin's Axiom. Then there exist two σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ of subsets of $\mathbf{R}$ such that*

(1) the Borel σ-algebra of $\mathbf{R}$ is contained in $\mathcal{S}_1 \cap \mathcal{S}_2$;

(2) both σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ are countably generated;

(3) both σ-algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ are invariant under the group Γ of all motions of $\mathbf{R}$;

(4) there exists a Γ-invariant measure μ_1 on $\mathcal{S}_1$ extending λ_0;

(5) there exists a Γ-invariant measure μ_2 on $\mathcal{S}_2$ extending λ_0;

(6) there is no nonzero σ-finite diffused measure on the σ-algebra generated by $\mathcal{S}_1 \cup \mathcal{S}_2$.

Proof. Let $\mathcal{B}(\mathbf{R})$ denote, as usual, the Borel σ-algebra of $\mathbf{R}$ and let $\mathcal{B}(\mathbf{R} \times \mathbf{T})$ denote the Borel σ-algebra of $\mathbf{R} \times \mathbf{T}$. Take the homomorphisms f_1 and f_2 of Theorem 1 and define

$$\mathcal{S}_1 = \{\{x : (x, f_1(x)) \in B\} : B \in \mathcal{B}(\mathbf{R} \times \mathbf{T})\};$$

$$\mathcal{S}_2 = \{\{x : (x, f_2(x)) \in B\} : B \in \mathcal{B}(\mathbf{R} \times \mathbf{T})\}.$$

A simple argument shows that these σ-algebras are the required ones.

Indeed, relations (1)–(5) are verified directly in view of Theorem 1. It remains to check the validity of relation (6).

Suppose, to the contrary, that there exists a nonzero σ-finite diffused measure μ on the σ-algebra $\mathcal{S}$ generated by the family $\mathcal{S}_1 \cup \mathcal{S}_2$ (observe that $\mathcal{S}$ is also countably generated and invariant under the group of all motions of $\mathbf{R}$). Because the homomorphism f_1 is measurable with respect to $\mathcal{S}_1$ and the homomorphism f_2 is measurable with respect to $\mathcal{S}_2$, we deduce that both f_1 and f_2 are measurable with respect to $\mathcal{S}$ (or, equivalently, with respect to μ). Consequently, the homomorphism $f_1 + f_2$ must be measurable with respect to μ, too, which contradicts the absolute nonmeasurability of $f_1 + f_2$ (see relation (5) of Theorem 1). The contradiction obtained finishes the proof.

Remark 3. One generalization of Theorem 2 for certain measure type functionals was established in [124].

Remark 4. Under Martin's Axiom, Theorems 1 and 2 can be generalized to the case of the n-dimensional Euclidean space $\mathbf{R}^n$ ($n \geq 1$) equipped with the group Γ_n which is generated by the family of all central symmetries of $\mathbf{R}^n$ (in particular, Γ_n contains the group of all translations of $\mathbf{R}^n$).

Remark 5. Actually, we do not need the full power of Martin's Axiom for obtaining Theorems 1 and 2. It suffices to utilize the corresponding properties of generalized Luzin sets, which are implied by this axiom.

The following example seems to be relevant in the context of considerations presented in this chapter.

Example 3. Let μ be a nonzero σ-finite measure on the real line $\mathbf{R}$ and let $f : \mathbf{R} \to \mathbf{R}$ be a function such that, for some nonzero σ-finite Borel measure ν on $\mathrm{ran}(f)$, the graph of f is $(\mu \otimes \nu)$-thick in the product set $\mathbf{R} \times \mathrm{ran}(f)$. Then f is measurable with respect to an appropriate extension μ' of μ (hence, f is not absolutely nonmeasurable). Indeed, we may suppose without loss of generality that ν is a Borel probability measure on $\mathrm{ran}(f)$

and we may apply once again the method of Kodaira and Kakutani [138]. For each set $Z \in \mathrm{dom}(\mu \otimes \nu)$, let us denote $Z' = \{x \in \mathbf{R} : (x, f(x)) \in Z\}$. Furthermore, introduce the family of sets $\mathcal{S}' = \{Z' : Z \in \mathrm{dom}(\mu \otimes \nu)\}$. Again, it can easily be seen that $\mathcal{S}'$ is a σ-algebra of subsets of $\mathbf{R}$. In addition, if $X \in \mathrm{dom}(\mu)$, then we trivially have

$$X \times \mathrm{ran}(f) \in \mathrm{dom}(\mu \otimes \nu), \quad X = \{x \in \mathbf{R} : (x, f(x)) \in X \times \mathrm{ran}(f)\},$$

whence it follows that $X \in \mathcal{S}'$. Consequently, we get $\mathrm{dom}(\mu) \subset \mathcal{S}'$. Now, for any set $Z \in \mathrm{dom}(\mu \otimes \nu)$, let us put $\mu'(Z') = (\mu \otimes \nu)(Z)$. A straightforward verification shows that the functional μ' is well-defined (by virtue of the thickness of the graph of our function f) and that μ' is a measure on $\mathcal{S}'$ extending the initial measure μ. The definition of μ' also implies that f turns out to be measurable with respect to μ'.

In particular, the previous argument shows that if the graph of a function $f : \mathbf{R} \to \mathbf{R}$ is $(\lambda \otimes \lambda)$-thick in the plane $\mathbf{R}^2$ (for examples of such functions, see, e.g., [78] or Theorem 5 from Chapter 10), then f can be made measurable with respect to an appropriate extension of λ.

In this context, the following question naturally arises.

Let μ be a measure on $\mathbf{R}$ extending λ and such that there exists a function acting from $\mathbf{R}$ into $\mathbf{R}$, whose graph is $(\mu \otimes \lambda)$-thick in the plane $\mathbf{R}^2$. Does there exist a group homomorphism from $\mathbf{R}$ into $\mathbf{R}$ whose graph is also $(\mu \otimes \lambda)$-thick in $\mathbf{R}^2$?

We do not know an answer to this question.

EXERCISES

1. Put $E = \mathbf{R}$ and let $\mathcal{M}$ be the class of all those measures on $\mathbf{R}$ which extend λ and are quasi-invariant under the group of all motions of $\mathbf{R}$.

Show that there exists a Vitali set Y in $\mathbf{R}$ whose characteristic function f_Y is not absolutely nonmeasurable with respect to $\mathcal{M}$.

2*. Let $(U, +)$ be an arbitrary commutative group and let $(V, +)$ be a divisible commutative group. Suppose that some partial homomorphism $\phi : U \to V$ is given (i.e., ϕ is a homomorphism acting from a subgroup of U into V).

By using the Zorn lemma, show that ϕ is extendible to a homomorphism acting from U into V.

3. Let $(U, +)$ be an arbitrary commutative group and let W be a divisible subgroup of U.

Applying the result of the previous exercise, prove that W is a direct summand in U. In other words, demonstrate that U admits a representation in the form

$$U = W + W' \qquad (W \cap W' = \{0\})$$

for some subgroup W' of U (in general, W' is not uniquely determined).

Show also that any direct summand in a divisible commutative group is divisible, too.

4. Verify the validity of relations (a), (b), and (c) formulated before Lemma 1 in this chapter.

5*. Denote by μ' the completion of the measure μ which is indicated before formulation of Theorem 1.

Prove that μ' possesses the uniqueness property, i.e., for an arbitrary σ-finite Γ-invariant measure θ with $\mathrm{dom}(\theta) = \mathrm{dom}(\mu')$, there exists a real coefficient $t \geq 0$ such that $\theta = t \cdot \mu'$ (in other words, any σ-finite Γ-invariant measure defined on the domain of μ' is proportional to μ').

Remark 6. In particular, we see that there are Γ-invariant strong extensions of λ possessing the uniqueness property (similarly to λ). Moreover, it is known that there are even nonseparable Γ-invariant extensions of λ with the same property (in this connection, see [118] and [125]).

6. Let E be a set and let $f : E \to \mathbf{R}$ be a function.

Show that the following two assertions are equivalent:

(a) f is absolutely nonmeasurable;

(b) the set $\mathrm{ran}(f)$ is universal measure zero and $\mathrm{card}(f^{-1}(t)) \leq \omega$ for all points $t \in \mathbf{R}$.

Deduce from this equivalence that an absolutely nonmeasurable function on E exists if and only if there is a universal measure zero subset X of $\mathbf{R}$ with $\mathrm{card}(X) = \mathrm{card}(E)$. In particular, if $\mathrm{card}(E) > \mathbf{c}$, then no real-valued function on E is absolutely nonmeasurable.

7*. Let S be an equivalence relation on $\mathbf{R}$, all equivalence classes of which are at most countable.

A mapping $f : \mathbf{R} \to \mathbf{R}$ is called a Vitali type function for the relation S if $\mathrm{ran}(f)$ is a selector of the partition of $\mathbf{R}$ canonically determined by S and $(r, f(r)) \in S$ whenever $r \in \mathbf{R}$.

Vitali's classical result (see Chapter 10) implies that if V is the Vitali equivalence relation on $\mathbf{R}$ (i.e., $(x, y) \in V \Leftrightarrow x - y \in \mathbf{Q}$), then every Vitali type function for V is nonmeasurable with respect to any translation invariant measure on $\mathbf{R}$ extending λ.

Verify that there are additive Vitali type functions for V.

Conclude that there exist some solutions of Cauchy's functional equation which are absolutely nonmeasurable with respect to the class of all translation invariant measures on $\mathbf{R}$ extending λ.

On the other hand, prove that if f is an arbitrary Vitali type function for V, then f is measurable with respect to a certain measure on $\mathbf{R}$ extending the Lebesgue measure λ.

8*. Assuming Martin's Axiom, demonstrate that there exist a generalized Luzin set $X \subset \mathbf{R}$ and an equivalence relation $S \subset \mathbf{R} \times \mathbf{R}$ satisfying the following two conditions:

(a) $\mathrm{card}(S(r)) = \omega$ for any $r \in \mathbf{R}$;

(b) X is a selector of the partition $\{S(r) : r \in \mathbf{R}\}$ of $\mathbf{R}$.

Let $h : \mathbf{R} \to \mathbf{R}$ be a Vitali type function for S such that $\mathrm{ran}(h) = X$.

Verify that the function h is absolutely nonmeasurable.

Infer from this fact that the validity of the result presented in Exercise 7 is essentially based on specific properties of the Vitali partition of $\mathbf{R}$.

Remark 7. It can be proved within **ZFC** theory that there exists a Vitali set absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on $\mathbf{R}$ (see [132]). It immediately follows from this result that there exist Vitali type functions for V which are absolutely nonmeasurable with respect to the same class of measures.

9. Show that there exists a function $g : \mathbf{R} \to \mathbf{R}$ having the following property: for any σ-finite diffused Borel measure μ on $\mathbf{R}$ and for any σ-finite measure ν on $\mathbf{R}$, the graph of g is a $(\mu \otimes \nu)$-thick subset of the plane $\mathbf{R} \times \mathbf{R}$.

In order to establish this fact, keep in mind the existence of a partition $\{X_t : t \in \mathbf{R}\}$ of $\mathbf{R}$ such that all X_t $(t \in \mathbf{R})$ are Bernstein sets in $\mathbf{R}$.

Try to construct a solution of Cauchy's functional equation possessing the same property.

10. Let $E \neq \{0\}$ be a separable Banach space. Demonstrate that E admits a representation in the form

$$E = X_1 + X_2 \qquad (X_1 \cap X_2 = \{0\}),$$

where X_1 and X_2 satisfy these two conditions:

(a) both X_1 and X_2 are vector spaces over $\mathbf{Q}$;

(b) both X_1 and X_2 are Bernstein subsets of E.

Applying such a representation, prove that there exists a group homomorphism $h : E \to \mathbf{T}$ having the following property: for any σ-finite diffused

Borel measure μ on E, the graph of h is a $(\mu \otimes \nu)$-thick subset of the product group $E \times \mathbf{T}$ (here ν denotes the Haar probability measure on $\mathbf{T}$).

Infer from this property that

(c) for every σ-finite diffused Borel measure μ on E, the function h becomes measurable with respect to an appropriate extension μ' of μ;

(d) if an initial σ-finite diffused Borel measure μ on E is invariant (quasi-invariant) under a subgroup G of E, then the measure μ' is invariant (quasi-invariant) under the same group G.

Remark 8. The results formulated in Exercises 9 and 10 strengthen some statements given in Chapter 10 (see, for instance, Theorem 5 from that chapter).

11. Assuming Martin's Axiom (**MA**), demonstrate that **R** admits a representation in the form

$$\mathbf{R} = Y_1 + Y_2 \qquad (Y_1 \cap Y_2 = \{0\}),$$

where Y_1 and Y_2 satisfy the following two conditions:

(a) both Y_1 and Y_2 are vector spaces over **Q**;

(b) both Y_1 and Y_2 are generalized Luzin sets (respectively, generalized Sierpiński sets) in **R**.

Deduce from this fact that there exist two generalized Luzin subsets L_1 and L_2 of **R** whose algebraic sum $L_1 + L_2$ is a Bernstein subset of **R**.

Remark 9. We thus see that, under Martin's Axiom, there are two universal measure zero sets in **R** (even two strong measure zero sets in **R**) whose algebraic sum is absolutely nonmeasurable with respect to the class of the completions of all nonzero σ-finite diffused Borel measures on **R**. Let us underline that this result can be established only under some additional set-theoretical assumptions, because there exist models of **ZFC** in which $\omega_1 < \mathbf{c}$ and the cardinalities of all universal measure zero subsets of **R** do not exceed ω_1 (see [157], [184]).

12*. Prove in **ZFC** theory that there exists a Lebesgue measure zero set $X \subset \mathbf{R}$ such that $X + X$ is not measurable in the Lebesgue sense.

For this purpose, use the technique of Hamel bases.

Remark 10. The existence of X was first obtained by Sierpiński [237]).

13*. Show in **ZFC** theory that **R** admits a representation in the form

$$\mathbf{R} = X + Y \qquad (X \cap Y = \{0\}),$$

where X and Y satisfy the following two conditions:

(a) both X and Y are vector spaces over $\mathbf{Q}$;

(b) both X and Y are of Lebesgue measure zero.

Remark 11. The result of Exercise 13 is due to Erdös, Kunen, and Mauldin (see [66]).

14*. Demonstrate that there exists a subset Z of $\mathbf{R}$ possessing the following three properties:

(a) Z is of first category in $\mathbf{R}$;

(b) Z is of Lebesgue measure zero;

(c) for any countable family $\{h_i : i \in I\}$ of translations of $\mathbf{R}$, the relation $\cap\{h_i + Z : i \in I\} \neq \emptyset$ holds true.

Assuming the Continuum Hypothesis (**CH**), starting with the above-mentioned properties of Z and applying the method of transfinite recursion, construct two subsets X and Y of $\mathbf{R}$ such that

(d) both X and Y are vector spaces over $\mathbf{Q}$;

(e) $X + Y = \mathbf{R}$ and $X \cap Y = \{0\}$;

(f) both X and Y are of first category in $\mathbf{R}$ and have Lebesgue measure zero.

15. As known, the real line $\mathbf{R}$ can be represented in the form $\mathbf{Q} + W$, where W is some vector space over $\mathbf{Q}$ and $\mathbf{Q} \cap W = \{0\}$. Actually, W is a Vitali subset of $\mathbf{R}$ and, simultaneously, is a hyperplane in $\mathbf{R}$ regarded as a vector space over $\mathbf{Q}$ (see Chapters 10 and 11). Thus, one can conclude that there exists an infinite countable (hence Borel) subgroup of $\mathbf{R}$ which is a direct summand in $\mathbf{R}$.

Suppose now that $\mathbf{R}$ is represented in the form $\mathbf{R} = X + Y$, where X and Y are some analytic (Suslin) subgroups of $\mathbf{R}$ and $X \cap Y = \{0\}$.

Demonstrate that either $X = \{0\}$ or $Y = \{0\}$.

16*. Prove that there exists a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the following three conditions:

(a) f is a solution of the Cauchy functional equation;

(b) f is a Sierpiński–Zygmund function (consequently, f turns out to be a nontrivial solution of the Cauchy functional equation);

(c) f is measurable with respect to some $\mathbf{R}$-quasi-invariant extension of the Lebesgue measure λ (in particular, f is not absolutely nonmeasurable).

Remark 12. The last exercise of this chapter shows that there exist additive Sierpiński–Zygmund functions which are rather good from the measure-theoretical point of view.

Chapter 15
Egorov type theorems

One of the earliest important results in real analysis and Lebesgue measure theory was obtained by Egorov [62], who discovered close relationships between the uniform convergence and the convergence almost everywhere of a sequence of real-valued Lebesgue measurable functions. This classical result (widely known now as Egorov's theorem) has numerous consequences and applications in mathematical analysis and measure theory. For example, it suffices to mention that another classical result in real analysis – Luzin's theorem on the structure of Lebesgue measurable functions – can easily be deduced by starting with the Egorov theorem.

Here we wish to discuss some aspects of Egorov's theorem and, in addition, to show that, for a sequence of nonmeasurable real-valued functions, there is no hope of getting a reasonable analogue of this theorem. In other words, we are going to demonstrate in our further considerations that there are some sequences of strange real-valued functions for which even weak analogues of Egorov type theorems fail to be true.

First of all, we want to give Egorov's theorem in a form slightly more general than those of its versions which are usually presented in standard courses of real analysis and measure theory. In order to do this, we need some auxiliary notions and facts.

Let E be a nonempty base (ground) set and let $\mathcal{S}$ be some class of subsets of E, satisfying the following two conditions:

(1) $\emptyset \in \mathcal{S}$ and $E \in \mathcal{S}$;

(2) $\mathcal{S}$ is closed under countable unions and countable intersections.

Suppose also that a functional $\nu : \mathcal{S} \to \mathbf{R}$ is given such that

(a) for any increasing (with respect to the inclusion relation) sequence of sets $\{X_n : n < \omega\} \subset \mathcal{S}$, we have

$$\nu(\cup\{X_n \ : \ n < \omega\}) \leq \sup\{\nu(X_n) \ : \ n < \omega\};$$

(b) for any decreasing (with respect to the inclusion relation) sequence of sets $\{Y_n : n < \omega\} \subset \mathcal{S}$, we have

$$\nu(\cap\{Y_n \ : \ n < \omega\}) \geq \inf\{\nu(Y_n) \ : \ n < \omega\}.$$

In this case, we say that $\mathcal{S}$ is an admissible class of subsets of E and ν is an admissible functional on $\mathcal{S}$.

Notice that, in mathematical analysis, there are many natural examples of admissible functionals. This can be confirmed by the following fairly standard example.

Example 1. Let E be a nonempty set and let $\mathcal{S}$ be some σ-algebra of subsets of E. Then it is obvious that $\mathcal{S}$ is an admissible class. Suppose, in addition, that ν is a finite measure on $\mathcal{S}$. Then it can easily be observed that ν is an admissible functional on E. Actually, in this case, the inequalities of (a) and (b) are reduced to the equalities.

Analogously to the concept of measurability of real-valued functions with respect to ordinary measures, the concept of measurability of real-valued functions with respect to admissible functionals may be introduced and investigated.

Namely, we say that a function $f : E \to \mathbf{R}$ is measurable with respect to an admissible functional ν on E (or, simply, f is ν-measurable) if, for each open interval $]a, b[\ \subset \mathbf{R}$, the relation $f^{-1}(]a, b[) \in \mathrm{dom}(\nu)$ holds true.

Obviously, the same definition can be introduced for partial functions acting from E into $\mathbf{R}$.

The properties of functions (partial functions) measurable with respect to admissible functionals turn out to be very similar to the properties of functions (partial functions) measurable in the usual sense. Exercises 2 and 3 of the present chapter more or less illustrate this fact.

Now, we are able to formulate and prove a direct analogue of Egorov's theorem for sequences of real-valued functions measurable with respect to an admissible functional.

Theorem 1. *Let E be a ground set and let ν be an admissible functional on E. Suppose, in addition, that a sequence $\{f_n : n < \omega\}$ of ν-measurable functions is given, pointwise convergent on E, and let f denote the corresponding limit function.*

Then, for each real $\varepsilon > 0$, there exists a set $X \in \mathrm{dom}(\nu)$ satisfying these two relations:

(1) $\nu(E) - \nu(X) \leq \varepsilon$;

(2) the sequence of functions $\{f_n|X \ : \ n < \omega\}$ converges uniformly to the function $f|X$.

Proof. For any natural number m, let us denote

$$E_{0,m} = \{x \in E \ : \ (\forall n \geq m)(|f_n(x) - f(x)| < 1)\}.$$

It is easy to check that the set $E_{0,m}$ belongs to $\mathrm{dom}(\nu)$ and

$$E_{0,0} \subset E_{0,1} \subset ... \subset E_{0,m} \subset ..., \qquad \cup\{E_{0,m} \ : \ m < \omega\} = E.$$

Consequently, there is a natural index m_0 such that $\nu(E) - \nu(E_{0,m_0}) < \varepsilon$. Let us put $X_0 = E_{0,m_0}$. Further, for each $m < \omega$, consider the set

$$E_{1,m} = \{x \in X_0 \ : \ (\forall n \geq m)(|f_n(x) - f(x)| < 1/2)\}.$$

Evidently, $E_{1,m}$ belongs to $\mathrm{dom}(\nu)$ and

$$E_{1,0} \subset E_{1,1} \subset ... \subset E_{1,m} \subset ..., \qquad \cup\{E_{1,m} \ : \ m < \omega\} = X_0.$$

Consequently, there is a natural index m_1 such that $\nu(E) - \nu(E_{1,m_1}) < \varepsilon$. Let us put $X_1 = E_{1,m_1}$. Continuing in this manner, we will be able to define by recursion a certain sequence $\{X_k : k < \omega\}$ of sets, all of which belong to $\mathrm{dom}(\nu)$ and satisfy the relations

(i) $X_0 \supset X_1 \supset ... \supset X_k \supset ...$;

(ii) for any $k < \omega$, we have $\nu(E) - \nu(X_k) < \varepsilon$;

(iii) for any $k < \omega$, there is a natural number m_k such that

$$(\forall n \geq m_k)(\forall x \in X_k)(|f_n(x) - f(x)| < 1/2^k).$$

Finally, we put $X = \cap\{X_k \ : \ k < \omega\}$. Then, by virtue of the definition of an admissible functional, we may write $\nu(E) - \nu(X) \leq \varepsilon$, and it can easily be verified that the sequence of the restricted functions $\{f_n|X \ : \ n < \omega\}$ converges uniformly to the restricted function $f|X$. This completes the proof of Theorem 1.

Obviously, the theorem proved above immediately implies the classical Egorov theorem [62]. It suffices to take as ν an arbitrary finite measure on E. In this connection, let us recall that for σ-finite measures the direct analogue of Egorov's theorem is not true in general (see Exercise 4 of the present chapter).

In conformity with Egorov's theorem, any convergent sequence of measurable real-valued functions converges uniformly on some large measurable subset of E (of course, "large" means here that the measure of the complement of this subset may be taken arbitrarily small). In particular, if a given finite Borel measure on a topological space E is nonzero, diffused, and inner regular, then we immediately obtain that every convergent sequence of measurable real-valued functions on E converges uniformly on an uncountable closed subset of E. Hence, if E is an uncountable Polish topological space equipped with a nonzero finite diffused Borel measure, then, for any convergent sequence of measurable real-valued functions on E, there exists a nonempty compact perfect subset of E (actually, a subset homeomorphic to the Cantor discontinuum) on which the sequence converges uniformly.

In connection with these observations, it makes sense to consider the following more general situation.

Let E be an arbitrary uncountable complete metric space without isolated points and let $\{f_n : n < \omega\}$ be a sequence of real-valued Borel functions on E, such that, for some constant $d \geq 0$, we have

$$(\forall n < \omega)(\forall x \in E)(|f_n(x)| \leq d).$$

In other words, our sequence of functions is uniformly bounded. Then one may ask whether there exist a nonempty perfect compact subset P of E and an infinite subset K of ω, for which the partial sequence of functions $\{f_n|P \ : \ n \in K\}$ is uniformly convergent on P.

Evidently, we may restrict our considerations to the case where the given space E is homeomorphic to the Cantor discontinuum (because, according to the well-known theorem from general topology, every complete metric space without isolated points contains a homeomorphic image of the Cantor discontinuum). Actually, we may suppose from the beginning that our E is an uncountable Polish topological space.

Also, in order to get a positive solution to the question formulated above, it suffices to demonstrate that there exist an infinite subset K of ω and a nonempty perfect compact subset P' of E, such that the sequence $\{f_n|P' \ : \ n \in K\}$ converges pointwise on P'. Indeed, suppose that this fact has already been established and equip the set P' with some Borel diffused probability measure μ. Then, evidently, we may apply Egorov's theorem to μ and to the sequence of functions $\{f_n|P' \ : \ n \in K\}$. In accordance with this theorem, there exists a Borel set $X \subset P'$ with $\mu(X) > 1/2$, for which the sequence of functions $\{f_n|X \ : \ n \in K\}$ converges uniformly. Since μ is diffused and $\mu(X) > 0$, we obviously obtain the relation $\mathrm{card}(X) \geq \omega_1$ and, consequently, $\mathrm{card}(X) = \mathbf{c}$ because X is Borel in P'. It is clear now that X contains a nonempty perfect compact subset P for which the sequence of functions $\{f_n|P \ : \ n \in K\}$ converges uniformly, too.

Mazurkiewicz was the first mathematician to prove that, for any uniformly bounded sequence of real-valued Borel functions given on an uncountable Polish space E, there exists a subset of E homeomorphic to the Cantor discontinuum, on which some subsequence of the sequence converges uniformly (see his remarkable work [179]).

In order to present a detailed proof of this interesting result, we need some auxiliary notions and simple statements concerning those notions.

Let E be an uncountable Polish space and let Φ be a family of real-valued functions defined on E. We say that the family Φ is semicompact if, for every sequence $\{\phi_n : n < \omega\} \subset \Phi$ and for each nonempty perfect set $P \subset E$, there exist an infinite subset K of ω and a nonempty perfect set P' contained in P, such that the partial sequence of functions $\{\phi_n : n \in K\}$ converges pointwise on P'.

We say that a family $\mathcal{S}$ consisting of some Borel subsets of E is semicompact if the corresponding family of characteristic functions $\{f_X : X \in \mathcal{S}\}$ is semicompact in the sense of the definition above.

The following auxiliary proposition yields a much more vivid description of semicompact families of Borel sets in E.

Lemma 1. *Let $\mathcal{S}$ be a family of Borel subsets of an uncountable Polish space E. Then these two assertions are equivalent:*

(1) the family $\mathcal{S}$ is semicompact;

(2) for any sequence $\{X_n : n < \omega\}$ of sets from $\mathcal{S}$ and for each nonempty perfect subset P of E, there exists an infinite set $K \subset \omega$ such that

$$\mathrm{card}((\cap\{X_n : n \in K\}) \cap P) > \omega \ \vee \ \mathrm{card}((\cap\{E \setminus X_n : n \in K\}) \cap P) > \omega.$$

Proof. Notice first that the implication (2) $\Rightarrow$ (1) is almost trivial because if, for example, we have $\mathrm{card}((\cap\{X_n : n \in K\}) \cap P) > \omega$ for a nonempty perfect set $P \subset E$ and for some infinite subset K of ω, then the set $(\cap\{X_n : n \in K\}) \cap P$ contains a nonempty perfect subset P', and the sequence of characteristic functions $\{f_{X_n} : n \in K\}$ converges pointwise on the set P' to the characteristic function $f_{P'}$ (actually, all the functions f_{X_n} $(n \in K)$ are identically equal to 1 on P').

Now, let us verify the implication (1) $\Rightarrow$ (2). Suppose that relation (1) is fulfilled. Let $\{X_n : n < \omega\}$ be an arbitrary sequence of sets from $\mathcal{S}$ and let P be a nonempty perfect subset of E. We may assume, without loss of generality, that $P = E$. Then, according to (1), there exists an infinite subset K of ω such that the corresponding sequence of characteristic functions $\{f_{X_n} : n \in K\}$ is convergent on an uncountable Borel subset Y of E. Let us denote

$$f(y) = \lim_{n \to +\infty, n \in K} f_{X_n}(y) \qquad (y \in Y).$$

Obviously, f is a Borel function on Y and $\mathrm{ran}(f) \subset \{0, 1\}$. Therefore, at least one of the sets

$$Y_0 = \{y \in Y : f(y) = 0\}, \qquad Y_1 = \{y \in Y : f(y) = 1\}$$

is uncountable. Suppose, for example, that $\mathrm{card}(Y_1) > \omega$. Then, by taking account of the formula

$$Y_1 = Y_1 \cap \mathrm{limsup}\{X_n \ : \ n \in K\} = Y_1 \cap \mathrm{liminf}\{X_n \ : \ n \in K\},$$

it can easily be checked that, for some infinite subset K_1 of K, the inequality

$$\mathrm{card}(\cap\{X_n \ : \ n \in K_1\}) > \omega$$

is satisfied. If $\mathrm{card}(Y_0) > \omega$, then an analogous argument applied to the sequence of characteristic functions $\{f_{E \setminus X_n} : n \in K\}$ yields the existence of an infinite subset K_0 of K for which the inequality

$$\mathrm{card}(\cap\{E \setminus X_n \ : \ n \in K_0\}) > \omega$$

is fulfilled. This establishes the validity of the implication (1) ⇒ (2) and finishes the proof of Lemma 1.

The next two auxiliary propositions also are not hard to prove.

Lemma 2. *Let Φ be a semicompact family of real-valued functions defined on an uncountable Polish space E. Then, for any real number $d \geq 0$, the family of functions $\{t\phi \ : \ |t| \leq d, \ \phi \in \Phi\}$ is semicompact, too.*

Lemma 3. *Let Φ_1 and Φ_2 be any two semicompact families of real-valued functions defined on an uncountable Polish space E. Then the family of functions $\{\phi_1 + \phi_2 \ : \ \phi_1 \in \Phi_1, \ \phi_2 \in \Phi_2\}$ is semicompact, too.*

It immediately follows from the above two lemmas (by using the method of induction) that if $d \geq 0$ and Φ_1, Φ_2,..., Φ_k are some semicompact families of real-valued functions on an uncountable Polish space E, then the family of all those functions ϕ which can be represented in the form

$$\phi = t_1\phi_1 + t_2\phi_2 + ... + t_k\phi_k,$$

where

$$|t_1| \leq d, \ |t_2| \leq d, ..., \ |t_k| \leq d, \quad \phi_1 \in \Phi_1, \ \phi_2 \in \Phi_2, ..., \phi_k \in \Phi_k,$$

is also semicompact.

Lemma 4. *Let Φ be a semicompact family of bounded real-valued functions on an uncountable Polish space E and let Φ^* denote the family of all* *those functions which are uniform limits of sequences of functions belonging to Φ. Then the family Φ^* is semicompact, too.*

Proof. Let $\{\phi_n^* : n < \omega\}$ be an arbitrary sequence of functions from the family Φ^*. By virtue of the definition of Φ^*, for every natural number n, there exists a function $\phi_n \in \Phi$ such that

$$||\phi_n^* - \phi_n|| \leq 1/(n+1).$$

Let us consider the family of functions $\{\phi_n : n < \omega\}$. According to our assumption, Φ is semicompact. Hence, for each nonempty perfect set P in E, there exist an infinite subset K of ω and a nonempty perfect subset P' of P, such that the partial sequence of functions $\{\phi_n|P' : n \in K\}$ converges pointwise on P' to some function ϕ defined on P'. Now, it can readily be verified that the corresponding sequence of functions $\{\phi_n^*|P' : n \in K\}$ also converges pointwise to ϕ. This completes the proof of Lemma 4.

The following auxiliary statement plays the key role in our further considerations.

Lemma 5. *The family of all Borel subsets of an uncountable Polish topological space E is semicompact.*

Proof. Let $\{X_n : n < \omega\}$ be an arbitrary sequence of Borel subsets of E and let P be a nonempty perfect set in E. Denote by $2^{<\omega}$ the family of all finite sequences whose terms belong to the set $2 = \{0, 1\}$. We are going to construct (by recursion) a dyadic family $\{P_\sigma : \sigma \in 2^{<\omega}\}$ of nonempty perfect sets in E and a sequence $\{n_k : k < \omega\}$ of natural numbers, such that

(a) $P_\emptyset$ is contained in P;

(b) for any $\sigma \in 2^{<\omega}$, we have

$$P_{\sigma 0} \cup P_{\sigma 1} \subset P_\sigma, \qquad P_{\sigma 0} \cap P_{\sigma 1} = \emptyset;$$

(c) for any nonempty $\sigma \in 2^{<\omega}$, we have

$$\mathrm{diam}(P_\sigma) \leq 1/(\mathrm{lh}(\sigma)),$$

where the symbol $\mathrm{lh}(\sigma)$ denotes the length of σ;

(d) the sequence $\{n_k : k < \omega\}$ is strictly increasing;

(e) for each nonzero $k < \omega$, the inclusion

$$\cup\{P_\sigma : \mathrm{lh}(\sigma) = k\} \subset X_{n_1} \cap X_{n_2} \cap ... \cap X_{n_k}$$

is fulfilled.

Pick a nonempty perfect set $P_\emptyset \subset P$ and a natural number n_0 arbitrarily.

Suppose now that the natural numbers $n_0 < n_1 < \ ... \ < n_k$ and the partial family $\{P_\sigma \ : \ \sigma \in 2^{<\omega}, \ \mathrm{lh}(\sigma) \leq k\}$ of nonempty perfect sets in E have already been defined.

Only two cases are possible here.

1. There exists a natural number $n > n_k$ for which all the sets

$$X_n \cap P_\sigma \qquad (\mathrm{lh}(\sigma) = k)$$

are uncountable. In this case, we may put $n_{k+1} = n$ and, for any $\sigma \in 2^{<\omega}$ with $\mathrm{lh}(\sigma) = k$, we can construct two nonempty perfect sets $P_{\sigma 0}$ and $P_{\sigma 1}$ satisfying the relations

$$P_{\sigma 0} \cap P_{\sigma 1} = \emptyset, \qquad P_{\sigma 0} \cup P_{\sigma 1} \subset P_\sigma \cap X_{n_{k+1}},$$

$$\mathrm{diam}(P_{\sigma 0}) \leq 1/(k+1), \qquad \mathrm{diam}(P_{\sigma 1}) \leq 1/(k+1).$$

So we see that the process of our construction can be continued.

2. For each natural number $n > n_k$, there exists a σ from $2^{<\omega}$ with $\mathrm{lh}(\sigma) = k$, such that $\mathrm{card}(P_\sigma \cap X_n) \leq \omega$. In this case, since the family $\{P_\sigma \ : \ \mathrm{lh}(\sigma) = k\}$ is finite, we can find an infinite subset M of ω and a $\sigma' \in 2^{<\omega}$ with $\mathrm{lh}(\sigma') = k$, such that $\mathrm{card}(P_{\sigma'} \cap X_n) \leq \omega$ for all numbers n belonging to M. From the latter relation we obtain

$$\mathrm{card}((\cap\{E \setminus X_n \ : \ n \in M\}) \cap P_{\sigma'}) > \omega,$$

which immediately gives the desired result (in view of Lemma 1).

Thus, we may restrict our considerations only to case 1. As mentioned above, in this case, the described construction can be continued and after ω many steps it yields a dyadic family $\{P_\sigma \ : \ \sigma \in 2^{<\omega}\}$ of nonempty perfect subsets of E. Now, putting

$$P' = \bigcap_{k<\omega} (\cup\{P_\sigma \ : \ \mathrm{lh}(\sigma) = k\}),$$

we get a nonempty perfect set P' such that

$$P' \subset (\cap\{X_{n_k} \ : \ 1 \leq k < \omega\}) \cap P.$$

Hence, we come to the inequality

$$\mathrm{card}((\cap\{X_{n_k} \ : \ 1 \leq k < \omega\}) \cap P) > \omega.$$

Using once again Lemma 1, we complete the proof of Lemma 5.

Now, taking into account all the preceding lemmas, we are able to formulate and prove the following result of Mazurkiewicz [179].

Theorem 2. *Let Φ be an arbitrary uniformly bounded family of real-valued Borel functions on an uncountable Polish space E. Then Φ is semicompact.*

Proof. Because our Φ is uniformly bounded, there exists a real number $d \geq 0$ such that $(\forall \phi \in \Phi)(||\phi|| \leq d)$. Let us denote by Ψ the family of all those functions ψ which satisfy the following two relations:

(1) $||\psi|| \leq d$;

(2) ψ is a linear combination of characteristic functions of some Borel subsets of E.

Then, according to Lemmas 2, 3, and 5, the family Ψ is semicompact. Also, it is clear that the original family Φ is contained in the closure of Ψ (with respect to the topology of uniform convergence on E). Hence, in view of Lemma 4, the family Φ is semicompact as well. This finishes the proof of Mazurkiewicz's theorem.

Let us observe that if $E = \mathbf{R}$, then Theorem 2 can be extended to an arbitrary family of uniformly bounded real-valued Lebesgue measurable functions on E and to an arbitrary family of uniformly bounded real-valued functions on E having the Baire property. Indeed, in order to obtain the corresponding results, it suffices to apply the following well-known fact:

For an arbitrary Lebesgue measurable (respectively, having the Baire property) real-valued function f on $\mathbf{R}$, there exists a real-valued Borel function g on $\mathbf{R}$ such that the set $\{x \in \mathbf{R} : f(x) \neq g(x)\}$ is of Lebesgue measure zero (respectively, of first category).

As demonstrated above, for any uniformly bounded sequence of real-valued functions possessing good descriptive properties, we have the pointwise convergence (and even the uniform convergence) of an appropriate subsequence on some nonempty perfect set, hence, on some set of cardinality continuum. However, various uniformly bounded sequences of real-valued functions are possible, which are extremely bad for the pointwise convergence. The following statement (essentially due to Sierpiński) shows that the existence of such sequences can be directly deduced from the existence of a Luzin set Z in an uncountable Polish space E, with $\mathrm{card}(Z) = \mathbf{c}$. In this connection, it is reasonable to recall here that the existence of a Luzin set of cardinality continuum is easily implied by **CH** (see Theorem 1 from Chapter 13).

Theorem 3. *Let Z be a Luzin subset of the Cantor space 2^ω, satisfying the equality* $\mathrm{card}(Z) = \mathbf{c}$. *Then there exists a sequence $\{X_n : n < \omega\}$ of* *subsets of 2^ω such that, for each infinite subset K of ω, the corresponding partial sequence of characteristic functions $\{f_{X_n} : n \in K\}$ converges pointwise only on a countable subset of 2^ω.*

Proof. For every $n \in \mathbf{N}$, let us denote $B_n = \{x \in 2^\omega : x_n = 1\}$. The sets B_n $(n < \omega)$ and their complements are clopen in the Cantor space 2^ω and generate a base of the standard product topology on 2^ω. It can easily be seen that, for any infinite subset K of ω, both intersections

$$\cap\{B_n \ : \ n \in K\}, \qquad \cap\,\{2^\omega \setminus B_n \ : \ n \in K\}$$

are nowhere dense closed subsets of 2^ω. Because Z is a Luzin set in 2^ω, we have

$$\mathrm{card}(\cap\{B_n : n \in K\} \cap Z) \leq \omega, \quad \mathrm{card}(\cap\{2^\omega \setminus B_n : n \in K\} \cap Z) \leq \omega.$$

Now, let $h : 2^\omega \to 2^\omega$ be an injective mapping such that $h(2^\omega) = Z$. We put

$$X_n = h^{-1}(B_n) \qquad (n < \omega).$$

Consider the sequence of characteristic functions $\{f_{X_n} : n < \omega\}$. We assert that this sequence is as required. Indeed, it immediately follows from the definition of the family of sets $\{X_n : n < \omega\}$ that, for any infinite subset K of ω, the intersections

$$\cap\{X_n \ : \ n \in K\}, \qquad \cap\,\{2^\omega \setminus X_n \ : \ n \in K\}$$

are at most countable. But from this fact we easily infer that the corresponding partial sequence of characteristic functions $\{f_{X_n} : n \in K\}$ can be convergent pointwise only on a countable subset of 2^ω. Theorem 3 has thus been proved.

Remark 1. Theorem 3 is established under the assumption of the existence of a Luzin set with cardinality equal to $\mathbf{c}$. As we know (see, e.g., Chapter 13), if Martin's Axiom and the negation of the Continuum Hypothesis hold, then there are no Luzin sets in the real line $\mathbf{R}$ (and in the Cantor discontinuum 2^ω). So, in such a case, the above argument does not work.

The last theorem of this chapter, presented below, shows that under Martin's Axiom the pointwise convergence of an appropriate subsequence of real-valued functions can be achieved for any subset with small cardinality (i.e., with cardinality strictly less than $\mathbf{c}$). Actually, in order to establish the desired result, we do not need the full power of Martin's Axiom. It suffices to apply one purely combinatorial assertion concerning certain families of infinite subsets of ω. This auxiliary combinatorial assertion is formulated in Exercise 10.

Theorem 4. *Assume Martin's Axiom (***MA***). Let E be an arbitrary set of cardinality* $\mathbf{c}$*, let X be a subset of E with* $\mathrm{card}(X) < \mathrm{card}(E)$*, and let $\{f_n : n < \omega\}$ be a uniformly bounded sequence of real-valued functions given on E. Then there exists an infinite subset M of ω such that the partial sequence of functions $\{f_n|X : n \in M\}$ converges pointwise on X.*

Proof. Let us put $\kappa = \mathrm{card}(X)$ and let $\{x_\xi : \xi < \kappa\}$ be some enumeration of all elements of X. By using the notation and result of Exercise 10, it is not hard to define recursively a family $\{M_\xi : \xi < \kappa\}$ of infinite subsets of ω, satisfying the following two conditions:

(a) $M_\zeta \preceq M_\xi$ for $\xi \leq \zeta < \kappa$;

(b) for each $\xi < \kappa$, the partial sequence of reals $\{f_n(x_\xi) : n \in M_\xi\}$ is convergent.

Now, applying the result of Exercise 10 once more, we can define an infinite subset M of ω such that $(\forall \xi < \kappa)(M \preceq M_\xi)$. Then it is readily verified that the partial sequence of functions $\{f_n|X : n \in M\}$ converges pointwise on the set X. Theorem 4 has thus been proved.

Remark 2. In addition to the above theorem, it should be mentioned that the method just described turns out to be useful in those questions of mathematical analysis which are concerned with various kinds of convergence of sequences of real-valued functions on a given set E. In fact, this method may be regarded as a certain generalization of the classical diagonalization method of Cantor. Notice also that purely combinatorial arguments (similar to the one presented above) have found numerous applications in real analysis (see, for instance, [38] and [47]).

EXERCISES

1. Give an example of an admissible functional on $\mathbf{R}$ which is not a measure on $\mathbf{R}$.

2. Let ν be an admissible functional on E, let $f : E \to \mathbf{R}$ and $g : E \to \mathbf{R}$ be any two ν-measurable functions, and let $t \in \mathbf{R}$.

Show that the three functions tf, $f + g$, and $f \cdot g$ are ν-measurable, too.

In addition, show that if $g(x) \neq 0$ for all $x \in E$, then the function f/g is also ν-measurable.

Finally, let X be an arbitrary set from $\mathrm{dom}(\nu)$.

Check that the restriction $f|X$ is a ν-measurable partial function acting from E into $\mathbf{R}$.

3. Let ν be an admissible functional on a ground set E and let a sequence $\{f_n : n < \omega\}$ of ν-measurable real-valued functions be given, pointwise convergent on E to some function f.

Demonstrate that the function f is also measurable with respect to ν.

4. Let λ denote, as usual, the standard Lebesgue measure on $\mathbf{R}$.

Give an example of a sequence $\{f_n : n < \omega\}$ of real-valued uniformly bounded λ-measurable (even continuous) functions on $\mathbf{R}$ which is convergent everywhere on $\mathbf{R}$ but there exists no unbounded subset X of $\mathbf{R}$ such that the sequence of the restricted functions $\{f_n|X : n < \omega\}$ is uniformly convergent on X.

5. Let E be a normal topological space and let μ be a finite inner regular Borel measure on E, i.e., for each Borel subset Y of E, one has the equality

$$\mu(Y) = \sup\{\mu(F) \ : \ F \subset Y \ \& \ F \text{ is closed in } E\}.$$

Denote by μ' the usual completion of μ and let $f : E \to \mathbf{R}$ be an arbitrary μ'-measurable function.

By applying the Tietze–Urysohn theorem on the existence of a continuous extension of a continuous real-valued function defined on a closed subset of E, show that, for every real number $\varepsilon > 0$, there exists a continuous function $g : E \to \mathbf{R}$ satisfying the relation

$$\mu'(\{x \in E \ : \ |f(x) - g(x)| \geq \varepsilon\}) < \varepsilon.$$

Infer from this fact that, for any μ'-measurable function $\phi : E \to \mathbf{R}$, there exists a sequence $\{\phi_n : n < \omega\}$ of continuous real-valued functions on E, convergent to ϕ almost everywhere (with respect to μ').

6. Let E be a normal topological space, μ be a finite inner regular Borel measure on E, and let μ' denote the completion of μ.

Starting with Egorov's theorem and applying the result of Exercise 5, prove the following Luzin type theorem:

For any μ'-measurable function $f : E \to \mathbf{R}$ and for each real $\varepsilon > 0$, there exists a continuous function $g : E \to \mathbf{R}$ such that

$$\mu'(\{x \in E \ : \ f(x) \neq g(x)\}) < \varepsilon.$$

Remark 3. The above relation expresses in a precise form that the given function f has the so-called C-property of Luzin. It is frequently said that all μ'-measurable functions possess this property (and the converse assertion is true, too).

7. Give the detailed proofs of Lemma 2 and Lemma 3.

8. Let E be an uncountable Polish topological space.

Prove the two analogues of Theorem 2 for real-valued functions on E possessing the Baire property and for real-valued functions on E measurable with respect to the completion of a fixed nonzero σ-finite diffused Borel measure on E.

In addition, give an example of a sequence $\{f_n : n < \omega\}$ of uniformly bounded real-valued Borel functions on $\mathbf{R}$, such that, for each infinite subset M of ω, the corresponding partial sequence $\{f_n : n \in M\}$ is convergent only on a first category subset of $\mathbf{R}$ being simultaneously of Lebesgue measure zero.

9. Formulate and prove the analogue of Theorem 3 in the situation when there exists a generalized Luzin subset of the Cantor space 2^ω (we recall that the existence of generalized Luzin sets in 2^ω follows, for instance, from Martin's Axiom; see Chapter 13).

In addition, formulate and prove two statements analogous to Theorem 3 under the assumption of the existence of a Sierpiński set in 2^ω (respectively, of a generalized Sierpiński set in 2^ω).

10*. For any two subsets M and K of ω, write $M \preceq K$ if the inequality $\mathrm{card}(M \setminus K) < \omega$ is valid.

Observe that the relation $\preceq$ is a pre-ordering on the family of all subsets of ω.

Suppose that Martin's Axiom holds. Let κ be an infinite cardinal strictly less than $\mathbf{c}$ (as usual, we identify κ with the smallest ordinal number having the same cardinality), and let $\{M_\xi : \xi < \kappa\}$ be a family of infinite sets in ω such that

$$(\forall \xi)(\forall \zeta)(\xi \le \zeta < \kappa \Rightarrow M_\zeta \preceq M_\xi).$$

Demonstrate that there exists an infinite subset M of ω satisfying the relation $(\forall \xi < \kappa)(M \preceq M_\xi)$.

In addition, show within **ZF** theory that if $\{M_n : n < \omega\}$ is a sequence of infinite subsets of ω such that

$$(\forall n)(\forall k)(n \le k < \omega \Rightarrow M_k \preceq M_n),$$

then there exists an infinite set $M \subset \omega$ satisfying the relation

$$(\forall n < \omega)(M \preceq M_n).$$

Remark 4. Some other combinatorial consequences of Martin's Axiom closely related to the one presented in Exercise 10 are discussed in [18] and [146].

11*. Let X be a Polish topological space equipped with a Borel probability measure μ, let $F : X \times \mathbf{R} \to \mathbf{R}$ be a Borel mapping, and suppose that for each point $x \in X$, there exists a limit $\lim_{t \to 0} F(x,t) = f(x)$.

Starting with the fact that the projection of a Borel subset of a Polish product space is an analytic set and taking into account the universal measurability of analytic (co-analytic) sets, prove the following parameterized version of Egorov's theorem:

For any real $\varepsilon > 0$, there is a closed set $Y \subset X$ such that $\mu(Y) > 1 - \varepsilon$ and the equality $\lim_{t \to 0} F(y,t) = f(y)$ holds uniformly with respect to a variable $y \in Y$.

Remark 5. The result formulated in Exercise 11 is due to Tolstov (see [267]).

12. Let $\{Y_n : n < \omega\}$ be a countable partition of the segment $[0,1]$ into λ-thick subsets of $[0,1]$ and let, for each $n < \omega$, the symbol g_n denote the characteristic function of $\cup\{Y_k : n \leq k < \omega\}$.

Check that

(a) the sequence of functions $\{g_n : n < \omega\}$ is decreasing and converges pointwise to zero;

(b) if Y is a subset of $[0,1]$ and $\{g_n|Y : n < \omega\}$ converges uniformly on Y, then $\lambda_*(Y) = 0$.

Remark 6. It was demonstrated by Weiss that the following Egorov type statement is independent from **ZFC** theory:

For any pointwise convergent sequence $\{f_n : n < \omega\}$ of real-valued functions on $[0,1]$ and for any real $\varepsilon > 0$, there exists a set $X \subset [0,1]$ with $\lambda^*(X) > 1 - \varepsilon$ such that $\{f_n|X : n < \omega\}$ converges uniformly on X.

Chapter 16
A difference between the Riemann and Lebesgue iterated integrals

If one is given a real-valued function f which is defined on a nondegenerate rectangle $[a, b] \times [c, d] \subset \mathbf{R}^2$, then one may consider the question of the existence of two iterated integrals

$$\int_c^d (\int_a^b f(x, y)dx)dy, \qquad \int_a^b (\int_c^d f(x, y)dy)dx$$

in the Riemann sense or in the Lebesgue sense. Further, if both iterated integrals exist, then one can ask whether they are equal to each other, i.e., whether the equality

$$\int_c^d (\int_a^b f(x, y)dx)dy = \int_a^b (\int_c^d f(x, y)dy)dx$$

holds true.

In the present chapter we are focused on these two questions and we will show that in the case of the Riemann integrable uniformly bounded partial functions

$$f(\cdot, y), \quad f(x, \cdot) \qquad (x \in [a, b],\ y \in [c, d])$$

both questions have positive answers. On the other hand, if the partial functions written above are assumed Lebesgue integrable, then the situation turns out to be radically different. Moreover, certain additional set-theoretical assumptions enter the scene and imply some unexpected (at least, at first sight) effects. A more detailed explanation of the phenomenon arising in this context will be given in the next chapter where Sierpiński's partition of the unit square $[0, 1]^2$ will be discussed with several interesting applications.

As usual, we will denote by λ $(= \lambda_1)$ the standard one-dimensional Lebesgue measure on the real line $\mathbf{R}$.

In what follows we need the dominated convergence theorem of Lebesgue (cf. [23], [28], [85], [194], [221]). This theorem plays a key role in integration theory and is helpful in many situations. A much more general result in this direction is also known, which is due to Vitali and establishes necessary and sufficient conditions for the commutativity of the operations of taking limits and integrals, respectively (see, e.g., [61], [194], [196]).

Theorem 1. *Let $(E, \mathcal{S}, \mu)$ be a σ-finite measure space and let $\{f_n : n \in \mathbf{N}\}$ be a sequence of μ-measurable real-valued functions on E satisfying the following two conditions:*

(1) for each point $x \in E$, the sequence $\{f_n(x) : n \in \mathbf{N}\}$ converges to some value $f(x)$;

(2) there exists a μ-integrable function $\phi : E \to \mathbf{R}$ such that

$$|f_n(x)| \leq |\phi(x)| \qquad (n \in \mathbf{N},\ x \in E).$$

Then the equality

$$\lim_{n\to+\infty} \int_E |f_n(x) - f(x)| d\mu(x) = 0$$

holds true and, in addition, these two assertions are valid:

(a) the function f is μ-integrable on E;

(b) $\lim_{n\to+\infty} \int_E f_n(x) d\mu(x) = \int_E f(x) d\mu(x)$.

Proof. It suffices to consider the case when $0 < \mu(E) < +\infty$.

First of all, notice that the function f, being a pointwise limit of a sequence of μ-measurable real-valued functions, is also μ-measurable. Further, condition (2) directly implies that

$$|f(x)| \leq |\phi(x)| \qquad (x \in E),$$

so f is also μ-integrable. Now, take any real $\varepsilon > 0$. There exists a real $\delta > 0$ such that

$$\int_X |\phi(x)| d\mu(x) < \varepsilon/3$$

whenever $X \in \mathcal{S}$ and $\mu(X) < \delta$ (this is the so-called absolute continuity of the μ-integral; cf. Exercise 2). According to Egorov's theorem (see Chapter 15), there exists a μ-measurable subset Y of E satisfying the following two relations:

(i) $\mu(Y) < \delta$;

(ii) the sequence $\{f_n : n \in \mathbf{N}\}$ uniformly converges on $E \setminus Y$.

Let n_0 be a natural number such that

$$|f_n(x) - f(x)| < \varepsilon/(3\mu(E))$$

whenever $n > n_0$ and $x \in E \setminus Y$. Then, for all natural numbers $n > n_0$, we can write

$$\int_E |f_n(x) - f(x)| d\mu(x) = \int_{E\setminus Y} |f_n(x) - f(x)| d\mu(x) + \int_Y |f_n(x) - f(x)| d\mu(x) \leq$$

$$\int_{E\setminus Y}|f_n(x)-f(x)|d\mu(x)+\int_Y|f_n(x)|d\mu(x)+\int_Y|f(x)|d\mu(x)\le$$

$$\int_{E\setminus Y}|f_n(x)-f(x)|d\mu(x)+2\int_Y|\phi(x)|d\mu(x)<\varepsilon/3+2\varepsilon/3=\varepsilon.$$

This yields the required result and finishes the proof.

The above theorem can be successfully applied to many questions concerning the Riemann (respectively, Lebesgue) integrability of limits of pointwise convergent sequences of real-valued functions. Here is a simple result in this direction (cf. [72]).

Theorem 2. *Let $\{f_n : n\in\mathbf{N}\}$ be a sequence of real-valued functions, all of which are defined on a closed interval $[a,b]\subset\mathbf{R}$, and suppose that*

(1) this sequence is uniformly bounded, i.e., $|f_n(x)|\le L$ for some fixed real constant $L\ge 0$ and for all points $x\in[a,b]$;

(2) every function f_n is Riemann (respectively, Lebesgue) integrable on $[a,b]$;

(3) there exists a pointwise limit

$$f(x)=\mathrm{lim}_{n\to+\infty}f_n(x)\qquad(x\in[a,b]).$$

Under these conditions, the following three assertions are valid:

(a) there exists a limit of the sequence of integrals

$$\{\int_a^b f_n(x)dx : n\in\mathbf{N}\};$$

(b) if the limit function f is Riemann integrable, then

$$\mathrm{lim}_{n\to+\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx;$$

(c) if all functions f_n $(n\in\mathbf{N})$ are Lebesgue integrable, then the limit function f is automatically Lebesgue integrable and the above equality remains true.

Actually, this statement is an easy special case of the Lebesgue dominated convergence theorem, so we omit a detailed proof of Theorem 2.

As an immediate application of the above theorem, we can present the following statement.

Theorem 3. *Let $[a,b]$ and $[c,d]$ be closed intervals on $\mathbf{R}$ and let*

$$f:[a,b]\times[c,d]\to\mathbf{R}$$

be a function satisfying these three conditions:

(1) for each $y\in[c,d]$, the partial function $f(\cdot,y)$ is Riemann (Lebesgue) integrable on $[a,b]$;

(2) there exists a constant $L \geq 0$ such that $|f(x,y)| \leq L$ for all points $(x,y) \in [a,b] \times [c,d]$;

(3) there exists a point $y_0 \in [c,d]$ such that

$$\lim_{y \to y_0} f(x,y) = \phi(x)$$

for each point $x \in [a,b]$.

Under the conditions (1)–(3), the following two assertions are valid:

(a) if ϕ is a Riemann integrable function on $[a,b]$, then

$$\lim_{y \to y_0} \int_a^b f(x,y)dx = \int_a^b \phi(x)dx;$$

(b) the above equality holds true in the case of the Lebesgue integrable functions $f(\cdot,y)$ ($y \in [c,d]$), and in this case the limit function ϕ is automatically Lebesgue integrable.

Proof. Let $\{t_n : n \in \mathbf{N}\}$ be an arbitrary sequence of points from $[c,d]$ converging to y_0. For each index $n \in \mathbf{N}$, denote

$$\phi_n(x) = f(x,t_n) \qquad (x \in [a,b]).$$

Applying Theorem 2 to the sequence of functions $\{\phi_n : n \in \mathbf{N}\}$, we get

$$\lim_{n \to +\infty} \int_a^b \phi_n(x)dx = \int_a^b \phi(x)dx.$$

Since $\{t_n : n \in \mathbf{N}\}$ was taken arbitrarily, we come to the desired equality

$$\lim_{y \to y_0} \int_a^b f(x,y)dx = \int_a^b \phi(x)dx$$

in both cases of Riemann or Lebesgue integrable functions. This completes the proof.

A slightly more complicated argument is needed to prove the next well-known result of classical mathematical analysis (see, e.g., [72]).

Theorem 4. *Let $[a,b]$ and $[c,d]$ be two closed intervals on $\mathbf{R}$ and let*

$$f : [a,b] \times [c,d] \to \mathbf{R}$$

be a function satisfying the following three conditions:

(1) for each $y \in [c,d]$, the partial function $f(\cdot,y)$ is Riemann (Lebesgue) integrable on $[a,b]$;

(2) there exists a partial derivative $f'_y(x,y)$ at every point (x,y) from the rectangle $[a,b] \times [c,d]$ and, for any $y \in [c,d]$, the function $f'_y(\cdot,y)$ is Riemann (Lebesgue) integrable on $[a,b]$;

(3) there exists a real constant $L \geq 0$ such that the inequality $|f'_y(x,y)| \leq L$ is satisfied for all points (x,y) from $[a,b] \times [c,d]$.

Let us define a function $\Phi : [c,d] \to \mathbf{R}$ by the formula

$$\Phi(y) = \int_a^b f(x,y)dx \qquad (y \in [c,d]).$$

Then Φ is differentiable at each point $y_0 \in [c,d]$ and the relation

$$\Phi'(y_0) = \int_a^b f'_y(x,y_0)dx$$

holds true.

Proof. Fix any $y_0 \in [c,d]$ and let $\{h_n : n \in \mathbf{N}\}$ be an arbitrary sequence of nonzero real numbers converging to 0. Obviously, for every index $n \in \mathbf{N}$, we may write

$$\frac{\Phi(y_0 + h_n) - \Phi(y_0)}{h_n} = \int_a^b \frac{f(x,y_0+h_n) - f(x,y_0)}{h_n}dx.$$

Using the classical Lagrange formula, we get

$$\frac{f(x,y_0+h_n) - f(x,y_0)}{h_n} = f'_y(x, y_0 + \theta_n h_n),$$

where $0 \leq \theta_n \leq 1$. Let us introduce the notation

$$f_n(x) = \frac{f(x,y_0+h_n) - f(x,y_0)}{h_n} \qquad (x \in [a,b]).$$

By virtue of condition (3), we may write

$$|f_n(x)| = |f'_y(x, y_0 + \theta_n h_n)| \leq L \qquad (x \in [a,b]).$$

It is also clear that

$$\lim_{n \to +\infty} f_n(x) = f'_y(x,y_0) \qquad (x \in [a,b]).$$

Now, tending n to $+\infty$ and applying Theorem 3 to the introduced sequence of functions $\{f_n : n \in \mathbf{N}\}$, we come to the relation

$$\lim_{n \to +\infty} \frac{\Phi(y_0 + h_n) - \Phi(y_0)}{h_n} = \int_a^b f'_y(x,y_0)dx \qquad (y_0 \in [c,d]).$$

Finally, since $\{h_n : n \in \mathbf{N}\}$ was taken arbitrarily, we conclude that there exists $\Phi'(y_0)$ and the required relation

$$\Phi'(y_0) = \int_a^b f'_y(x,y_0)dx \qquad (y_0 \in [c,d])$$

is valid. This finishes the proof.

The above results were proved simultaneously for Riemann and Lebesgue integrable real-valued functions. The next result is specific for Riemann integrable functions and has no analogue (at least, within **ZFC** set theory) for Lebesgue integrable functions (see [72] and Chapter 17).

Theorem 5. *Let* $[a, b]$ *and* $[c, d]$ *be any two closed bounded intervals on* $\mathbf{R}$ *and let*

$$f : [a, b] \times [c, d] \to \mathbf{R}$$

be a function satisfying the following conditions:

(1) there is a real constant $L \geq 0$ *such that* $|f(x, y)| \leq L$ *for all points* $(x, y) \in [a, b] \times [c, d]$*;*

(2) for each $x \in [a, b]$*, the partial function* $f(x, \cdot)$ *is Riemann integrable on* $[c, d]$*;*

(3) for each $y \in [c, d]$*, the partial function* $f(\cdot, y)$ *is Riemann integrable on* $[a, b]$*.*

Then there exist two Riemann iterated integrals

$$\int_c^d (\int_a^b f(x, y)dx)dy, \qquad \int_a^b (\int_c^d f(x, y)dy)dx$$

and the equality

$$\int_c^d (\int_a^b f(x, y)dx)dy = \int_a^b (\int_c^d f(x, y)dy)dx$$

holds true.

Proof. First of all, we need to introduce two real-valued functions

$$\Phi(y) = \int_a^b f(x, y)dx \qquad (y \in [c, d]),$$

$$\Psi(x) = \int_c^d f(x, y)dy \qquad (x \in [a, b]).$$

By virtue of conditions (2) and (3), the functions Φ and Ψ are well-defined.

We wish to show that both these functions are Riemann integrable. Let us do it for the function Φ. For this purpose, take any sequence

$$\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n, ...\}$$

of finite decompositions of $[c, d]$ into intervals such that the maximum of the lengths of all intervals belonging to $\mathcal{D}_n$ ($n \in \mathbf{N}$) tends to zero as n tends to infinity. Now, fix

$$\mathcal{D}_n = \{\Delta_{n,0}, \Delta_{n,1}, ..., \Delta_{n,j(n)}\}$$

in the above-mentioned sequence and take any family of points

$$y_{n,0} \in \Delta_{n,0}, \quad y_{n,1} \in \Delta_{n,1}, \quad . \ . \ . \ , \quad y_{n,j(n)} \in \Delta_{n,j(n)}.$$

By definition, the Riemann sum s_n for Φ corresponding to the decomposition $\mathcal{D}_n$ and to the sequence of points $(y_{n,0}, \ y_{n,1}, \ . \ . \ . \ , \ y_{n,j(n)})$ is equal to

$$\Phi(y_{n,0})\lambda_1(\Delta_{n,0}) + \Phi(y_{n,1})\lambda_1(\Delta_{n,1}) + \ . \ . \ . \ + \Phi(y_{n,j(n)})\lambda_1(\Delta_{n,j(n)}).$$

Clearly, this sum can be rewritten as

$$s_n = \int_a^b \sum\{f(x, y_{n,k})\lambda_1(\Delta_{n,k}) : 0 \leq k \leq j(n)\}dx.$$

Introducing the notation

$$f_n^*(x) = \sum\{f(x, y_{n,k})\lambda_1(\Delta_{n,k}) : 0 \leq k \leq j(n)\} \qquad (x \in [a, b]),$$

we come to the following expression of s_n:

$$s_n = \int_a^b f_n^*(x)dx.$$

By virtue of assumption (2), for each $x \in [a, b]$, the partial function $f(x, \cdot)$ is Riemann integrable on $[c, d]$, which means that the introduced sequence of functions $\{f_n^* : n \in \mathbf{N}\}$ pointwise converges on $[a, b]$ and

$$\lim_{n \to +\infty} f_n^*(x) = \int_c^d f(x, y)dy = \Psi(x),$$

in view of the definition of $\Psi(x)$. At the same time, using condition (1), we get

$$|f_n^*(x)| \leq L(d - c) \qquad (x \in [a, b]).$$

Therefore, according to Theorem 2, we infer that there exists a limit

$$\lim_{n \to +\infty} \int_a^b f_n^*(x)dx$$

and this limit is independent of the choice of $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n, ...\}$ and of the choice of $(y_{n,0}, y_{n,1}, \ . \ . \ . \ , y_{n,j(n)})$ for any $\mathcal{D}_n$. The last circumstance directly indicates that our function Φ is Riemann integrable on $[c, d]$ and

$$\int_c^d \Phi(y)dy = \int_a^b \Psi(x)dx,$$

where the right integral in the above equality can be understood in the Lebesgue sense.

Finally, by using a dual argument for Ψ, we come to the analogous result, which yields that, similarly to Φ, the function Ψ is Riemann integrable on $[a, b]$ and

$$\int_a^b \Psi(x)dx = \int_c^d \Phi(y)dy.$$

In other words, we have the required equality

$$\int_c^d (\int_a^b f(x,y)dx)dy = \int_a^b (\int_c^d f(x,y)dy)dx$$

for these two Riemann iterated integrals. Theorem 5 has thus been proved.

Let us repeat once more that Theorem 5 cannot be extended (within **ZFC** set theory) to the case of iterated integrals in the Lebesgue sense.

In various courses of mathematical analysis it is usually underlined that there is an essential advantage of the Lebesgue integral over the Riemann integral. In fact, one may observe a very good behavior of the Lebesgue integral with respect to standard operations of analysis and, first of all, with respect to the fundamental limit operation. Nevertheless, Theorem 5 shows us that sometimes the Riemann integral turns out to be preferable to the Lebesgue integral.

It should also be mentioned that there exist many Lebesgue nonmeasurable functions $f : [0, 1]^2 \to [0, 1]$ such that all partial functions

$$f(x, \cdot), \quad f(\cdot, y) \qquad (x \in [0, 1],\ y \in [0, 1])$$

are integrable in the Riemann sense (in this connection, see Exercise 6).

EXERCISES

1. Let $(E, \mathcal{S}, \nu)$ and $(E, \mathcal{S}, \mu)$ be two measure spaces. Recall that ν is absolutely continuous with respect to μ if, for every set $X \in \mathcal{S}$, the implication

$$\mu(X) = 0 \Rightarrow \nu(X) = 0$$

holds true.

Demonstrate that if ν is a finite measure, then the following two assertions are equivalent:

(a) ν is absolutely continuous with respect to μ;

(b) for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that the relation

$$X \in \mathcal{S} \ \& \ \mu(X) < \delta$$

implies $\nu(X) < \varepsilon$.

Give an example of two σ-finite measure spaces $(E, \mathcal{S}, \nu)$ and $(E, \mathcal{S}, \mu)$ for which the equivalence (a) $\Leftrightarrow$ (b) fails to be true.

2. Let $(E, \mathcal{S}, \mu)$ be a σ-finite measure space and let $f : E \to \mathbf{R}$ be any μ-integrable function.

Show that, for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that the relation

$$X \in \mathcal{S} \ \& \ \mu(X) < \delta$$

implies

$$|\int_X f(x)d\mu(x)| \leq \int_X |f(x)|d\mu(x) < \varepsilon$$

(the absolute continuity of the μ-integral).

For this purpose, define

$$\nu(X) = \int_X |f(x)|d\mu(x) \qquad (X \in \mathcal{S})$$

and apply Exercise 1 to the measures μ and ν.

3. Give a detailed proof of Theorem 2.

4. Deduce the dominated convergence theorem of Lebesgue from Fubini's theorem on the product of two σ-finite measures.

For this purpose, denoting by λ the standard Lebesgue measure on $\mathbf{R}$, consider a σ-finite measure space $(E, \mathcal{S}, \mu)$, the product measure $\mu \otimes \lambda$ on $E \times \mathbf{R}$, and take into account that, for any real-valued μ-integrable function $f \geq 0$ on E, the following equality holds true:

$$\int_E f(x)d\mu(x) = (\mu \otimes \lambda)(L(f)),$$

where the set $L(f)$ is defined by the formula

$$L(f) = \{(x, t) \in E \times \mathbf{R} : 0 \leq t \leq f(x)\}.$$

5*. Let $(E, \mathcal{S})$ be a measurable space and let $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of probability measures, all of which are given on the σ-algebra $\mathcal{S}$. Suppose that, for every set $X \in \mathcal{S}$, there exists $\lim_{n \to +\infty} \mu_n(X)$ and denote this limit by $\mu(X)$.

Prove that μ is also a probability measure on E with $\mathrm{dom}(\mu) = \mathcal{S}$.

6*. Work in **ZFC** theory and construct an example of a Lebesgue non-measurable function

$$f : [0, 1]^2 \to \{0, 1\}$$

such that

(a) for each $x \in [0, 1]$, the partial function $f(x, \cdot)$ coincides with the characteristic function of a singleton contained in $[0, 1]$;

(b) for each $y \in [0,1]$, the partial function $f(\cdot, y)$ coincides with the characteristic function of a singleton contained in $[0,1]$.

In order to show the existence of such a function f, construct by the method of transfinite recursion a set $Z \subset [0,1]^2$ satisfying the following three conditions:

(c) Z meets every λ_2-measurable subset of $[0,1]^2$ with strictly positive measure;

(d) $\mathrm{card}(Z \cap (\{x\} \times [0,1])) = 1$ for any point $x \in [0,1]$;

(e) $\mathrm{card}(Z \cap ([0,1] \times \{y\})) = 1$ for any point $y \in [0,1]$.

Further, denoting by f the characteristic function of Z, verify that f satisfies both conditions (a) and (b) and is λ_2-nonmeasurable; conclude that all partial functions

$$f(x, \cdot), \quad f(\cdot, y) \qquad (x \in [0,1],\ y \in [0,1])$$

are uniformly bounded, integrable in the Riemann sense, and

$$\int_0^1 (\int_0^1 f(x,y)dx)dy = \int_0^1 (\int_0^1 f(x,y)dy)dx = 0.$$

7*. Let $n \geq 1$ be a natural number, let $(E, \mathcal{S}, \mu)$ be a σ-finite measure space, and let $f_1, f_2, ..., f_n$ be any μ-integrable real-valued functions on E.

Prove that the inequality

$$(\int_E |f_1(t)f_2(t)...f_n(t)|^{1/n} dt)^n \leq \int_E |f_1(t)|dt \int_E |f_2(t)|dt ... \int_E |f_n(t)|dt$$

holds true and specify the case when the above inequality is reduced to the equality.

Argue in two steps. First, demonstrate that if the inequality is valid for n functions, then it is also valid for $2n$ functions. Then demonstrate that if the inequality is valid for $n+1$ functions, then it is also valid for n functions (Cauchy's classical method).

Chapter 17
Sierpiński's partition of the Euclidean plane

In this chapter we discuss several results and statements closely connected with Sierpiński's partition of the Euclidean plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. It turns out that these results and statements can be successfully applied in various fields of mathematics (especially, in real analysis, measure theory, and general topology).

Let ω denote, as usual, the least infinite ordinal number and let ω_1 denote the least uncountable ordinal number. It is a well-known fact that Sierpiński was the first mathematician who considered, in his classical paper [236], a partition $\{A, B\}$ of the product set $\omega_1 \times \omega_1$, defined as follows:

$$A = \{(\xi, \zeta) : \xi \leq \zeta < \omega_1\}, \qquad B = \{(\xi, \zeta) : \omega_1 > \xi > \zeta\}.$$

He observed that, for any two ordinal numbers $\xi < \omega_1$ and $\zeta < \omega_1$, the inequalities $\mathrm{card}(A^\zeta) \leq \omega$ and $\mathrm{card}(B_\xi) \leq \omega$ are valid, where

$$A^\zeta = \{\xi : (\xi, \zeta) \in A\}, \qquad B_\xi = \{\zeta : (\xi, \zeta) \in B\}.$$

In other words, each of the sets A and B can be represented as the union of a countable family of "curves" lying in the product set $\omega_1 \times \omega_1$. This property of the partition $\{A, B\}$ implies many interesting and important consequences. For instance, it immediately follows from the existence of $\{A, B\}$ that if the Continuum Hypothesis (**CH**) holds true, then there exists a partition $\{A', B'\}$ of the Euclidean plane $\mathbf{R}^2$, satisfying these two relations:

(1) for each straight line L in $\mathbf{R}^2$ parallel to the line $\mathbf{R} \times \{0\}$, the inequality $\mathrm{card}(A' \cap L) \leq \omega$ is fulfilled;

(2) for each straight line M in $\mathbf{R}^2$ parallel to the line $\{0\} \times \mathbf{R}$, the inequality $\mathrm{card}(B' \cap M) \leq \omega$ is fulfilled.

Moreover, Sierpiński demonstrated that if a covering $\{A', B'\}$ of $\mathbf{R}^2$ with the above-mentioned properties (1) and (2) does exist, then **CH** holds true.

Indeed, suppose that $\{A', B'\}$ is such a covering of $\mathbf{R}^2$. Choose a subset X of $\mathbf{R}$ having cardinality ω_1 and put $Z = (X \times \mathbf{R}) \cap B'$. Then, according to relation (2), we get $\mathrm{card}(Z) \leq \omega \cdot \omega_1 = \omega_1$. On the other hand, let us show that $\mathrm{pr}_2(Z) = \mathbf{R}$. In order to do this, take an arbitrary point $y \in \mathbf{R}$ and consider the straight line $\mathbf{R} \times \{y\}$. Relation (1) implies that $\mathrm{card}(A' \cap (\mathbf{R} \times \{y\})) \leq \omega$. At the same time, we obviously have

$$\mathrm{card}((X \times \mathbf{R}) \cap (\mathbf{R} \times \{y\})) = \omega_1.$$

Hence there exists $t \in \mathbf{R}$ such that $(t, y) \notin A'$ and $(t, y) \in X \times \mathbf{R}$. Because $\{A', B'\}$ is a covering of $\mathbf{R}^2$, we infer that $(t, y) \in B'$ and, consequently, $(t, y) \in Z$ and $y \in \mathrm{pr}_2(Z)$, which yields the desired equality $\mathrm{pr}_2(Z) = \mathbf{R}$. We thus obtain

$$\mathbf{c} = \mathrm{card}(\mathbf{R}) \leq \mathrm{card}(Z) \leq \omega_1$$

and, finally, $\mathbf{c} = \omega_1$.

In other words, Sierpiński showed that **CH** is equivalent to the statement that there exists a partition $\{A', B'\}$ of the Euclidean plane $\mathbf{R}^2$, satisfying relations (1) and (2).

Let us mention an important consequence of the existence of a Sierpiński partition $\{A', B'\}$ of $\mathbf{R}^2$. For this purpose, consider two sets

$$A'' = [0, 1]^2 \cap A', \qquad B'' = [0, 1]^2 \cap B'.$$

Then we get a partition $\{A'', B''\}$ of $[0, 1]^2$ with the properties very similar to the ones of $\{A', B'\}$. Let us introduce the following two functions:

$f =$ the characteristic function of A'';

$g =$ the characteristic function of B''.

It can easily be observed that there exist the Lebesgue iterated integrals

$$\int_0^1 (\int_0^1 f(x, y)dy)dx, \qquad \int_0^1 (\int_0^1 f(x, y)dx)dy,$$

$$\int_0^1 (\int_0^1 g(x, y)dy)dx, \qquad \int_0^1 (\int_0^1 g(x, y)dx)dy,$$

but we have

$$\int_0^1 (\int_0^1 f(x, y)dy)dx = \int_0^1 (\int_0^1 g(x, y)dx)dy = 1,$$

$$\int_0^1 (\int_0^1 f(x, y)dx)dy = \int_0^1 (\int_0^1 g(x, y)dy)dx = 0$$

and, consequently,

$$\int_0^1 (\int_0^1 f(x, y)dy)dx \neq \int_0^1 (\int_0^1 f(x, y)dx)dy,$$

$$\int_0^1 (\int_0^1 g(x,y)dy)dx \neq \int_0^1 (\int_0^1 g(x,y)dx)dy.$$

Thus, we infer that the classical Fubini theorem does not hold for each of the functions f and g. But these functions, obviously, are nonnegative and bounded on $[0,1]^2$. Therefore, both f and g are nonmeasurable in the Lebesgue sense (cf. the situation for Riemann iterated integrals that was discussed in Chapter 16).

Remark 1. We see that **CH** implies the existence of a function f acting from $[0,1]^2$ into $[0,1]$ such that its Lebesgue iterated integrals do exist but differ from each other. It is not hard to verify that **CH** is not necessary for this conclusion. For instance, Martin's Axiom also implies the existence of such a function (and, moreover, we do not need here the whole power of **MA**; it suffices to assume that each subset of $\mathbf{R}$ whose cardinality is strictly less than $\mathbf{c}$ is measurable in the Lebesgue sense). On the other hand, it was demonstrated in [76] that there are models of set theory in which, for every function $g : [0,1]^2 \to [0,1]$, the existence of both iterated integrals

$$\int_0^1 (\int_0^1 g(x,y)dx)dy, \qquad \int_0^1 (\int_0^1 g(x,y)dy)dx$$

implies the equality between them. For some further results concerning iterated integrals and tightly connected with Sierpiński's partition of $\mathbf{R}^2$, see [231]. Roughly speaking, we do not have any equivalent of **CH** in terms of iterated integrals. Below, we shall see that there is a beautiful equivalent of **CH** in terms of differentiability of real-valued functions.

We now formulate one statement (also interesting from the viewpoint of measure theory) which is based on some properties of Sierpiński's partition $\{A, B\}$ of the product set $\omega_1 \times \omega_1$.

(i) If $\mathcal{P}(\omega_1)$ is the σ-algebra of all subsets of ω_1, then the product σ-algebra $\mathcal{P}(\omega_1) \otimes \mathcal{P}(\omega_1)$ coincides with the σ-algebra $\mathcal{P}(\omega_1 \times \omega_1)$ of all subsets of $\omega_1 \times \omega_1$.

In order to establish this result, it is sufficient to consider an arbitrary subset X of the real line $\mathbf{R}$ with $\mathrm{card}(X) = \omega_1$ and to apply the well-known fact that the graph of any real-valued function on X is a measurable subset of the product space

$$(X, \mathcal{P}(X)) \times (\mathbf{R}, \mathcal{B}(\mathbf{R})) = (X \times \mathbf{R}, \mathcal{P}(X) \otimes \mathcal{B}(\mathbf{R})).$$

Notice that an argument establishing the above equality relies essentially on the Axiom of Choice because the existence of an embedding of ω_1 into

R cannot be proved in **ZF** & **DC** theory. Moreover, as has been shown by Shelah [228] and Raisonnier [210], the existence of such an embedding implies in **ZF** & **DC** theory the existence of a subset of **R** nonmeasurable in the Lebesgue sense.

Starting with the equality $\mathcal{P}(\omega_1 \times \omega_1) = \mathcal{P}(\omega_1) \otimes \mathcal{P}(\omega_1)$, one can directly obtain the following important statement:

(ii) There does not exist a nonzero σ-finite diffused measure μ defined on the whole σ-algebra $\mathcal{P}(\omega_1)$.

Let us recall that this classical statement is due to Ulam [270], who established the nonexistence of such a measure in another way, by applying a transfinite matrix of a special type (for details, see, e.g., [18], [192], [202], or [270]). In order to prove this statement by using the corresponding properties of the partition $\{A, B\}$, suppose for a moment that such a measure μ does exist and apply the Fubini theorem to the product measure $\mu \otimes \mu$ and to the sets A and B of Sierpiński's partition. Because all horizontal sections of A are at most countable and all vertical sections of B are also at most countable, we get the equalities $(\mu \otimes \mu)(A) = (\mu \otimes \mu)(B) = 0$ and, consequently, $(\mu \otimes \mu)(A \cup B) = (\mu \otimes \mu)(\omega_1 \times \omega_1) = 0$, which yields a contradiction with our assumption that μ is not identically equal to zero. Thus, ω_1 is not a real-valued measurable cardinal.

Actually, the real-valued nonmeasurability of ω_1 is historically the first nontrivial fact which concerns some important combinatorial properties of uncountable cardinals and which can be established within **ZFC** theory.

Remark 2. We see, in particular, that if **CH** holds, then the cardinality of the continuum **c** is not real-valued measurable, either. It is reasonable to recall here that the latter result was first obtained by Banach and Kuratowski in their joint paper [16]. In this context, it should also be noticed that the method of [16] gives a more general result. Namely, let us consider the family F of all functions acting from ω into ω. Let f and g be any two functions from F. We put $f \preceq g$ if and only if there exists a natural number $n = n(f, g)$ such that the inequality $f(m) \leq g(m)$ is valid for all natural numbers $m \geq n$. Obviously, the relation $\preceq$ is a pre-ordering of F. Now, assuming that **CH** holds, it is not difficult to define a subset $E = \{f_\xi \ : \ \xi < \omega_1\}$ of F satisfying the following two conditions:

(a) if f is an arbitrary function from F, then there exists an ordinal $\xi < \omega_1$ such that $f \preceq f_\xi$;

(b) for any ordinals ξ and ζ such that $\xi < \zeta < \omega_1$, the relation $f_\zeta \preceq f_\xi$ is not valid.

Evidently, each of conditions (a) and (b) implies $\mathrm{card}(E) = \omega_1$.

Further, for any two natural numbers m and n, we put

$$E_{m,n} = \{f_\xi \ : \ f_\xi(m) \le n\}.$$

So we get a double family of sets $(E_{m,n})_{m<\omega,\ n<\omega}$, which is usually called a Banach–Kuratowski matrix. It is easy to check that, for every $m < \omega$, we have the series of inclusions $E_{m,0} \subset E_{m,1} \subset \ ... \ \subset E_{m,n} \subset ...$ and the equality $E = \cup\{E_{m,n} \ : \ n < \omega\}$. Also, conditions (a) and (b) immediately imply that if f is an arbitrary function from F, then the intersection

$$E_{0,f(0)} \cap E_{1,f(1)} \cap \ ... \ \cap E_{m,f(m)} \cap \ ...$$

is at most countable. From these properties of the Banach–Kuratowski matrix it is not hard to deduce that there does not exist a nonzero σ-finite diffused measure on E defined simultaneously for all sets $E_{m,n}$, where $m < \omega$ and $n < \omega$. In addition, it can easily be seen that the analogous result is true for many other functionals (much more general than measures). Namely, let ν be a real-valued positive (i.e., nonnegative) function defined on some class of subsets of E, closed under finite intersections. We say (cf. the corresponding definition presented in Chapter 15) that ν is an admissible functional on E if the following three conditions hold:

(1) the family of all countable subsets of E is contained in $\mathrm{dom}(\nu)$ and, for any countable set $Z \subset E$, we have the equality $\nu(Z) = 0$;

(2) if $\{X_n \ : \ n < \omega\}$ is an increasing (with respect to inclusion) family of sets, such that $X_n \in \mathrm{dom}(\nu)$ for all $n < \omega$, then the set $\cup\{X_n \ : \ n < \omega\}$ also belongs to $\mathrm{dom}(\nu)$ and $\nu(\cup\{X_n \ : \ n < \omega\}) \le \sup\{\nu(X_n) \ : \ n < \omega\}$;

(3) if $\{Y_n \ : \ n < \omega\}$ is a decreasing (with respect to inclusion) family of sets, such that $Y_n \in \mathrm{dom}(\nu)$ for all $n < \omega$, then the set $\cap\{Y_n \ : \ n < \omega\}$ also belongs to $\mathrm{dom}(\nu)$ and $\nu(\cap\{Y_n \ : \ n < \omega\}) \ge \inf\{\nu(Y_n) \ : \ n < \omega\}$.

Evidently, if ν is a finite diffused measure on E, then ν satisfies conditions (1), (2), and (3). In general, an admissible functional ν need not have any additive properties similar to the corresponding properties of usual measures. However, the Banach–Kuratowski method works for such functionals, too, and one can conclude that there does not exist a nonzero admissible functional on E defined simultaneously for all sets of a given Banach–Kuratowski matrix.

Let us return to a Sierpiński partition of the Euclidean plane $\mathbf{R}^2$ and consider some other interesting results related to it. For instance, we have the following "geometric" fact:

(iii) Assuming **CH**, there exists a function $\phi : \mathbf{R} \to \mathbf{R}$ such that

$$\mathbf{R}^2 = \cup\{g_n(\Gamma_\phi) \ : \ n < \omega\}$$

where Γ_ϕ denotes the graph of ϕ and g_n $(n < \omega)$ are some motions of the plane $\mathbf{R}^2$, each of which either is a translation of $\mathbf{R}^2$ or is a rotation of $\mathbf{R}^2$ (about a point) whose angle is equal to $\pm\pi/2$.

The proof of this result is not difficult and we leave it to the reader as a useful exercise.

Let now X and Y be any two sets. We recall (see, e.g., Chapter 0) that a set-valued mapping is an arbitrary function of the type $F : X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ denotes, as usual, the family of all subsets of Y. According to a well-known definition from general set theory, a subset Z of X is independent with respect to F if, for any two distinct elements $z \in Z$ and $z' \in Z$, we have the relations $z \notin F(z')$ and $z' \notin F(z)$.

It can easily be shown that there exists a set-valued mapping

$$F \ : \ \omega_1 \to [\omega_1]^{\leq\omega}$$

such that no two-element subset of ω_1 is independent with respect to F (as usual, the symbol $[\omega_1]^{\leq\omega}$ stands for the family of all at most countable subsets of ω_1). Actually, the desired set-valued mapping F may trivially be defined as follows:

$$F(\zeta) = A^\zeta \qquad (\zeta < \omega_1)$$

where A is the first component of Sierpiński's partition $\{A, B\}$ of the product set $\omega_1 \times \omega_1$.

In connection with this simple fact, let us remark that if a set-valued mapping $F : \omega_1 \to [\omega_1]^{<\omega}$ is given, then there always exists a subset Ξ of ω_1 satisfying the following two conditions:

(*) $\mathrm{card}(\Xi) = \omega_1$;

(**) Ξ is independent with respect to F.

The reader can easily derive this result from the so-called $\triangle$-system lemma (see, e.g., [18] or [146]). It is also reasonable to point out here that the $\triangle$-system lemma is a theorem of the theory **ZF** & **DC**.

Finally, let us mention that an analogous result (concerning the existence of large independent subsets) holds true for uncountable cardinal numbers, but the proof of this generalized result due to Hajnal and Erdös is more difficult and needs a delicate additional argument.

There are many other interesting statements and facts which are related to Sierpiński's partition of $\omega_1 \times \omega_1$ or can be obtained by using certain properties of this partition (see, e.g., [44], [47], [73], [129], [140], [190], [191], [246], [247], and [251]).

Here we wish to consider an application of Sierpiński's partition in real analysis. Namely, we shall present one theorem of Morayne [190] that establishes an interesting connection of this partition with the existence of some strange mappings acting from $\mathbf{R}$ onto $\mathbf{R} \times \mathbf{R}$. In order to prove the above-mentioned theorem, we need several auxiliary notions and propositions.

Let f be a partial function acting from $\mathbf{R}$ into $\mathbf{R}$, and let X be a subset of $\mathbf{R}$. We say that f satisfies the Banach condition on X if the set

$$\{y \in f(X) \ : \ \mathrm{card}(f^{-1}(y) \cap X) > \omega\}$$

is of Lebesgue measure zero.

We also recall that a partial function f acting from a metric space (X, d) into $\mathbf{R}$ satisfies the Lipschitz condition if there exists a real $L \geq 0$ such that

$$|f(x) - f(y)| \leq Ld(x, y) \qquad (x \in \mathrm{dom}(f), \ y \in \mathrm{dom}(f)).$$

In this case, the real L is usually called a Lipschitz constant for f.

It is not hard to prove the following auxiliary statement.

Lemma 1. *Let (X, d) be a metric space, Y be a subset of X, and let f be a function acting from Y into $\mathbf{R}$ and satisfying the Lipschitz condition with a Lipschitz constant $L \geq 0$.*

Then f can be extended to a function $g : X \to \mathbf{R}$ which fulfils this condition, too, with the same Lipschitz constant L.

Proof. Assume that $Y \neq \emptyset$ and, for any point $x \in X$, define

$$g(x) = \inf\{f(y) + Ld(x, y) \ : \ y \in Y\}.$$

In this way we obtain a mapping g acting from X into $\mathbf{R}$. Let us check that g is the required extension of f. Fix an arbitrary $x \in Y$. Obviously, for each $y \in Y$, we have

$$g(x) \leq f(y) + Ld(x, y).$$

In particular, putting $y = x$, we get $g(x) \leq f(x)$. On the other hand, the relation

$$|f(x) - f(y)| \leq Ld(x, y) \qquad (y \in Y)$$

implies that

$$f(x) \leq f(y) + Ld(x, y) \qquad (y \in Y)$$

and, hence, $f(x) \leq g(x)$. So we obtain the equality $g(x) = f(x)$ and, consequently, g is an extension of f.

Now, let x_1 and x_2 be two arbitrary points from X and let $\varepsilon > 0$. There exist points $y_1 \in Y$ and $y_2 \in Y$ such that

$$f(y_1) + Ld(x_1, y_1) - \varepsilon \leq g(x_1) \leq f(y_2) + Ld(x_1, y_2),$$

$$f(y_2) + Ld(x_2, y_2) - \varepsilon \leq g(x_2) \leq f(y_1) + Ld(x_2, y_1).$$

Then we may write

$$g(x_2) - g(x_1) \leq L(d(x_2, y_1) - d(x_1, y_1)) + \varepsilon \leq Ld(x_1, x_2) + \varepsilon,$$

$$g(x_1) - g(x_2) \leq L(d(x_1, y_2) - d(x_2, y_2)) + \varepsilon \leq Ld(x_1, x_2) + \varepsilon,$$

and, finally, $|g(x_1) - g(x_2)| \leq Ld(x_1, x_2) + \varepsilon$. Because ε is an arbitrary strictly positive real number, we conclude that

$$|g(x_1) - g(x_2)| \leq Ld(x_1, x_2),$$

which completes the proof of Lemma 1.

Actually, we need only a very special case of this lemma when $X = \mathbf{R}$. In this case, the proof can be done directly.

Lemma 2. *Let f be a partial function acting from $\mathbf{R}$ into $\mathbf{R}$, and suppose that f is differentiable at all points of* $\mathrm{dom}(f)$*, i.e., for every point $t \in \mathrm{dom}(f)$, there exists a derivative $f'_{\mathrm{dom}(f)}(t)$ relative to the set* $\mathrm{dom}(f)$*. Then* $\mathrm{dom}(f)$ *can be represented in the form* $\mathrm{dom}(f) = \cup\{P_i : i \in I\}$*, where* $\mathrm{card}(I) \leq \omega$ *and each set P_i $(i \in I)$ has the property that $f|P_i$ satisfies the Lipschitz condition.*

Proof. First, let us denote $D = \mathrm{dom}(f)$ and, for every natural number $n > 0$, define the set $D_n \subset D$ by

$$D_n = \{t \in D \ : \ (\forall t' \in D)(|t' - t| \leq 1/n \Rightarrow |f(t') - f(t)| \leq n|t' - t|)\}.$$

Then, taking into account the assumption that f is differentiable relative to D, it is not hard to check the equality $D = \cup\{D_n : 0 < n < \omega\}$. Further, for any natural number $n > 0$, we may write $D_n = \cup\{D_{nk} : k < \omega\}$, where $(\forall k < \omega)(\mathrm{diam}(D_{nk}) \leq 1/n)$. Now, it immediately follows from the definition of D_n that all the restrictions

$$f|D_{nk} \qquad (0 < n < \omega, \ k < \omega)$$

satisfy the Lipschitz condition. So we can put

$$\{P_i \ : \ i \in I\} = \{D_{nk} \ : \ 0 < n < \omega, \ k < \omega\},$$

and Lemma 2 is thus proved.

The next auxiliary proposition is well known in real analysis and is due to Banach (see, e.g., [194], [225]).

Lemma 3. *Let f be a continuous real-valued function of finite variation, defined on some segment $[a,b] \subset \mathbf{R}$. For every $y \in \mathbf{R}$, put $\phi_f(y) = +\infty$ if* $\mathrm{card}(f^{-1}(y)) \geq \omega$, *and* $\phi_f(y) = \mathrm{card}(f^{-1}(y))$ *if* $\mathrm{card}(f^{-1}(y)) < \omega$.

Then the function $\phi_f : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ is integrable in the Lebesgue sense, and the relation

$$\int_{\mathbf{R}} \phi_f(y)dy = var_{[a,b]}(f) < +\infty$$

is valid. In particular, for almost all (with respect to the Lebesgue measure λ) points $y \in \mathbf{R}$, one has the inequality $\mathrm{card}(f^{-1}(y)) < \omega$.

Lemma 4. *Let f be a function acting from $\mathbf{R}$ into $\mathbf{R}$, and let*

$$D = \{t \in \mathbf{R} \ : \ f'(t) \text{ exists}\}.$$

Then f satisfies the Banach condition on D.

Proof. In view of Lemmas 1 and 2, it suffices to demonstrate that, for each closed interval $[a,b] \subset \mathbf{R}$, any function $g : [a,b] \to \mathbf{R}$ satisfying the Lipschitz condition on $[a,b]$, fulfills the Banach condition on the same interval. But this immediately follows from Lemma 3 because g is continuous and of finite variation on $[a,b]$.

Now, we are ready to formulate and prove the result of Morayne [190]. Actually, this result yields a purely analytic equivalent of the Continuum Hypothesis.

Theorem 1. *The following two assertions are equivalent:*

(1) the Continuum Hypothesis (**CH**)*;*

(2) there exists a surjection $f = (f_1, f_2) : \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ such that, for any point $t \in \mathbf{R}$, at least one of the coordinate functions f_1 and f_2 is differentiable at t.

Proof. We first establish the implication (1) ⇒ (2). Suppose that **CH** holds. Then we may consider a Sierpiński type partition $\{A, B\}$ of $\mathbf{R} \times \mathbf{R}$ such that $(\forall x \in \mathbf{R})(\mathrm{card}(A(x)) \leq \omega)$ and $(\forall y \in \mathbf{R})(\mathrm{card}(B(y)) \leq \omega)$, where

$$A(x) = \{y \in \mathbf{R} \ : \ (x,y) \in A\}, \qquad B(y) = \{x \in \mathbf{R} \ : \ (x,y) \in B\}.$$

Further, we introduce a function $\phi : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$\phi(t) = t \cdot \sin(t) \qquad (t \in \mathbf{R}).$$

It is evident, from the geometrical point of view, that, for any two numbers $u \in \mathbf{R}$ and $v \in \mathbf{R}$, the sets $\phi^{-1}(u) \cap\ ]-\infty,-1]$ and $\phi^{-1}(v) \cap [1,+\infty[$ are countably infinite. So we may write

$$\phi^{-1}(u) \cap\ ]-\infty,-1] = \{t_1^u, t_2^u, ..., t_n^u, ...\},$$

$$\phi^{-1}(v) \cap [1,+\infty[\ = \{s_1^v, s_2^v, ..., s_n^v, ...\}.$$

At the same time, we can represent the two countable sets

$$A(u) = \{y \in \mathbf{R} \ :\ (u,y) \in A\}, \qquad B(v) = \{x \in \mathbf{R} \ :\ (x,v) \in B\}$$

as $A(u) = \{a_1^u, a_2^u, ..., a_n^u, ...\}$ and $B(v) = \{b_1^v, b_2^v, ..., b_n^v, ...\}$, respectively.

Let now t be an arbitrary point of $\mathbf{R}$.

If $t \in\ ]-\infty,1[$, then we put $f_1(t) = \phi(t)$.

If $t \in [1,+\infty[$, then $t = s_n^v$ for some real v and natural n. In this case, we define $f_1(t) = f_1(s_n^v) = b_n^v$.

Analogously, if $t \in\ ]-1,+\infty[$, then we put $f_2(t) = \phi(t)$.

If $t \in\ ]-\infty,-1]$, then $t = t_n^u$ for some real u and natural n. In this case, we define $f_2(t) = f_2(t_n^u) = a_n^u$.

Finally, we introduce a mapping $f : \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ by the formula

$$f(x) = (f_1(x), f_2(x)) \qquad (x \in \mathbf{R}).$$

Let us verify that f is the required function. For this purpose, take any point $t \in \mathbf{R}$. Since $\mathbf{R} =\ ]-\infty,1[\ \cup\ ]-1,+\infty[$, there exists an open neighborhood $W(t)$ of t such that

$$W(t) \subset\ ]-\infty,1[\quad \vee \quad W(t) \subset\ ]-1,+\infty[$$

and, hence, at least one of the coordinate functions f_1 and f_2 coincides with the function ϕ on $W(t)$. But ϕ is differentiable everywhere on $\mathbf{R}$. Consequently, at least one of the functions f_1 and f_2 is differentiable on $W(t)$ and, in particular, differentiable at the point $t \in W(t)$.

Also, it can easily be checked that f is a surjection. Indeed, let (u,v) be an arbitrary point of $\mathbf{R}^2$. Since the equality $\mathbf{R}^2 = A \cup B$ holds, the point (u,v) belongs either to A or to B. We may assume, without loss of generality, that $(u,v) \in A$. Then

$$v \in A(u) = \{a_1^u, a_2^u, ..., a_n^u, ...\},$$

$$\phi^{-1}(u) \cap\ ]-\infty,-1]\ = \{t_1^u, t_2^u, ..., t_n^u, ...\}$$

and, for some natural number n, we must have $v = a_n^u$. Putting $t = t_n^u$ and taking into account the definition of f, we get

$$f(t) = (f_1(t), f_2(t)) = (\phi(t), a_n^u) = (u, v),$$

which shows that our f is a surjection.

In this way we have established the implication (1) $\Rightarrow$ (2).

Suppose now that assertion (2) is valid, i.e., for some surjective mapping $f = (f_1, f_2) \ : \ \mathbf{R} \to \mathbf{R} \times \mathbf{R}$, the equality $\mathbf{R} = D_1 \cup D_2$ is fulfilled, where

$$D_1 = \{t \in \mathbf{R} \ : \ f_1'(t) \text{ exists}\}, \qquad D_2 = \{t \in \mathbf{R} \ : \ f_2'(t) \text{ exists}\}.$$

In conformity with Lemma 4, the coordinate functions f_1 and f_2 satisfy the Banach condition on the sets D_1 and D_2, respectively. Hence the sets

$$K_1 = \{y \in f_1(D_1) \ : \ \mathrm{card}(f_1^{-1}(y) \cap D_1) > \omega\},$$

$$K_2 = \{y \in f_2(D_2) \ : \ \mathrm{card}(f_2^{-1}(y) \cap D_2) > \omega\}$$

are of Lebesgue measure zero. Let us put $M_1 = \mathbf{R} \setminus K_1$ and $M_2 = \mathbf{R} \setminus K_2$. Then, for the sets M_1 and M_2, we have $\mathrm{card}(M_1) = \mathrm{card}(M_2) = \mathbf{c}$. Now, we define

$$A = f(D_1) \cap (M_1 \times M_2), \qquad B = f(D_2) \cap (M_1 \times M_2).$$

Keeping in mind the fact that f is a surjection, we get $A \cup B = M_1 \times M_2$. Also, it immediately follows from the definition of M_1 and M_2 that

$$(\forall x \in M_1)(\mathrm{card}(A(x)) \leq \omega), \qquad (\forall y \in M_2)(\mathrm{card}(B(y)) \leq \omega).$$

In other words, we obtain a covering $\{A, B\}$ of the product set $M_1 \times M_2$, having properties quite similar to those of Sierpiński's partition of $\omega_1 \times \omega_1$. But the product set $M_1 \times M_2$ may be identified (in the purely set-theoretical sense) with $\mathbf{R} \times \mathbf{R}$. So we conclude that **CH** must be true (cf. the argument presented at the beginning of this chapter; see also Exercise 1). The proof of Theorem 1 is thus finished.

Remark 3. One interesting generalization of the above theorem was obtained in [44].

The function f considered in Theorem 1 is singular from the point of view of Lebesgue measurability (Exercise 7 illustrates this circumstance).

We want to finish this chapter with one statement closely related to the Sierpiński partition of $\omega_1 \times \omega_1$. This statement does not require any additional set-theoretical hypotheses and establishes a certain relationship between Sierpiński type partitions and the nonmeasurability in the Lebesgue sense.

Theorem 2. *In* **ZF** & **DC** *theory, the assertion "there exists a bijection from* **R** *onto a well-ordered set" implies the assertion "there exists a subset of* **R** *nonmeasurable in the Lebesgue sense".*

Proof. Obviously, the existence of a bijection between **R** and some well-ordered set is equivalent to the existence of a well-ordering of **R** and, actually, means that **R** can be represented as an injective family of points $\mathbf{R} = \{x_\xi \ : \ \xi < \alpha\}$, where α denotes some ordinal number of cardinality continuum. Also, it is clear that if we want to prove the existence of Lebesgue nonmeasurable subsets of **R**, it suffices for us to establish the existence of subsets of the plane $\mathbf{R}^2$, nonmeasurable with respect to the standard two-dimensional Lebesgue measure λ_2 on $\mathbf{R}^2$. Let λ denote, as usual, the standard Lebesgue measure on **R**. Let $\beta \leq \alpha$ be the least ordinal for which $\lambda^*(\{x_\xi \ : \ \xi < \beta\}) > 0$, where λ^* is the outer measure associated with λ. If the set $\{x_\xi \ : \ \xi < \beta\}$ is nonmeasurable with respect to λ, then we are done. Otherwise, we may write

$$\{x_\xi \ : \ \xi < \beta\} \in \mathrm{dom}(\lambda), \qquad \lambda(\{x_\xi \ : \ \xi < \beta\}) > 0$$

and, according to the definition of β, for each ordinal number $\gamma < \beta$, we have $\lambda(\{x_\xi \ : \ \xi < \gamma\}) = 0$. Now, define a subset Z of $\mathbf{R}^2$ as follows:

$$Z = \{(x_\xi, x_\zeta) \ : \ \xi \leq \zeta < \beta\}.$$

We assert that Z is nonmeasurable with respect to λ_2. Indeed, suppose for a moment that $Z \in \mathrm{dom}(\lambda_2)$. Then, considering all vertical and horizontal sections of Z and applying the classical Fubini theorem to Z, we get, on the one hand, the relation $\lambda_2(Z) > 0$ and, on the other hand, the equality $\lambda_2(Z) = 0$. Because this is impossible, we conclude that Z is not λ_2-measurable, which also implies the existence of a Lebesgue nonmeasurable subset of the real line **R**. Theorem 2 has thus been proved.

Actually, Theorem 2 and the result of Exercise 12 show us that the existence of a well-ordering of the real line **R** immediately yields the existence of subsets of **R** having a very bad descriptive structure from the points of view of the Lebesgue measurability and Baire property.

In this connection, let us recall that the existence of a totally imperfect subset of **R** of cardinality continuum also implies (within the same **ZF** & **DC** theory) the existence of a Lebesgue nonmeasurable subset of **R** and the existence of a subset of $\mathbf{R}$ without the Baire property (see Exercise 5 from Chapter 10).

EXERCISES

1. Let E_1 and E_2 be two sets such that $\text{card}(E_1) = \text{card}(E_2) = \mathbf{c}$.
Check that the following two assertions are equivalent:
(i) the Continuum Hypothesis (**CH**);
(ii) there exists a covering (partition) $\{A, B\}$ of the product set $E_1 \times E_2$, satisfying the relations

$$(\forall y \in E_2)(\text{card}(A \cap (E_1 \times \{y\})) \leq \omega),$$

$$(\forall x \in E_1)(\text{card}(B \cap (\{x\} \times E_2)) \leq \omega).$$

2. Define a function $f : [-1,1]^2 \to \mathbf{R}$ by putting
$f(x,y) = (xy)/((x^2 + y^2)^2)$ if $(x,y) \neq (0,0)$;
$f(x,y) = 0$ if $(x,y) = (0,0)$.
Demonstrate that
(a) the function f is of first Baire class (hence Lebesgue measurable) but is not Lebesgue integrable;
(b) the iterated integrals for f do exist and

$$\int_{-1}^{1}(\int_{-1}^{1} f(x,y)dx)dy = \int_{-1}^{1}(\int_{-1}^{1} f(x,y)dy)dx = 0.$$

Remark 4. In connection with this exercise, let us note that a much more general result is presented in the old paper by Fichtenholz [71].

3. Give a detailed proof of the equality $\mathcal{P}(\omega_1 \times \omega_1) = \mathcal{P}(\omega_1) \otimes \mathcal{P}(\omega_1)$.

4. Let E be a ground set with $\text{card}(E) > \omega$, for which a Banach–Kuratowski matrix $(E_{m,n})_{m<\omega,\ n<\omega}$ of subsets of E does exist.

Show that there is no nonzero admissible functional ν on E satisfying the relation $(\forall m < \omega)(\forall n < \omega)(E_{m,n} \in \text{dom}(\nu))$.

5*. By starting with Sierpiński's partition of $\mathbf{R}^2$, give a proof of the assertion (iii).

6*. Give a detailed proof of Lemma 3.

7. Let $f = (f_1, f_2)$ be any surjection from $\mathbf{R}$ onto $\mathbf{R} \times \mathbf{R}$ having the property that, for each $t \in \mathbf{R}$, at least one of the coordinate functions f_1 and f_2 is differentiable at t.

Demonstrate that f is not measurable in the Lebesgue sense.

Remark 5. In connection with this exercise, let us notice that a somewhat stronger result is obtained in [190].

8. Let $n \geq 2$ be a natural number. Recall that the n-dimensional Euclidean space $\mathbf{R}^n$ consists of all n-sequences x of the form

$$x = (x_1, x_2, ..., x_n) \qquad (x_i \in \mathbf{R},\ i = 1, 2, ..., n).$$

For any $t \in \mathbf{R}$ and for each natural index $i \in [1, n]$, denote

$$\Gamma_i(t) = \{x \in \mathbf{R}^n \ : \ x_i = t\}.$$

Obviously, $\Gamma_i(t)$ is an affine hyperplane in the space $\mathbf{R}^n$.

Verify that the following two assertions are equivalent:

(a) the Continuum Hypothesis (**CH**);

(b) there is a partition $\{A_1, A_2, ..., A_n\}$ of $\mathbf{R}^n$ such that, for any $t \in \mathbf{R}$ and for each natural index $i \in [1, n]$, the inequality $\mathrm{card}(\Gamma_i(t) \cap A_i) \leq \omega$ is satisfied.

Remark 6. The above result generalizes the case of $\mathbf{R}^2$ and is also due to Sierpiński. The following simple exercise shows that, for infinite-dimensional analogues of a Euclidean space, the situation is substantially different.

9*. Consider the infinite-dimensional topological vector space

$$\mathbf{R}^\omega = R_1 \times R_2 \times ... \times R_n \times ...,$$

where $(\forall i \in \omega \setminus \{0\})(R_i = \mathbf{R})$, and denote by $\Gamma_i(t)$ the affine hyperplane in this space, corresponding to an index $i \in \{1, 2, ..., n, ...\}$ and to a real number t, i.e., $\Gamma_i(t) = \{x \in \mathbf{R}^\omega \ : \ x_i = t\}$.

Show that there does not exist a covering $\{A_i \ : \ i \in \omega \setminus \{0\}\}$ of $\mathbf{R}^\omega$ such that, for any $i \in \omega \setminus \{0\}$ and for each $t \in \mathbf{R}$, the relation

$$\mathrm{card}(\Gamma_i(t) \cap A_i) < \mathbf{c}$$

is fulfilled.

Consequently, there is no covering $\{B_i : i \in \omega \setminus \{0\}\}$ of $\mathbf{R}^\omega$ such that, for any $i \in \omega \setminus \{0\}$ and for any $t \in \mathbf{R}$, the relation $\mathrm{card}(\Gamma_i(t) \cap B_i) \leq \omega$ is satisfied.

10. Let $n \geq 2$ be an arbitrary natural number.

Prove that the following two assertions are equivalent:

(a) the Continuum Hypothesis (**CH**);

(b) there exists a surjection $f = (f_1, f_2, ..., f_n) : \mathbf{R} \to \mathbf{R}^n$ such that, for each point $t \in \mathbf{R}$, at least one of the coordinate functions $f_1,\ f_2,\ \dots,\ f_n$ is differentiable at t.

11. Show that there is no surjection $f = (f_1, f_2, ..., f_n, ...) : \mathbf{R} \to \mathbf{R}^\omega$ having the property that, for any point $t \in \mathbf{R}$, at least one of the coordinate functions $f_1,\ f_2,\ \dots,\ f_n,\ \dots$ is differentiable at t.

Remark 7. The results presented in the preceding two exercises are due to Morayne (see [190] and [191]).

12. Work in **ZF** & **DC** theory and verify that the assertion "there exists a bijection from **R** onto some ordinal number" implies the assertion "there exists a subset of **R** not possessing the Baire property".

13*. Let Z be a subset of the plane $\mathbf{R}^2$ such that $\mathrm{card}(Z \cap L) < \omega$ for every straight line L in $\mathbf{R}^2$ which is parallel to the line $\mathbf{R} \times \{0\}$. Let μ be an arbitrary σ-finite measure on $\mathbf{R}^2$ quasi-invariant under the group of all translations of $\mathbf{R}^2$.

Prove that if $Z \in \mathrm{dom}(\mu)$, then $\mu(Z) = 0$.

14. Demonstrate that there exists no covering $\{X, Y\}$ of the plane $\mathbf{R}^2$ satisfying the following conditions:

(a) $\mathrm{card}(X \cap L) < \omega$ for all straight lines L in $\mathbf{R}^2$ which are parallel to the line $\mathbf{R} \times \{0\}$;

(b) $\mathrm{card}(Y \cap M) < \mathbf{c}$ for all straight lines M in $\mathbf{R}^2$ which are parallel to the line $\{0\} \times \mathbf{R}$.

15*. Show that the Continuum Hypothesis (**CH**) is equivalent to the existence of a partition $\{X, Y, Z\}$ of the Euclidean space $\mathbf{R}^3$ such that

(a) $\mathrm{card}(X \cap L) < \omega$ for all straight lines L in $\mathbf{R}^3$ which are parallel to the line $\mathbf{R} \times \{0\} \times \{0\}$;

(b) $\mathrm{card}(Y \cap M) < \omega$ for all straight lines M in $\mathbf{R}^3$ which are parallel to the line $\{0\} \times \mathbf{R} \times \{0\}$;

(c) $\mathrm{card}(Z \cap N) < \omega$ for all straight lines N in $\mathbf{R}^3$ which are parallel to the line $\{0\} \times \{0\} \times \mathbf{R}$.

Remark 8. The geometric equivalent of **CH** formulated in Exercise 15 was found by Sierpiński (see, e.g., [247]).

16*. Let $(G, +)$ be an arbitrary uncountable commutative group and let 0 denote the neutral element of G.

Prove that there exist three subgroups $G_1,\ G_2,\ G_3$ of G satisfying these four relations:

(a) $\mathrm{card}(G_1) = \mathrm{card}(G_2) = \mathrm{card}(G_3) = \omega_1$;
(b) $G_1 \cap (G_2 + G_3) = \{0\}$;
(c) $G_2 \cap (G_1 + G_3) = \{0\}$;
(d) $G_3 \cap (G_1 + G_2) = \{0\}$.

In other words, this result shows that any uncountable commutative group G contains a direct sum of three uncountable subgroups of G.

Moreover, prove that any uncountable commutative group G contains a direct sum of an ω_1-sequence of uncountable subgroups of G.

17. Let $(\Gamma, \cdot)$ be a group and let X be a subset of Γ. This X is called a Γ-negligible set in Γ if the following two conditions hold:

(a) there exists a nonzero σ-finite left Γ-invariant measure μ on Γ such that $X \in \mathrm{dom}(\mu)$;

(b) for every σ-finite left Γ-quasi-invariant measure ν on Γ, one has the implication $X \in \mathrm{dom}(\nu) \Rightarrow \nu(X) = 0$.

Starting with the results of Exercises 15 and 16, demonstrate that if $(G, +)$ is an uncountable commutative group, then G admits a representation in the form $G = X \cup Y \cup Z$, where X, Y, and Z are some pairwise disjoint G-negligible subsets of G.

18*. Recall that a group $(\Gamma, \cdot)$ is solvable if there exists a finite family $\{\Gamma_i : 1 \leq i \leq n\}$ of subgroups of Γ such that

(a) $\Gamma_1 = \{e\}$, where e is the neutral element of Γ;

(b) $\Gamma_n = \Gamma$;

(c) for each natural index $i \in [1, n-1]$, the group Γ_i is a normal subgroup of Γ_{i+1} and the quotient group Γ_{i+1}/Γ_i is commutative.

By using the result of Exercise 17, prove that if $(\Gamma, \cdot)$ is an arbitrary uncountable solvable group, then this Γ can be represented in the form $\Gamma = X \cup Y \cup Z$, where X, Y, and Z are three pairwise disjoint Γ-negligible subsets of Γ.

Chapter 18
Bad functions defined on second category sets

According to commonly used terminology, if E is a topological space which is not of first category on itself, then E is said to be of second category (see Chapter 0).

Analogously, if X is a subset of E and X is not of first category in E, then one simply says that X is of second category (in E).

Obviously, if $X \subset E$ is of first category in E and $f : X \to \mathbf{R}$ is an arbitrary function, then f can be extended to a function $f^* : E \to \mathbf{R}$ possessing the Baire property. Indeed, it suffices to put $f^*(x) = 0$ for all points $x \in E \setminus X$.

In this chapter we are interested in the following naturally arising question:

Let E be a topological space without isolated points and let $X \subset E$ be of second category in E. Does there exist a function $f : X \to \mathbf{R}$ that cannot be extended to a function $f^* : E \to \mathbf{R}$ having the Baire property?

It is reasonable first to examine this question for the classical case $E = \mathbf{R}$ and then to try to consider a more general situation.

It turns out that, for $E = \mathbf{R}$, the answer is positive and this result is essentially due to Novikov (see [198]).

In his above-mentioned work [198] Novikov dealt only with the question of the existence of a function $f : X \to \mathbf{R}$ that cannot be extended to a Borel function $f^* : \mathbf{R} \to \mathbf{R}$, where X is an arbitrary second category subset of $\mathbf{R}$. However, slightly modifying his argument, it is not difficult to obtain the required result in terms of the Baire property.

Now, we are going to discuss thoroughly this remarkable result of descriptive set theory (let us emphasize that it does not appeal to extra set-theoretical axioms, i.e., belongs to **ZFC** theory).

We need several auxiliary statements. First, let us recall the precise formulation of the Kuratowski–Ulam theorem, which can be regarded as a reasonable topological version of the Fubini theorem (see, e.g., [105], [149] or [202]).

Theorem 1. *Let E_1 and E_2 be two topological spaces, let E_2 possess a countable base, and let $Z \subset E_1 \times E_2$. The following three assertions are valid:*

(1) if Z is of first category in the product space $E_1 \times E_2$, then for almost all points $x \in E_1$ the section $Z(x) = \{y \in E_2 : (x, y) \in Z\}$ is of first category in the space E_2;

(2) if Z possesses the Baire property in the product space $E_1 \times E_2$, then for almost all points $x \in E_1$ the section $Z(x)$ possesses the Baire property in the space E_2;

(3) if Z possesses the Baire property in $E_1 \times E_2$ and, for almost all $x \in E_1$, the set $Z(x)$ is of first category in E_2, then Z is of first category in $E_1 \times E_2$.

As pointed out in preceding chapters, the Kuratowski–Ulam theorem has many interesting consequences and applications in set-theoretical topology and mathematical analysis. One nontrivial application of this theorem will be given later in Chapter 22, where some questions concerning the descriptive structure of generalized derivatives are discussed.

In the present chapter we need only the following special case of the Kuratowski–Ulam theorem for $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

Lemma 1. *Let Z be a subset of the plane $\mathbf{R} \times \mathbf{R}$ such that*

(1) $\mathrm{pr}_2(Z)$ is of second category in $\mathbf{R}$;

(2) for any $y \in \mathrm{pr}_2(Z)$, the set $\{x \in \mathbf{R} : (x, y) \in Z\}$ is of second category in $\mathbf{R}$.

Then the set Z is of second category in $\mathbf{R} \times \mathbf{R}$.

Obviously, this lemma is a direct consequence of Theorem 1.

Lemma 2. *Let A be an arbitrary subset of the plane $\mathbf{R} \times \mathbf{R}$. Denote*

$$A^* = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y \in \mathrm{cl}(A(x))\},$$

where $\mathrm{cl}(A(x))$ stands, as usual, for the closure of $A(x)$ in $\mathbf{R}$.

If A is an analytic subset of $\mathbf{R} \times \mathbf{R}$, then A^* is also analytic in $\mathbf{R} \times \mathbf{R}$. In particular, if A is Borel in $\mathbf{R} \times \mathbf{R}$, then A^* is analytic in $\mathbf{R} \times \mathbf{R}$.

Proof. According to the definition of A^*, we can write

$$(x, y) \in A^* \Leftrightarrow (\forall n < \omega)(\exists z \in \mathbf{R})(|y - z| < 1/(n + 1) \ \&\ (x, z) \in A).$$

For any $n < \omega$, define the set D_n in the space $\mathbf{R}^3$ by the formula

$$D_n = \{(x, y, z) : |y - z| < 1/(n + 1) \ \&\ (x, z) \in A\}.$$

It can easily be seen that D_n is an analytic subset of $\mathbf{R}^3$. This implies at once that the set

$$A^* = \bigcap_{n<\omega} \mathrm{pr}_{\mathbf{R}\times\mathbf{R}} D_n$$

is analytic in $\mathbf{R} \times \mathbf{R}$, and the proof is completed.

Lemma 3. *Let B be a Borel subset of $\mathbf{R} \times \mathbf{R}$ such that*

$$(\forall x \in \mathbf{R})(\mathrm{card}(B(x)) \leq \omega).$$

Let us denote

$$B' = \{x \in \mathbf{R} \ : \ B \cap (\{x\} \times \mathbf{R}) \text{ is nowhere dense in } \{x\} \times \mathbf{R}\}.$$

Then the set B' is also Borel in $\mathbf{R}$.

Proof. Fix a countable base $\{U_n : n < \omega\}$ of open subsets of $\mathbf{R}$. For any natural number n, define the set $I(n) = \{m < \omega : U_m \subset U_n\}$. Clearly, we can write

$$x \in B' \Leftrightarrow (\forall n < \omega)(\exists m \in I(n))((\{x\} \times U_m) \cap B = \emptyset)$$

or, equivalently,

$$x \in B' \Leftrightarrow (\forall n < \omega)(\exists m \in I(n))\neg(\exists y)(y \in U_m \ \& \ (x, y) \in B).$$

It suffices to establish that all sets

$$\{x \in \mathbf{R} : \neg(\exists y)(y \in U_m \ \& \ (x, y) \in B)\} \qquad (m < \omega)$$

are Borel in $\mathbf{R}$. We observe that, for each $m < \omega$, the set

$$T_m = \{x \in \mathbf{R} : (\exists y)((x, y) \in (\mathbf{R} \times U_m) \cap B)\}$$

coincides with the projection on $\mathbf{R}$ of some Borel subset of $\mathbf{R}^2$, all vertical sections of which are at most countable. Consequently (see Chapter 0), the set T_m is also Borel in $\mathbf{R}$. Clearly, the same is true for the set $\mathbf{R} \setminus T_m$. We thus get the required result.

Lemma 4. *Let $X \subset \mathbf{R}$ be a set of second category in $\mathbf{R}$. There exists a set $Z \subset \mathbf{R} \times \mathbf{R}$ such that*

(1) $\mathrm{pr}_1(Z)$ is a subset of X and is also of second category in $\mathbf{R}$;

(2) for any point $x \in \mathrm{pr}_1(Z)$, the set $Z(x)$ is nowhere dense in $\mathbf{R}$;

(3) Z is of second category in $\mathbf{R} \times \mathbf{R}$.

Proof. The argument presented below is very similar to that given in the proof of Theorem 2 from Chapter 17. Let us represent our set X in the form of an injective transfinite α-sequence of points:

$$X = \{x_\xi : \xi < \alpha\},$$

where $\alpha \geq \omega_1$ is some ordinal number. Let $\beta \leq \alpha$ denote the least ordinal number such that the set $\{x_\xi : \xi < \beta\}$ is of second category. Then we have

$$(\forall \xi < \beta)(\text{the set } \{x_\zeta : \zeta < \xi\} \text{ is of first category in } \mathbf{R}).$$

Let us put

$$X_0 = \{x_\xi : \xi < \beta\}, \qquad S = \{(x_\xi, x_\zeta) : \zeta \leq \xi < \beta\}.$$

Evidently, we have the equalities

$$\mathrm{pr}_1(S) = X_0, \qquad \mathrm{pr}_2(S) = X_0,$$

and, for any point $x \in \mathrm{pr}_1(S)$, the set $S(x)$ is of first category. Applying Sierpiński's classical argument to S (cf. Chapter 17) and taking into account Lemma 1, we infer that S does not possess the Baire property, hence S is of second category. At the same time, it is evident that S admits a representation $S = \cup\{Z_n : n < \omega\}$, where each set Z_n has the following property:

$$(\forall x \in \mathrm{pr}_1(Z_n))(Z_n(x) \text{ is nowhere dense in } \mathbf{R}).$$

Obviously, there exists $n_0 < \omega$ such that Z_{n_0} is of second category in $\mathbf{R} \times \mathbf{R}$. We define $Z = Z_{n_0}$. It is not difficult to verify that relations (1), (2), and (3) of the lemma are satisfied for Z, which finishes the proof.

We now are ready to establish the following statement.

Theorem 2. *Let $X \subset \mathbf{R}$ be an arbitrary set of second category in $\mathbf{R}$. Then there exists a function $f : X \to \mathbf{R}$ which does not admit any extension $f^* : \mathbf{R} \to \mathbf{R}$ possessing the Baire property.*

Proof. In view of the preceding lemma, a set $Z \subset \mathbf{R} \times \mathbf{R}$ can be found such that

(1) $\mathrm{pr}_1(Z) \subset X$ and $\mathrm{pr}_1(Z)$ is of second category in $\mathbf{R}$;
(2) Z is of second category in $\mathbf{R} \times \mathbf{R}$;
(3) for any $x \in \mathrm{pr}_1(Z)$, the set $Z(x)$ is nowhere dense in $\mathbf{R}$.

It suffices to show that there exists a function $f : \mathrm{pr}_1(Z) \to \mathbf{R}$ which does not admit an extension $f^* : \mathbf{R} \to \mathbf{R}$ having the Baire property.

For every point $x \in \mathrm{pr}_1(Z)$, denote by $D(x)$ a subset of $Z(x)$ that is at most countable and everywhere dense in $Z(x)$. Further, consider the set

$$T = \bigcup_{x \in \mathrm{pr}_1(Z)} (\{x\} \times D(x)).$$

Notice that $\mathrm{pr}_1(T) = \mathrm{pr}_1(Z)$. Also, the set

$$T^* = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y \in \mathrm{cl}(T(x))\}$$

contains Z and hence is of second category in $\mathbf{R} \times \mathbf{R}$. Obviously, we may represent T as a countable union of the graphs of functions acting from $\mathrm{pr}_1(Z)$ into $\mathbf{R}$. In other words, we may write

$$T = \cup\{f_n : n < \omega\},$$

where $f_n : \mathrm{pr}_1(Z) \to \mathbf{R}$ for each $n < \omega$.

Now, we assert that at least one function f_n cannot be extended to a function $f_n^* : \mathbf{R} \to \mathbf{R}$ possessing the Baire property.

Suppose to the contrary that every f_n admits such an extension f_n^*. Then, according to the well-known theorem from general topology (see, e.g., [149], [202] or Exercise 13 from Chapter 0), for any $n < \omega$, we can find an F_σ-set $P_n \subset \mathbf{R}$ of first category, such that the restriction $f_n^*|(\mathbf{R} \setminus P_n)$ is continuous. Let us denote

$$P = \cup\{P_n : n < \omega\}.$$

The set P is also of type F_σ and of first category in $\mathbf{R}$. In addition, all restrictions $f_n^*|(\mathbf{R} \setminus P)$, where $n < \omega$, are continuous. Let us observe that the set $K = \mathrm{pr}_1(Z) \setminus P$ is of second category in $\mathbf{R}$, the set

$$B = \bigcup_{x \in \mathbf{R} \setminus P} (\{x\} \times \{f_0^*(x), f_1^*(x), ..., f_n^*(x), ...\})$$

is Borel in $\mathbf{R} \times \mathbf{R}$ and, in view of the definition of all functions f_n^*, we have

$$\bigcup_{x \in K} (\{x\} \times D(x)) \subset B.$$

Further, putting $H = \bigcup_{x \in K}(\{x\} \times D(x))$ and taking into account the equality

$$T = H \cup (\bigcup_{x \in \mathrm{pr}_1(Z) \cap P} (\{x\} \times D(x))),$$

we see that the set

$$H^* = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y \in \mathrm{cl}(H(x))\}$$

is of second category in $\mathbf{R} \times \mathbf{R}$. By virtue of Lemma 3, the set B' is Borel in $\mathbf{R}$ and the inclusion $K \subset B'$ holds. We thus get the relation

$$H \subset B \cap (B' \times \mathbf{R}).$$

Notice now that the set $L = B \cap (B' \times \mathbf{R})$ is Borel in $\mathbf{R}^2$ and, by Lemma 2, the set L^* is analytic in $\mathbf{R}^2$. Also, all vertical sections of L^* are nowhere dense. Remembering that L^* has the Baire property (see Chapter 0), we conclude by the Kuratowski–Ulam theorem that L^* is of first category in $\mathbf{R} \times \mathbf{R}$. But this is impossible since $H^* \subset L^*$ and, as mentioned above, H^* is of second category. The contradiction obtained completes the proof.

We wish to underline once more that Theorem 2 is a result of **ZFC** theory and essentially relies on some profound facts from classical descriptive set theory.

Obviously, we may replace the real line $\mathbf{R}$ by any nonempty Polish space E without isolated points, and prove the analogous result for E.

Now, let us give our attention to the general situation where a topological space E without isolated points is given with its second category subset X. Does there exist a function $f : X \to \mathbf{R}$ that is not extendible to a function possessing the Baire property?

It turns out that we cannot positively answer this question within **ZFC** theory. Indeed, assuming the existence of some large cardinal, Kunen proved the consistency with **ZFC** of the following statement.

There exists a topological space W satisfying these four conditions:

(1) $\mathrm{card}(W) = \omega_1$;

(2) W is Hausdorff and contains no isolated points;

(3) W is a Baire space;

(4) no nonempty open set $U \subset W$ admits a representation in the form of the union of two disjoint subsets each of which is everywhere dense in U.

We recall (see Exercise 4 from Chapter 2) that a topological space E is resolvable if it admits a representation $E = A \cup B$, where A and B are disjoint and everywhere dense subsets of E.

Clearly, if E is resolvable, then it does not contain isolated points.

Relation (4) above says that no nonempty open subspace of the Kunen space W is resolvable.

Exercise 5 of this chapter shows that any subset of the Kunen space W possesses the Baire property. Hence, any real-valued function defined on W has the Baire property. Therefore, the answer to the question posed above is trivially negative for such a space.

So we must introduce some natural restrictions on a general topological space E if we want our question for E to be solved positively. Those restrictions can be formulated in purely topological terms, but it is desirable to formulate some conditions only in terms of the σ-ideal $\mathcal{K}(E)$ of all first category subsets of E, not touching the inner properties of the topology of E. Notice that, in many cases, those properties are not preserved under taking subspaces.

The following definition is useful for our further considerations.

Let E be a nonempty set equipped with a σ-algebra $\mathcal{S}$ of its subsets and let $\mathcal{I}$ be a σ-ideal of subsets of E such that $\mathcal{I} \subset \mathcal{S}$. The members of $\mathcal{I}$ are usually called small sets (or negligibles) with respect to the measurable structure $(E, \mathcal{S})$.

The triple $(E, \mathcal{S}, \mathcal{I})$ is called a measurable space with negligibles (see, for instance, [75]).

Of course, we have the following two widely known examples of such spaces $(E, \mathcal{S}, \mathcal{I})$.

Example 1. Let E be a topological space of second category on itself, let $\mathcal{S} = \mathcal{B}a(E)$ be the σ-algebra of all subsets of E having the Baire property in E, and let $\mathcal{I} = \mathcal{K}(E)$ be the σ-ideal of all first category subsets of E. Then the triple $(E, \mathcal{S}, \mathcal{I})$ can be regarded as a measurable space with negligibles.

Example 2. Let E be a set, $\mathcal{S}$ be a σ-algebra of subsets of E, and let μ be a nonzero σ-finite complete measure with $\mathrm{dom}(\mu) = \mathcal{S}$. Denote by $\mathcal{I} = \mathcal{I}(\mu)$ the σ-ideal of all μ-measure zero sets in E. Then the triple $(E, \mathcal{S}, \mathcal{I})$ is a measurable space with negligibles.

It is easy to see that the question posed at the beginning of this chapter can be reformulated in the following more general form:

Let $(E, \mathcal{S}, \mathcal{I})$ be a measurable space with negligibles and let X be a subset of E not belonging to $\mathcal{I}$. Does there exist a function $f : X \to \mathbf{R}$ which cannot be extended to an $\mathcal{S}$-measurable function $f^* : E \to \mathbf{R}$?

A rather natural approach for solving this question is presented in the proof of the next statement.

Theorem 3. *Let $(E, \mathcal{S}, \mathcal{I})$ be a measurable space with negligibles, X be a subset of E, and let*

$$\mathcal{S}_X = \{Y \cap X : Y \in \mathcal{S}\}.$$

Suppose that $\mathcal{S}_X \neq \mathcal{P}(X)$. Then there exists a function $g : X \to \mathbf{R}$ which cannot be extended to an $\mathcal{S}$-measurable function $g_1 : E \to \mathbf{R}$.

In particular, if $\mathcal{S}_X \neq \mathcal{P}(X)$ for all non-small sets $X \subset E$, then there are real-valued functions on any non-small set in E which do not admit an $\mathcal{S}$-measurable extension defined on the entire set E.

Proof. The argument is very easy. Because $\mathcal{S}_X \neq \mathcal{P}(X)$, there exists a set $Z \subset X$ not belonging to $\mathcal{S}_X$. Let

$$g_Z \; : \; X \to \{0, 1\}$$

denote the characteristic function of Z. Clearly, g_Z is not $\mathcal{S}_X$-measurable. We assert that $g = g_Z$ is the required one. Indeed, suppose to the contrary that there exists a function $g_1 : E \to \mathbf{R}$ extending g_Z and measurable with respect to $\mathcal{S}$. Then, for any open set $U \subset \mathbf{R}$, we can write

$$g_Z^{-1}(U) = g_1^{-1}(U) \cap X.$$

Because $g_1^{-1}(U) \in \mathcal{S}$, we get $g_Z^{-1}(U) \in \mathcal{S}_X$. Therefore, g_Z must be $\mathcal{S}_X$-measurable, which contradicts the definition of g_Z. The contradiction obtained finishes the proof.

Actually, the same argument establishes a much stronger statement, namely, the characteristic function g_Z cannot be extended to a partial $\mathcal{S}$-measurable real-valued function.

Returning to the problem of the existence of a real-valued partial function on a topological space E which does not admit an extension defined on E and possessing the Baire property, we will see later that this problem can be solved positively in the case when we are able to prove that any second category subset X of E includes a set without the Baire property (with respect to X). Some exercises for the present chapter yield certain results in this direction. They are based on Theorem 3 formulated and proved above.

EXERCISES

1. Let E be a topological space. Recall that a family $\{U_i : i \in I\}$ of nonempty open subsets of E is a π-base in E if, for each nonempty open set $U \subset E$, there exists an index $i \in I$ such that $U_i \subset U$.

Evidently, any base of E is also a π-base in E but the converse assertion is not true, in general.

Let E_1 be an arbitrary topological space and let E_2 be a topological space with a countable π-base.

For a set $Z \subset E_1 \times E_2$, verify the validity of relations (1), (2), and (3) of Theorem 1.

2. Let E be a topological space of second category, satisfying the Suslin condition, and let $\{U_i : i \in I\}$ be some π-base in E.

Show that there exists a base $\mathcal{B}$ of the σ-ideal $\mathcal{K}(E)$ of all first category subsets of E, such that $\mathrm{card}(\mathcal{B}) \leq (\mathrm{card}(I))^{\omega}$.

3. Let A be an arbitrary analytic set in the Euclidean plane $\mathbf{R}^2$.

Check that, for every natural number n, the set

$$A_n = \{x \in \mathbf{R} \ : \ \mathrm{card}(A(x)) \geq n\}$$

is analytic in $\mathbf{R}$.

Deduce from this fact that the set $A_{\omega} = \{x \in \mathbf{R} : \mathrm{card}(A(x)) \geq \omega\}$ is also analytic in $\mathbf{R}$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a Sierpiński–Zygmund function.

Assuming that each set in $\mathbf{R}$ of cardinality strictly less than $\mathbf{c}$ is of first category in $\mathbf{R}$, show that for any second category set $X \subset \mathbf{R}$, the function $f|X$ cannot be extended to a function $f^* : \mathbf{R} \to \mathbf{R}$ possessing the Baire property.

5. Suppose that in a topological space E no nonempty open set U admits a representation in the form of the union of two disjoint everywhere dense subsets of U.

Demonstrate that any subset X of E with $\mathrm{int}(X) = \emptyset$ is nowhere dense in E.

Deduce from this fact that in such a space E each subset can be represented as the union of an open set and a nowhere dense set. Consequently, each subset of E has the Baire property in E.

6. Check that there exist a Baire space E without isolated points and satisfying the Suslin condition (even separable) and a subspace X of E of second category in E, such that the Suslin condition does not hold for X.

7. Let E be an arbitrary topological space, A be a subset of E, and let e be a point in E. One says that A is of second category at e if, for every neighborhood $U(e)$ of e, the set $U(e) \cap A$ is of second category in E.

Let X be an arbitrary second category subset of E.

Show that there exists a set $Y \subset X$ satisfying the following two relations:

(a) Y is of second category at each point $y \in Y$;

(b) $X \setminus Y$ is of first category in E.

For this purpose, use Exercise 30 from Chapter 0.

Also, demonstrate that

(c) $\mathcal{K}(Y) = \{Z \cap Y : Z \in \mathcal{K}(E)\}$;

(d) $\mathcal{B}a(Y) = \{Z \cap Y : Z \in \mathcal{B}a(E)\}$.

8*. Let E be an infinite ground set and let $\{X_j : j \in J\}$ be a family of subsets of E such that

(a) $\mathrm{card}(J) \leq \mathrm{card}(E)$;

(b) $(\forall j \in J)(\mathrm{card}(X_j) = \mathrm{card}(E))$.

Prove that there exists a family $\{Y_i : i \in I\}$ of subsets of E satisfying these three conditions:

(c) $\mathrm{card}(I) > \mathrm{card}(E)$;

(d) $\mathrm{card}(Y_i \cap Y_{i'}) < \mathrm{card}(E)$ for any $i \in I,\ i' \in I,\ i \neq i'$ (in other words, the family $\{Y_i : i \in I\}$ is almost disjoint);

(e) $(\forall i \in I)(\forall j \in J)(\mathrm{card}(X_j \cap Y_i) = \mathrm{card}(E))$.

For this purpose, use the method of transfinite induction (cf. Exercise 7 from Chapter 10).

Remark 1. The above result generalizes the classical Sierpiński theorem on large almost disjoint families of sets (see [247]). In particular, one can directly obtain from Exercise 8 that there exists some almost disjoint family $\{Y_i : i \in I\}$ of infinite subsets of ω with $\mathrm{card}(I) > \omega$. Moreover, it is easy to strengthen the last fact and prove (even within **ZF** theory) that there exists an almost disjoint family of infinite subsets of ω, whose cardinality is equal to $\mathbf{c}$. This result is also due to Sierpiński and has interesting applications in various mathematical questions (cf. Exercise 13 of the present chapter).

9*. Let E be a topological space satisfying the following relations:

(1) $\mathrm{card}(E) = \omega_1$;

(2) E is of second category and has no isolated points;

(3) there exists a base of $\mathcal{K}(E)$ whose cardinality does not exceed ω_1.

Show that there exists a subset of E which does not possess the Baire property.

Moreover, prove that, for each second category subset X of E, there exists a set $Y \subset X$ without the Baire property (in E).

For this purpose, keeping in mind the results of Exercises 7 and 8, apply the Banach theorem on open sets of first category (see Exercise 30 from Chapter 0).

Further, show that, for any second category subset X of E, there exists a function $g : X \to \mathbf{R}$ not extendible to a function $g^* : E \to \mathbf{R}$ possessing the Baire property.

In particular, conclude that under **CH** the analogue of Theorem 2 holds true for any topology $\mathcal{T}$ on $\mathbf{R}$ extending the standard Euclidean topology of $\mathbf{R}$ and such that the σ-ideal of all first category sets (with respect to $\mathcal{T}$) contains all singletons in $\mathbf{R}$ and possesses a base whose cardinality does not exceed $\mathbf{c}$.

Remark 2. It is useful to compare the above exercise with Kunen's result formulated in this chapter.

10. Let E be a set with $\mathrm{card}(E) = \omega_1$ and let μ be a nonzero σ-finite complete diffused measure defined on some σ-algebra of subsets of E. Denote by $\mathcal{S}$ the domain of μ and by $\mathcal{I} = \mathcal{I}(\mu)$ the σ-ideal of all μ-measure zero sets in E.

Applying to the measurable space with negligibles $(E, \mathcal{S}, \mathcal{I})$ the classical Ulam theorem on the non-real-valued measurability of ω_1, show that for any set $X \notin \mathcal{I}$, there exists a function $g : X \to \mathbf{R}$ not extendible to an $\mathcal{S}$-measurable function $g^* : E \to \mathbf{R}$.

11*. Let E be a topological space of second category and suppose that the following two conditions are satisfied:

(a) for an arbitrary family $\mathcal{F} \subset \mathcal{K}(E)$ with $\mathrm{card}(\mathcal{F}) < \mathrm{card}(E)$, one has $\cup\mathcal{F} \in \mathcal{K}(E)$, i.e., the σ-ideal $\mathcal{K}(E)$ is $\mathrm{card}(E)$-additive;

(b) there exists a base of $\mathcal{K}(E)$ whose cardinality is less than or equal to $\mathrm{card}(E)$.

Let $\{X_i : i \in I\} \subset \mathcal{K}(E)$ be a point-finite family of sets, i.e.,

$$(\forall x \in E)(\mathrm{card}(\{i \in I : x \in X_i\}) < \omega),$$

and suppose in addition that $\cup\{X_i : i \in I\} = E$.

Prove that there exist two disjoint sets $I_1 \subset I$ and $I_2 \subset I$ such that both sets $\cup\{X_i : i \in I_1\}$ and $\cup\{X_i : i \in I_2\}$ do not possess the Baire property in E.

12*. Let E be a topological space of second category, satisfying the conditions (a) and (b) of Exercise 11, let E_1 be a metric space, and let

$$\Phi \ : \ E \to \mathcal{P}(E_1)$$

be a set-valued mapping such that $\Phi(x)$ is nonempty and compact for any $x \in E$. Suppose also that, for each open set $U \subset E_1$, the set

$$\{x \in E : \Phi(x) \cap U \neq \emptyset\}$$

possesses the Baire property in E.

Fix a real $\varepsilon > 0$ and demonstrate that there exists a ball $B \subset E_1$ with $\mathrm{diam}(B) \leq \varepsilon$, such that the set

$$\{x \in E : \Phi(x) \cap B \neq \emptyset\}$$

is of second category in E.

Assume additionally that E satisfies the Suslin condition and prove that there exists a selector f of Φ possessing the Baire property, i.e., there exists a function

$$f : E \to E_1$$

possessing the Baire property and such that $f(x) \in \Phi(x)$ for all $x \in E$.

13*. Let E be a Banach space and let E_1 be a Banach subspace of E. Recall that E_1 admits a complemented space if there exists a Banach subspace E_2 of E such that E is a direct topological sum of E_1 and E_2. In this case, E_1 is also called a direct topological summand in E.

For example, every Hilbert subspace H_1 of an arbitrary Hilbert space H is a direct topological summand in H.

Let l_∞ denote the Banach space of all bounded real-valued sequences and let c_0 denote the Banach subspace of l_∞ consisting of all sequences tending to zero (notice that l_∞ is nonseparable and c_0 is separable).

Demonstrate that c_0 does not admit a complemented space in l_∞ (the old result of Sobczyk).

Do this in the following manner. Start with the observation that the conjugate space $(l_\infty)^*$ contains a countable subfamily separating the points in l_∞. Consequently, each vector subspace F of l_∞ possesses the same property, i.e., F^* contains a countable subfamily which separates the points in F. Assuming to the contrary that F is a complemented space for c_0, infer first that F must be isomorphic to the Banach factor-space l_∞/c_0.

On the other hand, show that $(l_\infty/c_0)^*$ does not contain a countable subfamily separating the points in l_∞/c_0.

For this purpose, take an arbitrary uncountable almost disjoint family $\{A_i : i \in I\}$ of infinite subsets of ω (see Exercise 8 and Remark 1) and consider the characteristic functions

$$f_{A_i} : \omega \to \{0, 1\} \qquad (i \in I)$$

of these subsets. Observe that all functions f_{A_i} belong to l_∞. Let

$$\phi : l_\infty \to l_\infty/c_0$$

denote the canonical surjective homomorphism of Banach spaces and let

$$x_i = \phi(f_{A_i}) \qquad (i \in I).$$

Further, for any functional $g \in (l_\infty/c_0)^*$ and for each index i from I, put $a_i = \mathrm{sgn}(g(x_i))$.

Check that if a set $I_0 \subset I$ is finite, then $||\sum_{i \in I_0} a_i x_i|| \leq 1$ in view of the almost disjointness of $\{A_i : i \in I\}$.

Deduce from this fact that

$$||g|| \geq |g(\sum_{i \in I_0} a_i x_i)| = \sum_{i \in I_0} |g(x_i)|,$$

which implies that the family $\{|g(x_i)| : i \in I\}$ of real numbers is summable, i.e.,

$$\sum \{|g(x_i)| : i \in I\} < +\infty.$$

Consequently, there are only countably many members of this family which differ from zero.

Finally, conclude that if $\{g_k : k < \omega\}$ is an arbitrary countable family of functionals from $(l_\infty/c_0)^*$, then there exists an index $i_0 \in I$ satisfying the relations

$$g_k(x_{i_0}) = 0 \qquad (k < \omega).$$

In other words, $\{g_k : k < \omega\}$ does not separate the points $x_{i_0} \neq 0$ and 0 in the space l_∞/c_0.

The obtained contradiction shows that c_0 is not a direct topological summand in l_∞.

Remark 3. As the reader could see, Exercise 13 yields a nontrivial application of almost disjoint families of infinite subsets of ω to a concrete problem arising in the theory of Banach spaces.

14. Observe that if a Banach space E_1 is a direct topological summand in a Banach space E and a Banach space E_2 is a direct topological summand in E_1, then E_2 is a direct topological summand in E.

Deduce from this fact and the result of Exercise 13 that the Banach space c consisting of all convergent real-valued sequences does not admit a complemented space in l_∞.

15. Applying Exercise 8 of this chapter, prove that if E is a set and μ is a nonatomic probability measure defined on the family $\mathcal{P}(E)$ of all subsets of E, then μ is not perfect (cf. Exercise 1 and Remark 4 from Chapter 8).

On the other hand, check that if ν is a two-valued probability measure defined on the same family $\mathcal{P}(E)$, then ν is perfect.

16*. Recall that a cardinal **a** is two-valued measurable if there is a two-valued probability diffused measure defined on the family of all subsets of **a**. Recall also that the existence of two-valued measurable cardinals cannot be established within **ZFC** set theory (see, e.g., [97], [103]).

Let $(G, \cdot)$ be a nondiscrete topological group of second category (not necessarily Hausdorff) and let card(G) be strictly less than the smallest two-valued measurable cardinal.

Demonstrate that there exists a subset of G not having the Baire property.

Argue as follows. Suppose to the contrary that all subsets of G possess the Baire property and assume, without loss of generality, that card(G) takes the minimum value. Using Exercise 1 from Chapter 10, verify the validity of the following two relations:

(a) card$(G) > \omega$;

(b) if $X \subset G$ and card$(X) <$ card(G), then X is of first category in G.

Denoting by α the least ordinal number with card$(\alpha) =$ card(G), construct a partition $\{X_\xi : \xi < \alpha\}$ of G such that

(c) card$(X_\xi) <$ card(G) for all $\xi < \alpha$;

(d) for any set $\Xi \subset \alpha$ and for any $g \in G$, the symmetric difference

$$(g \cdot (\cup\{X_\xi : \xi \in \Xi\}))\triangle(\cup\{X_\xi : \xi \in \Xi\})$$

has cardinality strictly less than card(G), so is of first category.

Infer from (b), (c), and (d) that card(G) is a two-valued measurable cardinal, which yields a contradiction.

17. Let $(G, \cdot)$ be a nondiscrete topological group (not necessarily Hausdorff) and let card(G) be strictly less than the smallest two-valued measurable cardinal.

Starting with Exercise 16, show that the disjunction of the following two assertions holds true:

(a) G is a resolvable topological space;

(b) G is a first category topological space.

Chapter 19
Sup-measurable and weakly sup-measurable functions

It is well known that the concept of measurability of sets and functions plays an important role in various fields of classical and modern mathematical analysis (also, in probability theory, in the theory of random processes, and in general topology).

For functions of several variables, a related notion of sup-measurability was introduced and investigated (see, e.g., [5], [9], [47], [51], [82], [111], [123], [126], [128], [143], [144], [218], [233], [234] and references given therein). It turned out that this notion can successfully be applied to some topics of mathematical analysis and, in particular, to the theory of ordinary differential equations (for more information concerning applications of sup-measurable mappings in the above-mentioned theory, see [144] and Chapter 21 of the present book).

Here we would like to introduce and examine the following three classes of functions acting from $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$:

(1) the class of sup-continuous mappings;

(2) the class of sup-measurable mappings;

(3) the class of weakly sup-measurable mappings.

We begin with the definitions of sup-continuous and sup-measurable functions.

We say that a mapping $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is sup-continuous (sup-measurable) with respect to the second variable y if, for every continuous (Lebesgue measurable) function $\phi : \mathbf{R} \to \mathbf{R}$, the superposition $\Phi_\phi : \mathbf{R} \to \mathbf{R}$ given by the formula

$$\Phi_\phi(x) = \Phi(x, \phi(x)) \qquad (x \in \mathbf{R})$$

is also continuous (Lebesgue measurable).

Actually, the sup-continuity notion yields nothing new, because the class of all sup-continuous mappings coincides with the class of all continuous mappings acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. For the sake of completeness, we give below the proof of this simple and, probably, well-known fact.

Theorem 1. *Let Φ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. Then the following two assertions are equivalent:*
(1) Φ is continuous;
(2) Φ is sup-continuous.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. So it remains to establish only the converse implication (2) $\Rightarrow$ (1). Let Φ be sup-continuous, and suppose to the contrary that Φ is not continuous. Then there exist a point (x_0, y_0) of $\mathbf{R} \times \mathbf{R}$, a real number $\varepsilon > 0$, and a sequence of points

$$\{(x_n, y_n) \ : \ n \in \mathbf{N}, \ n > 0\} \subset \mathbf{R} \times \mathbf{R}$$

such that
(a) $\lim_{n \to +\infty} (x_n, y_n) = (x_0, y_0)$;
(b) $|\Phi(x_n, y_n) - \Phi(x_0, y_0)| > \varepsilon$ for all $n \in \mathbf{N} \setminus \{0\}$.

We may assume, without loss of generality, that the sequence of points $\{x_n : n \in \mathbf{N}, \ n > 0\} \subset \mathbf{R}$ is injective and $x_n \neq x_0$ for each $n \in \mathbf{N} \setminus \{0\}$. Indeed, if

$$f_n \ : \ \mathbf{R} \to \mathbf{R} \qquad (n = 1, 2, ...)$$

denotes the function identically equal to y_n, then $\Phi_{f_n} : \mathbf{R} \to \mathbf{R}$ is a continuous function and $\Phi_{f_n}(x_n) = \Phi(x_n, y_n)$. Therefore, for some strictly positive real number $\delta = \delta(x_n)$ and for all points x belonging to the open interval $]x_n - \delta, \ x_n + \delta[$, we have the inequality $|\Phi_{f_n}(x) - \Phi(x_0, y_0)| > \varepsilon$ or, equivalently,

$$|\Phi(x, y_n) - \Phi(x_0, y_0)| > \varepsilon.$$

From this fact it immediately follows that the above-mentioned sequence of points $\{x_n : n \in \mathbf{N}, \ n > 0\}$ can be chosen to be injective and satisfying the relation $(\forall n \in \mathbf{N} \setminus \{0\})(x_n \neq x_0)$.

Now, it is not difficult to define a continuous function $f : \mathbf{R} \to \mathbf{R}$ such that

$$(\forall n \in \mathbf{N})(f(x_n) = y_n).$$

For this function f, we get the continuous superposition $\Phi_f : \mathbf{R} \to \mathbf{R}$. Since $\lim_{n \to +\infty} x_n = x_0$, we must have the equality

$$\lim_{n \to +\infty} \Phi_f(x_n) = \Phi_f(x_0)$$

and, consequently,

$$\lim_{n \to +\infty} \Phi(x_n, y_n) = \Phi(x_0, y_0),$$

which is impossible. This contradiction finishes the proof of Theorem 1.

A completely different situation occurs for sup-measurable mappings. On the one hand, simple examples show that if $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable mapping, then it need not be sup-measurable (see, e.g., Exercise 1). On the other hand, it turns out that there exist (under some additional set-theoretical axioms) various sup-measurable mappings which are not measurable in the Lebesgue sense. In order to present this result, let us first formulate and prove one simple auxiliary statement.

Lemma 1. *Suppose that Ψ is a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. Then the following two assertions are equivalent:*

(1) Ψ is sup-measurable;

(2) for every continuous function $\psi : \mathbf{R} \to \mathbf{R}$, the function Ψ_ψ is Lebesgue measurable.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Let us show that the converse implication (2) $\Rightarrow$ (1) is true, too. Let Ψ satisfy (2) and let ψ be an arbitrary Lebesgue measurable function acting from $\mathbf{R}$ into $\mathbf{R}$. Using Luzin's C-property of ψ, we can find a countable disjoint covering $\{X_k : k < \omega\}$ of $\mathbf{R}$ and a countable family $\{\psi_k : k < \omega\}$ of functions acting from $\mathbf{R}$ into $\mathbf{R}$, such that

(a) all sets X_k $(1 \leq k < \omega)$ are closed in $\mathbf{R}$ and X_0 is of Lebesgue measure zero;

(b) all functions ψ_k $(1 \leq k < \omega)$ are continuous;

(c) for each index $k < \omega$, the restriction of ψ to X_k coincides with the restriction of ψ_k to X_k.

Let us denote by f_k the characteristic function of X_k. Then it is not difficult to check the equality

$$\Psi_\psi = \sum_{k<\omega} f_k \cdot \Psi_{\psi_k}.$$

According to our assumption, all superpositions Ψ_{ψ_k} $(1 \leq k < \omega)$ are Lebesgue measurable. In addition, the function $f_0 \cdot \Psi_{\psi_0}$ is equivalent to zero. Thus, we easily conclude that the superposition Ψ_ψ is Lebesgue measurable, too. Lemma 1 has thus been proved.

Now, we wish to formulate and prove the following statement.

Theorem 2. *Let λ denote the standard Lebesgue measure on $\mathbf{R}$ and let $[\mathbf{R}]^{<\mathbf{c}}$ be the family of all subsets of $\mathbf{R}$ whose cardinalities are strictly less than $\mathbf{c}$. There exists a subset Z of $\mathbf{R} \times \mathbf{R}$ such that*

(1) no three distinct points of Z belong to a straight line (in other words, Z is a set of points in general position in the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$);

(2) Z is a Lebesgue nonmeasurable subset of $\mathbf{R} \times \mathbf{R}$;

(3) if the inclusion $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$ holds, then the characteristic function of Z is sup-measurable.

Proof. The argument is very similar to that applied in the construction of a Sierpiński–Zygmund function (see Chapter 8).

Obviously, we can identify $\mathbf{c}$ with the first ordinal number α such that $\mathrm{card}(\alpha) = \mathbf{c}$. Let λ_2 denote the standard two-dimensional Lebesgue measure on the plane $\mathbf{R} \times \mathbf{R}$ and let $\{Z_\xi \ : \ \xi < \alpha\}$ be the family of all Borel subsets of $\mathbf{R} \times \mathbf{R}$ having strictly positive λ_2-measure. In addition, let $\{\phi_\xi \ : \ \xi < \alpha\}$ be the family of all continuous functions acting from $\mathbf{R}$ into $\mathbf{R}$. As usual, we identify any function from $\mathbf{R}$ into $\mathbf{R}$ with its graph lying in the plane $\mathbf{R}^2$. Now, using the method of transfinite recursion, we are going to define an α-sequence of points $\{(x_\xi, y_\xi) : \xi < \alpha\} \subset \mathbf{R} \times \mathbf{R}$, satisfying the following four conditions:

(a) if $\xi < \alpha$, $\zeta < \alpha$, and $\xi \neq \zeta$, then $x_\xi \neq x_\zeta$;

(b) for each $\xi < \alpha$, the point (x_ξ, y_ξ) belongs to the set Z_ξ;

(c) for each $\xi < \alpha$, the point (x_ξ, y_ξ) does not belong to the union of the family $\{\phi_\zeta \ : \ \zeta \leq \xi\}$;

(d) for each $\xi < \alpha$, no three distinct points of the set $\{(x_\zeta, y_\zeta) : \zeta \leq \xi\}$ belong to a straight line.

Suppose that, for an ordinal $\xi < \alpha$, the partial ξ-sequence of points

$$\{(x_\zeta, y_\zeta) \ : \ \zeta < \xi\} \subset \mathbf{R} \times \mathbf{R}$$

has already been defined. Let us consider the set Z_ξ. We have the inequality $\lambda_2(Z_\xi) > 0$, so according to the classical Fubini theorem, we can write

$$\lambda(\{x \in \mathbf{R} \ : \ Z_\xi(x) \in \mathrm{dom}(\lambda) \ \& \ \lambda(Z_\xi(x)) > 0\}) > 0,$$

where $Z_\xi(x)$ denotes the section of Z_ξ corresponding to a point $x \in \mathbf{R}$. Taking account of the latter formula, we see that there exists an element

$$x_\xi \in \mathbf{R} \setminus \{x_\zeta \ : \ \zeta < \xi\}$$

for which $\lambda(Z_\xi(x_\xi)) > 0$. In particular, we get $\mathrm{card}(Z_\xi(x_\xi)) = \mathbf{c}$. Consequently, there exists an element

$$y_\xi \in Z_\xi(x_\xi) \setminus \cup\{\phi_\zeta(x_\xi) \ : \ \zeta \leq \xi\}.$$

Moreover, y_ξ can be chosen in such a way that the corresponding point (x_ξ, y_ξ) does not belong to the union of all straight lines having at least two common points with the set $\{(x_\zeta, y_\zeta) \ : \ \zeta < \xi\}$.

Thus, the point $(x_\xi, y_\xi) \in \mathbf{R} \times \mathbf{R}$ is defined.

Proceeding in this manner, we are able to construct the α-sequence $\{(x_\xi, y_\xi) : \xi < \alpha\}$ satisfying conditions (a), (b), (c), and (d). Finally, let us put

$$Z = \{(x_\xi, y_\xi) : \xi < \alpha\}$$

and let Φ denote the characteristic function of Z (obviously, Z is considered as a subset of the plane $\mathbf{R} \times \mathbf{R}$). Notice that Z can also be regarded as the graph of a partial function acting from $\mathbf{R}$ into $\mathbf{R}$. Hence the inner λ_2-measure of Z is equal to zero. On the other hand, the construction of Z immediately yields that Z is a λ_2-thick subset of the plane $\mathbf{R}^2$. Consequently, Z is nonmeasurable in the Lebesgue sense and the same is true for its characteristic function Φ.

It remains to check that Φ is a sup-measurable mapping under the assumption $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$. Let us take any continuous function $\phi : \mathbf{R} \to \mathbf{R}$. Then $\phi = \phi_\xi$ for some ordinal $\xi < \alpha$. Now, we can write

$$\{x \in \mathbf{R} : \Phi(x, \phi_\xi(x)) \neq 0\} = \{x \in \mathbf{R} : (x, \phi_\xi(x)) \in Z\}$$

and it easily follows from condition (c) that

$$\mathrm{card}(\{x \in \mathbf{R} : (x, \phi_\xi(x)) \in Z\}) < \mathbf{c}.$$

Because the inclusion $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$ holds, we obtain that the function $\Phi_{\phi_\xi} = \Phi_\phi$ almost vanishes (with respect to λ) and, in particular, Φ_ϕ is λ-measurable. Applying Lemma 1, we conclude that Φ is sup-measurable and this finishes the proof.

Remark 1. It is reasonable to stress here that the set Z (and, consequently, its characteristic function Φ) is defined within **ZFC** theory. We used an additional set-theoretical hypothesis only to establish that Φ is a sup-measurable mapping. Let us also recall that the first construction of a Lebesgue nonmeasurable subset of the Euclidean plane, no three points of which belong to a straight line, is due to Sierpiński (see, for instance, [202]).

Remark 2. In connection with Theorem 2, the question naturally arises whether it is possible to establish within **ZFC** the existence of a sup-measurable mapping that is not measurable in the Lebesgue sense (see, e.g., [111]). This question was solved negatively by Shelah and Roslanowski. Namely, they have shown in [218] that the statement "all sup-measurable mappings are Lebesgue measurable" is consistent with **ZFC** theory.

We now introduce the notion of a weakly sup-measurable function.

Let Φ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. We shall say that Φ is weakly sup-measurable if, for any continuous function $\phi : \mathbf{R} \to \mathbf{R}$ differentiable almost everywhere (with respect to λ), the superposition Φ_ϕ is Lebesgue measurable.

From the view-point of the theory of first-order ordinary differential equations, the notion of a weakly sup-measurable mapping is more preferable than the notion of a sup-measurable mapping, because any solution of a first-order ordinary differential equation must be continuous everywhere and differentiable almost everywhere.

Clearly, Theorem 2 can be formulated in terms of weakly sup-measurable mappings. In this connection, the following question seems to be of interest: does there exist a weakly sup-measurable mapping that is not sup-measurable? In order to give a partial answer to this question, we need one auxiliary statement due to Jarník (see [95]).

Lemma 2. *There exists a continuous function $f : \mathbf{R} \to \mathbf{R}$ which is nowhere approximately differentiable.*

We recall that Lemma 2 was proved in Chapter 7 of this book, devoted to one special construction of a nowhere approximately differentiable function.

We also want to recall that, in fact, Jarník proved in [95] that the set of all those functions from the Banach space $C([0,1])$, which are nowhere approximately differentiable, is residual in $C([0,1])$, i.e., is the complement of a first category set. Nevertheless, in our further considerations, we need only one such function.

Theorem 3. *Suppose that*

(1) $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$;

(2) for any cardinal number $\kappa < \mathbf{c}$ and for any family $\{X_\xi : \xi < \kappa\}$ of λ-measure zero subsets of $\mathbf{R}$, we have $\cup\{X_\xi : \xi < \kappa\} \neq \mathbf{R}$.

Then there exists a weakly sup-measurable mapping $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which is not sup-measurable.

Proof. We can identify $\mathbf{c}$ with the first ordinal number α such that $\mathrm{card}(\alpha) = \mathbf{c}$. Let f be a function from Lemma 2. Let $\{B_\xi : \xi < \alpha\}$ be some Borel base of the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$ and let $\{\phi_\xi : \xi < \alpha\}$ be the family of all continuous functions acting from $\mathbf{R}$ into $\mathbf{R}$ and differentiable almost everywhere in $\mathbf{R}$. We are going to construct (by transfinite recursion) an injective α-sequence

$$\{(x_\xi, y_\xi) : \xi < \alpha\} \subset \mathbf{R} \times \mathbf{R}$$

of points belonging to the graph of f. Suppose that, for an ordinal $\xi < \alpha$, the partial ξ-sequence $\{(x_\zeta, y_\zeta) : \zeta < \xi\}$ has already been defined. Notice that, for each ordinal $\zeta \le \xi$, the closed set

$$P_\zeta = \{x \in \mathbf{R} \ : \ \phi_\zeta(x) = f(x)\}$$

is of Lebesgue measure zero. Indeed, assuming otherwise, i.e., $\lambda(P_\zeta) > 0$, we can find a density point x of P_ζ belonging to P_ζ such that there exists an approximate derivative $f'_{ap}(x) = \phi'_\zeta(x)$. But this is impossible in view of the main property of f. Consequently, $\lambda(P_\zeta) = 0$ for all ordinal numbers $\zeta \le \xi$, and the set

$$\mathbf{R} \setminus ((\cup\{B_\zeta \ : \ \zeta \le \xi\}) \cup (\cup\{P_\zeta \ : \ \zeta \le \xi\}) \cup \{x_\zeta \ : \ \zeta < \xi\})$$

is not empty. Let x_ξ be an arbitrary point from this set and let $y_\xi = f(x_\xi)$.

Proceeding in such a manner, we are able to define the required family of points $\{(x_\xi, y_\xi) \ : \ \xi < \alpha\}$. Now, we put

$$Z = \{(x_\xi, y_\xi) : \xi < \alpha\}, \qquad X = \{x_\xi : \xi < \alpha\}$$

and denote by Φ the characteristic function of Z. Then it can easily be seen that Φ is a weakly sup-measurable mapping (cf. the proof of Theorem 2). On the other hand, let us consider the superposition Φ_f. Obviously, we have

$$\Phi_f(x) = 1 \Leftrightarrow (x, f(x)) \in Z \Leftrightarrow x \in X.$$

It follows from our construction that X is a Sierpiński type subset of the real line $\mathbf{R}$ (for the definition and various properties of Sierpiński sets, see, e.g., [43], [149], [184], [192], [202] or Chapter 13 of this book, which is specially devoted to Luzin and Sierpiński sets). In particular, as shown in that chapter, X is not measurable in the Lebesgue sense and, therefore, Φ_f is not Lebesgue measurable, either. We thus conclude that Φ is not a sup-measurable mapping. This completes the proof of the theorem.

Remark 3. It is well known that assumptions (1) and (2) of Theorem 3 are logically independent (see, for instance, [147]). Slightly changing the argument presented above, one can show (under the assumptions of Theorem 3) that there exists a weakly sup-measurable mapping which is not sup-measurable and, in addition, is not Lebesgue measurable. We do not know whether the assertion of Theorem 3 (i.e., the existence of a weakly sup-measurable mapping which is not sup-measurable) can be proved within **ZFC** theory.

Remark 4. Evidently, the notion of sup-measurability can be formulated in terms of the Baire property instead of measurability in the Lebesgue sense. More precisely, we say that a function $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is sup-measurable in the sense of the Baire property if, for any function $\phi : \mathbf{R} \to \mathbf{R}$ possessing the Baire property, the superposition

$$x \to \Phi(x, \phi(x)) \qquad (x \in \mathbf{R})$$

possesses the Baire property, too. In a similar way, the notion of weak sup-measurability (in the sense of the Baire property) can be introduced. It is not difficult to verify that, for functions with the Baire property, the direct analogue of Theorem 2 holds true. The corresponding analogue of Theorem 3 also holds (in this case, we do not need Lemma 2; it suffices to apply the existence of a continuous nowhere differentiable function acting from $\mathbf{R}$ into $\mathbf{R}$). The corresponding details are left to the reader as a useful exercise.

There is an important class of λ_2-measurable real-valued functions on $\mathbf{R}^2$ which are also sup-measurable. Namely, Exercise 2 of this chapter shows that all functions of two variables which satisfy the Carathéodory conditions are good from the point of view of sup-measurability. In addition, such functions play a significant role in the theory of first-order ordinary differential equations. The following definition introduces a slightly more general class of functions.

Let $(X, \mathcal{S}, \mu)$ be a space with a nonzero σ-finite complete measure, let Y be a topological space, and let $f : X \times Y \to \mathbf{R}$ be a function. We say that f almost satisfies the Carathéodory conditions if

(*) for almost all (with respect to μ) points $x \in X$, the partial function $f(x, \cdot)$ is continuous on Y;

(**) for all points $y \in Y$, the partial function $f(\cdot, y)$ is μ-measurable.

Obviously, if f satisfies the Carathéodory conditions, then it almost satisfies these conditions. The converse assertion is not true, in general. However, it can easily be verified that if f almost satisfies the Carathéodory conditions, then there exists a function $g : X \times Y \to \mathbf{R}$ satisfying these conditions, such that $f(x, \cdot) = g(x, \cdot)$ for almost all points $x \in X$.

Some useful additional information about functions which almost satisfy the Carathéodory conditions is given in Exercises 13 and 14 of this chapter. Exercise 16 highlights the role of such functions in certain delicate questions of mathematical analysis that are closely connected with measurable selectors. Here we wish to consider a result which is much deeper than that presented in Exercise 16. For this purpose, we need the notion of a C-set. This notion was introduced by Luzin and Kolmogorov many years ago and was investigated by several authors (see, e.g., [163], [197], and [227]).

Let E be a metric space. We define the family of all C-sets in E as the smallest class containing all open subsets of E and closed under the operation of taking the complement and under the (A)-operation. Because the (A)-operation includes countable unions and countable intersections, we see that the class of all C-sets in E forms a certain σ-algebra containing all analytic subsets of E (in particular, the Borel σ-algebra of E is contained in the class of all C-subsets of E). It immediately follows from the definition that all C-sets are universally measurable and possess the Baire property in the restricted sense (see Chapter 0).

Now, let E' be a metric space and let $f : E \to E'$ be a mapping. We say that f is C-measurable if, for each open subset U of E', the pre-image $f^{-1}(U)$ is a C-set in E.

The definition of the (A)-operation directly implies that the pre-image of the result of this operation over a given family of sets coincides with the result of the same operation over the family of pre-images of sets. Taking this fact into account, one may readily infer that, for any C-measurable mapping f acting from E into E' and for any C-set Z in E', the pre-image $f^{-1}(Z)$ is a C-set in E.

In particular, the composition of two C-measurable functions is a C-measurable function, too (this is an important feature of C-measurable functions, which has no analogue in the class of Lebesgue measurable functions).

Let now X and Y be any two Polish topological spaces and let

$$f : X \times Y \to \mathbf{R}$$

be a Borel mapping such that

$$(\forall x \in X)(\exists y \in Y)(f(x, y) = \inf_{t \in Y} f(x, t)).$$

Let us define

$$f^*(x) = \inf_{t \in Y} f(x, t) \qquad (x \in X).$$

Notice that, for each $a \in \mathbf{R}$, we have

$$\{x \in X : f^*(x) < a\} = \{x \in X : (\exists t \in Y)(f(x, t) < a)\}.$$

Because the original function f is Borel, we conclude that the second set in the above equality is analytic, i.e., the function f^* is measurable with respect to the σ-algebra generated by all analytic subsets of X (hence f^* is C-measurable as well). In the product space $X \times \mathbf{R} \times Y$ consider the set

$$B = \{(x, z, y) \ : \ f(x, y) = z\}.$$

Obviously, this set is Borel (as a homeomorphic image of the graph of a Borel function). According to the classical uniformization theorem of Luzin, Jankov, and von Neumann (which is an easy consequence of the theorem of Kuratowski and Ryll-Nardzewski [154] on measurable selectors), there exists a function $h : \mathrm{pr}_{X\times\mathbf{R}}(B) \to Y$ such that

(a) h is measurable with respect to the σ-algebra generated by all analytic subsets of $\mathrm{dom}(h)$;

(b) the graph of h is contained in B.

Further, our assumption on f and the definition of f^* imply the following two relations:

(c) $\mathrm{pr}_X(\mathrm{dom}(h)) = X$;

(d) for any $x \in X$, we have $(x, f^*(x)) \in \mathrm{dom}(h)$.

Consequently, we may write

$$(\forall(x,z) \in \mathrm{dom}(h))((x,z,h(x,z)) \in B),$$

$$(\forall(x,z) \in \mathrm{dom}(h))(f(x,h(x,z)) = z),$$

$$(\forall x \in X)(f(x,h(x,f^*(x))) = f^*(x)).$$

Now, define a function $g : X \to \mathbf{R}$ by the formula

$$g(x) = h(x, f^*(x)) \qquad (x \in X).$$

Then g is C-measurable (as the composition of two C-measurable functions) and we get

$$f(x, g(x)) = f^*(x) = \inf_{t\in Y} f(x,t)$$

for all points $x \in X$ (cf. Exercise 16).

EXERCISES

1. Give an example of a function $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which is Lebesgue measurable but is not sup-measurable.

More precisely, demonstrate that the existence of such examples follows directly from the widely known fact that the composition of two Lebesgue measurable functions (acting from $\mathbf{R}$ into $\mathbf{R}$) need not be Lebesgue measurable.

2. Let Φ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. Suppose that this mapping satisfies the so-called Carathéodory conditions, i.e.,

(1) for each $x \in \mathbf{R}$, the partial function $y \to \Phi(x, y)$ $(y \in \mathbf{R})$ is continuous;

(2) for each $y \in \mathbf{R}$, the partial function $x \to \Phi(x, y)$ $(x \in \mathbf{R})$ is Lebesgue measurable.

Show that

(a) Φ is measurable with respect to the usual two-dimensional Lebesgue measure λ_2 on the plane $\mathbf{R}^2$ (more precisely, Φ is measurable with respect to the product of the σ-algebras $\mathrm{dom}(\lambda)$ and $\mathcal{B}(\mathbf{R})$);

(b) Φ is sup-measurable.

3. Let Φ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. Suppose that the following two conditions hold:

(1) for each $x \in \mathbf{R}$, the partial function $y \to \Phi(x, y)$ $(y \in \mathbf{R})$ is a continuous mapping from $\mathbf{R}$ into $\mathbf{R}$;

(2) for each $y \in \mathbf{R}$, the partial function $x \to \Phi(x, y)$ $(x \in \mathbf{R})$ is a Borel mapping from $\mathbf{R}$ into $\mathbf{R}$.

Demonstrate that Φ is a Borel mapping from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$.

By using the method of transfinite induction, give an example of a function Ψ acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$ such that

(a) for any points $x_0 \in \mathbf{R}$ and $y_0 \in \mathbf{R}$, the partial functions

$$y \to \Psi(x_0, y) \ \ (y \in \mathbf{R}), \qquad x \to \Psi(x, y_0) \ \ (x \in \mathbf{R})$$

are upper semicontinuous (hence Borel);

(b) Ψ is not measurable with respect to the two-dimensional Lebesgue measure λ_2 on $\mathbf{R} \times \mathbf{R}$.

4. Let $n > 1$ be a natural number and let Φ be a mapping acting from the n-dimensional Euclidean space $\mathbf{R}^n$ into $\mathbf{R}$. Suppose also that Φ satisfies the following condition: for each natural index $i \in [1, n]$ and for any points $x_1 \in \mathbf{R}$, ..., $x_{i-1} \in \mathbf{R}$, $x_{i+1} \in \mathbf{R}$, ..., $x_n \in \mathbf{R}$, the partial function

$$x \to \Phi(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n) \qquad (x \in \mathbf{R})$$

is continuous.

Prove, using induction on n, that Φ is a Borel mapping acting from $\mathbf{R}^n$ into $\mathbf{R}$. More precisely, verify that the Baire order of Φ is less than or equal to $n-1$ (cf. Theorem 4 of Chapter 2).

5*. Let $\mathbf{T}$ denote the one-dimensional unit torus, i.e., the set

$$\mathbf{T} = \{(x, y) \in \mathbf{R} \times \mathbf{R} \ : \ x^2 + y^2 = 1\}$$

is regarded as a commutative compact topological group with respect to the usual group operation and the Euclidean topology. Denote by e the neutral element of this group and consider the product group $\mathbf{T}^\omega$ which is a commutative compact topological group, too. Equip this product group with the Haar probability measure μ. In fact, μ is the product measure of the countable family of measures, each of which coincides with the Haar probability measure on $\mathbf{T}$. Further, denote by G the subset of $\mathbf{T}^\omega$ consisting of all those elements $\{x_n : n \in \omega\} \in \mathbf{T}^\omega$ for which

$$\mathrm{card}(\{n \in \omega : x_n \neq e\}) < \omega.$$

Obviously, G is an everywhere dense Borel subgroup of $\mathbf{T}^\omega$. Finally, let $\{D_i : i \in I\}$ be the injective family of all G-orbits in $\mathbf{T}^\omega$.

Check that $\mathrm{card}(I) = \mathbf{c}$, where $\mathbf{c}$ denotes, as usual, the cardinality of the continuum.

Prove that there exists $J \subset I$ such that the set $D = \cup\{D_j : j \in J\}$ is not measurable with respect to the completion of μ.

Deduce from this fact that there exists a mapping $\Psi : \mathbf{T}^\omega \to \mathbf{R}$ satisfying the following three relations:

(a) $\mathrm{ran}(\Psi) = \{0, 1\}$;

(b) Ψ is constant with respect to each variable x_n $(n \in \omega)$; in particular, Ψ is continuous with respect to each x_n;

(c) Ψ is nonmeasurable with respect to the completion of μ.

Remark 5. The result presented in Exercise 5 shows that, for some standard infinite-dimensional spaces, the situation is essentially different from the one described in Exercise 4.

6. Verify that the result of Exercise 5 has a direct analogue in terms of the Baire property.

7. By assuming Martin's Axiom and using an argument similar to the proof of Theorem 2, demonstrate that there is a mapping $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying the following two conditions:

(a) for every Lebesgue measurable function $\phi : \mathbf{R} \to \mathbf{R}$, both superpositions

$$x \to \Phi(\phi(x), x) \qquad x \to \Phi(x, \phi(x)) \qquad (x \in \mathbf{R})$$

are also Lebesgue measurable;

(b) Φ is not measurable in the Lebesgue sense.

Remark 6. Actually, for the existence of a function Φ of the previous exercise, one does not need the whole power of Martin's Axiom. It suffices to apply a certain set-theoretical hypothesis weaker than Martin's Axiom (cf. Theorem 3). On the other hand, it is not difficult to prove that if a mapping $\Psi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ has the property that, for any two Borel functions f and g acting from $\mathbf{R}$ into $\mathbf{R}$, the superposition

$$x \to \Psi(f(x), g(x)) \qquad (x \in \mathbf{R})$$

is Lebesgue measurable, then Ψ is Lebesgue measurable, too (see, e.g., [111]). The next exercise presents a slightly more general result.

8. Let Φ be a function acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$ and such that the superposition

$$x \to \Phi(f(x), g(x)) \qquad (x \in \mathbf{R})$$

is Lebesgue measurable for all continuous functions f and g acting from $\mathbf{R}$ into $\mathbf{R}$.

Show that Φ is Lebesgue measurable (cf. Lemma 1 of this chapter).

9. Prove the analogue of Theorem 2 for the Baire property.

10. Prove the analogue of Theorem 3 for the Baire property.

11. Let k be a strictly positive integer. Suppose that

(a) any subset X of $\mathbf{R}$ with $\mathrm{card}(X) < \mathbf{c}$ has the Baire property, i.e., is of first category;

(b) for any family $\{X_i : i \in I\}$ with $\mathrm{card}(I) < \mathbf{c}$, consisting of first category subsets of $\mathbf{R}$, we have $\mathbf{R} \neq \cup\{X_i : i \in I\}$.

Demonstrate that there exists a mapping $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying the following two relations:

(1) for every k-times continuously differentiable function $g : \mathbf{R} \to \mathbf{R}$, the superposition

$$x \to \Phi(x, g(x)) \qquad (x \in \mathbf{R})$$

has the Baire property;

(2) there is a $(k-1)$-times continuously differentiable function $h : \mathbf{R} \to \mathbf{R}$ such that the superposition

$$x \to \Phi(x, h(x)) \qquad (x \in \mathbf{R})$$

does not possess the Baire property.

Formulate and prove the analogous result in terms of the Lebesgue measure λ.

12. Let Φ be an arbitrary function acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$.

Verify that the following two assertions are equivalent:

(a) Φ is sup-measurable in the sense of the Baire property;

(b) for every Borel function ϕ acting from $\mathbf{R}$ into $\mathbf{R}$, the superposition

$$x \to \Phi(x, \phi(x)) \qquad (x \in \mathbf{R})$$

has the Baire property.

Also, determine the precise Baire order of functions ϕ which is sufficient for the equivalence of these two assertions.

13*. Let $(X, \mathcal{S}, \mu)$ be a space with a complete probability measure, Y be a topological space with a countable base, and let $f : X \times Y \to [0,1]$ be a function satisfying the Carathéodory conditions. Pick a countable base $\{U_n : n \in \mathbf{N}\}$ of Y and consider the family F of all those functions ϕ which can be represented in the form $\phi = q_n \cdot \psi_{U_n}$, where

$$n \in \mathbf{N}, \quad q_n \in \mathbf{Q} \cap [0,1], \quad \psi_{U_n} = \text{ the characteristic function of } U_n.$$

Because the family F is countable, one may write $F = \{\phi_k : k \in \mathbf{N}\}$. Further, for any $k \in \mathbf{N}$, define

$$X_k = \{x \in X \ : \ (\forall y \in Y)(\phi_k(y) \leq f(x,y))\}.$$

Fix a countable subset Y_0 of Y everywhere dense in Y and, for each $k \in \mathbf{N}$, put

$$X'_k = \{x \in X \ : \ (\forall y \in Y_0)(\phi_k(y) \leq f(x,y))\}.$$

Check that the set X'_k is μ-measurable and the equality $X_k = X'_k$ holds; therefore, X_k is a μ-measurable subset of X.

Show that

$$f(x,y) = \sup\nolimits_{k \in \mathbf{N}} \psi_{X_k}(x)\phi_k(y)$$

for all $x \in X$ and $y \in Y$ (here ψ_{X_k} denotes the characteristic function of X_k).

Deduce from this fact that

(a) the function f is measurable with respect to the product of the σ-algebras $\mathcal{S}$ and $\mathcal{B}(Y)$, where $\mathcal{B}(Y)$ denotes, as usual, the Borel σ-algebra of Y;

(b) the function f is sup-measurable, i.e., for any μ-measurable mapping $h : X \to Y$, the superposition $x \to f(x, h(x))$ $(x \in X)$ is μ-measurable, too.

Suppose, in addition, that X is a Hausdorff topological space and μ is the completion of a Radon probability measure on X.

By applying a Luzin type theorem on the structure of μ-measurable real-valued functions, prove that, for each real $\varepsilon > 0$, there exists a compact set $P \subset X$ with $\mu(P) > 1 - \varepsilon$, such that the restriction of f to the product set $P \times Y$ is lower semicontinuous.

Extend the results presented above to the functions which almost satisfy the Carathéodory conditions.

14. Let X be a Hausdorff topological space, let μ be the completion of a Radon probability measure on X, and let Y be a topological space with a countable base.

By using the results of Exercise 13, show that, for an arbitrary function $f : X \times Y \to [0, 1]$, the following two assertions are equivalent:

(a) f almost satisfies the Carathéodory conditions;

(b) for every real $\varepsilon > 0$, there exists a compact set $P \subset X$ such that $\mu(P) > 1 - \varepsilon$ and the restriction of f to the product set $P \times Y$ is continuous.

Remark 7. The equivalence of these two assertions is usually called the Scorza Dragoni theorem.

15*. Let $(X, \mathcal{S}, \mu)$ be a space with a complete σ-finite measure, Y be a locally compact topological space with a countable base, and let Z be a subset of the product space $X \times Y$, measurable with respect to the product of the σ-algebras $\mathcal{S}$ and $\mathcal{B}(Y)$.

Demonstrate that the set $\mathrm{pr}_1(Z)$ is measurable with respect to μ.

For this purpose, apply Choquet's theorem on capacities (see, e.g., [43], [57] or [196]).

16*. Let $(X, \mathcal{S}, \mu)$ be a space with a complete probability measure, let Y be a nonempty compact metric space, and let $f : X \times Y \to [0, 1]$ be a function satisfying the Carathéodory conditions. For each point $x \in X$, denote $F(x) = \{y \in Y : f(x, y) = \inf_{t \in Y} f(x, t)\}$.

Check that $F(x)$ is a nonempty closed subset of Y; so one may consider the associated set-valued mapping $F : X \to \mathcal{P}(Y)$.

Prove that, for any open set $U \subset Y$, the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is μ-measurable.

Derive from this fact, by using the theorem of Kuratowski and Ryll-Nardzewski on measurable selectors (see [150] or [154]), that there exists a μ-measurable mapping $h : X \to Y$ such that

$$f(x, h(x)) = \inf_{y \in Y} f(x, y)$$

for all points $x \in X$.

Formulate and prove the analogous statement for functions almost satisfying the Carathéodory conditions.

17. Consider the diagonal $\{(x, y) \ : \ x \in \mathbf{R}, \ y \in \mathbf{R}, \ x = y\}$ of the Euclidean plane $\mathbf{R}^2$ and let Z be a subset of this diagonal, nonmeasurable with respect to the standard one-dimensional Lebesgue measure on it. Introduce a function $f : \mathbf{R}^2 \to \mathbf{R}$ by putting

$f(x, y) = 0$ if $(x, y) \notin Z$, and $f(x, y) = -1$ if $(x, y) \in Z$.

Verify that

(a) for each point $x \in \mathbf{R}$, the partial function $f(x, \cdot) : \mathbf{R} \to \mathbf{R}$ is lower semicontinuous;

(b) for each point $y \in \mathbf{R}$, the partial function $f(\cdot, y) : \mathbf{R} \to \mathbf{R}$ is lower semicontinuous;

(c) the function $x \to f(x, x)$ $(x \in \mathbf{R})$ is not Lebesgue measurable (consequently, f is not sup-measurable);

(d) f is measurable with respect to the two-dimensional Lebesgue measure λ_2 on $\mathbf{R}^2$.

18. Suppose that there exists a Luzin subset of the Euclidean plane $\mathbf{R}^2$.

Demonstrate that, in this case, there exists a mapping $\Phi : \mathbf{R}^2 \to \mathbf{R}$ satisfying the following two relations:

(a) Φ does not have the Baire property;

(b) for any function $\phi : \mathbf{R} \to \mathbf{R}$ whose graph is a first category subset of the plane (in particular, for any $\phi : \mathbf{R} \to \mathbf{R}$ possessing the Baire property), the superposition $x \to \Phi(x, \phi(x))$ $(x \in \mathbf{R})$ possesses the Baire property.

Deduce the analogous result (in terms of the Lebesgue measurability) from the existence of a Sierpiński subset of the plane $\mathbf{R}^2$.

19*. Denote by the symbol $M_0 = M_0[0, 1]$ the family of all Lebesgue measurable functions acting from $[0, 1]$ into $\mathbf{R}$. Obviously, there is a canonical equivalence relation $\equiv$ in M_0 defined by the formula

$$f \equiv g \Leftrightarrow f \text{ and } g \text{ coincide almost everywhere on } [0, 1].$$

Also, denote by M the quotient set with respect to this equivalence relation (i.e., M is the family of all equivalence classes with respect to $\equiv$). If f is an arbitrary function from the original family M_0, then the symbol $[f]$ will stand for the class of all those functions which are equivalent to f. The natural algebraic operations in M_0 are compatible with the relation $\equiv$ and, consequently, induce the corresponding algebraic operations in M. Therefore, M becomes an algebra over the field $\mathbf{R}$.

Further, for any two elements $[f] \in M$ and $[g] \in M$, put

$$d([f], [g]) = \int_0^1 \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} dt.$$

Check that this definition is correct (i.e., it does not depend on the choice of f and g) and that the function $d : M \times M \to \mathbf{R}$ obtained in this way turns out to be a metric on M.

Show that

(a) the pair (M, d) is a Polish topological vector space;

(b) there exists no nonzero linear continuous functional defined on the entire space M (in other words, the conjugate space M^* is trivial).

Now, extend the equivalence relation $\equiv$ introduced above onto the family of all Lebesgue measurable partial functions acting from $[0,1]$ into $\mathbf{R}$. Namely, for any two such functions f and g, put $f \equiv g$ iff

$$\lambda(\mathrm{dom}(f) \triangle \mathrm{dom}(g)) = 0 \ \& \ f(x) = g(x) \text{ for almost all } x \in \mathrm{dom}(f) \cap \mathrm{dom}(g).$$

Further, denote by the symbol M' the family of all equivalence classes with respect to $\equiv$. Evidently, one gets a canonical embedding $j : M \to M'$ and so one may identify M with the subset $j(M)$ of M'.

Suppose now that some mapping (operator) $H : M' \to M'$ is given and satisfies the following two conditions:

(1) for each element $[f] \in M'$ and for any Lebesgue measurable partial function g such that $[g] = H([f])$, one has the equality

$$\lambda(\mathrm{dom}(f) \triangle \mathrm{dom}(g)) = 0;$$

(2) if $[f] \in M'$, all partial functions f_n $(n < \omega)$ are the restrictions of f to pairwise disjoint Lebesgue measurable subsets of $\mathbf{R}$ whose union coincides with $\mathrm{dom}(f)$ and

$$H([f_n]) = [g_n] \qquad (n < \omega),$$

$$\mathrm{dom}(g_n) \cap \mathrm{dom}(g_m) = \emptyset \qquad (n < \omega, \ m < \omega, \ n \neq m),$$

then one has the equality $H([f]) = [g]$, where g denotes the common extension of all partial functions g_n $(n < \omega)$ with

$$\mathrm{dom}(g) = \cup\{\mathrm{dom}(g_n) \ : \ n < \omega\}.$$

In this case, H is called an admissible operator acting from the family M' into itself.

Suppose that such an operator H is given.

Under the assumption that the σ-ideal $\mathcal{I}(\lambda)$ is $\mathbf{c}$-additive, establish the existence of a mapping $\Phi : [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfying the following relation:

For every Lebesgue measurable partial function $f : [0,1] \to \mathbf{R}$, the superposition $\Phi_f : [0,1] \to \mathbf{R}$ is a Lebesgue measurable partial function, too, and the equality $H([f]) = [\Phi_f]$ holds true.

In other words, any admissible operator $H : M' \to M'$ is representable in the form of some superposition operator Φ (this result is essentially due to Krasnoselskii and Pokrovskii).

Also, check that descriptive properties of this Φ can be bad, namely, Φ can be nonmeasurable with respect to the standard two-dimensional Lebesgue measure λ_2 on $\mathbf{R}^2$ (cf. Theorem 2 of this chapter).

Finally, formulate and prove an appropriate analogue of the preceding result for partial functions acting from $[0,1]$ into $\mathbf{R}$ and possessing the Baire property (under the assumption that the σ-ideal $\mathcal{K}(\mathbf{R})$ of all first category subsets of $\mathbf{R}$ is $\mathbf{c}$-additive).

20*. Let X be a set equipped with a complete σ-finite measure μ, let Y be a compact metric space, and let $\Phi_0 : X \times Y \to \mathbf{R}$ be a partial mapping measurable with respect to the product of the σ-algebras $\mathrm{dom}(\mu)$ and $\mathcal{B}(Y)$.

Applying Exercise 15 of this chapter, the Tietze–Urysohn theorem on extensions of continuous real-valued functions, and the theorem of Kuratowski and Ryll-Nardzewski on measurable selectors, demonstrate that the following two assertions are equivalent:

(a) for each element $x \in X$, the partial function $\Phi_0(x, \cdot)$ is uniformly continuous on its domain;

(b) there exists a mapping $\Phi : X \times Y \to \mathbf{R}$ extending Φ_0 and satisfying the Carathéodory conditions.

21*. Let X be a topological space equipped with a σ-finite inner regular Borel measure μ and let Y be a compact metric space. Denote

μ' = the completion of μ;

$\mathcal{S}' = \mathrm{dom}(\mu')$;

$\mathcal{F}(Y)$ = the family of all nonempty closed subsets of Y.

Let $\Psi : X \to \mathcal{F}(Y)$ be a set-valued mapping.

Prove that the following three assertions are equivalent:

(a) Ψ is μ'-measurable as a function acting from the measure space $(X, \mathcal{S}', \mu')$ into the space $\mathcal{F}(Y)$ endowed with the Hausdorff metric;

(b) Ψ is lower $\mathcal{S}'$-measurable, i.e., $\{x \in X : \Psi(x) \cap U \neq \emptyset\} \in \mathcal{S}'$ for all open sets $U \subset Y$;

(c) Ψ is upper $\mathcal{S}'$-measurable, i.e., $\{x \in X : \Psi(x) \cap F \neq \emptyset\} \in \mathcal{S}'$ for all closed sets $F \subset Y$.

Chapter 20
Generalized step-functions and superposition operators

The previous chapter was devoted to some properties of sup-measurable and weakly sup-measurable functions of two variables. In this chapter, for a given σ-ideal of sets, we introduce the notion of a real-valued generalized step-function and investigate generalized step-functions in connection with the problem of sup-measurability of certain functions of two variables, regarded as superposition operators.

Let $\mathbf{R}$ denote, as usual, the real line and let $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function of two variables. According to the material of Chapter 19, this Φ can be treated as a superposition operator defined as follows: for any function $f : \mathbf{R} \to \mathbf{R}$, we put

$$(\Phi(f))(x) = \Phi(x, f(x)) \qquad (x \in \mathbf{R}).$$

Sometimes, Φ is also called the Nemytskii superposition operator (cf. [142], [234]).

Let λ $(= \lambda_1)$ denote the standard Lebesgue measure on $\mathbf{R}$. In many cases, it is important to know whether a given superposition operator Φ preserves the class $L(\mathbf{R}, \mathbf{R})$ of all real-valued Lebesgue measurable functions on $\mathbf{R}$ (i.e., $\Phi(f)$ is λ-measurable whenever f is λ-measurable). There are various sufficient conditions under which Φ maps $L(\mathbf{R}, \mathbf{R})$ into itself. In particular, we know (see again Chapter 19) that if Φ is λ-measurable with respect to the first variable and continuous with respect to the second variable (the so-called Carathéodory classical conditions), then Φ preserves $L(\mathbf{R}, \mathbf{R})$ or, in short, Φ is sup-measurable. In such a case, Φ is also λ_2-measurable, where λ_2 stands for the two-dimensional Lebesgue measure on the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

Other conditions for the sup-measurability of Φ can be found, e.g., in [233].

We have already shown in Chapter 19 that, under some additional set-theoretical axioms, there exist sup-measurable operators Φ, which are not

λ_2-measurable.

In this chapter our attention will be focused on the following problem:

Give a characterization of all those functions $f \in L(\mathbf{R}, \mathbf{R})$ for which there exists a superposition operator Φ having rather nice descriptive properties and such that $\Phi(f)$ does not belong to $L(\mathbf{R}, \mathbf{R})$.

In order to present a solution of this problem, we need several auxiliary notions and propositions. First of all, let us formulate the following classical statement from descriptive set theory.

Lemma 1. *Let E be a Polish topological space, E' be a metric space, and let $\phi : E \to E'$ be a continuous mapping whose range is uncountable. Then there exists a set $C \subset E$ homeomorphic to the Cantor discontinuum, such that the restriction $\phi|C$ is injective (consequently, $\phi|C$ is a homeomorphism between C and $\phi(C)$).*

For the proof of Lemma 1, see, e.g., [149]. In fact, this lemma directly implies that the cardinality of any uncountable analytic subset of a Polish space is equal to the cardinality of the continuum ($= \mathbf{c}$).

Recall that $f \in L(\mathbf{R}, \mathbf{R})$ is a step-function if $\mathrm{card}(\mathrm{ran}(f)) \leq \omega$, i.e., the range of f is at most countable.

Clearly, if $f \in L(\mathbf{R}, \mathbf{R})$ is a step-function, then there exists a countable partition $\{X_j : j \in J\}$ of $\mathbf{R}$ consisting of Lebesgue measurable sets such that, for any $j \in J$, the restriction $f|X_j$ is a constant function.

We shall say that $f \in L(\mathbf{R}, \mathbf{R})$ is a generalized step-function if there exists at least one step-function $g \in L(\mathbf{R}, \mathbf{R})$ such that f and g are equivalent with respect to the measure λ, i.e.,

$$\{x \in \mathbf{R} : f(x) \neq g(x)\} \in \mathcal{I}(\lambda).$$

The next lemma yields a characterization of generalized step-functions. This characterization will be helpful for our purposes.

Lemma 2. *If $f \in L(\mathbf{R}, \mathbf{R})$, then the following two assertions are equivalent:*

(1) f is not a generalized step-function;

(2) there exists a set $Y \subset \mathbf{R}$ with $\lambda^*(Y) > 0$ such that the restriction $f|Y$ is injective.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let us prove the implication (1) $\Rightarrow$ (2). Suppose that $f \in L(\mathbf{R}, \mathbf{R})$ satisfies (1). Let us denote

$$T_0 = \{t \in \mathrm{ran}(f) : \lambda(f^{-1}(t)) > 0\}.$$

In view of the σ-finiteness of λ, we have the inequality $\mathrm{card}(T_0) \leq \omega$. Because f is not a generalized step-function, we also have

$$\lambda(\mathbf{R} \setminus f^{-1}(T_0)) > 0.$$

Moreover, applying to f Luzin's classical C-property, we readily infer that there exists a closed set $P \subset \mathbf{R} \setminus f^{-1}(T_0)$ with $\lambda(P) > 0$ for which the restriction $f|P$ is continuous and $\mathrm{card}(\mathrm{ran}(f|P)) > \omega$. Let us put

$$h = f|P, \qquad T = \mathrm{ran}(h).$$

Then $\lambda(h^{-1}(t)) = 0$ for each point $t \in T$.

Denote by α the least ordinal number of cardinality continuum and let $\{P_\xi : \xi < \alpha\}$ be an injective enumeration of all closed subsets of P having strictly positive λ-measure. Construct, by using the method of transfinite recursion, a family of points

$$\{y_\xi : \xi < \alpha\} \subset P.$$

Namely, take an arbitrary ordinal $\xi < \alpha$ and suppose that the partial family $\{y_\zeta : \zeta < \xi\}$ has already been defined. Keeping in mind Lemma 1, it is not difficult to check that

$$P_\xi \setminus \cup\{h^{-1}(h(y_\zeta)) : \zeta < \xi\} \neq \emptyset.$$

Hence, there exists a point y belonging to $P_\xi \setminus \cup\{h^{-1}(h(y_\zeta)) : \zeta < \xi\}$. We then put $y_\xi = y$.

By proceeding in this manner, the required family of points $\{y_\xi : \xi < \alpha\}$ will be constructed. Now, denote

$$Y = \{y_\xi : \xi < \alpha\}.$$

It immediately follows from our construction that Y is a partial selector of the disjoint family of sets $\{h^{-1}(t) : t \in T\}$. This implies that the restriction $h|Y$ (consequently, the restriction $f|Y$) is injective. Moreover, since we have $P_\xi \cap Y \neq \emptyset$ for each $\xi < \alpha$, we easily deduce that $\lambda^*(Y) = \lambda(P) > 0$. This completes the proof of Lemma 2.

Lemma 3. *If $f \in L(\mathbf{R}, \mathbf{R})$ is not a generalized step-function, then there exists a λ-nonmeasurable set $X \subset \mathbf{R}$ for which the restriction $f|X$ is injective.*

Proof. According to Lemma 2, there exists a set $Y \subset \mathbf{R}$ with $\lambda^*(Y) > 0$ such that $f|Y$ is an injection. If Y is not measurable in the Lebesgue sense, then we are done. Suppose now that $Y \in \mathrm{dom}(\lambda)$ and hence $\lambda(Y) > 0$. It is well known (see, e.g., Chapter 10 of this book) that Y contains a subset nonmeasurable with respect to λ. Take any such subset and denote it by X. Clearly, $f|X$ is an injection, and the proof is finished.

Theorem 1. *Let $f \in L(\mathbf{R}, \mathbf{R})$ and suppose that f is not a generalized step-function. Then there exists a superposition operator*

$$\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

satisfying the following five relations:

(1) $\mathrm{ran}(\Phi) = \{0, 1\}$*;*

(2) for any $x \in \mathbf{R}$, the partial function $\Phi(x, \cdot)$ is lower semicontinuous;

(3) for any $y \in \mathbf{R}$, the partial function $\Phi(\cdot, y)$ is lower semicontinuous;

(4) Φ is a λ_2-measurable operator;

(5) the function $\Phi(f)$ is not λ-measurable.

Proof. According to Lemma 3, there exists a λ-nonmeasurable set $X \subset \mathbf{R}$ for which the restriction $f|X$ is injective. Define the required superposition operator Φ as follows:

$$\Phi(x, y) = 0 \qquad (x \in X,\ y = f(x)),$$

$$\Phi(x, y) = 1 \qquad (x \in \mathbf{R} \setminus X,\ y = f(x)),$$

$$\Phi(x, y) = 1 \qquad (x \in \mathbf{R},\ y \in \mathbf{R},\ y \neq f(x)).$$

For this Φ, relations (1), (2), and (3) are verified directly. Further, because the graph of f is a λ_2-measure zero subset of $\mathbf{R}^2$, we infer that Φ is equivalent to a constant function and, consequently, Φ is λ_2-measurable. Finally, we have

$$\Phi(x, y) = 0 \Leftrightarrow (x \in X \ \&\ y = f(x)),$$

whence it follows that $(\Phi(f))^{-1}(0) = X$ and, therefore, $\Phi(f)$ is not λ-measurable. Theorem 1 has thus been proved.

Theorem 2. *Let $f \in L(\mathbf{R}, \mathbf{R})$ and suppose that f is not a generalized step-function. Then there exists a superposition operator $\Psi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that*

(1) $ran(\Psi) = \{1, 2\}$,

(2) for any $x \in \mathbf{R}$, the partial function $\Psi(x, \cdot)$ is lower semicontinuous;

(3) for any $y \in \mathbf{R}$, the partial function $\Psi(\cdot, y)$ is lower semicontinuous;

(4) Ψ is a λ_2-nonmeasurable operator;

(5) the function $\Psi(f)$ is λ-nonmeasurable.

Proof. By using the method of transfinite recursion and applying a fairly standard argument (cf. the proof of Theorem 5 from Chapter 10), an injective function $g : \mathbf{R} \to \mathbf{R}$ can be defined whose graph is λ_2-thick in $\mathbf{R}^2$ and does not intersect the graph of f. Let χ_g denote the characteristic function of the graph of g (of course, this graph is considered as a subset of $\mathbf{R}^2$). We put

$$\Psi = \Phi + 1 - \chi_g,$$

where Φ is the superposition operator of Theorem 1. It is easy to verify that Ψ is the required superposition operator, i.e., Ψ satisfies all relations (1)–(5) of Theorem 2.

Remark 1. If a superposition operator $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is λ-measurable with respect to the first variable, then $\Phi(f)$ is λ-measurable for every generalized step-function $f \in L(\mathbf{R}, \mathbf{R})$. We thus see (in view of Theorem 1) that the generalized step-functions are exactly those functions $f \in L(\mathbf{R}, \mathbf{R})$ for which every superposition operator Φ, Lebesgue measurable with respect to the first variable, yields Lebesgue measurable $\Phi(f)$.

Remark 2. If a superposition operator $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is lower semicontinuous (more generally, Borel) with respect to the first variable and continuous with respect to the second variable, then Φ is a Borel mapping from $\mathbf{R}^2$ into $\mathbf{R}$, hence Φ is also sup-measurable (see Exercise 3 of Chapter 19). A much stronger result is presented in Exercise 1 of this chapter.

Remark 3. Theorems 1 and 2 admit direct analogues for functions possessing the Baire property. Those analogues can be proved by the same scheme as for Lebesgue measurable functions. Only one essential moment should be mentioned. Namely, the proofs of Theorems 1 and 2 are based on Luzin's theorem concerning the structure of λ-measurable functions. Because we cannot apply Luzin's theorem to functions possessing the Baire property, we must replace this theorem by an appropriate similar statement. The required statement is well known in general topology (see [149], [202] or Exercise 13 from Chapter 0) and is formulated as follows. Let E_1 be a topological space, E_2 be a topological space with a countable base, and let $f : E_1 \to E_2$ be a mapping possessing the Baire property; then there exists a first category set $Z \subset E_1$ such that the restriction $f|(E_1 \setminus Z)$ is continuous. In this statement we may assume, without loss of generality, that Z is an F_σ-subset of E_1, hence $E_1 \setminus Z$ is a G_δ-set in E_1. If the original space E_1 is Polish, then $E_1 \setminus Z$ is also Polish (by virtue of the Alexandrov theorem). Consequently, if E_1 is a Polish space and $E_2 = \mathbf{R}$, we are able to apply Lemma 1 to the continuous function $f|(E_1 \setminus Z)$.

Under some additional set-theoretical axioms, Lemma 2 admits a significant generalization and we want to consider some abstract version of this lemma.

Fix an uncountable ground set E and a σ-ideal $\mathcal{I}$ of subsets of E, containing all singletons in E.

We shall say that $g : E \to \mathbf{R}$ is a step-function if $\mathrm{card}(\mathrm{ran}(g)) \leq \omega$.

We shall say that $f : E \to \mathbf{R}$ is a generalized step-function with respect to $\mathcal{I}$ if there exists at least one step-function $g : E \to \mathbf{R}$ for which we have

$$\{x \in E : f(x) \neq g(x)\} \in \mathcal{I},$$

i.e., f and g are $\mathcal{I}$-equivalent functions.

Recall that a family of sets $\mathcal{B} \subset \mathcal{I}$ forms a base of $\mathcal{I}$ if, for any set $Y \in \mathcal{I}$, there exists a set $Z \in \mathcal{B}$ such that $Y \subset Z$.

The following statement is valid.

Theorem 3. *Let $\mathrm{card}(E) = \omega_1$, let $\mathcal{I}$ be a σ-ideal of subsets of E, containing all singletons in E and possessing a base whose cardinality does not exceed ω_1, and let $f : E \to \mathbf{R}$ be a function. Then these two assertions are equivalent:*

(1) f is not a generalized step-function with respect to $\mathcal{I}$;

(2) there exists a set $X \subset E$ such that $X \notin \mathcal{I}$ and the restriction $f|X$ is injective.

Proof. The implication (2) $\Rightarrow$ (1) is evident. Let us establish the validity of the implication (1) $\Rightarrow$ (2). Suppose that f satisfies (1) and introduce the following two sets:

$$T_0 = \{t \in \mathrm{ran}(f) : f^{-1}(t) \notin \mathcal{I}\}, \qquad T_1 = \{t \in \mathrm{ran}(f) : f^{-1}(t) \in \mathcal{I}\}.$$

According to our assumption, there exists a base $\mathcal{B} = \{B_\xi : \xi < \omega_1\}$ of the given σ-ideal $\mathcal{I}$. Only two cases are possible.

1. $\mathrm{card}(T_0) = \omega_1$. In this case we may write

$$T_0 = \{t_\xi : \xi < \omega_1\},$$

where $t_\xi \neq t_\zeta$ for all distinct $\xi < \omega_1$ and $\zeta < \omega_1$.

Consider the family of sets $\{f^{-1}(t_\xi) \setminus B_\xi : \xi < \omega_1\}$. Obviously, we have

$$f^{-1}(t_\xi) \setminus B_\xi \neq \emptyset$$

for each ordinal $\xi < \omega_1$. Let us choose an element $x_\xi \in f^{-1}(t_\xi) \setminus B_\xi$ for any $\xi < \omega_1$, and let us put

$$X = \{x_\xi : \xi < \omega_1\}.$$

From the definition of X it immediately follows that the restriction $f|X$ is an injection. Moreover, we have $X \setminus B_\xi \neq \emptyset$ whenever $\xi < \omega_1$. The latter circumstance implies at once that the set X does not belong to $\mathcal{I}$.

2. $\mathrm{card}(T_0) \leq \omega$. In this case we obtain $\mathrm{card}(T_1) = \omega_1$ and $f^{-1}(T_1) \notin \mathcal{I}$ (because our f is not a generalized step-function with respect to $\mathcal{I}$). Let us construct, by using the method of transfinite recursion, an ω_1-sequence $\{x_\xi : \xi < \omega_1\}$ of points of $f^{-1}(T_1)$. Suppose that, for an ordinal $\xi < \omega_1$, the partial family of points

$$\{x_\zeta : \zeta < \xi\} \subset f^{-1}(T_1)$$

has already been determined. Clearly,

$$(\cup\{f^{-1}(f(x_\zeta)) : \zeta < \xi\}) \cup B_\xi \in \mathcal{I}.$$

Therefore,

$$f^{-1}(T_1) \setminus ((\cup\{f^{-1}(f(x_\zeta)) : \zeta < \xi\}) \cup B_\xi) \neq \emptyset.$$

Choose any element x from the above nonempty set and put $x_\xi = x$.

Proceeding in this manner, we are able to construct the required ω_1-sequence $\{x_\xi : \xi < \omega_1\}$. Finally, put

$$X = \{x_\xi : \xi < \omega_1\}.$$

In view of our construction, X is a partial selector of the disjoint family of sets $\{f^{-1}(t) : t \in T_1\}$. Hence, the restriction of f to X is injective. Furthermore, $X \setminus B_\xi \neq \emptyset$ for all ordinals $\xi < \omega_1$, whence it follows that X does not belong to $\mathcal{I}$. This completes the proof of Theorem 3.

Remark 4. Assume the Continuum Hypothesis (**CH**) and take as $\mathcal{I}$ the σ-ideal $\mathcal{I}(\lambda)$ of all Lebesgue measure zero subsets of $\mathbf{R}$. Let $f : \mathbf{R} \to \mathbf{R}$ be a function distinct from all generalized step-functions with respect to $\mathcal{I}(\lambda)$. Suppose also that the graph of f is of λ_2-measure zero. Then it is not hard to show that, for such an f, there always exists a superposition operator $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying relations (1)–(5) of Theorem 1. In this connection, let us underline that f does not need to be a λ-measurable function.

Remark 5. Let E be a nonempty set, $\mathcal{I}$ be a σ-ideal of subsets of E, and let $\mathcal{S}$ be a σ-algebra of subsets of E such that $\mathcal{I} \subset \mathcal{S}$. Elements of $\mathcal{S}$ are usually called measurable sets in E and elements of $\mathcal{I}$ are called negligible sets in E. The triple $(E, \mathcal{S}, \mathcal{I})$ is called a measurable space with negligibles (see, e.g., [75]). If $X \subset E$ and $X \notin \mathcal{I}$, then, in general, we cannot assert that X contains at least one subset not belonging to $\mathcal{S}$. However, in some situations the specific features of a given σ-ideal $\mathcal{I}$ imply that any non-negligible set in E includes a nonmeasurable set. For example, assume again that $\mathrm{card}(E) = \omega_1$ and that E is a topological space of second category, all singletons of which are of first category. Let $\mathcal{I} = \mathcal{K}(E)$ denote the σ-ideal of all first category subsets of E and suppose that this σ-ideal possesses a base whose cardinality does not exceed ω_1. Denote also by $\mathcal{B}a(E)$ the σ-algebra of all subsets of E having the Baire property (recall that $\mathcal{K}(E) \subset \mathcal{B}a(E)$). Then, for any set $X \subset E$, the following two assertions are equivalent:

(a) $X \notin \mathcal{K}(E)$ (i.e., X is not of first category in E);

(b) there exists a set $Y \subset X$ such that $Y \notin \mathcal{B}a(E)$ (i.e., Y does not have the Baire property in E).

The proof of the equivalence of (a) and (b) can be found in [116] where some related results are also presented (cf. Exercise 9 from Chapter 18). Notice once more that this equivalence relies on the inner properties of the σ-ideal $\mathcal{K}(E)$ and does not touch the structure of the σ-algebra $\mathcal{B}a(E)$.

Let us continue our consideration of the question of measurability of functions obtained by using the superposition operator which is induced by a given function of two variables. In this connection, we would like to discuss some related measurability properties of functions of two variables.

Let $\Phi : \mathbf{R}^2 \to \mathbf{R}$ be again a function of two variables and let F be a class of functions acting from $\mathbf{R}$ into $\mathbf{R}$. As in Chapter 19, for any $f \in F$, we denote by Φ_f the function acting from $\mathbf{R}$ into $\mathbf{R}$ and defined by

$$\Phi_f(x) = \Phi(x, f(x)) \qquad (x \in \mathbf{R}).$$

We have already mentioned that, in some sense, Φ plays the role of a superposition operator whose domain coincides with the given class F of functions.

A general problem arises to describe those conditions on Φ under which various nice properties of functions from F are preserved by Φ. For example, suppose that $F = L(\mathbf{R}, \mathbf{R})$ is the class of all real-valued Lebesgue measurable functions on $\mathbf{R}$. Then it is natural to try to characterize those operators Φ for which $L(\mathbf{R}, \mathbf{R})$ is preserved (i.e., the Lebesgue measurability of functions of one variable is preserved by Φ). We already know that, in general, the Lebesgue measurability of Φ (regarded as a function of two variables) does not guarantee the Lebesgue measurability of Φ_f for $f \in L(\mathbf{R}, \mathbf{R})$. Also, it is commonly known that if Φ satisfies the so-called Carathéodory conditions, then it preserves the class $L(\mathbf{R}, \mathbf{R})$.

In this context, the next example is relevant.

Example 1. For any Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$, there exists a function $\Phi : \mathbf{R}^2 \to \mathbf{R}$ satisfying the Carathéodory conditions and there is a continuous function $g : \mathbf{R} \to \mathbf{R}$ such that

$$f(x) = g'(x) = \Phi(x, g(x))$$

for almost all $x \in \mathbf{R}$. Indeed, according to the classical Luzin theorem on the existence of primitives (see [168], [225] or Appendix 1 of this book), there exists a continuous function $g : \mathbf{R} \to \mathbf{R}$ such that $g'(x) = f(x)$ for almost all $x \in \mathbf{R}$. Let us define

$$\Phi(x, y) = f(x) + y - g(x) \qquad (x \in \mathbf{R},\ y \in \mathbf{R}).$$

Then Φ is measurable with respect to x and affine with respect to y (hence, Φ satisfies the Lipschitz condition with respect to y). Obviously, we also have $f(x) = g'(x) = \Phi(x, g(x))$ for almost all $x \in \mathbf{R}$. We thus conclude that any real-valued Lebesgue measurable function can be simultaneously regarded as the derivative (almost everywhere) of a continuous function and as the image of the same continuous function, under an appropriate superposition operator satisfying the Carathéodory conditions.

As mentioned earlier, several works were devoted to constructions of a Lebesgue nonmeasurable function Φ that, however, preserves the class $L(\mathbf{R}, \mathbf{R})$. All those constructions were based on some extra set-theoretical axioms. In this connection, the problem was posed whether it is possible to construct analogous Φ within **ZFC** theory. Roslanowski and Shelah [218] established that the existence of such a function Φ cannot be proved in **ZFC** theory. The similar question for the Baire property was considered by Ciesielski and Shelah in [51].

Let us continue our discussion concerning various measurability and sup-measurability properties of functions.

Let E be a nonempty set, $\mathcal{S}$ be a σ-algebra of subsets of E, and let $\mathcal{I}$ be a σ-ideal of subsets of E, such that $\mathcal{I} \subset \mathcal{S}$. It will be assumed in our further considerations that $\mathcal{I}$ contains all one-element subsets of E and that the pair $(\mathcal{S}, \mathcal{I})$ satisfies the countable chain condition, i.e., every disjoint family of sets belonging to $\mathcal{S} \setminus \mathcal{I}$ is at most countable.

Let F be a class of real-valued $\mathcal{S}$-measurable functions on E. Suppose also that a function $\Phi : E \times \mathbf{R} \to \mathbf{R}$ is given. In accordance with the above consideration, this Φ will be treated as a superposition operator for the class F, i.e., by using Φ, we obtain from any function $f \in F$ the function Φ_f defined by

$$\Phi_f(x) = \Phi(x, f(x)) \qquad (x \in E).$$

We shall say that Φ is sup-measurable with respect to F if, for each $f \in F$, the corresponding function Φ_f is $\mathcal{S}$-measurable.

Starting with the class F, it is reasonable to consider other classes F' of $\mathcal{S}$-measurable real-valued functions, containing F and such that each operator Φ sup-measurable with respect to F remains also sup-measurable with respect to F'. In this case, we shall say that F' extends F with preserving the sup-measurability property. It is also reasonable to try to characterize maximal extensions of F for which this property is still preserved.

It will be demonstrated below that, in some natural situations, it is possible to describe such maximal extensions in terms of $(\mathcal{S}, \mathcal{I})$ and F.

Fix a class F of $\mathcal{S}$-measurable real-valued functions. We shall say that $f \in F^*$ if there exist a countable disjoint covering $\{E_n : n < \omega\} \subset \mathcal{S}$ of E and a countable family $\{f_n : 1 \leq n < \omega\} \subset F$, such that

$$E_0 \in \mathcal{I}, \quad f|E_n = f_n|E_n \quad (1 \leq n < \omega).$$

Clearly, we have the inclusion $F \subset F^*$. In some cases, this inclusion is reduced to the equality. For instance, if F is the family of all $\mathcal{S}$-measurable functions, then $F^* = F$.

Example 2. Let F denote the class of all constant real-valued functions on E. Then it is easy to see that F^* coincides with the class of all those real-valued functions on E which are $\mathcal{I}$-equivalent to step-functions (we recall that a step-function on E is any real-valued $\mathcal{S}$-measurable function whose range is at most countable).

In the sequel, we need the following simple auxiliary statement.

Lemma 4. *Let $h : E \to \mathbf{R}$ be an $\mathcal{S}$-measurable function and let F be some family of $\mathcal{S}$-measurable real-valued functions. Then h does not belong to F^* if and only if there exists a set $A \in \mathcal{S} \setminus \mathcal{I}$ possessing the following property: for any subset B of A belonging to $\mathcal{S} \setminus \mathcal{I}$ and for any function $f \in F$, the relation $f|B \neq h|B$ is fulfilled.*

The proof of Lemma 4 can easily be obtained by using the Zorn Lemma or the method of transfinite induction, taking into account the countable chain condition for the pair $(\mathcal{S}, \mathcal{I})$.

The next lemma is almost trivial.

Lemma 5. *Let Φ be a sup-measurable operator with respect to F. Then Φ is also sup-measurable with respect to F^*.*

We omit an easy proof of this lemma. Notice only that, for its validity, the countable chain condition is not necessary.

Let F be a family of $\mathcal{S}$-measurable real-valued functions.

We shall say that a family G of real-valued functions on E is fundamental for F if every function f from F is $\mathcal{I}$-equivalent to some function g from G.

It can easily be shown that the next auxiliary statement is true.

Lemma 6. *An operator Φ is sup-measurable with respect to a class F if and only if Φ is sup-measurable with respect to some class G fundamental for F.*

Example 3. The class of all Borel functions (acting from $\mathbf{R}$ into $\mathbf{R}$) is fundamental for the class of all Lebesgue measurable functions (acting from $\mathbf{R}$ into $\mathbf{R}$). The same class of Borel functions is also fundamental for the class of all those functions that act from $\mathbf{R}$ into $\mathbf{R}$ and possess the Baire property.

We recall that a family $\mathcal{B} \subset \mathcal{S} \setminus \mathcal{I}$ is a pseudo-base for a space $(E, \mathcal{S}, \mathcal{I})$ if every set $X \in \mathcal{S} \setminus \mathcal{I}$ contains at least one member of $\mathcal{B}$.

Because the given pair $(\mathcal{S}, \mathcal{I})$ satisfies the countable chain condition, every set $X \in \mathcal{S} \setminus \mathcal{I}$ contains a subset Y such that $X \setminus Y \in \mathcal{I}$ and Y is representable as the union of a countable disjoint family of members of a pseudo-base $\mathcal{B}$. This circumstance implies the validity of the next auxiliary proposition.

Lemma 7. *Let $\mathcal{B}$ be a pseudo-base for a space $(E, \mathcal{S}, \mathcal{I})$ such that* $\mathrm{card}(\mathcal{B}) \geq 2$. *Then there exists a family G of $\mathcal{S}$-measurable real-valued functions, fundamental for the family of all $\mathcal{S}$-measurable real-valued functions and satisfying the inequality* $\mathrm{card}(G) \leq (\mathrm{card}(\mathcal{B}))^{\omega}$.

The proof of Lemma 7 is left to the reader.

Lemma 8. *Suppose that the following three relations are satisfied for a given space $(E, \mathcal{S}, \mathcal{I})$:*

(1) there exists a pseudo-base $\mathcal{B}$ for $(E, \mathcal{S}, \mathcal{I})$ containing at least two members and such that $(\mathrm{card}(\mathcal{B}))^{\omega} \leq \mathrm{card}(E)$*;*

(2) for any set $X \in \mathcal{S} \setminus \mathcal{I}$ and for any family $\{X_\theta : \theta \in \Theta\} \subset \mathcal{I}$ with $\mathrm{card}(\Theta) < \mathrm{card}(E)$, *we have*

$$X \setminus \cup\{X_\theta : \theta \in \Theta\} \neq \emptyset;$$

(3) each subset of E with cardinality strictly less than $\mathrm{card}(E)$ *belongs to $\mathcal{I}$.*

Let F be a family of $\mathcal{S}$-measurable real-valued functions and let h be any $\mathcal{S}$-measurable real-valued function not belonging to F^*. Then there exists a superposition operator

$$\Phi : E \times \mathbf{R} \to \mathbf{R}$$

such that Φ is sup-measurable with respect to F, but is not sup-measurable with respect to the one-element class $\{h\}$.

Proof. Lemma 7 and relation (1) readily imply that there exists a family G of $\mathcal{S}$-measurable real-valued functions, fundamental for F and such that

$$\mathrm{card}(G) \leq \mathrm{card}(E).$$

We may also assume, without loss of generality, that every function from G is $\mathcal{I}$-equivalent to some function from F.

Let α denote the least ordinal number whose cardinality is equal to $\mathrm{card}(E)$ and let $\{g_\xi : \xi < \alpha\}$ be an enumeration of all functions from G. Taking into account the relation $h \notin F^*$ and applying Lemma 4, we can find a set $A \in \mathcal{S} \setminus \mathcal{I}$ such that h differs from any $f \in F$ on each $\mathcal{S}$-measurable subset of A not belonging to the ideal $\mathcal{I}$. Obviously, the same is true for all functions from G, i.e., for any $g \in G$, the function h differs from g on each $\mathcal{S}$-measurable subset of A not belonging to $\mathcal{I}$. We may assume in the sequel (without loss of generality) that $A = E$.

Let $\{B_\xi : \xi < \alpha\}$ be an enumeration of all members from the pseudobase $\mathcal{B}$. Applying the method of transfinite recursion, let us construct two injective disjoint α-sequences

$$\{x_\xi : \xi < \alpha\}, \qquad \{x'_\xi : \xi < \alpha\}$$

of points of the given space E. Suppose that, for an ordinal $\xi < \alpha$, the partial families $\{x_\zeta : \zeta < \xi\}$ and $\{x'_\zeta : \zeta < \xi\}$ have already been determined. For any ordinal $\zeta < \xi$, denote

$$A_\zeta = \{z \in E : g_\zeta(z) = h(z)\}.$$

Then it is clear that $A_\zeta \in \mathcal{I}$. Consider the set

$$P_\xi = B_\xi \setminus ((\cup\{A_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\} \cup \{x'_\zeta : \zeta < \xi\}).$$

By virtue of the relations (2) and (3), we have $\mathrm{card}(P_\xi) = \mathrm{card}(E)$. Therefore, we can choose two distinct points x and x' from P_ξ. Finally, we put

$$x_\xi = x, \qquad x'_\xi = x'.$$

Proceeding in this manner, we will be able to construct the required α-sequences of points $\{x_\xi : \xi < \alpha\}$ and $\{x'_\xi : \xi < \alpha\}$.

Now, we define a function $\Phi : E \times \mathbf{R} \to \mathbf{R}$ as follows:

$\Phi(x_\xi, h(x_\xi)) = 1$ for each ordinal $\xi < \alpha$;

$\Phi(x'_\xi, h(x'_\xi)) = -1$ for each ordinal $\xi < \alpha$;
$\Phi(x, t) = 0$ for all other pairs $(x, t) \in E \times \mathbf{R}$.

Let us show that Φ is sup-measurable with respect to F and, at the same time, the function Φ_h is not $\mathcal{S}$-measurable. Indeed, take an arbitrary $f \in F$ and find a function $g \in G$ which is $\mathcal{I}$-equivalent to f. Clearly, g coincides with some g_η where $\eta < \alpha$. Further, introduce the set

$$Z = \{z \in E : \Phi(z, g(z)) \neq 0\}.$$

For any $z \in Z$, the disjunction

$$\Phi(z, g(z)) = 1 \quad \vee \quad \Phi(z, g(z)) = -1$$

must be valid. This implies that either $z = x_\xi$ and $g(z) = h(z)$, or $z = x'_\xi$ and $g(z) = h(z)$. It follows directly from our construction that, in both cases above, $\xi \leq \eta$. Consequently, the cardinality of Z must be strictly less than $\mathrm{card}(E)$. Hence, in view of relation (3), the function Φ_g must be $\mathcal{S}$-measurable, and the same is true for Φ_f.

On the other hand, the definition of Φ easily yields that the function Φ_h cannot be $\mathcal{S}$-measurable. Indeed, for any set $B \in \mathcal{B}$, we have from our construction that

$$\{-1, 1\} \subset \mathrm{ran}(\Phi_h | B).$$

Remembering that $\mathcal{B}$ is a pseudo-base for $(E, \mathcal{S}, \mathcal{I})$, we obtain that the sets $\Phi_h^{-1}(-1)$ and $\Phi_h^{-1}(1)$ are $\mathcal{S}$-thick in E, i.e., they intersect all members from $\mathcal{S} \setminus \mathcal{I}$. This fact immediately implies that both these sets are not $\mathcal{S}$-measurable and hence Φ_h is not $\mathcal{S}$-measurable, too. The proof is thus completed.

Keeping in mind the preceding lemmas, we are able to formulate the following statement.

Theorem 4. *Let a space $(E, \mathcal{S}, \mathcal{I})$ be given, let F be a class of $\mathcal{S}$-measurable real-valued functions, and let the assumptions of Lemma 8 be fulfilled. Then the class F^* is the largest extension of F which preserves the sup-measurability property.*

Lemma 8 and hence Theorem 4 were proved under assumptions of somewhat set-theoretical flavor. These assumptions are known to be consistent with **ZFC** theory for those natural measurable spaces with negligible sets, which are extensively studied in real analysis. We now give several examples illustrating the theorem obtained above. Let us begin with the following very simple example.

Example 4. Let $E = \mathbf{R}$, let $\mathcal{S} = \mathrm{dom}(\lambda)$ be the σ-algebra of all Lebesgue measurable sets in $\mathbf{R}$, and let $\mathcal{I} = \mathcal{I}(\lambda)$ be the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$. Denote by F the class of all real-valued constant functions on $\mathbf{R}$. Obviously, there are many real-valued functions h on $\mathbf{R}$ not belonging to F^* (for instance, any strictly monotone function acting from $\mathbf{R}$ into $\mathbf{R}$ can be taken as h). So, we obtain that there exists a superposition operator $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, which is Lebesgue measurable with respect to the first variable, but produces λ-nonmeasurable functions of type Φ_h. Actually, this result needs no additional set-theoretical assumptions.

The next example is less trivial.

Example 5. Again, let $E = \mathbf{R}$, let $\mathcal{S} = \mathrm{dom}(\lambda)$ be the σ-algebra of all Lebesgue measurable sets in $\mathbf{R}$, and let $\mathcal{I} = \mathcal{I}(\lambda)$ be the σ-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$. We denote by F the family of all real-valued continuous functions on $\mathbf{R}$ differentiable almost everywhere (with respect to the Lebesgue measure). Let h be a real-valued continuous function on $\mathbf{R}$ such that it is nowhere approximately differentiable. Then $h \notin F^*$. Therefore, under the corresponding set-theoretical assumptions on $(E, \mathcal{S}, \mathcal{I})$, there exists a superposition operator Φ sup-measurable with respect to F, for which the function Φ_h is not Lebesgue measurable (cf. Chapter 19 where a somewhat similar approach was used).

Example 6. Let $E = \mathbf{R}$, let $\mathcal{S} = \mathcal{B}a(\mathbf{R})$ be the σ-algebra of all those sets in $\mathbf{R}$ which possess the Baire property, and let $\mathcal{I} = \mathcal{K}(\mathbf{R})$ be the σ-ideal of all first category subsets of $\mathbf{R}$. We denote by F the family of all real-valued continuous functions f on $\mathbf{R}$ having the property that every nonempty open subinterval of $\mathbf{R}$ contains at least one point at which f is differentiable. Take any real-valued continuous function h on $\mathbf{R}$ that is nowhere differentiable. Then it is not hard to demonstrate that $h \notin F^*$. Therefore, under the corresponding set-theoretical assumptions on $(E, \mathcal{S}, \mathcal{I})$, there exists a superposition operator Φ sup-measurable with respect to F, for which the function Φ_h does not possess the Baire property (cf. again Chapter 19).

Example 7. It is not difficult to show that the existence of a Sierpiński subset of the Euclidean plane $\mathbf{R}^2$ implies the existence of a superposition operator $\Phi : \mathbf{R}^2 \to \mathbf{R}$ which is sup-measurable with respect to the class $L(\mathbf{R}, \mathbf{R})$ but is not Lebesgue measurable as a function of two variables. Indeed, it suffices to take as Φ the characteristic function of a Sierpiński set on the plane (we shall say that such a Φ determines a Sierpiński superposition operator). Moreover, it can be observed that the same Φ yields Lebesgue measurable functions Φ_f for all those functions $f : \mathbf{R} \to \mathbf{R}$ whose graphs are sets of Lebesgue measure zero in $\mathbf{R}^2$. Obviously, there are many λ-nonmeasurable functions among those f. An analogous situation holds in terms of category and the Baire property. In this case, the existence of a Luzin subset of the plane $\mathbf{R}^2$ is needed for constructing a relevant example.

In connection with Example 7, the next statement is of some interest.

Theorem 5. *Let $h : \mathbf{R} \to \mathbf{R}$ be a function whose graph has strictly positive outer Lebesgue measure in $\mathbf{R}^2$. Then, under the Continuum Hypothesis (**CH**), there exists a Sierpiński superposition operator Φ such that Φ_h is not Lebesgue measurable.*

Proof. As usual, let λ denote the standard one-dimensional Lebesgue measure on $\mathbf{R}$, let λ_2 denote the standard two-dimensional Lebesgue measure on $\mathbf{R}^2$, and let $\Gamma \subset \mathbf{R}^2$ be the graph of h. By virtue of the assumption of this theorem, Γ is not contained in a set of λ_2-measure zero.

Let $\{X_\xi : \xi < \omega_1\}$ be the family of all Borel sets in $\mathbf{R}$ of λ-measure zero and let $\{B_\xi : \xi < \omega_1\}$ be the family of all Borel sets in $\mathbf{R}^2$ of λ_2-measure zero. We shall construct, by applying the method of transfinite recursion, an injective family $\{x_\xi : \xi < \omega_1\}$ of points of $\mathbf{R}$. Suppose that, for an ordinal $\xi < \omega_1$, the partial family of points $\{x_\zeta : \zeta < \xi\}$ has already been defined. Consider the set

$$T_\xi = (\cup\{B_\zeta : \zeta < \xi\}) \cup (\cup\{X_\zeta \times \mathbf{R} : \zeta < \xi\}) \cup (\cup\{\{x_\zeta\} \times \mathbf{R} : \zeta < \xi\}).$$

Clearly, we have the equality $\lambda_2(T_\xi) = 0$. Hence, $\Gamma \setminus T_\xi \neq \emptyset$ and there exists a point $(x, y) \in \Gamma \setminus T_\xi$. We put $x_\xi = x$.

Proceeding in this way, we will be able to construct the required family of points $\{x_\xi : \xi < \omega_1\}$. It immediately follows from our construction that

(i) the set $\{x_\xi : \xi < \omega_1\}$ is a Sierpiński subset of the real line $\mathbf{R}$;

(ii) the set $\{(x_\xi, h(x_\xi)) : \xi < \omega_1\}$ is a Sierpiński subset of the plane $\mathbf{R}^2$.

Let Φ denote the characteristic function of the latter subset of $\mathbf{R}^2$. Then Φ is a Sierpiński superposition operator. At the same time, considering the function Φ_h, we easily observe that

$$\Phi_h^{-1}(1) = \{x_\xi : \xi < \omega_1\}.$$

Thus, Φ_h is not Lebesgue measurable since no Sierpiński subset of $\mathbf{R}$ is λ-measurable (see Theorem 5 from Chapter 13). This completes the proof.

Remark 6. Actually, the argument presented above yields (under **CH**) a more general result. Namely, for any set $\Gamma \subset \mathbf{R}^2$ of strictly positive outer

λ_2-measure, there exists a partial function h acting from $\mathbf{R}$ into $\mathbf{R}$ and having the following three properties:

(1) the graph of h is contained in Γ;

(2) the graph of h is a Sierpiński subset of $\mathbf{R}^2$;

(3) the domain of h is a Sierpiński subset of $\mathbf{R}$.

EXERCISES

1. Let X be a topological space, Y be a separable metric space, Z be a metric space, and let $\Phi : X \times Y \to Z$ be a mapping satisfying these two conditions:

(a) for each $x \in X$, the partial mapping $\Phi(x, \cdot)$ is continuous;

(b) for each $y \in Y$, the partial mapping $\Phi(\cdot, y)$ is Borel.

Show that Φ is a Borel mapping (cf. the proof of Theorem 4 from Chapter 2).

2. Give a detailed proof of Lemma 4.

3. Establish the validity of Lemma 5 without assuming the countable chain condition.

4. Give a proof of Lemma 6 and, in addition, verify that the countable chain condition is not needed for this lemma.

5. Check the validity of Lemma 7.

6. Under **CH**, formulate and prove an appropriate analogue of the result of Remark 6 (in terms of category, Baire property, and Luzin sets).

Also, assuming Martin's Axiom, establish analogous statements in terms of generalized Sierpiński sets and in terms of generalized Luzin sets.

Chapter 21
Ordinary differential equations with bad right-hand sides

In this chapter we wish to consider some set-theoretical questions concerning the existence and uniqueness of solutions of ordinary differential equations. In particular, we deal here with those ordinary differential equations of first order, the right-hand sides of which are nonmeasurable in the Lebesgue sense, but for which we are able to establish the theorem on the existence and uniqueness of a solution with nice descriptive properties.

The results presented in this chapter are substantially based on the material of Chapter 19 (see also [123]).

The existence of a solution of a first-order ordinary differential equation with a continuous right-hand side is stated by the famous Peano theorem (see, e.g., [206]). Exercise 1 shows that this classical theorem does not rely on any form of the Axiom of Choice and, in fact, is a result of **ZF** set theory (cf. also [252]).

There are many simple examples of ordinary differential equations whose right-hand sides are not as good as their solutions. For instance, let us take the following ordinary differential equation:

$$y' = |y|.$$

Then it is easy to verify that

(1) the right-hand side of this equation is continuous on $\mathbf{R} \times \mathbf{R}$ but is not differentiable at each point of $\mathbf{R} \times \{0\}$;

(2) all solutions of this equation are analytic;

(3) for any initial condition, there exists a unique solution of this equation, satisfying the condition.

In our further considerations we shall show that some differential equations

$$y' = \Phi(x, y)$$

of first order are possible, for which Φ is nonmeasurable in the Lebesgue sense but relations (2) and (3) hold true.

We begin with an old remarkable result of Orlicz (see [199]) stating that, for almost each (in the category sense) function Φ from the Banach space $C_b(\mathbf{R} \times \mathbf{R})$ consisting of all bounded continuous real-valued functions defined on $\mathbf{R} \times \mathbf{R}$, the corresponding Cauchy problem

$$y' = \Phi(x, y) \qquad (y(x_0) = y_0, \quad x_0 \in \mathbf{R}, \quad y_0 \in \mathbf{R})$$

has a unique solution. In order to present this result we, first of all, want to recall the purely topological Kuratowski lemma on closed projections (see, e.g., [64], [149] or Chapter 0).

Namely, if X and Y are some topological spaces and, in addition, Y is quasi-compact, then the canonical projection

$$\mathrm{pr}_1 \; : \; X \times Y \to X$$

is a closed mapping, i.e., for each closed subset A of $X \times Y$, the image $\mathrm{pr}_1(A)$ is closed in X. Several applications of the Kuratowski lemma were discussed in Chapter 0. Here we are going to consider an application of this lemma to the theory of ordinary differential equations. Actually, we need here a slightly more general version of the lemma.

Let us recall that a topological space E is σ-quasi-compact if it can be represented in the form

$$E = \cup\{E_n \; : \; n < \omega\},$$

where all sets E_n $(n < \omega)$ are quasi-compact subspaces of E.

Now, the following slight generalization of the Kuratowski lemma is true.

Lemma 1. *Let X be a topological space and let Y be a σ-quasi-compact space. Let, as above, pr_1 denote the canonical projection from $X \times Y$ into X. Then, for each F_σ-subset A of $X \times Y$, the image $\mathrm{pr}_1(A)$ is an F_σ-subset of X.*

Proof. In fact, the Kuratowski lemma easily implies this result. Indeed, since Y is σ-quasi-compact, we may write

$$Y = \cup\{Y_n \; : \; n < \omega\},$$

where all Y_n $(n < \omega)$ are quasi-compact subspaces of Y. Then, for any set $A \subset X \times Y$, we have the equality

$$\mathrm{pr}_1(A) = \cup\{\mathrm{pr}_1(A \cap (X \times Y_n)) \; : \; n < \omega\}.$$

Suppose now that A is an F_σ-subset of $X \times Y$. Then A can be represented in the form

$$A = \cup\{A_m \ : \ m < \omega\},$$

where all sets A_m $(m < \omega)$ are closed in $X \times Y$. Therefore, we obtain

$$\mathrm{pr}_1(A) = \cup\{\mathrm{pr}_1(A_m \cap (X \times Y_n)) \ : \ m < \omega, \ n < \omega\}.$$

Now, every set

$$A_m \cap (X \times Y_n) \qquad (m < \omega, \ n < \omega)$$

is closed in the product space $X \times Y_n$ and every space Y_n is quasi-compact. Hence, by the Kuratowski lemma, the set $\mathrm{pr}_1(A_m \cap (X \times Y_n))$ is closed in X. Consequently, $\mathrm{pr}_1(A)$ is an F_σ-subset of X. This completes the proof of Lemma 1.

Now, let us return to the Banach space $C_b(\mathbf{R} \times \mathbf{R})$. For each function Φ from this space, we can consider the ordinary differential equation

$$y' = \Phi(x, y)$$

and, for any point $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, we can investigate the corresponding Cauchy problem of finding a solution $y \ : \ \mathbf{R} \to \mathbf{R}$ of this equation, satisfying the initial condition $y(x_0) = y_0$.

It is well known (see, e.g., [206] or Exercise 1) that such a solution does always exist and, since Φ is bounded, any solution is global, i.e., it is defined on the whole real line $\mathbf{R}$. On the other hand, we cannot assert, in general, the uniqueness of a solution. There are simple examples of continuous bounded real-valued functions Φ on $\mathbf{R}\times\mathbf{R}$ for which the corresponding Cauchy problem admits at least two distinct solutions (in this connection, let us mention the celebrated work by Lavrentieff [159] where a much stronger result was obtained).

Actually, we need some additional properties of the original function Φ in order to have the uniqueness of a solution of the differential equation

$$y' = \Phi(x, y) \qquad (y(x_0) = y_0, \ x_0 \in \mathbf{R}, \ y_0 \in \mathbf{R}).$$

For instance, if Φ satisfies the so-called local Lipschitz condition with respect to the second variable y, then we have a unique solution for every Cauchy problem corresponding to Φ.

It makes sense to recall here that Φ satisfies the local Lipschitz condition with respect to y if, for any point $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, there exist a neighborhood $V(x_0, y_0)$ of this point and a positive real number

$$L = L(\Phi, (x_0, y_0))$$

such that

$$|\Phi(x, y_1) - \Phi(x, y_2)| \leq L|y_1 - y_2|$$

for all points (x, y_1) and (x, y_2) belonging to $V(x_0, y_0)$.

Let us denote by $Lip_l(\mathbf{R} \times \mathbf{R})$ the family of all those functions from $C_b(\mathbf{R} \times \mathbf{R})$ which satisfy the local Lipschitz condition with respect to y. Then, obviously, $Lip_l(\mathbf{R} \times \mathbf{R})$ is a vector subspace of $C_b(\mathbf{R} \times \mathbf{R})$. Notice also that $Lip_l(\mathbf{R} \times \mathbf{R})$ is an everywhere dense subset of $C_b(\mathbf{R} \times \mathbf{R})$. Indeed, this fact is almost trivial from the geometrical point of view. Thus, we can conclude that, for all functions Φ belonging to some everywhere dense subset of $C_b(\mathbf{R} \times \mathbf{R})$, the Cauchy problem

$$y' = \Phi(x, y) \qquad (y(x_0) = y_0, \ \ x_0 \in \mathbf{R}, \ \ y_0 \in \mathbf{R})$$

has a unique solution. Orlicz essentially improved this result and showed that it holds true for almost all (in the Baire category sense) functions from the Banach space $C_b(\mathbf{R} \times \mathbf{R})$.

More precisely, one can formulate the following statement.

Theorem 1. *The set of all those functions from $C_b(\mathbf{R} \times \mathbf{R})$ for which the corresponding Cauchy problem has a unique solution (for any point (x_0, y_0) from $\mathbf{R} \times \mathbf{R}$) is an everywhere dense G_δ-subset of $C_b(\mathbf{R} \times \mathbf{R})$.*

Proof. Let us denote by the symbol U the family of all those functions from $C_b(\mathbf{R} \times \mathbf{R})$ for which the corresponding Cauchy problem has a unique solution (for any point (x_0, y_0) belonging to $\mathbf{R} \times \mathbf{R}$). As mentioned above, the set U is everywhere dense in $C_b(\mathbf{R} \times \mathbf{R})$. Therefore, it remains to prove that U is a G_δ-subset of $C_b(\mathbf{R} \times \mathbf{R})$. In order to show this, let us first rewrite the Cauchy problem in the equivalent integral form:

$$y(x) = \int_{x_0}^{x} \Phi(t, y(t))dt \ + \ y_0.$$

Further, for any two rational numbers $\varepsilon > 0$ and q, let us denote by $P(\varepsilon, q)$ the set of all those elements

$$(\Phi, x_0, y_0) \in C_b(\mathbf{R} \times \mathbf{R}) \times \mathbf{R} \times \mathbf{R}$$

for which there exist at least two real-valued continuous functions ϕ_1 and ϕ_2 such that

$$\mathrm{dom}(\phi_1) = \mathrm{dom}(\phi_2) = \mathbf{R},$$

$$\phi_1(x) = \int_{x_0}^{x} \Phi(t, \phi_1(t))dt \ + \ y_0 \qquad (x \in \mathbf{R}),$$

$$\phi_2(x) = \int_{x_0}^{x} \Phi(t, \phi_2(t))dt \ + \ y_0 \qquad (x \in \mathbf{R}),$$

$$|\phi_1(q) - \phi_2(q)| \geq \varepsilon.$$

It is not difficult to establish that $P(\varepsilon, q)$ is a closed subset of the product space $C_b(\mathbf{R} \times \mathbf{R}) \times \mathbf{R} \times \mathbf{R}$. Indeed, suppose that a sequence

$$\{(\Phi^{(n)}, x_0^{(n)}, y_0^{(n)}) \ : \ n \in \mathbf{N}\}$$

of elements of $P(\varepsilon, q)$ converges to some element

$$(\Phi, x_0, y_0) \in C_b(\mathbf{R} \times \mathbf{R}) \times \mathbf{R} \times \mathbf{R}.$$

Then we obviously have

$$\lim_{n\to+\infty} x_0^{(n)} = x_0, \qquad \lim_{n\to+\infty} y_0^{(n)} = y_0,$$

and the sequence of functions

$$\{\Phi^{(n)} \ : \ n \in \mathbf{N}\} \subset C_b(\mathbf{R} \times \mathbf{R})$$

converges uniformly to the function Φ. We may assume without loss of generality that

$$(\forall n \in \mathbf{N})(||\Phi^{(n)}|| \leq ||\Phi|| + 1).$$

For every natural number n, let $\phi_1^{(n)}$ and $\phi_2^{(n)}$ denote two real-valued continuous functions satisfying the following relations:

$$\mathrm{dom}(\phi_1^{(n)}) = \mathrm{dom}(\phi_2^{(n)}) = \mathbf{R},$$

$$\phi_1^{(n)}(x) = \int_{x_0^{(n)}}^{x} \Phi^{(n)}(t, \phi_1^{(n)}(t))dt \ + \ y_0^{(n)} \qquad (x \in \mathbf{R}),$$

$$\phi_2^{(n)}(x) = \int_{x_0^{(n)}}^{x} \Phi^{(n)}(t, \phi_2^{(n)}(t))dt \ + \ y_0^{(n)} \qquad (x \in \mathbf{R}),$$

$$|\phi_1^{(n)}(q) - \phi_2^{(n)}(q)| \geq \varepsilon.$$

Then it is not hard to verify that all functions from the family

$$\{\phi_1^{(n)} \ : \ n \in \mathbf{N}\} \cup \{\phi_2^{(n)} \ : \ n \in \mathbf{N}\}$$

are equicontinuous. More precisely, for each function ϕ from this family and for any two points $x' \in \mathbf{R}$ and $x'' \in \mathbf{R}$, we have the inequality

$$|\phi(x') - \phi(x'')| \leq (||\Phi|| + 1)|x' - x''|.$$

So, applying the classical Ascoli–Arzelá theorem (see, e.g., [206]), we can readily derive that there exists an infinite subset K of $\mathbf{N}$ for which the partial sequences of functions

$$\{\phi_1^{(n)} \ : \ n \in K\}, \qquad \{\phi_2^{(n)} \ : \ n \in K\}$$

converge uniformly (on each bounded subinterval of $\mathbf{R}$) to some functions ϕ_1 and ϕ_2, respectively. Also, it can easily be checked that, for ϕ_1 and ϕ_2, we have the analogous relations

$$\mathrm{dom}(\phi_1) = \mathrm{dom}(\phi_2) = \mathbf{R},$$

$$\phi_1(x) = \int_{x_0}^{x} \Phi(t, \phi_1(t))dt \ + \ y_0 \qquad (x \in \mathbf{R}),$$

$$\phi_2(x) = \int_{x_0}^{x} \Phi(t, \phi_2(t))dt \ + \ y_0 \qquad (x \in \mathbf{R}),$$

$$|\phi_1(q) - \phi_2(q)| \geq \varepsilon.$$

Thus we see that

$$(\Phi, x_0, y_0) \in P(\varepsilon, q),$$

and hence $P(\varepsilon, q)$ is closed in the product space $C_b(\mathbf{R} \times \mathbf{R}) \times \mathbf{R} \times \mathbf{R}$.

Now, let us put

$$P = \cup\{P(\varepsilon, q) \ : \ \varepsilon > 0, \ \varepsilon \in \mathbf{Q}, \ q \in \mathbf{Q}\}.$$

Then it is clear that a function $\Psi \in C_b(\mathbf{R} \times \mathbf{R})$ does not belong to the set U if and only if there exist a rational number $\varepsilon > 0$, a rational number q, and some points $x_0 \in \mathbf{R}$ and $y_0 \in \mathbf{R}$, such that (Ψ, x_0, y_0) belongs to the set $P(\varepsilon, q)$. In other words, we may write

$$C_b(\mathbf{R} \times \mathbf{R}) \setminus U = \mathrm{pr}_1(P)$$

where

$$\mathrm{pr}_1 \ : \ C_b(\mathbf{R} \times \mathbf{R}) \times \mathbf{R} \times \mathbf{R} \to C_b(\mathbf{R} \times \mathbf{R})$$

denotes the canonical projection. It immediately follows from the definition of the set P that P is an F_σ-subset of the product space $C_b(\mathbf{R}\times\mathbf{R})\times\mathbf{R}\times\mathbf{R}$. In addition, the Euclidean plane $\mathbf{R} \times \mathbf{R}$ is a σ-compact space. So, applying Lemma 1, we conclude that $\mathrm{pr}_1(P)$ is an F_σ-subset of $C_b(\mathbf{R} \times \mathbf{R})$ and, consequently, U is a G_δ-subset of $C_b(\mathbf{R} \times \mathbf{R})$. This finishes the proof of Theorem 1.

Remark 1. Evidently, the Banach space $C_b(\mathbf{R} \times \mathbf{R})$ is not separable. Let E denote the subset of this space, consisting of all those functions that are constant at infinity. In other words, $\Phi \in E$ if and only if there exists a constant $M = M(\Phi) \in \mathbf{R}$ such that, for any $\varepsilon > 0$, a positive real number $a = a(\Phi, \varepsilon)$ can be found for which we have

$$(\forall x)(\forall y)((x, y) \in \mathbf{R} \times \mathbf{R} \setminus [-a, a] \times [-a, a] \Rightarrow |\Phi(x, y) - M| < \varepsilon).$$

Notice that E is a closed vector subspace of $C_b(\mathbf{R} \times \mathbf{R})$ and hence E is a Banach space, as well. Moreover, one can easily verify that E is separable. Clearly, a direct analogue of Theorem 1 holds true for E. Actually, in [199] Orlicz deals with the space E. Several analogues of Theorem 1, for other spaces similar to $C_b(\mathbf{R} \times \mathbf{R})$ or E, are discussed in [3].

Remark 2. Unfortunately, the set U considered above has a bad algebraic structure. In particular, U is not a subgroup of the additive group of $C_b(\mathbf{R} \times \mathbf{R})$ and, consequently, U is not a vector subspace of $C_b(\mathbf{R} \times \mathbf{R})$. Indeed, suppose for a while that U is a subgroup of $C_b(\mathbf{R} \times \mathbf{R})$. Then U must be a proper subgroup of $C_b(\mathbf{R} \times \mathbf{R})$. Let us take an arbitrary function $\Psi \in C_b(\mathbf{R} \times \mathbf{R}) \setminus U$. Obviously, we may write

$$U \cap (\{\Psi\} + U) = \emptyset.$$

But both sets U and $\{\Psi\} + U$ are the complements of some first category subsets of $C_b(\mathbf{R} \times \mathbf{R})$. Therefore, their intersection $U \cap (\{\Psi\} + U)$ must be the complement of a first category subset of $C_b(\mathbf{R} \times \mathbf{R})$, too, and hence

$$U \cap (\{\Psi\} + U) \neq \emptyset.$$

We have thus obtained a contradiction which yields that U cannot be a subgroup of $C_b(\mathbf{R} \times \mathbf{R})$. For some other properties of U interesting from the set-theoretical and algebraic points of view, see, e.g., [3].

Theorem 1 proved above shows us that, for many functions from the space $C_b(\mathbf{R} \times \mathbf{R})$, we have the existence and uniqueness of a solution of the Cauchy problem. In fact, this is one of the most important results in the theory of ordinary differential equations. Naturally, we may consider a more general class of functions

$$\Phi \ : \ \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

not necessarily continuous or Lebesgue measurable and investigate for such functions the corresponding Cauchy problem from the point of view of the existence and uniqueness of a solution.

For this purpose, let us recall that, as shown in Chapter 19 of this book, there exists a subset Z of the plane $\mathbf{R} \times \mathbf{R}$, satisfying the following four relations:

(1) no three distinct points of Z belong to a straight line;

(2) Z is the graph of some partial function acting from $\mathbf{R}$ into $\mathbf{R}$;

(3) Z is a λ_2-thick subset of the plane $\mathbf{R} \times \mathbf{R}$, where λ_2 denotes the standard two-dimensional Lebesgue measure on $\mathbf{R} \times \mathbf{R}$;

(4) for any Borel mapping $\phi : \mathbf{R} \to \mathbf{R}$, the intersection of Z with the graph of ϕ has cardinality strictly less than the cardinality of the continuum $\mathbf{c}$.

We denote by Φ the characteristic function of the above-mentioned set Z. Then, obviously, Φ is a Lebesgue nonmeasurable function and, furthermore, if $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$, then Φ is sup-measurable as well.

Now, starting with the function Φ described above, we wish to consider an ordinary differential equation

$$y' = \Psi(x, y)$$

with a Lebesgue nonmeasurable right-hand side Ψ, and we are going to show that, in some situations, it is possible to obtain the existence and uniqueness of a solution of this equation (for any initial condition).

First of all, we need to determine the class of functions to which a solution must belong. It is natural to take the class $AC_l(\mathbf{R})$ consisting of all locally absolutely continuous real-valued functions on $\mathbf{R}$. In other words, $\psi \in AC_l(\mathbf{R})$ if and only if, for each point $x \in \mathbf{R}$, there exists a neighborhood $V(x)$ of x such that the restriction $\psi|V(x)$ is absolutely continuous. Another characterization of locally absolutely continuous functions on $\mathbf{R}$ is the following: a function ψ belongs to $AC_l(\mathbf{R})$ if and only if there exists a Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$ such that f is locally integrable and

$$\psi(x) = \int_0^x f(t)dt + \psi(0)$$

for any point $x \in \mathbf{R}$.

Let Ψ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$, and let us fix a point $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$. We say that the corresponding Cauchy problem

$$y' = \Psi(x, y) \qquad (y(x_0) = y_0)$$

has a unique solution (in the class $AC_l(\mathbf{R})$) if there exists a unique function $\psi \in AC_l(\mathbf{R})$ satisfying the relations

(a) $\psi'(x) = \Psi(x, \psi(x))$ for almost all (with respect to the Lebesgue measure λ) points $x \in \mathbf{R}$;

(b) $\psi(x_0) = y_0$.

For example, if our mapping Ψ is bounded, Lebesgue measurable with respect to x, and satisfies locally the Lipschitz condition with respect to y, then, for each $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, the corresponding Cauchy problem has a unique solution. The reader can easily verify this fact by using the standard argument. Notice that, in this example, Ψ is necessarily Lebesgue measurable and sup-measurable (cf. Exercise 2 from Chapter 19). Notice also that an analogue of Theorem 1 holds true for a certain class of Banach spaces consisting of mappings acting from $\mathbf{R}\times\mathbf{R}$ into $\mathbf{R}$, which are Lebesgue measurable with respect to x and continuous with respect to y.

The next statement shows that the existence and uniqueness of a solution can be fulfilled even for some ordinary differential equations whose right-hand sides are extremely bad, e.g., are nonmeasurable in the Lebesgue sense.

Theorem 2. *There exists a Lebesgue nonmeasurable mapping*

$$\Psi \ : \ \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

such that the Cauchy problem

$$y' = \Psi(x, y) \qquad (y(x_0) = y_0)$$

has a unique solution for any point $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$.

Proof. Let Z be the subset of $\mathbf{R} \times \mathbf{R}$, constructed in Chapter 19 (see Theorem 2 therein). Denote by Φ the characteristic function of Z and fix a real number t. Further, put

$$\Psi(x, y) = \Phi(x, y) + t \qquad (x \in \mathbf{R},\ y \in \mathbf{R}).$$

We assert that Ψ is the required mapping. Indeed, Ψ is Lebesgue nonmeasurable because Φ is Lebesgue nonmeasurable. Let now (x_0, y_0) be an arbitrary point of the plane $\mathbf{R} \times \mathbf{R}$. Consider a function $\psi : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$\psi(x) = tx + (y_0 - tx_0) \qquad (x \in \mathbf{R}).$$

The graph of this function is a straight line, so it has at most two common points with the set Z. Consequently, the function

$$\Psi_\psi \ : \ \mathbf{R} \to \mathbf{R}$$

is equal to t for almost all (with respect to the Lebesgue measure λ) points from $\mathbf{R}$. We also have $\psi'(x) = t$ for all $x \in \mathbf{R}$. In other words, ψ is a solution of the Cauchy problem

$$y' = \Psi(x, y) \qquad (y(x_0) = y_0).$$

It remains to show that ψ is a unique solution from the class $AC_l(\mathbf{R})$. For this purpose, let us take an arbitrary solution ϕ of the same Cauchy problem, belonging to $AC_l(\mathbf{R})$. Then, for almost all points $x \in \mathbf{R}$, we have the equality

$$\phi'(x) = \Phi(x, \phi(x)) + t.$$

It immediately follows from this equality that the function Φ_ϕ is measurable in the Lebesgue sense. But, as we know,

$$\mathrm{card}(\{x \in \mathbf{R} \ : \ \Phi_\phi(x) \neq 0\}) < \mathbf{c}.$$

So we obtain that Φ_ϕ is equivalent to zero and hence $\phi'(x) = t$ for almost all $x \in \mathbf{R}$. Therefore, we can conclude that

$$\phi(x) = tx + (y_0 - tx_0) \qquad (x \in \mathbf{R}).$$

This completes the proof of Theorem 2.

Remark 3. The preceding theorem was proved within **ZFC** theory. In this connection, let us stress once more that the function Ψ of Theorem 2 is Lebesgue nonmeasurable and, under a certain set-theoretical hypothesis, is also sup-measurable (hence weakly sup-measurable). At the same time, we already know that it is impossible to establish in **ZFC** theory the existence of a sup-measurable mapping which is not measurable in the Lebesgue sense (recall that this result is due to Roslanowski and Shelah [218]).

Now, we are going to demonstrate that, under the set-theoretical assumption

$$[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda),$$

Theorem 1 of Orlicz can be generalized to Banach spaces of mappings acting from $\mathbf{R}\times\mathbf{R}$ into $\mathbf{R}$, essentially larger than the classical space $C_b(\mathbf{R}\times\mathbf{R})$ (notice that all spaces of real-valued bounded mappings considered in this chapter are assumed to be equipped with the norm of uniform convergence).

More precisely, we can formulate and prove the next result.

Theorem 3. *Suppose that $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$. Then there exists a Banach space B_0 of mappings acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$, satisfying the following relations:*

(1) $C_b(\mathbf{R} \times \mathbf{R}) \subset B_0$;

(2) there are sup-measurable but Lebesgue nonmeasurable functions belonging to B_0;

(3) the analogue of Theorem 1 holds true for B_0, i.e., the family of all functions $\Psi \in B_0$ such that the ordinary differential equation

$$y' = \Psi(x, y)$$

has a unique solution for any initial condition $y(x_0) = y_0$ is an everywhere dense G_δ-subset of B_0.

Proof. Let $\Phi = \Phi_0$ be again the characteristic function of the set Z described in the proof of Theorem 2 from Chapter 19. Obviously, Φ does not belong to the vector space $C_b(\mathbf{R} \times \mathbf{R})$. Denote by B_0 the vector space of functions generated by the set $\{\Phi_0\} \cup C_b(\mathbf{R} \times \mathbf{R})$. Clearly, each function Ψ belonging to B_0 can be represented in the form

$$\Psi = \Psi_1 + t_1\Phi_0,$$

where $\Psi_1 \in C_b(\mathbf{R} \times \mathbf{R})$ and $t_1 \in \mathbf{R}$. Moreover, because B_0 is the direct sum of the vector spaces $C_b(\mathbf{R} \times \mathbf{R})$ and $\{t\Phi_0 \ : \ t \in \mathbf{R}\}$, such a representation is unique. We equip B_0 with the norm of uniform convergence. Taking account of the fact that Φ_0 is Lebesgue nonmeasurable, we may write

$$\mathrm{dist}(\Phi_0, C_b(\mathbf{R} \times \mathbf{R})) > 0.$$

In other words, B_0 can be regarded as a direct topological sum of the two Banach spaces $C_b(\mathbf{R} \times \mathbf{R})$ and $\{t\Phi_0 \ : \ t \in \mathbf{R}\}$. Consequently, we may identify B_0 with the product space $C_b(\mathbf{R} \times \mathbf{R}) \times \mathbf{R}$.

Let now Ψ_1 be an arbitrary function from $C_b(\mathbf{R} \times \mathbf{R})$ such that the corresponding ordinary differential equation

$$y' = \Psi_1(x, y)$$

has a unique solution for any initial condition $y(x_0) = y_0$. Then it is not difficult to check (by using the properties of our function Φ_0) that, for each real number t_1, the ordinary differential equation

$$y' = \Psi_1(x, y) + t_1\Phi_0(x, y)$$

has also a unique solution for any initial condition $y(x_0) = y_0$. Conversely, if a function

$$\Psi = \Psi_1 + t_1\Phi_0$$

from the space B_0 (where $\Psi_1 \in C_b(\mathbf{R} \times \mathbf{R})$) is such that the ordinary differential equation

$$y' = \Psi(x, y)$$

possesses a unique solution for every initial condition, then the ordinary differential equation

$$y' = \Psi_1(x, y)$$

possesses a unique solution for every initial condition, too.

Let us recall that the symbol U denotes (in this chapter) the family of all functions Ψ_1 from $C_b(\mathbf{R} \times \mathbf{R})$ such that the differential equation $y' = \Psi_1(x, y)$ has a unique solution for any initial condition. Denote now by V the analogous family for the space B_0, i.e., let V be the family of all functions Ψ from B_0 such that the differential equation $y' = \Psi(x, y)$ has a unique solution for any initial condition. Then, taking account of the preceding argument, we can assert that

$$V = U + \{t\Phi_0 \ : \ t \in \mathbf{R}\}.$$

According to Theorem 1, U is an everywhere dense G_δ-subset of the Banach space $C_b(\mathbf{R} \times \mathbf{R})$, so we easily conclude that V is an everywhere dense G_δ-subset of the Banach space B_0. Theorem 3 has thus been proved.

Remark 4. Let B be a Banach space of bounded sup-measurable mappings, for which the analogue of Theorem 1 is valid, i.e., the family of all $\Psi \in B$ such that the differential equation $y' = \Psi(x, y)$ has a unique solution for any initial condition $y(x_0) = y_0$ is an everywhere dense G_δ-subset of B. It is not difficult to see that the class of all such Banach spaces B is sufficiently wide. In particular, it follows from Theorem 3 that there is a space B belonging to this class and containing a Lebesgue nonmeasurable mapping. In this connection, it would be interesting to obtain a characterization (description) of the above-mentioned class of Banach spaces.

Finally, we would like to point out once more that some logical and set-theoretical aspects of the classical Cauchy–Peano theorem on the existence of solutions of ordinary differential equations are discussed in the paper by Simpson [252].

EXERCISES

1*. Let W be a nonempty open subset of the plane $\mathbf{R} \times \mathbf{R}$ and let

$$\Phi : W \to \mathbf{R}$$

be a function. Fix a point $(x_0, y_0) \in W$. Recall that a differentiable function $f \ : \ ]t_1, t_2[\ \to \mathbf{R}$ is a (local) solution of the ordinary differential equation

$$y' = \Phi(x, y) \qquad (y(x_0) = y_0)$$

if $x_0 \in \ ]t_1, t_2[$, the graph of f is contained in W and

$$f(x_0) = y_0, \quad f'(x) = \Phi(x, f(x)) \qquad (x \in \ ]t_1, t_2[).$$

In this case, one says that f is a solution of the Cauchy problem for the given function Φ and for the initial condition $(x_0, y_0) \in W$.

The Peano theorem states that if Φ is continuous on W, then a solution always exists for any initial condition from W.

Demonstrate that this theorem is provable within **ZF** theory.

Remark 5. In connection with this exercise, see also the paper by Simpson [252] where some more precise results are presented.

2. Give an example of a function Φ from the space $C_b(\mathbf{R} \times \mathbf{R})$, for which there exists an initial condition $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$ such that the corresponding Cauchy problem

$$y' = \Phi(x, y) \qquad (y(x_0) = y_0)$$

possesses at least two distinct solutions.

3*. Prove that the analogue of Theorem 1 remains true for any Banach space E of bounded mappings acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$, for which there exists an everywhere dense set $D \subset E$ such that each function from D is Lebesgue measurable with respect to x and satisfies locally the Lipschitz condition with respect to y.

4. Let n be an arbitrary natural number and let

$$a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$$

be a polynomial of degree n (over the field $\mathbf{R}$).

Show that there exists a mapping $\Psi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfying the following relations:

(a) Ψ is nonmeasurable in the Lebesgue sense;

(b) for any initial condition $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, the differential equation

$$y' = \Psi(x, y)$$

has a unique solution ψ with $\psi(x_0) = y_0$;

(c) all solutions ψ of the above-mentioned differential equation are of the form

$$\psi(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a \qquad (x \in \mathbf{R}),$$

where $a \in \mathbf{R}$.

5. By assuming the hypothesis $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$, give an example of a Banach space B_1 of functions acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$, satisfying the following four relations:

(1) $C_b(\mathbf{R} \times \mathbf{R}) \subset B_1$;

(2) there are Lebesgue measurable sup-measurable discontinuous functions belonging to B_1;

(3) there are Lebesgue nonmeasurable sup-measurable functions belonging to B_1;

(4) the analogue of Theorem 1 holds true for B_1.

Chapter 22
Nondifferentiable functions from the point of view of category and measure

In some preceding sections of this book we were concerned with various continuous but nondifferentiable functions acting from **R** into **R** (see, e.g., Chapters 0 and 7). In the present chapter, we wish to discuss one general approach to such functions from the viewpoint of category and measure. Briefly speaking, our goal is to demonstrate that, for a given generalized notion of derivative (introduced within **ZF** & **DC** theory), the set of real-valued continuous nondifferentiable functions (with respect to this notion) turns out to be sufficiently large.

We begin with an approach based on the concept of Baire category. More precisely, it is based on the important theorem of Kuratowski and Ulam from general topology (for the formulation and proof of this theorem see, e.g., [149], [202], or Chapter 18 of the present book). We have already mentioned that the Kuratowski–Ulam theorem can be interpreted as a reasonable purely topological analogue of the classical Fubini theorem from measure theory. It is commonly known that the Fubini theorem is fundamental for all of measure theory. Moreover, this theorem has many applications in real analysis, probability theory, and other fields of mathematics. Also, it is well known that the Kuratowski–Ulam theorem possesses a number of nontrivial applications in general topology and mathematical analysis (some of them are pointed out in [149] and [202]).

In our further considerations, the main role is played by the following statement.

Theorem 1. *Let E_1 and E_2 be any two topological spaces with countable bases (or, more generally, with countable π-bases) and let E_3 be a topological space. Let Z be a subset of the product space $E_1 \times E_2$. Suppose that a certain mapping $\Phi : Z \to E_3$ is given and satisfies these two conditions:*

(1) the partial function Φ acting from the topological space $E_1 \times E_2$ into the topological space E_3 has the Baire property, i.e., for any open set V from

E_3, the pre-image $\Phi^{-1}(V)$ has the Baire property in $E_1 \times E_2$;

(2) for almost all (in the sense of category) points $x \in E_1$, the domain of the partial mapping $\Phi(x, .)$ given by

$$\Phi(x, .)(y) = \Phi(x, y)$$

is a first category set in the space E_2.

Then the following relations hold:

(a) Z is a first category subset of the product space $E_1 \times E_2$;

(b) for almost all (in the sense of category) points $y \in E_2$, the set $\{x : (x, y) \in dom(\Phi)\}$ is of first category in the space E_1; roughly speaking, almost each point $y \in E_2$ is almost singular with respect to the partial mapping $\Phi(., y)$.

Proof. The argument leading to the required result is very simple. Indeed, according to the Kuratowski–Ulam theorem, relation (a) implies relation (b). Therefore, it suffices to establish the validity of relation (a). By virtue of condition (1), the partial function Φ has the Baire property, so the set $Z = \Phi^{-1}(E_3)$ has the Baire property in the product space $E_1 \times E_2$. Using condition (2) and the Kuratowski–Ulam theorem once more, we finish the proof.

In connection with Theorem 1, the natural question arises: how can condition (2) be checked for a concrete partial mapping Φ?

The following situation can be frequently met in mathematical analysis and it will be crucial for us in the sequel. Suppose that E_2 is a Polish topological vector space, E_3 is a topological vector space with a countable base, and suppose that our partial mapping Φ satisfies condition (1) and the next condition:

($2'$) for almost each (in the sense of category) point $x \in E_1$, the partial mapping $\Phi(x, .)$ is linear and discontinuous on its domain.

Then it can be shown that Φ satisfies condition (2), as well. Indeed, for almost all points $x \in E_1$, the function $\Phi(x, .)$ has the Baire property and is linear and discontinuous on the vector space

$$Z(x) = \{y : (x, y) \in Z\}.$$

Let us prove that, for the points x indicated above, the set $Z(x)$ is of first category in the space E_2. Suppose otherwise, i.e., suppose that $Z(x)$ is a second category set with the Baire property. Then we may apply to $Z(x)$ the well-known Banach–Kuratowski–Pettis theorem from the theory of topological groups (see, for example, [107], [149] or Exercise 1 of Chapter

10). This theorem is a topological analogue of the classical Steinhaus property of Lebesgue measurable sets with strictly positive measure. Namely, according to this theorem, the set

$$Z(x) - Z(x) = \{y - z \ : \ y \in Z(x), \ z \in Z(x)\}$$

contains a nonempty open subset of the topological vector space E_2 (more precisely, the set $Z(x) - Z(x)$ is a neighborhood of zero of E_2). But since the set $Z(x)$ is a vector space, too, we come to the equality

$$Z(x) - Z(x) = Z(x)$$

and, finally, we obtain $Z(x) = E_2$. Hence the function $\Phi(x, .)$ is defined on the whole Polish topological vector space E_2 and is linear on this space. Now, by taking account of the fact that the function $\Phi(x, .)$ has the Baire property, it is not difficult to prove (by using the same Banach–Kuratowski–Pettis theorem) that $\Phi(x, .)$ is a continuous mapping. But this contradicts the choice of the point x. The contradiction obtained shows us that the set $Z(x)$ must be of first category in the space E_2. Therefore, condition (2) is satisfied for our partial mapping Φ.

Remark 1. Theorem 1 may be considered as one of the possible formalizations of a well-known principle in mathematical analysis that is frequently called "the principle of condensation of singularities." Among various works devoted to this principle, the most famous is the classical paper of Banach and Steinhaus [17]. It is easy to see that the Banach–Steinhaus principle of condensation of singularities is closely connected with Theorem 1 and can be obtained as a consequence of the Kuratowski–Ulam theorem. Indeed, let us take $E_1 = \mathbf{N}$, where the set $\mathbf{N}$ of all natural numbers is equipped with the discrete topology, and let E_2 be an arbitrary Banach space. Suppose that E_3 is also a Banach space and a double sequence of continuous linear operators

$$L_{m,n} : E_2 \to E_3 \qquad (m, n \in \mathbf{N})$$

is given such that, for any $m \in \mathbf{N}$, we have

$$\sup_{n\in\mathbf{N}} ||L_{m,n}|| = +\infty.$$

Let us define a partial mapping Φ from the product space $E_1 \times E_2$ into the space E_3 by the following formula:

$$\Phi(m, x) = \lim_{n\to+\infty} L_{m,n}(x).$$

It is clear that this partial mapping has the Baire property and, for each $m \in \mathbf{N}$, the partial mapping $\Phi(m, .)$ is defined on a first category subset of the space E_2. Hence the domain of the partial mapping Φ is also a first category set in the product space $E_1 \times E_2$. Now, we may apply the Kuratowski–Ulam theorem and, evidently, as a result we obtain that, for almost all elements $x \in E_2$, the set $\{m : (m, x) \in \mathrm{dom}(\Phi)\}$ is empty. But, actually, this is the Banach–Steinhaus principle of condensation of singularities.

Remark 2. The general scheme of applications of Theorem 1 is as follows. First of all, we must check that a given partial mapping Φ has the Baire property. Obviously, Φ has this property if it is a Borel mapping or, more generally, if it is a measurable mapping with respect to the σ-algebra generated by some family of analytic sets (such situations are typical in modern analysis). Now, assume that our partial mapping Φ of two variables has the Baire property. Then the second step is to check that the corresponding partial mappings of one variable are defined on sets of first category. This will be valid if E_2 and E_3 are Polish topological vector spaces and if, for almost all elements $x \in E_1$, the corresponding mappings $\Phi(x, .)$ are linear and discontinuous on their domains (notice that if the given space E_3 is a normed vector space, then we need to check the linearity and the unboundedness of the corresponding partial mappings). Finally, we can apply Theorem 1.

Below, we wish to present an application of Theorem 1 in a more concrete situation. Namely, we will be interested in a certain type of a generalized derivative.

Let c_0 denote the separable Banach space consisting of all real-valued sequences converging to zero. Let $\mathbf{R}$ denote the real line and let $[0, 1]$ be the closed unit interval in $\mathbf{R}$. Suppose that a mapping

$$\phi : [0, 1] \to c_0$$

is given. Evidently, we may write $\phi = \{\phi_n : n \in \mathbf{N}\}$, where

$$\phi_n : [0, 1] \to \mathbf{R} \qquad (n \in \mathbf{N}).$$

Let us assume that the mapping ϕ satisfies the following condition: for each point $x \in [0, 1]$ and for each index $n \in \mathbf{N}$, the value $\phi_n(x)$ is not equal to zero. Moreover, let us assume (without loss of generality) that

$$0 \ < \ \phi_n(0) \ \leq \ 1, \qquad 0 \ > \ \phi_n(1) \ \geq \ -1$$

for all natural numbers n. If f is a real-valued function defined on the segment $[0, 1]$ and a point x belongs to this segment, then the real number

$$\lim_{n \to +\infty} \frac{f(x + \phi_n(x)) - f(x)}{\phi_n(x)}$$

is called the ϕ-derivative of f at x (if this limit exists, of course). In our further considerations, we denote the limit mentioned above by the symbol $f'_\phi(x)$.

Let us envisage the special case when

$$E_1 = [0, 1], \quad E_2 = C[0, 1], \quad E_3 = \mathbf{R}.$$

In this case we introduce a partial mapping Φ acting from the product space $E_1 \times E_2$ into the space E_3 and defined by the formula

$$\Phi(x, f) = f'_\phi(x).$$

Suppose that the original function ϕ has the Baire property. We assert that, in such a case, the partial mapping Φ has the Baire property, too. Indeed, it suffices to observe that, for every natural number n, the two mappings

$$(x, f) \to f(x + \phi_n(x)) - f(x), \qquad (x, f) \to \phi_n(x)$$

have the Baire property. For the second mapping, this is obvious since the function ϕ_n has the Baire property. Further, the mapping $(x, f) \to f(x)$ is continuous and the mapping $(x, f) \to f(x + \phi_n(x))$ can be represented as the following superposition:

$$(x, f) \to (x, \phi_n(x), f) \to (x + \phi_n(x), f) \to f(x + \phi_n(x)).$$

In this superposition the first mapping has the Baire property and the two other mappings are continuous. Therefore, we conclude that the superposition also has the Baire property. Let us notice, by the way, that the same result can be established in a different manner. Namely, if in the function of two variables

$$(x, f) \to f(x + \phi_n(x)) - f(x)$$

we fix a point x, then we obtain a continuous function of one variable, and if in the same function of two variables we fix a second variable f, then we obtain a function of one variable having the Baire property. So we see that, for our function of two variables, the conditions very similar to Carathéodory's classical conditions (i.e., the measurability with respect to one variable and the continuity with respect to another variable) are fulfilled. From this fact it immediately follows that our function of two variables has the Baire property (in this connection, see also [176] where a general problem concerning the measurability of functions of two or more variables is investigated in detail).

Taking the preceding remarks into account, we conclude that the partial mapping $(x, f) \to f'_\phi(x)$ has the Baire property. Moreover, it is easy to see that if a point x is fixed, then this partial mapping yields a linear discontinuous function of one variable f. Consequently, we can apply Theorem 1 and formulate the following statement.

Theorem 2. *If a mapping $\phi : [0,1] \to c_0$ has the Baire property, then almost each function from the Banach space $C[0,1]$ does not possess a ϕ-derivative almost everywhere on the segment $[0,1]$.*

Actually, the proof of Theorem 2 is already obtained by the argument presented above.

Remark 3. We want to point out that the standard operations used in classical mathematical analysis are, as a rule, of projective type, i.e., these operations are described completely by some projective sets lying in certain Polish topological spaces. In many natural situations, it can happen that the graph of a partial mapping Φ from Theorem 1 is a projective subset of the corresponding Polish product space. Then, according to important results of Solovay and Martin, we must appeal to some additional set-theoretical axioms for the validity of an appropriate version of Theorem 1. For example, suppose that Φ satisfies only condition (2) of Theorem 1, the graph of Φ lies in a Polish product space $E_1 \times E_2 \times E_3$, and this graph is a continuous image of the complement of an analytic subset of a Polish topological space. Then if we wish to preserve the assertion of Theorem 1 for Φ, we need the existence of a two-valued measurable cardinal or Martin's Axiom with the negation of the Continuum Hypothesis. Analogously, if the graph of our partial mapping Φ is a projective subset of a Polish product space, belonging to a higher projective class, then we need the Axiom of Projective Determinacy (**PD**) or a similar set-theoretical axiom (for more details, see [18], [97], [105]). Actually, suppose that we work in the following theory:

$$\mathbf{ZF} \ \& \ \mathbf{DC} \ \& \ (\text{each subset of } \mathbf{R} \text{ has the Baire property}).$$

Then the assertion of Theorem 1 will be true for all Polish topological spaces E_1, E_2, E_3 and for all partial mappings Φ acting from $E_1 \times E_2$ into E_3 and satisfying condition (2) of this theorem. See, e.g., [113] where the theory indicated above is applied to some questions connected with the existence of generalized derivatives of various types. In particular, it is established in [113] that if we work in the above-mentioned theory, then almost each function from the space $C[0,1]$ does not possess a generalized derivative almost everywhere on the segment $[0,1]$. Obviously, such an approach can also be applied to special types of generalized derivatives, for instance, to the so-called path derivatives (for the definition and basic properties of path derivatives, see, e.g., [36]). In addition, let us stress that the direct analogue of the classical Banach–Mazurkiewicz theorem (which was considered in Chapter 0) cannot be established for all generalized derivatives, since there is (in **ZF** & **DC** theory) a certain notion of a generalized derivative having the property that, for any continuous function $f : [0, 1] \to \mathbf{R}$, there exists at least one point x from the segment $[0, 1]$, such that f is differentiable at x in the sense of this generalized derivative (cf. [113]).

Further, the following natural question arises: does there exist a reasonable analogue of the above result in terms of measure theory?

In other words, does there exist a Borel diffused probability measure μ on the space $C[0, 1]$ such that, for any generalized derivative introduced within **ZF** & **DC** theory, almost all (with respect to μ) functions from $C[0, 1]$ are not differentiable, in the sense of this derivative, at almost all (with respect to λ) points of $[0, 1]$?

At the present time, this question remains open.

Here we give a construction of the classical Wiener measure μ_w on $C[0, 1]$ and demonstrate that, for the derivative in the usual sense, μ_w yields a positive answer to this question. We would like to recall that historically the Wiener measure appeared as a certain mathematical model of the Brownian motion (for an interesting survey of this phenomenon, see, e.g., [28] and, especially, [162]).

Notice that the construction of Wiener's measure is not easy and needs a number of auxiliary facts and statements. By the way, these facts and statements turn out to be very useful for the general theory of stochastic processes.

To begin, we first of all wish to recall some simple notions from probability theory and Kolmogorov's celebrated theorem on mutually consistent finite-dimensional probability distributions.

Let E be a ground (base) set, let $\mathcal{S}$ be a σ-algebra of subsets of E, and let μ be a probability measure on $\mathcal{S}$. So we are dealing with the basic probability space $(E, \mathcal{S}, \mu)$. In our further constructions we assume, as a rule, that μ is a complete measure. This does not restrict the generality of our considerations, because we can always replace μ by its completion.

Let f be a partial function acting from E into $\mathbf{R}$. We say that f is a random variable if f is measurable with respect to the σ-algebra $\mathcal{S}$ (i.e., for any open set $U \subset \mathbf{R}$, the pre-image $f^{-1}(U)$ belongs to $\mathcal{S}$) and the equality $\mu(E \setminus \mathrm{dom}(f)) = 0$ holds true.

For any random variable f, we may define the Borel probability measure

μ_f on $\mathbf{R}$, putting

$$\mu_f(X) = \mu(f^{-1}(X)) \qquad (X \in B(\mathbf{R})).$$

The measure μ_f is usually called the probability distribution of a random variable f. Actually, the measure μ_f is defined in such a way that it becomes the homomorphic image of the measure μ under the homomorphism f, so one may write $\mu_f = \mu \circ f^{-1}$.

Obviously, μ_f is uniquely determined by the function $F_f : \mathbf{R} \to [0, 1]$ such that

$$F_f(x) = \mu(\{e \in E \ : \ f(e) < x\}) \qquad (x \in \mathbf{R}).$$

This function is also called the distribution of f. It is increasing and satisfies the following three relations:

(a) $\lim_{t\to-\infty} F_f(t) = 0$;

(b) $\lim_{t\to+\infty} F_f(t) = 1$;

(c) $(\forall x \in \mathbf{R})(\lim_{t\to x-} F_f(t) = F_f(x))$, i.e., the function F_f is continuous from the left.

Let $(E, \mathcal{S}, \mu)$ be again a probability space and let $f : E \to \mathbf{R}$ be a random variable.

We recall that $\int_E f(e)d\mu(e)$ denotes the (mathematical) expectation of f, of course, under the assumption that this integral exists. We also recall the simple formula

$$\int_E f(e)d\mu(e) = \int_{\mathbf{R}} xdF_f(x).$$

More generally, for any Borel function $\phi : \mathbf{R} \to \mathbf{R}$, we have the equality

$$\int_E \phi(f(e))d\mu(e) = \int_{\mathbf{R}} \phi(x)dF_f(x)$$

under the assumption that the corresponding integrals exist.

In many cases, it may happen that the distribution μ_f of a random variable f can be defined with the aid of its density. In this connection, we recall that a Lebesgue measurable function

$$p_f \ : \ \mathbf{R} \to [0, +\infty[$$

is a density of μ_f (of F_f) if, for each Borel set $X \subset \mathbf{R}$, one has

$$\mu_f(X) = \int_X p_f(x)dx.$$

This means that the measure μ_f is absolutely continuous with respect to the Lebesgue measure λ on $\mathbf{R}$. Evidently, any two densities of μ_f are equivalent with respect to λ. In addition, if p_f exists, then one can write

$$\int_E f(e)d\mu(e) = \int_{\mathbf{R}} xp_f(x)dx$$

and, more generally,

$$\int_E \phi(f(e))d\mu(e) = \int_{\mathbf{R}} \phi(x)p_f(x)dx$$

for every Borel function $\phi : \mathbf{R} \to \mathbf{R}$ such that the corresponding integrals exist.

The most standard example of a probability distribution is the normal (or Gaussian) distribution. For the real line $\mathbf{R}$, the density of the so-called centered normal distribution is given by the formula

$$p_f(x) = (2\pi)^{-1/2}(1/\sigma)\exp(-x^2/2\sigma^2) \qquad (x \in \mathbf{R})$$

where $\sigma > 0$ is a real constant. It can easily be checked in this case that

$$\int_E f^2(e)d\mu(e) = \int_{\mathbf{R}} x^2 p_f(x)dx = \sigma^2.$$

Taking the derivative (with respect to a parameter σ) in the last equality, we obtain

$$\int_E f^4(e)d\mu(e) = \int_{\mathbf{R}} x^4 p_f(x)dx = d\sigma^4,$$

where d is some strictly positive constant whose precise value is not interesting for us at this moment.

We now wish to recall Kolmogorov's theorem on the existence of a probability measure with given finite-dimensional distributions (see, e.g., [28], [60], [196], [212]). This theorem plays the fundamental role in the theory of stochastic processes.

Let T be an arbitrary set of indices. Consider a family $\{R_t : t \in T\}$ where, for each index $t \in T$, the set R_t coincides with the real line $\mathbf{R}$.

Suppose that, for any finite set $\tau = \{t_1, ..., t_n\} \subset T$, a Borel probability measure μ_τ on the product space

$$R_\tau = R_{t_1} \times ... \times R_{t_n}$$

is given in such a way that the entire family $\{\mu_\tau : \tau \in [T]^{<\omega}\}$ of probability measures is consistent, i.e., for any two finite subsets τ and τ' of T such that $\tau \subset \tau'$, we have the equality

$$\mu_\tau = \mu_{\tau'} \circ pr^{-1}_{\tau',\tau},$$

where $\mathrm{pr}_{\tau',\tau} : R_{\tau'} \to R_\tau$ denotes the canonical projection from $R_{\tau'}$ onto R_τ. Further, consider the product space

$$\mathbf{R}^T = \prod_{t \in T} R_t$$

with the σ-algebra $\mathcal{S}$ generated by the family of mappings $\{\mathrm{pr}_t : t \in T\}$ where, for each index $t \in T$, the mapping $\mathrm{pr}_t : \mathbf{R}^T \to R_t$ coincides with the canonical projection from $\mathbf{R}^T$ onto R_t.

In other words, $\mathcal{S}$ is defined as the smallest σ-algebra of subsets of $\mathbf{R}^T$, such that all mappings pr_t $(t \in T)$ are measurable with respect to $\mathcal{S}$. This $\mathcal{S}$ is also frequently called the cylindrical σ-algebra in the space $\mathbf{R}^T$, generated by the family of linear functionals $\{\mathrm{pr}_t : t \in T\}$.

Kolmogorov's extension theorem states that there exists a unique probability measure μ_T defined on the cylindrical σ-algebra $\mathcal{S}$ of $\mathbf{R}^T$ and satisfying the relations

$$\mu_\tau = \mu_T \circ \mathrm{pr}^{-1}_{T,\tau} \qquad (\tau \in [T]^{<\omega}),$$

where, for each finite set $\tau \subset T$, the mapping $\mathrm{pr}_{T,\tau} : \mathbf{R}^T \to R_\tau$ is the canonical projection from $\mathbf{R}^T$ onto R_τ.

The original measures μ_τ are usually called the finite-dimensional distributions of μ_T.

The proof of the Kolmogorov theorem is not very difficult. Indeed, using the consistency conditions, we first define the functional μ_T on the cylindrical algebra (consisting of all finite unions of elementary subsets of $\mathbf{R}^T$) in such a manner that the equalities

$$\mu_\tau = \mu_T \circ \mathrm{pr}^{-1}_{T,\tau}$$

would be fulfilled for all finite sets $\tau \subset T$. Then we have to show that this functional is countably additive on the above-mentioned algebra. This is not hard because all finite-dimensional spaces R_τ are Radon, i.e., for any Borel set $X \subset R_\tau$ and for each $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $\mu_\tau(X \setminus K) < \varepsilon$. Finally, utilizing the classical Carathéodory theorem, we can extend our functional μ_T onto the whole cylindrical σ-algebra $\mathcal{S}$ (for details, see, e.g., [28], [196] or [212]).

Remark 4. There are various generalizations of Kolmogorov's theorem. For example, this theorem may be regarded as a particular case of the statement asserting the existence of a projective limit of a given projective system of Radon probability measures. Furthermore, there are some abstract versions of Kolmogorov's theorem in terms of the so-called compact classes of sets introduced by Marczewski. For more details, see again [28], [196], [212]. It is interesting to notice that, by using some generalized version of Kolmogorov's theorem, the well-known Riesz theorem concerning representations of all continuous linear functionals on the space $C(K)$, where K is a compact topological space, can be deduced.

For our further purposes, we need only the special case of Kolmogorov's theorem when $T = [0,1]$. Let us fix a finite set

$$\tau = \{t_1, ..., t_n\} \subset [0,1] \setminus \{0\}.$$

Clearly, we may suppose that $0 < t_1 < ... < t_n$. Define a Borel probability measure μ_τ on R_τ by the formula

$$\mu_\tau(X) = \int_X p_\tau(x_1, ..., x_n)dx_1...dx_n \qquad (X \in \mathcal{B}(R_\tau)),$$

where the density p_τ satisfies the relation

$$p_\tau(x_1, ..., x_n) = (2\pi)^{-n/2}(t_1(t_2 - t_1)...(t_n - t_{n-1}))^{-1/2}\cdot$$

$$\exp((-1/2)(x_1^2/t_1 + (x_2 - x_1)^2/(t_2 - t_1) + ... + (x_n - x_{n-1})^2/(t_n - t_{n-1})))$$

for all points $(x_1, ..., x_n) \in R_\tau$. If τ is a finite subset of $[0,1]$ whose minimal element coincides with 0, then we put

$$\mu_\tau = \mu_0 \otimes \mu_{\tau \setminus \{0\}},$$

where μ_0 is the Borel probability measure on R_0 concentrated at the origin of R_0 (the so-called Dirac measure). It is not difficult to check the consistency of the family of probability measures

$$\{\mu_\tau \ : \ \tau \text{ is a finite subset of } [0,1]\}$$

(see Exercise 9 of this chapter). Applying Kolmogorov's extension theorem to this family of measures, we get the probability measure μ_w on the product space $\mathbf{R}^{[0,1]}$. We shall demonstrate below that the latter measure canonically induces the required Wiener measure on the classical Banach space $C[0,1] \subset \mathbf{R}^{[0,1]}$ (in this connection, notice that the initial measure μ_w also is called the Wiener measure on the product space $\mathbf{R}^{[0,1]}$).

In order to obtain the main result of this chapter, we need some simple but important notions from the general theory of stochastic processes.

Let $(E, \mathcal{S}, \mu)$ be a space endowed with a probability measure and let T be a set of indices (parameters).

We say that a partial function of two variables $H : E \times T \to \mathbf{R}$ is a stochastic (random) process if all partial functions

$$H(\cdot, t) \ : \ E \to \mathbf{R} \qquad (t \in T)$$

are random variables on the basic probability space $(E, \mathcal{S}, \mu)$.

In this case, for any $e \in E$, the partial function $H(e, \cdot) \ : \ T \to \mathbf{R}$ is called the trajectory of a given process H, corresponding to e.

Suppose that T is equipped with a σ-algebra $\mathcal{S}'$ of its subsets, i.e., the pair $(T, \mathcal{S}')$ turns out to be a measurable space.

We say that a stochastic process H is measurable if H (regarded as a partial function on $E \times T$) is measurable with respect to the product σ-algebra of $\mathcal{S}$ and $\mathcal{S}'$.

Further, suppose that some two stochastic processes H and G are given on $E \times T$.

We say that they are stochastically equivalent if, for each $t \in T$, the random variables $H(\cdot, t)$ and $G(\cdot, t)$ are equivalent (i.e., coincide almost everywhere with respect to μ).

Stochastically equivalent processes have very similar properties and, as a rule, are identified. However, in certain problems of probability theory (e.g., in those where special features of trajectories of a given process play an essential role) such an identification turns out to be useless.

Assume now that a set T of parameters is a topological space.

We say that a stochastic process $H : E \times T \to \mathbf{R}$ is stochastically continuous at a point $t_0 \in T$ if, for each real $\varepsilon > 0$, we have

$$\lim_{t \to t_0} \mu(\{e \in E \ : \ |H(e, t) - H(e, t_0)| > \varepsilon\}) = 0.$$

Furthermore, we say that a process H is stochastically continuous if H is stochastically continuous at all points $t \in T$.

Notice that if H_1 and H_2 are any two stochastically equivalent processes, then H_1 is stochastically continuous if and only if H_2 is stochastically continuous.

Lemma 1. *Let $T = [0, 1]$ be equipped with the usual Euclidean topology and let $H : E \times T \to \mathbf{R}$ be a stochastic process. Then the following two conditions are equivalent:*

(1) H is stochastically continuous;

(2) for any real $\varepsilon > 0$*, we have*

$$\lim_{d \to 0+} \sup_{t \in T, t' \in T, |t-t'|<d} \mu(\{e \in E \ : \ |H(e,t) - H(e,t')| > \varepsilon\}) = 0.$$

Proof. Suppose that condition (1) is fulfilled. Fix $\varepsilon > 0$ and $\delta > 0$. For each $t \in T$, there exists an open neighborhood $V(t)$ of t such that

$$\sup_{t' \in V(t)} \mu(\{e \in E \ : \ |H(e,t') - H(e,t)| > \varepsilon/2\}) < \delta/2.$$

The family $\{V(t) \ : \ t \in T\}$ forms an open covering of $T = [0,1]$. Because $[0,1]$ is compact, there exists a Lebesgue number $d > 0$ for this covering, i.e., d has the property that any subinterval of $[0,1]$ with diameter $2d$ is contained in one of the sets of the covering. Consequently, if

$$t \in T, \quad t' \in T, \quad |t - t'| < d,$$

then $t' \in \ ]t-d, t+d[$ and, for some $r \in T$, we get

$$]t-d, t+d[\ \subset V(r), \quad t \in V(r), \quad t' \in V(r).$$

Thus, for almost all $e \in E$, we may write

$$\{e \ : \ |H(e,t') - H(e,t)| > \varepsilon\} \subset$$

$$\{e \ : \ |H(e,t') - H(e,r)| > \varepsilon/2\} \cup \{e \ : \ |H(e,t) - H(e,r)| > \varepsilon/2\}$$

and, taking into account the definition of $V(r)$, we obtain

$$\mu(\{e \in E \ : \ |H(e,t') - H(e,t)| > \varepsilon\}) < \delta/2 + \delta/2 = \delta.$$

This establishes the implication (1) $\Rightarrow$ (2). Notice now that the converse implication (2) $\Rightarrow$ (1) is trivial, and the lemma has thus been proved.

The simple result presented in Exercise 13 and describing some sufficient condition for stochastic continuity can directly be applied to the Wiener measure μ_w introduced above.

Namely, consider the probability space $(\mathbf{R}^{[0,1]}, \mathcal{S}, \mu_w)$ and the stochastic process

$$W \ : \ \mathbf{R}^{[0,1]} \times [0,1] \to \mathbf{R}$$

canonically associated with μ_w and defined by the formula

$$W(\cdot, t) = \mathrm{pr}_t \qquad (t \in [0,1]).$$

In particular, we see that $\mathrm{dom}(W) = \mathbf{R}^{[0,1]} \times [0,1]$. Choose any two points t_1 and t_2 from $[0,1]$ such that $0 < t_1 < t_2$. According to the definition of μ_w, the two-dimensional distribution of the random vector $(W(\cdot, t_1), W(\cdot, t_2))$ is given by the corresponding density

$$p_{t_1,t_2} \; : \; \mathbf{R}^2 \to \mathbf{R}$$

where, for all $(x_1, x_2) \in \mathbf{R}^2$, we have

$$p_{t_1,t_2}(x_1, x_2) =$$

$$(1/2\pi)(t_1(t_2 - t_1))^{-1/2}\exp((-1/2)(x_1^2/t_1 + (x_2 - x_1)^2/(t_2 - t_1))).$$

Consider the random variable $W(\cdot, t_1) - W(\cdot, t_2)$. It is easy to see that the density $p : \mathbf{R} \to \mathbf{R}$ of this variable is defined by the formula

$$p(x) = (2\pi(t_2 - t_1))^{-1/2}\exp(-x^2/2(t_2 - t_1)) \qquad (x \in \mathbf{R}).$$

Indeed, this immediately follows from the general fact stating that if (f_1, f_2) is a random vector whose density of distribution is $q_{(f_1,f_2)} : \mathbf{R}^2 \to \mathbf{R}$, then the density of distribution of $f_1 - f_2$ is $q_{f_1 - f_2} : \mathbf{R} \to \mathbf{R}$, where

$$q_{f_1-f_2}(x) = \int_{\mathbf{R}} q_{(f_1,f_2)}(x + y, y)dy \qquad (x \in \mathbf{R}).$$

Now, if t and t' are any two points from $[0,1]$, we may write

$$\mu_w(\{e \in \mathbf{R}^{[0,1]} \; : \; |W(e,t) - W(e,t')| > \varepsilon\}) \leq$$

$$(1/\varepsilon^2)\int (W(e,t) - W(e,t'))^2 d\mu_w(e) = |t - t'|/\varepsilon^2.$$

This shows us that the process W is stochastically continuous.

W is usually called the standard Wiener process. Let us remark that W may be regarded as a canonical example of a Gaussian process (for further information about Gaussian processes, see, e.g., [39], [60], [196], and [212]).

Let us return to a general probability space $(E, \mathcal{S}, \mu)$ and assume that T is a set of parameters equipped with some σ-algebra $\mathcal{S}'$ of its subsets. Consider two stochastic processes

$$H \; : \; E \times T \to \mathbf{R}, \qquad G \; : \; E \times T \to \mathbf{R}.$$

We shall say that G is a measurable modification of H if the following two conditions are fulfilled:

(a) H and G are stochastically equivalent;

(b) G is a measurable process, i.e., G regarded as a partial function acting from $E \times T$ into $\mathbf{R}$ is measurable with respect to the product σ-algebra of $\mathcal{S}$ and $\mathcal{S}'$.

In particular, if $(T, \mathcal{S}', \nu)$ is a probability space and G is a measurable modification of H, then we also say that G is a $(\mu \otimes \nu)$-measurable modification of H. But, in various cases, it is more convenient to define a $(\mu \otimes \nu)$-measurable modification of H as a stochastically equivalent process measurable with respect to the completion of the product measure $\mu \otimes \nu$.

Suppose that a set T of parameters is a topological space.

We shall say that a stochastic process $H : E \times T \to \mathbf{R}$ is separable if there are a μ-measure zero set $A \subset E$ and a countable set $Q \subset T$, such that, for any element $e \in E \setminus A$ and for any point $t \in \mathrm{dom}(H(e, \cdot))$, there exists a sequence $\{t_n \ : \ n \in \mathbf{N}\} \subset Q \cap \mathrm{dom}(H(e, \cdot))$ converging to t and having the property

$$\lim_{n \to +\infty} H(e, t_n) = H(e, t).$$

From this definition follows at once that Q is everywhere dense in T, so T is separable (as a topological space).

The above-mentioned set Q is called a set of separability of H.

Lemma 2. *Let T coincide with the unit segment $[0, 1]$ equipped with the standard Lebesgue measure λ, and let $H = \{H(\cdot, t) : t \in T\}$ be a stochastically continuous process. Then there exists a process $G = \{G(\cdot, t) : t \in T\}$ satisfying these four relations:*

(1) H and G are stochastically equivalent;

(2) G is measurable;

(3) G is separable and one of its sets of separability coincides with

$$Q = \{k/2^m \ : \ k \in \mathbf{N}, \ m \in \mathbf{N}, \ k/2^m \leq 1\};$$

(4) there exists a μ-measurable set E' with $\mu(E') = 1$ such that, for any point $t \in Q$, we have $H(\cdot, t)|E' = G(\cdot, t)|E'$.

In particular, G turns out to be a measurable separable modification of the stochastic process H.

Proof. In view of Lemma 1, for each real $\varepsilon > 0$, we can write

$$\lim_{d \to 0+} \sup_{t \in T, t' \in T, |t - t'| < d} \mu(\{e \in E \ : \ |H(e, t) - H(e, t')| > \varepsilon\}) = 0.$$

Consequently, for any integer $n > 0$, there exists a finite family of reals

$$0 = t_0^n < t_1^n < \ \ldots \ < t_{k(n)}^n = 1$$

belonging to Q and satisfying the conditions

(a) the length of each segment $[t_i^n, t_{i+1}^n]$ is less than $1/n$;

(b) if t and t' belong to some segment $[t_i^n, t_{i+1}^n]$, then

$$\mu(\{e \in E \ : \ |H(e,t) - H(e,t')| > 1/n\}) < 1/2^n.$$

Moreover, we may choose the above-mentioned families

$$Q_n = \{t_i^n \ : \ i = 0, 1, ..., k(n)\}$$

in such a manner that the following conditions would be fulfilled, too:

(c) for any $n \in \mathbf{N} \setminus \{0\}$, the set Q_n is contained in the set Q_{n+1};

(d) $Q = \cup\{Q_n \ : \ n \in \mathbf{N}, \ n > 0\}$.

Now, let us put $E' = \cap\{\mathrm{dom}(H(\cdot, t)) \ : \ t \in Q\}$. Obviously, we have the equality $\mu(E') = 1$. Further, for each integer $n > 0$, define a function

$$G_n \ : \ E' \times [0,1] \to \mathbf{R}$$

by the relations

$$G_n(e,t) = H(e, t_i^n) \qquad (t \in [t_i^n, t_{i+1}^n[),$$

$$G_n(e,1) = H(e,1).$$

Evidently, the partial function G_n is measurable with respect to the product σ-algebra of $\mathcal{S}$ and $\mathcal{B}([0,1])$. Furthermore, the series

$$\sum_{n>0} \mu(\{e \in E' \cap \mathrm{dom}(H(\cdot,t)) \ : \ |H(e,t) - G_n(e,t)| > 1/n\})$$

is convergent for any point $t \in [0,1]$. Hence, for each $t \in [0,1]$, we get

$$\lim_{n \to +\infty} G_n(\cdot, t) = H(\cdot, t)$$

almost everywhere in E (with respect to μ, of course). Let us put

$$G(e,t) = \mathrm{limsup}_{n \to +\infty} G_n(e,t)$$

for all those pairs $(e,t) \in E' \times T$ for which the above-mentioned limsup does exist. In this way, we obtain a partial mapping $G : E \times T \to \mathbf{R}$.

The definition of G implies at once that G is a measurable stochastic process stochastically equivalent to H and, for any point $t \in Q$, we have

$$G(\cdot, t)|E' = H(\cdot, t)|E'.$$

Let now t be an arbitrary point from $[0,1] \setminus Q$. Then there exists an increasing sequence $\{t^n_{i(n)} \ : \ n \in \mathbf{N}, \ n > 0\} \subset [0,1]$ such that

$$t^n_{i(n)} \in Q_n, \quad t \in [t^n_{i(n)}, t^n_{i(n)+1}[, \quad \lim_{n \to +\infty} t^n_{i(n)} = t.$$

In virtue of the definition of G, we easily obtain

$$G(e,t) \in \mathrm{cl}(\{G(e, t^n_{i(n)}) \ : \ n \in \mathbf{N} \setminus \{0\}\})$$

for any point $e \in E' \cap \mathrm{dom}(G(\cdot, t))$. This completes the proof of the lemma.

Remark 5. The process G of Lemma 2 is usually called a separable modification of the original process H. Notice that the existence of a separable modification of a given process can be established in a much more general situation than in that described by Lemma 2. For our further purposes, this lemma is completely sufficient. More deep results may be found in [196] and [212]. It is interesting to mention here that the most general theorem concerning the existence of a separable modification of a stochastic process essentially relies on the notion of a von Neumann topology (multiplicative lifting). For details, see, e.g., [212] where such an approach is developed.

Lemma 3. *Let $(\alpha_n)_{n \in \mathbf{N}}$ and $(\beta_n)_{n \in \mathbf{N}}$ be two sequences of strictly positive real numbers, such that*

$$\sum_{n \in \mathbf{N}} \alpha_n < +\infty, \qquad \sum_{n \in \mathbf{N}} \beta_n < +\infty,$$

and let $\{f_n : n \in \mathbf{N}\}$ be a sequence of random variables on $(E, \mathcal{S}, \mu)$ satisfying the relations

$$\mu(\{e \in E \ : \ |f_n(e)| > \alpha_n\}) < \beta_n \qquad (n \in \mathbf{N}).$$

Then there exists a μ-measure zero set $A \subset E$ such that, for any point $e \in E \setminus A$, the series $\sum_{n \in \mathbf{N}} |f_n(e)|$ is convergent.

Proof. For each natural number n, let us denote

$$A_n = \{e \in E \ : \ |f_n(e)| > \alpha_n\}.$$

Then, according to our assumption,

$$\mu(A_n) < \beta_n \qquad (n \in \mathbf{N}).$$

Let us put

$$A = \cap_{n\in\mathbf{N}}(\cup_{m\in\mathbf{N},m>n}A_m).$$

Then we obviously have $\mu(A) = 0$. Take any point e from $E \setminus A$. There exists a natural number k for which $e \notin \cup_{m\in\mathbf{N},m>k}A_m$. This means that, for every integer $m > k$, the inequality $|f_m(e)| \leq \alpha_m$ is fulfilled. Hence the series $\sum_{n\in\mathbf{N}} |f_n(e)|$ is convergent, and the proof is completed.

Lemma 4. *Let H be a stochastic process such that* $\mathrm{dom}(H) = E \times [0,1]$ *and*

$$\int_E |H(\cdot, t + r) - H(\cdot, t)|^4 d\mu \leq d \cdot r^2$$

for all $t \in [0,1]$ and $t+r \in [0,1]$, where $d > 0$ is some constant. Then there exists a stochastic process G satisfying these five relations:

(1) G and H are stochastically equivalent;

(2) G is measurable;

(3) G is separable with the set of separability

$$Q = \{k/2^m \ : \ k \in \mathbf{N}, \ m \in \mathbf{N}, \ k/2^m \leq 1\};$$

(4) for any point $t \in Q$, we have $H(\cdot, t) = G(\cdot, t)$;

(5) almost all (with respect to μ) trajectories of G are continuous real-valued functions defined on the whole segment $[0,1]$.

Proof. First of all, we may write

$$\mu(\{e \in E \ : \ |H(e, t + r) - H(e, t)| > |r|^{1/5}\}) \leq (|r|^{-4/5})dr^2 = d|r|^{6/5},$$

for any $t \in [0,1]$ and $t + r \in [0,1]$. This immediately implies that H is stochastically continuous. Applying Lemma 2, we can find a process

$$G \ : \ E \times [0,1] \to \mathbf{R}$$

satisfying relations (1)–(4). Indeed, relations (1)–(3) are fulfilled by virtue of Lemma 2, and relation (4) is valid because of $\mathrm{dom}(H) = E \times [0,1]$. Let us denote

$$\Phi_m = \sup_{0\leq k<2^m} |G(\cdot, (k + 1)/2^m) - G(\cdot, k/2^m)|,$$

where k and m are assumed to be natural numbers. Obviously, Φ_m is a random variable. Furthermore, we have

$$\mu(\{e \in E \ : \ \Phi_m(e) > 2^{-m/5}\}) \leq$$

$$\sum_{0\le k<2^m} \mu(\{e \in E \ : \ |G(e,(k+1)/2^m) - G(e,k/2^m)| > 2^{-m/5}\})$$
$$\le 2^m d 2^{-6m/5} = d2^{-m/5}.$$

In view of Lemma 3, the series $\sum_{m\in\mathbf{N}} \Phi_m$ is convergent almost everywhere in E, i.e., there exists a μ-measure zero set A such that

$$\sum_{m\in\mathbf{N}} \Phi_m(e) < +\infty$$

for all elements $e \in E \setminus A$. Now, we fix $n \in \mathbf{N}$ and readily observe that if $t \in [0,1]$, $t' \in [0,1]$ and $|t - t'| < 2^{-n}$, then, for some integer $k \ge 0$, the rational number $k/2^n$ is less than or equal to 1 and

$$|t - k/2^n| < 1/2^n, \quad |t' - k/2^n| < 1/2^n.$$

Evidently,

$$|G(\cdot,t) - G(\cdot,t')| \le |G(\cdot,t) - G(\cdot,k/2^n)| + |G(\cdot,t') - G(\cdot,k/2^n)|.$$

If, in addition, $t \in Q$ and $t' \in Q$, then it can directly be checked that

$$|G(\cdot,t) - G(\cdot,k/2^n)| \le \sum_{m\in\mathbf{N},\ m>n} \Phi_m,$$

$$|G(\cdot,t') - G(\cdot,k/2^n)| \le \sum_{m\in\mathbf{N},\ m>n} \Phi_m,$$

which yields the relation

$$|G(\cdot,t) - G(\cdot,t')| \le 2(\sum_{m\in\mathbf{N},\ m>n} \Phi_m).$$

Utilizing the separability of G, we infer that there exists a μ-measure zero set B having the following property: if e is an arbitrary element from $E\setminus(A\cup B)$ and t and t' are any two points such that

$$t \in \mathrm{dom}(G(e,\cdot)), \quad t' \in \mathrm{dom}(G(e,\cdot)), \quad |t - t'| < 1/2^n,$$

then

$$|G(e,t) - G(e,t')| \le 2(\sum_{m\in\mathbf{N},\ m>n} \Phi_m(e)).$$

But we know that, for $e \in E\setminus(A\cup B)$, the series $\sum_{m\in\mathbf{N}} \Phi_m(e)$ is convergent. Thus, we conclude that the trajectory $G(e,\cdot)$ is uniformly continuous. This immediately implies that $G(e, \cdot)$ is a restriction of a continuous real-valued function defined on $[0, 1]$. So we may extend G to a new process in such a way that all trajectories of this process, corresponding to the elements from $E \setminus (A \cup B)$, turn out to be continuous on $[0, 1]$. It can easily be seen that the new process (denoted by the same symbol G) is separable and measurable as well. Indeed, the separability of G holds trivially and the measurability of G follows from the fact that G is measurable with respect to $e \in E$ and is continuous with respect to $t \in [0, 1]$. Lemma 4 has thus been proved.

We now are ready to establish the following result.

Theorem 3. *The Wiener measure μ_w induces a Borel probability measure μ on the space $C[0, 1]$, with properties analogous to the corresponding properties of μ_w.*

Proof. Indeed, let us again start with the probability measure space $(\mathbf{R}^{[0,1]}, \mathcal{S}, \mu_w)$ and with the standard Wiener process $W = (\mathrm{pr}_t)_{t \in [0,1]}$ for this space. In view of the preceding lemma, there exists a process G for the same space, such that

(1) W and G are stochastically equivalent;

(2) G is measurable;

(3) G is separable with the set of separability

$$Q = \{k/2^m \ : \ k \in \mathbf{N}, \ m \in \mathbf{N}, \ k/2^m \leq 1\};$$

(4) for any point $t \in Q$, we have $W(\cdot, t) = G(\cdot, t)$;

(5) almost all trajectories of G are continuous real-valued functions on $[0, 1]$.

Let E' denote the set of all those elements $e \in E = \mathbf{R}^{[0,1]}$ for which the trajectory $G(e, \cdot)$ is continuous on $[0, 1]$. Obviously, $\mu_w(E') = 1$. Define a mapping $\phi : E' \to C[0, 1]$ by the formula

$$\phi(e) = G(e, \cdot) \qquad (e \in E').$$

Observe that ϕ is measurable with respect to μ_w (this fact easily follows, e.g., from the result of Exercise 6). So we can put $\mu = \mu_w \circ \phi^{-1}$. Because μ is a homomorphic image of μ_w, we have

$$\mu(X) = \mu_w(\{e \in E \ : \ G(e, \cdot) \in X\})$$

for each Borel subset X of $C[0, 1]$. In particular, if $a > 0$ and t and t' are any two points of $[0, 1]$, then

$$\mu(\{f \in C[0, 1] \ : \ |f(t) - f(t')| < a\}) = \mu_w(\{e \in E \ : \ |G(e, t) - G(e, t')| < a\})$$

$$= \mu_w(\{e \in E \ : \ |W(e,t) - W(e,t')| < a\}).$$

This finishes the proof of Theorem 3.

In a certain sense, we may identify μ and μ_w. So it will be convenient to preserve the same notation μ_w for the obtained measure μ. In other words, we consider μ_w as a Borel probability measure on the space $C[0,1]$.

At last, we are able to return to the question of the differentiability of continuous real-valued functions on $[0,1]$ (from the point of view of μ_w). Namely, the following statement is true.

Theorem 4. *Almost all (with respect to μ_w) functions from $C[0,1]$ are nondifferentiable almost everywhere on $[0,1]$ (with respect to λ).*

Proof. Let us introduce the set

$$D = \{(f,t) \in C[0,1] \times [0,1] \ : \ f \text{ is differentiable at } t\}.$$

It can easily be checked that the set D is $(\mu_w \otimes \lambda)$-measurable in the product space $C[0,1] \times [0,1]$. So, taking into account the Fubini theorem, it suffices to show that, for each $t \in [0,1]$, the set

$$D_t = \{f \in C[0,1] \ : \ f \text{ is differentiable at } t\}$$

is of μ_w-measure zero. In order to do this, we first observe that the inclusion

$$D_t \subset \cup_{n \in \mathbf{N}} \{f \in C[0,1] \ : \ \mathrm{limsup}_{|r| \to 0+} |f(t+r) - f(t)|/|r| < n\}$$

is satisfied. Hence, it suffices for us to prove for any natural number n that $\mu_w(D_{t,n}) = 0$, where

$$D_{t,n} = \{f \in C[0,1] \ : \ \mathrm{limsup}_{|r| \to 0+} |f(t+r) - f(t)|/|r| < n\}.$$

Further, one can easily verify that $D_{t,n} \subset \cup_{\delta \in \mathbf{Q},\ \delta > 0} D_{t,n,\delta}$, where

$$D_{t,n,\delta} = \cap_{r \in \mathbf{Q},\ 0 < |r| < \delta} D_{t,n,\delta,r}$$

and

$$D_{t,n,\delta,r} = \{f \in C[0,1] \ : \ 0 < |r| < \delta, \ |f(t+r) - f(t)|/|r| < n\}.$$

Thus, it remains to demonstrate that $\mu_w(D_{t,n,\delta}) = 0$. But, for every r satisfying the inequalities $0 < |r| < \delta$, we may write

$$\mu_w(D_{t,n,\delta,r}) = \mu_w(\{f \in C[0,1] \ : \ |f(t+r) - f(t)| < n|r|\}) \leq$$

$$(2\pi|r|)^{-1/2}\int_{-n|r|}^{n|r|}\exp(-x^2/2|r|)dx =$$

$$(2\pi)^{-1/2}\int_{-n|r|^{1/2}}^{n|r|^{1/2}}\exp(-y^2/2)dy = O(|r|^{1/2}).$$

This immediately implies the desired result, since $|r| > 0$ can be chosen arbitrarily small.

Remark 6. A more general statement obtained by Wiener and Lévy holds true; namely, they proved that almost all (with respect to μ_w) functions from $C[0,1]$ are nowhere differentiable on $[0,1]$. Briefly speaking, almost all trajectories of the modified Wiener process are nowhere differentiable on $[0,1]$. For extensive information concerning the relationships between stochastic processes and Brownian motion, we refer the reader to the fundamental monograph by Lévy [162].

Remark 7. As mentioned earlier, the standard Wiener process is a very particular case of a Gaussian stochastic process. Gaussian processes form a natural class of stochastic processes, which have many interesting properties (see, e.g., [39], [196], [212], [253]) and are important from the point of view of their numerous applications.

EXERCISES

1. Let F be an increasing function acting from $\mathbf{R}$ into $\mathbf{R}$ and satisfying these three relations:

(a) $\lim_{t\to-\infty}F(t) = 0$;

(b) $\lim_{t\to+\infty}F(t) = 1$;

(c) $(\forall x \in \mathbf{R})(\lim_{t\to x-}F(t) = F(x))$.

Show that there exist a probability space $(E, \mathcal{S}, \mu)$ and a random variable $f : E \to \mathbf{R}$ such that $F = F_f$.

2. Recall that a probability measure μ is separable if the topological weight of the metric space canonically associated with μ does not exceed ω (in other words, the above-mentioned metric space is separable).

For instance, the classical Lebesgue measure λ on the unit segment $[0,1]$ is separable.

Verify that this fact is a trivial consequence of the following statement: the completion of any probability measure given on a countably generated σ-algebra of sets is separable.

Show the validity of this statement.

Check that any homomorphic image of a separable measure is separable, too.

Give an example of a topological space T and of a Borel probability measure on T which is not separable (e.g., take as T the commutative compact topological group $\{0,1\}^{\omega_1}$).

Remark 8. In connection with the result of Exercise 2, let us notice that there exist nonseparable measures on the segment $[0,1]$ extending the Lebesgue measure λ on $[0,1]$. Moreover, there are nonseparable extensions of the standard Lebesgue probability measure on the unit circle $\mathbf{S}_1$ which are invariant under the group of all rotations of this circle about its center (for more information, see, e.g., [90], [100], [112], [138], and [203]).

3. Let $(E, \mathcal{S}, \mu)$ be a probability space and let T be a topological space equipped with its Borel σ-algebra $\mathcal{B}(T)$.

Any μ-measurable partial mapping $f : E \to T$ satisfying the condition $\mu(E \setminus \mathrm{dom}(f)) = 0$ is usually called a T-valued random variable on E.

The Borel probability measure μ_f on T defined by the formula

$$\mu_f(X) = \mu(f^{-1}(X)) \qquad (X \in \mathcal{B}(T))$$

is called the distribution of f in T, and one writes $\mu_f = \mu \circ f^{-1}$.

Show that there exist a probability space $(E, \mathcal{S}, \mu)$, a topological space T, and a Borel probability measure ν on T, such that there is no T-valued random variable f on E for which $\mu_f = \nu$.

4. Let $(E, \mathcal{S}, \mu)$ be a probability space and let f be a random variable on this space.

Prove the formula

$$\int_E \phi(f(e))d\mu(e) = \int_{\mathbf{R}} \phi(x)dF_f(x)$$

for every Borel function $\phi : \mathbf{R} \to \mathbf{R}$ such that the corresponding integrals exist.

In particular, deduce that, for every natural number n, the equality

$$\int_E (f(e))^n d\mu(e) = \int_{\mathbf{R}} x^n dF_f(x)$$

holds true (if these integrals exist).

5. Check that the cylindrical σ-algebra $\mathcal{S}$ of the topological product space $\mathbf{R}^T$ coincides with its Borel σ-algebra if and only if $\mathrm{card}(T) \leq \omega$.

6*. Let X be a set and let $\{f_i : i \in I\}$ be a family of real-valued functions defined on X. Recall that this family separates the points of X if, for any two distinct points x and y from X, there exists an index $i \in I$ such that $f_i(x) \neq f_i(y)$.

Let now X be a Polish topological space and let $\{f_i : i \in I\}$ be a countable family of Borel real-valued functions on X, separating the points of X. Denote by $\mathcal{A} = \mathcal{S}(\{f_i : i \in I\})$ the smallest σ-algebra of subsets of X, for which all functions f_i $(i \in I)$ become measurable. Consider a mapping $f : X \to \mathbf{R}^I$ defined by the formula

$$f(x) = (f_i(x))_{i \in I} \qquad (x \in X).$$

Observe that, since $\mathrm{card}(I) \leq \omega$, the space $\mathbf{R}^I$ is isomorphic to one of the spaces $\mathbf{R}^\omega$, $\mathbf{R}^n$ $(n \in \mathbf{N})$.

Verify that

(a) f is injective and Borel;

(b) $\mathcal{A} = \{f^{-1}(Z) \ : \ Z \in \mathcal{B}(\mathbf{R}^I)\}$.

By using the classical theorem from descriptive set theory, stating that the image of a Borel subset of a Polish space under an injective Borel mapping into a Polish space is also Borel (see [105], [149] or Chapter 0 of this book), infer from (a) and (b) the equality $\mathcal{A} = \mathcal{B}(X)$.

In particular, consider the separable Banach space $C[0,1]$ of all continuous real-valued functions on the segment $[0,1]$ and take as I a countable subset of $[0,1]$ everywhere dense in $[0,1]$. For each index $i \in I$, define a mapping $f_i : C[0,1] \to \mathbf{R}$ by the formula

$$f_i(\phi) = \phi(i) \qquad (\phi \in C[0,1]).$$

Conclude from the result presented above that $\mathcal{A} = \mathcal{B}(C[0,1])$.

Give also a direct proof of this equality, without the aid of the mentioned result.

7. Show that, in the formulation of Kolmogorov's theorem, it suffices to assume only the consistency conditions of the form $\mu_\tau = \mu_{\tau'} \circ \mathrm{pr}^{-1}_{\tau',\tau}$, where τ and τ' are any finite subsets of T for which

$$\tau \subset \tau', \qquad \mathrm{card}(\tau' \setminus \tau) = 1.$$

8. By starting with the equality

$$\int_{-\infty}^{+\infty} \exp(-ax^2/2)dx = (2\pi/a)^{1/2} \qquad (a > 0),$$

demonstrate that

$$\int_{-\infty}^{+\infty} \exp((-1/2)((a-x)^2/c + (x-b)^2/d))dx$$

$$= (2\pi cd/(c+d))^{1/2} \cdot \exp((-1/2)((a-b)^2/(c+d))),$$

where a, b, c, d are strictly positive real numbers.

9. By using the results of Exercises 7 and 8, prove the consistency of the family of measures

$$\{\mu_\tau \ : \ \tau \text{ is a finite subset of } [0,1]\},$$

where μ_τ are defined after Remark 4 in this chapter.

10. Put $E = T = [0,1]$ and equip $[0,1]$ with the standard Lebesgue measure λ.

Give an example of a nonmeasurable stochastic process H such that $\mathrm{dom}(H) = E \times T$ and all trajectories $H(e,\cdot)$ $(e \in E)$ and all random variables $H(\cdot,t)$ $(t \in T)$ belong to the first Baire class (see Chapter 2).

11. Suppose that the unit segment $[0,1]$ is equipped with the Lebesgue measure λ.

Give an example of a measurable stochastic process H with

$$\mathrm{dom}(H) = [0,1] \times [0,1],$$

which is stochastically continuous but almost all of its trajectories are discontinuous.

12. Show that Lemma 1 of this chapter holds true in a more general situation, namely, when T is an arbitrary nonempty compact metric space.

13. Let $T = [0,1]$ and let $H : E \times T \to \mathbf{R}$ be a stochastic process. Suppose also that, for some real number $\alpha > 0$, there exists a function

$$\phi \ : \ [0,1] \to \ [0,+\infty[$$

satisfying the following two conditions:

(1) $\lim_{d\to 0+}\phi(d) = 0$;

(2) for all t and t' from $[0,1]$, one has

$$\int_E |H(e,t) - H(e,t')|^\alpha d\mu(e) \le \phi(|t-t'|).$$

Check that the process H is stochastically continuous.

14. Prove that if (f_1, f_2) is a random vector whose density of distribution is $q_{(f_1,f_2)} : \mathbf{R}^2 \to \mathbf{R}$, then the density of distribution of $f_1 - f_2$ is

$$q_{f_1-f_2} : \mathbf{R} \to \mathbf{R},$$

where

$$q_{f_1-f_2}(x) = \int_{\mathbf{R}} q_{(f_1,f_2)}(x+y, y)dy \qquad (x \in \mathbf{R}).$$

15. Let E be an infinite-dimensional separable Hilbert space over $\mathbf{R}$.

Demonstrate that E is isomorphic to the space $L_2[0,1]$ of all Lebesgue measurable square integrable real-valued functions on the unit segment $[0,1]$, equipped with the canonical inner product

$$\langle \phi, \psi \rangle = \int_0^1 \phi(x)\psi(x)dx \qquad (\phi \in L_2[0,1], \ \psi \in L_2[0,1]).$$

As usual, identify all those functions from $L_2[0,1]$ which coincide almost everywhere on $[0,1]$.

For each point $t \in [0,1]$, denote by f_t the characteristic function (i.e., indicator) of the interval $[0,t] \subset [0,1]$.

Verify that $f_t \in L_2[0,1]$ and that a mapping $f : [0,1] \to L_2[0,1]$ defined by

$$f(t) = f_t \qquad (t \in [0,1])$$

satisfies the following three relations:

(a) f is injective and continuous;

(b) for any four points t_1, t_2, t_3, t_4 from $[0,1]$ such that

$$t_1 < t_2 \leq t_3 < t_4,$$

the line segments $[f(t_1), f(t_2)]$ and $[f(t_3), f(t_4)]$ in $L_2[0,1]$ are perpendicular to each other;

(c) f is nowhere differentiable on $[0,1]$.

Notice that, in view of relation (a), the set $f([0,1])$ is a curve in $L_2[0,1]$ homeomorphic to $[0,1]$ (it is called the Wiener curve).

Remark 9. The last exercise shows that, for an infinite-dimensional separable Hilbert space E, the construction of a continuous nowhere differentiable function acting from $[0,1]$ into E is much easier than the classical construction of a continuous nowhere differentiable function acting from $[0,1]$ into $\mathbf{R}$ (cf. also Theorem 2 of Chapter 0).

Chapter 23
Absolute null subsets of the plane with bad orthogonal projections

In Chapters 10 and 13 we were dealing with several kinds of "small" subsets of the real line $\mathbf{R}$ and, more generally, with analogous subsets of an uncountable Polish topological space E, instead of $\mathbf{R}$. Among "small" sets the so-called absolute null sets (or universal measure zero sets) are of special interest from the point of view of topological measure theory.

In this chapter we will prove that, under Martin's Axiom, there exists an absolute null subset of the Euclidean plane $\mathbf{R}^2$, the orthogonal projections of which on all straight lines in $\mathbf{R}^2$ are absolutely nonmeasurable. Also, it will be shown that a similar but substantially weaker result is valid within the framework of **ZFC** set theory.

Among various set-theoretical operations commonly used in real analysis, the standard projection operation is very important but has a somewhat unpleasant property. Namely, the orthogonal projection of a subset Z of the plane $\mathbf{R}^2$ on a straight line lying in $\mathbf{R}^2$ may be of a much more complicated structure than the structure of Z. There are many examples confirming this circumstance. For instance, if Z is a Borel subset of $\mathbf{R}^2$, then the orthogonal projection of Z on the real line $\mathbf{R}$ is, in general, a non-Borel analytic (Suslin) subset of $\mathbf{R}$, and this fact turned out to be a starting point for the emergence and further development of classical descriptive set theory (see, e.g., [18], [23], [37], [97], [101], [105], [149], [153], [167], [169], [198], [248]).

Also, simple examples show that the projection of a Lebesgue measurable subset of $\mathbf{R}^2$ may be a Lebesgue nonmeasurable set in $\mathbf{R}$. In the present chapter we consider an analogous phenomenon for absolute null subsets of $\mathbf{R}^2$.

Recall that a measure μ defined on some σ-algebra of subsets of $\mathbf{R}$ (respectively, of $\mathbf{R}^2$) is diffused or continuous if μ vanishes on all singletons of $\mathbf{R}$ (respectively, of $\mathbf{R}^2$).

According to the standard definition (see Chapter 13), a subset U of $\mathbf{R}$ (respectively, of $\mathbf{R}^2$) is an absolute null set or universal measure zero set if, for every σ-finite continuous Borel measure μ on $\mathbf{R}$ (respectively, on $\mathbf{R}^2$), the equality $\mu^*(U) = 0$ holds true, where μ^* denotes the outer measure canonically associated with μ.

Equivalently, U is an absolute null set if and only if there exists no nonzero σ-finite continuous Borel measure on U, where U is assumed to be equipped with the induced topology.

The above definition shows that the absolute null subsets of $\mathbf{R}$ (of $\mathbf{R}^2$) are ultimately small with respect to the class $\mathcal{M}(\mathbf{R})$ (class $\mathcal{M}(\mathbf{R}^2)$) of the completions of all nonzero σ-finite continuous Borel measures on $\mathbf{R}$ (on $\mathbf{R}^2$). In particular, these subsets are absolutely measurable with respect to the two above-mentioned classes, i.e., are measurable with respect to any measure belonging to $\mathcal{M}(\mathbf{R})$ ($\mathcal{M}(\mathbf{R}^2)$).

There are several delicate constructions of uncountable absolute null subsets of $\mathbf{R}$ (of $\mathbf{R}^2$). For more details about those constructions, see, e.g., [105], [149], [184], [283], and Exercise 8 of the present chapter.

A subset X of the real line $\mathbf{R}$ is called absolutely nonmeasurable (with respect to the class $\mathcal{M}(\mathbf{R})$) if there exists no measure μ belonging to $\mathcal{M}(\mathbf{R})$ such that $X \in \mathrm{dom}(\mu)$.

This definition shows that absolutely nonmeasurable subsets of $\mathbf{R}$ are extremely bad relative to the class $\mathcal{M}(\mathbf{R})$. It makes sense to notice here that these subsets of $\mathbf{R}$ can be characterized in purely topological terms, as follows.

Recall that a subset B of $\mathbf{R}$ is a Bernstein set if, for each nonempty perfect set $P \subset \mathbf{R}$, the relations $P \cap B \neq \emptyset$ and $P \cap (\mathbf{R} \setminus B) \neq \emptyset$ are satisfied.

Such a set B was first constructed by Bernstein [19] in 1908. In his argument Bernstein essentially relies on an uncountable form of the Axiom of Choice (**AC**) and uses the method of transfinite recursion. Much later, it was recognized that no countable form of **AC** is enough for obtaining B. The importance of Bernstein sets in various topics of real analysis, general topology, measure theory, and the theory of Boolean algebras is well known (see, e.g., [19], [145], [149], [192], and [202]).

Lemma 1. *Let X be a subset of the real line $\mathbf{R}$. The following two assertions are equivalent:*

(1) X is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$;

(2) X is a Bernstein set in $\mathbf{R}$.

The proof of this lemma is not difficult and is left to the reader (see Exercise 9 from Chapter 10).

Some Bernstein sets may possess additional properties of purely algebraic nature.

Example 1. Consider the real line $\mathbf{R}$ as a vector space W over the field $\mathbf{Q}$ of all rational numbers. Any basis of W is usually called a Hamel basis of $\mathbf{R}$, because such a basis was first constructed by Hamel in [87]. There exists a Bernstein set in $\mathbf{R}$ which simultaneously is a Hamel basis of $\mathbf{R}$ (see, for instance, [145] and [192]; cf. also Exercise 10 from Chapter 11).

Example 2. Let G be a group of transformations of $\mathbf{R}$ with $\mathrm{card}(G) \leq \mathbf{c}$, where $\mathbf{c}$ denotes the cardinality of the continuum. There exists a Bernstein set $B \subset \mathbf{R}$ which is almost invariant under the group G. The latter means that for each transformation $g \in G$ the inequality

$$\mathrm{card}(B \triangle g(B)) < \mathbf{c}$$

is valid, where the symbol $\triangle$ denotes the operation of symmetric difference of two sets. In particular, taking the group Γ of all homotheties of $\mathbf{R}$ with center 0, one obtains a Bernstein subset of $\mathbf{R}$ which simultaneously is almost invariant under Γ.

Lemma 1 and the previous example allow us to demonstrate the existence of a small subset of $\mathbf{R}^2$ whose projection on every straight line l in $\mathbf{R}^2$ is absolutely nonmeasurable in l. In what follows, the symbol λ_1 stands for the ordinary one-dimensional Lebesgue measure on l and the symbol λ_2 stands for the ordinary two-dimensional Lebesgue measure on $\mathbf{R}^2$.

Theorem 1. *There exists a set $T \subset \mathbf{R}^2$ with $\lambda_2(T) = 0$ such that the orthogonal projection of T on every straight line l in $\mathbf{R}^2$ is absolutely nonmeasurable in l.*

Proof. Take a Bernstein set $X \subset \mathbf{R}$ which is almost invariant under the group Γ (see Example 2), and in $\mathbf{R}^2$ consider the following set:

$$T = (X \times \{0\}) \cup (\{0\} \times X).$$

This T is contained in the union of the two lines $\mathbf{R} \times \{0\}$ and $\{0\} \times \mathbf{R}$, so we trivially have $\lambda_2(T) = 0$. Now, let l be a straight line in $\mathbf{R}^2$ and let θ denote the angle between l and $\mathbf{R} \times \{0\}$. We may assume, without loss of generality, that l passes through the origin $(0, 0)$ and that $0 < \theta < \pi/2$. It is not difficult to verify that the orthogonal projection of T on l is congruent to the set

$$T^* = \cos(\theta)X \cup \sin(\theta)X \subset \mathbf{R}.$$

By virtue of the definition of X, we have the inequalities

$$\mathrm{card}((\cos(\theta)X) \triangle X) < \mathbf{c}, \qquad \mathrm{card}((\sin(\theta)X) \triangle X) < \mathbf{c},$$

whence it follows that $\mathrm{card}(T^* \triangle X) < \mathbf{c}$. Remembering that X is a Bernstein subset of $\mathbf{R}$, we readily conclude that T^* is also a Bernstein set in $\mathbf{R}$, which completes the proof of Theorem 1.

Example 3. Let H be a Hamel basis of $\mathbf{R}$ which simultaneously is a Bernstein set in $\mathbf{R}$ (see Example 1). We may assume, without loss of generality, that $H = \{h_\xi : \xi < \alpha\}$, where α denotes the least ordinal number of cardinality $\mathbf{c}$. According to the definition of H, any nonzero element $x \in \mathbf{R}$ admits a unique representation

$$x = q_1 h_{\xi_1} + q_2 h_{\xi_2} + ... + q_n h_{\xi_n},$$

where $n = n(x) \neq 0$ and $q_1 = q_1(x)$, $q_2 = q_2(x)$,..., $q_n = q_n(x)$ are some nonzero rational numbers, and $(\xi_1, \xi_2, ..., \xi_n)$ is a strictly increasing sequence of ordinals, all of which are strictly less than α. Further, let us put

$$K' = \{x \in \mathbf{R} : q_n(x) > 0\}.$$

Obviously, we may write

$$K' \cup (-K') = \mathbf{R} \setminus \{0\}, \qquad K' \cap (-K') = \emptyset.$$

Moreover, since $H \subset K'$ and $-H \subset \mathbf{R} \setminus K'$, we conclude that both K' and $-K'$ are Bernstein sets in $\mathbf{R}$. Now, denoting

$$K = (K' \times \{0\}) \cup (\{0\} \times (-K')),$$

we infer that $\lambda_2(K) = 0$. At the same time, considering in $\mathbf{R}^2$ the straight line

$$l = \{(x, y) : x - y = 0\},$$

we can easily deduce that the orthogonal projection of K on l coincides with the set $l \setminus \{(0, 0)\}$, so is λ_1-measurable in l. This fact explains why in the proof of Theorem 1 we appealed to the aid of an almost Γ-invariant Bernstein subset of the real line $\mathbf{R}$.

The natural question arises whether it is possible to strengthen Theorem 1 and to establish the existence of an absolute null subset of $\mathbf{R}^2$ (with respect to the class $\mathcal{M}(\mathbf{R}^2)$), the orthogonal projections of which on all straight lines in $\mathbf{R}^2$ are absolutely nonmeasurable in those lines. In this context, let us immediately remark that such a generalization of Theorem 1 is not realizable within **ZFC** set theory. Indeed, a model of **ZFC** was constructed in which the Continuum Hypothesis (**CH**) fails to be true and in which all uncountable absolute null subsets of $\mathbf{R}^2$ have cardinalities equal to ω_1, where ω_1 denotes the least uncountable cardinal number (for more details, see [157] and [184]). Since the cardinality of an arbitrary Bernstein set is $\mathbf{c}$, in the above-mentioned model of **ZFC** there exists no absolute null subset of $\mathbf{R}^2$ whose orthogonal projection on $\mathbf{R} \times \{0\}$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

Nevertheless, by using Martin's Axiom (**MA**), it becomes possible to substantially strengthen Theorem 1 in terms of absolute null subsets of $\mathbf{R}^2$. For this purpose, we need to recall the notion of a generalized Luzin set in $\mathbf{R}$.

A set $X \subset \mathbf{R}$ is a generalized Luzin subset of $\mathbf{R}$ if $\mathrm{card}(X) = \mathbf{c}$ and, for every first category set $F \subset \mathbf{R}$, the inequality $\mathrm{card}(F \cap X) < \mathbf{c}$ is satisfied.

It is well known that, under Martin's Axiom, there exist generalized Luzin subsets of $\mathbf{R}$ (see Chapter 13). In our further consideration, two generalized Luzin sets in $\mathbf{R}$ with certain specific properties will play a key role.

Lemma 2. *Assuming Martin's Axiom, every generalized Luzin subset of $\mathbf{R}$ is an absolute null set in $\mathbf{R}$.*

Lemma 3. *The product of two absolute null subsets of $\mathbf{R}$ is an absolute null subset of $\mathbf{R}^2$.*

Lemmas 2 and 3 are well known, so we omit their detailed proofs here. Actually, Lemma 3 is contained in Exercise 4 of Chapter 13 and Lemma 2 is readily implied by the following two assertions:

(i) assuming Martin's Axiom, any set $X \subset \mathbf{R}$ with $\mathrm{card}(X) < \mathbf{c}$ is an absolute null subset of $\mathbf{R}$;

(ii) every σ-finite Borel measure on $\mathbf{R}$ is concentrated on a first category subset of $\mathbf{R}$.

In connection with assertion (i), observe that it follows immediately from Theorem 8 of Chapter 13.

In connection with assertion (ii), see [39] or [131] where a more general result than (ii) is discussed for σ-finite Borel measures on metric spaces.

Lemma 4. *Under Martin's Axiom, there exists an absolute null subset Z of $\mathbf{R}^2$ such that, for every straight line l in $\mathbf{R}^2$, the equality $\mathrm{card}(l \cap Z) = \mathbf{c}$ holds true.*

Proof. Denote by α the least ordinal number of cardinality $\mathbf{c}$ and define

$\{l_\xi : \xi < \alpha\}$ = the family of all straight lines in $\mathbf{R}^2$ not parallel to the coordinate axes $\mathbf{R} \times \{0\}$ and $\{0\} \times \mathbf{R}$;

$\{B_\xi : \xi < \alpha\}$ = the family of all those Borel subsets of $\mathbf{R}$ which are of first category in $\mathbf{R}$.

According to the definition of $\{l_\xi : \xi < \alpha\}$, every straight line l in $\mathbf{R}^2$ given by the equation

$$y = ax + b \qquad (a \in \mathbf{R},\ b \in \mathbf{R},\ a \neq 0)$$

belongs to $\{l_\xi : \xi < \alpha\}$, and we may additionally suppose that l occurs in $\{l_\xi : \xi < \alpha\}$ continuum many times.

Now, by using the method of transfinite recursion, construct two injective families

$$\{x_\xi : \xi < \alpha\} \subset \mathbf{R}, \qquad \{y_\xi : \xi < \alpha\} \subset \mathbf{R}.$$

Assume that, for an ordinal number $\xi < \alpha$, the partial families $\{x_\zeta : \zeta < \xi\}$ and $\{y_\zeta : \zeta < \xi\}$ of points in $\mathbf{R}$ have already been determined. Consider the line l_ξ. The canonical equation corresponding to this line is of the form

$$y = a_\xi x + b_\xi \qquad (a_\xi \in \mathbf{R},\ b_\xi \in \mathbf{R},\ a_\xi \neq 0).$$

Using Martin's Axiom and keeping in mind the relation $a_\xi \neq 0$, it is not difficult to check that there exists a point $x' \in \mathbf{R}$ satisfying the following two conditions:

$$x' \notin (\cup\{B_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\};$$

$$a_\xi x' + b_\xi \notin (\cup\{B_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\}.$$

We then put $x_\xi = x'$ and $y_\xi = a_\xi x' + b_\xi$.

Proceeding in this manner, we obtain the required two injective α-sequences $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$ of points of $\mathbf{R}$. Further, we define

$$X = \{x_\xi : \xi < \alpha\}, \quad Y = \{y_\xi : \xi < \alpha\}.$$

It immediately follows from our construction that both X and Y are generalized Luzin subsets of $\mathbf{R}$.

By virtue of Lemmas 2 and 3, the product set $Z' = X \times Y$ is an absolute null subset of $\mathbf{R}^2$.

Also, it can easily be seen that every line l_ξ $(\xi < \alpha)$ meets Z' in continuum many points.

Finally, let g be a rotation of $\mathbf{R}^2$ about the origin $(0,0)$, whose corresponding angle is θ, where $0 < \theta < \pi/2$, and let

$$Z = Z' \cup g(Z').$$

Then Z is an absolute null subset of $\mathbf{R}^2$, too, and has continuum many common points with every straight line lying in $\mathbf{R}^2$. This completes the proof of the lemma.

As a straightforward consequence of Lemma 4, we obtain that the orthogonal projection of the absolute null set Z on any line l in $\mathbf{R}^2$ coincides with l. In this context, it should be mentioned that, under Martin's Axiom, the existence of an absolute null subset of $\mathbf{R}^2$, the orthogonal projection of which on every line $l \subset \mathbf{R}^2$ coincides with l, was also shown by Zindulka (see [284]).

Now, we can formulate and prove the main result of this chapter.

Theorem 2. *Assuming Martin's Axiom, there exists an absolute null subset T of $\mathbf{R}^2$, the orthogonal projection of which on every straight line $l \subset \mathbf{R}^2$ is an absolutely nonmeasurable subset of l.*

Proof. Let Z be as in Lemma 4. We shall construct a set $T \subset Z$ with the desired properties by using the method of transfinite recursion.

In what follows the symbol $l(z, z')$ will denote the straight line passing through two distinct points z and z' in $\mathbf{R}^2$.

Also, for any point $t \in \mathbf{R}^2$ and for any straight line $l \subset \mathbf{R}^2$, we will denote by the symbol $\mathrm{pr}_l(t)$ the orthogonal projection of t on l.

As before, let α be the least ordinal number of cardinality $\mathbf{c}$.

Let $\{(l_\xi, P_\xi) : \xi < \alpha\}$ be a transfinite α-sequence of pairs such that, for each $\xi < \alpha$, the symbol l_ξ denotes some straight line in $\mathbf{R}^2$ and the symbol P_ξ denotes some nonempty perfect subset of l_ξ.

Let l be an arbitrary straight line in $\mathbf{R}^2$. We may suppose that the family $\{(l_\xi, P_\xi) : \xi < \alpha\}$ satisfies the following two conditions:

(a) the set $\Xi(l) = \{\xi < \alpha : l_\xi = l\}$ is nonempty;

(b) for every nonempty perfect set $P \subset l$, the set $\{\xi \in \Xi(l) : P_\xi = P\}$ is also nonempty.

Let us now define, with the aid of transfinite recursion, two disjoint injective families of points

$$\{t_\xi : \xi < \alpha\} \subset Z, \quad \{t'_\xi : \xi < \alpha\} \subset Z.$$

Suppose that, for an ordinal $\xi < \alpha$, the two partial families

$$\{t_\zeta : \zeta < \xi\} \subset Z, \quad \{t'_\zeta : \zeta < \xi\} \subset Z$$

have already been determined. Take the pair (l_ξ, P_ξ) and introduce the following notation:

$T_\xi = \{t_\zeta : \zeta < \xi\}$;

$T'_\xi = \{t'_\zeta : \zeta < \xi\}$;

$\mathcal{L}_\xi$ = the family of all those straight lines in $\mathbf{R}^2$ which pass through one of the points from $T_\xi \cup T'_\xi$ and, simultaneously, are perpendicular to one of the straight lines from $\{l_\zeta : \zeta \leq \xi\}$;

S_ξ = the set of all points $z \in \mathbf{R}^2$ such that $\mathrm{pr}_{l_\xi}(z) \in P_\xi$.

Keeping in mind the relations

$$\mathrm{card}(\xi) < \mathrm{card}(\alpha) = \mathbf{c}, \qquad \mathrm{card}(T_\xi \cup T'_\xi) < \mathbf{c},$$

we immediately get the inequality $\mathrm{card}(\mathcal{L}_\xi) < \mathbf{c}$. In addition, remembering the property of Z described in the formulation of Lemma 4, we obtain that every straight line in $\mathbf{R}^2$ intersecting P_ξ and perpendicular to l_ξ is entirely contained in the set S_ξ and has continuum many common points with Z.

These circumstances imply the existence of two points

$$t \in S_\xi \cap Z, \qquad t' \in S_\xi \cap Z$$

satisfying the following two conditions:

(c) $\mathrm{pr}_{l_\xi}(t) \neq \mathrm{pr}_{l_\xi}(t')$ and the straight line $l(t, t')$ is not perpendicular to any straight line from the family $\{l_\zeta : \zeta < \xi\}$;

(d) $t \notin \cup\mathcal{L}_\xi$ and $t' \notin \cup\mathcal{L}_\xi$.

We then put $t_\xi = t$ and $t'_\xi = t'$. Proceeding in this manner, we come to the two disjoint injective α-sequences

$$\{t_\xi : \xi < \alpha\} \subset Z, \qquad \{t'_\xi : \xi < \alpha\} \subset Z.$$

Finally, we define

$$T = \{t_\xi : \xi < \alpha\}, \qquad T' = \{t'_\xi : \xi < \alpha\},$$

and claim that T is as required.

Indeed, first of all, T is an absolute null set in $\mathbf{R}^2$, because T is a subset of the absolute null set Z.

Let l be an arbitrary straight line in $\mathbf{R}^2$. There exists an ordinal $\xi < \alpha$ such that $l = l_\xi$. From the transfinite construction described above it follows that

(e) the orthogonal projection $\mathrm{pr}_l(T)$ of T on l has common points with every nonempty perfect subset of l and the orthogonal projection $\mathrm{pr}_l(T')$ of T' on l also has common points with every nonempty perfect subset of l;

(f) $\mathrm{card}(\mathrm{pr}_l(T) \cap \mathrm{pr}_l(T')) < \mathbf{c}$.

Indeed, to show the validity of (e), it suffices to apply (a) and (b).

To show the validity of (f), it suffices to observe that if two ordinal numbers $\zeta < \alpha$ and $\eta < \alpha$ are such that $\max(\zeta, \eta) > \xi$, then the line $l(t_\zeta, t'_\eta)$ cannot be perpendicular to $l = l_\xi$ (see (c) and (d)). Consequently, if a point x belongs to the set $\mathrm{pr}_l(T) \cap \mathrm{pr}_l(T')$, then

$$x = \mathrm{pr}_l(t_\zeta) = \mathrm{pr}_l(t'_\eta),$$

where

$$t_\zeta \in T, \quad t'_\eta \in T', \quad \zeta \le \xi, \quad \eta \le \xi,$$

whence it follows that the cardinality of the set $\mathrm{pr}_l(T) \cap \mathrm{pr}_l(T')$ does not exceed $\mathrm{card}(\xi) + 1$.

The relations (e) and (f) directly imply that both $\mathrm{pr}_l(T)$ and $\mathrm{pr}_l(T')$ are Bernstein subsets of l, so $\mathrm{pr}_l(T)$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(l)$. Theorem 2 has thus been proved.

Remark 1. In the literature, the notion of a strong measure zero set was introduced by Borel many years ago and provides an interesting representative of the so-called small sets (cf. [157], [184], or Chapter 13). Recall that a subset X of $\mathbf{R}$ has strong measure zero if, for every sequence $\{\varepsilon_n : n \in \mathbf{N}\}$ of strictly positive real numbers, there exists a sequence $\{\Delta_n : n \in \mathbf{N}\}$ of open intervals in $\mathbf{R}$ which collectively cover X and

$$\lambda_1(\Delta_n) < \varepsilon_n \qquad (n \in \mathbf{N}).$$

The analogous notion makes sense for the plane $\mathbf{R}^2$ (in the corresponding definition, open intervals should be replaced by open squares and λ_1 should be replaced by λ_2; cf. Exercise 24 from Chapter 13). It is not difficult to show that every strong measure zero set is an absolute null set (see Exercise 9 from Chapter 13). However, in contrast to absolute null sets in $\mathbf{R}$ (in $\mathbf{R}^2$), the existence of uncountable strong measure zero subsets of $\mathbf{R}$ (of $\mathbf{R}^2$) cannot be established within the framework of **ZFC** set theory (see again [157] and[184]). At the same time, under Martin's Axiom any generalized Luzin set in $\mathbf{R}$ (in $\mathbf{R}^2$) has strong measure zero. In fact, for strong measure zero subsets of $\mathbf{R}^2$ no analogue of Theorem 2 is true, because if Z is a strong measure zero subset of $\mathbf{R}^2$, then the orthogonal projection of Z on the coordinate axis $\mathbf{R} \times \{0\}$ is a strong measure zero subset of $\mathbf{R} \times \{0\}$, so trivially is λ_1-measurable.

Remark 2. Recall that a set $Z \subset \mathbf{R}^2$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R}^2)$ if there exists no measure μ belonging to this class such that $Z \in \mathrm{dom}(\mu)$. Actually, the absolutely nonmeasurable sets with respect to $\mathcal{M}(\mathbf{R}^2)$ are identical with the Bernstein subsets of $\mathbf{R}^2$ (this fact is a direct analogue of Lemma 1 and its proof does not differ from the proof of Lemma 1; the same argument works for any uncountable Polish topological space). If Z is an arbitrary subset of $\mathbf{R}^2$ absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R}^2)$ and l is an arbitrary straight line in $\mathbf{R}^2$, then the orthogonal projection of Z on l coincides with the whole of l. Indeed, take any point $t \in l$ and consider the straight line l' perpendicular to l and passing through t. Since l' is a nonempty perfect subset of $\mathbf{R}^2$ and Z is a Bernstein set in $\mathbf{R}^2$, we obviously have $Z \cap l' \neq \emptyset$. Consequently, $t \in \mathrm{pr}_l(Z)$ and so $l = \mathrm{pr}_l(Z)$. In particular, we see that the orthogonal projection of an absolutely nonmeasurable subset of $\mathbf{R}^2$ on any straight line l in $\mathbf{R}^2$ turns out to be absolutely measurable with respect to the class of all measures defined on various σ-algebras of subsets of l.

EXERCISES

1. Let E be a topological space, all singletons in which are Borel subsets of E. We recall that a set $X \subset E$ is absolute null (or universal measure zero) if, for any σ-finite continuous Borel measure μ on E, the equality $\mu^*(X) = 0$ holds true, where μ^* denotes the outer measure canonically associated with μ.

In particular, E itself is an absolute null space if and only if there exists no nonzero σ-finite continuous Borel measure on E.

Check that every nonseparable metric space E contains an uncountable absolute null subset.

2. Assuming the Continuum Hypothesis (**CH**), give an example of a nonseparable Hausdorff hereditarily Lindelöf topological space E such that

(a) $\mathrm{card}(E) = \mathbf{c}$;

(b) there exists no uncountable absolute null subset of E.

For this purpose, consider any Sierpiński set in $\mathbf{R}$ equipped with the topology induced by the density topology $\mathcal{T}_d$ of $\mathbf{R}$.

3. Prove the existence of a Bernstein set $B \subset \mathbf{R}$ having the property described in Example 2 of this chapter.

4*. Assume Martin's Axiom (**MA**) and demonstrate that there exists a set $X \subset \mathbf{R}$ satisfying the following three conditions:

(a) $\mathrm{card}(X) = \mathbf{c}$;

(b) X is an absolute null subset of $\mathbf{R}$;

(c) there is a continuous injection $f : X \to \mathbf{R}$ such that $f(X)$ is a Bernstein subset of $\mathbf{R}$.

Conclude from (c) that $f(X)$ is absolutely nonmeasurable with respect to the class of the completions of all nonzero σ-finite continuous Borel measures on $\mathbf{R}$.

5*. Under Martin's Axiom (**MA**), show that there exists a set $Y \subset \mathbf{R}$ satisfying the following three conditions:

(a) $\mathrm{card}(Y) = \mathbf{c}$;

(b) Y is an absolute null subset of $\mathbf{R}$;

(c) there is a continuous injection $f : Y \to \mathbf{R}$ such that no subset of $f(Y)$ having cardinality $\mathbf{c}$ is absolute null.

For this purpose, apply Exercise 23 from Chapter 13.

6*. Introduce the notation

$$\kappa = \inf\{\mathrm{card}(E) : E \text{ is a subset of } \mathbf{R} \text{ with } \lambda^*(E) > 0\}.$$

Clearly, one has the inequality $\kappa \geq \omega_1$.

Let E be a set of cardinality κ, let $\mathcal{S}$ be a countably generated σ-algebra of subsets of E such that all singletons in E belong to $\mathcal{S}$, and let ν be a nonzero σ-finite continuous measure with $\mathrm{dom}(\nu) = \mathcal{S}$.

Prove that, for every set $X \in \mathcal{S}$ with $\mathrm{card}(X) < \kappa$, the equality $\nu(X) = 0$ is valid.

For this purpose assume, without loss of generality, that

$$E \subset \mathbf{R}, \qquad \lambda^*(E) > 0,$$

and that ν is a nonzero diffused finite measure on $\mathcal{S}$. Then suppose to the contrary that there exists a ν-measurable set $X \subset E$ with $\mathrm{card}(X) < \kappa$, such that $\nu(X) > 0$. Undoubtedly, one can assume that $\nu(X) = 1$. The family of sets

$$\mathcal{S}_X = \{X \cap Y \ : \ Y \in \mathcal{S}\}$$

is a countably generated σ-algebra of subsets of X. Denote by $\{X_n : n < \omega\}$ a countable subfamily of $\mathcal{S}_X$ generating $\mathcal{S}_X$ and separating the points in X. Further, define a mapping

$$\phi \ : \ X \to 2^\omega$$

by the formula

$$\phi(x) = (i_n)_{n<\omega} \qquad (x \in X),$$

where $i_n = 1$ if $x \in X_n$, and $i_n = 0$ if $x \notin X_n$. (The mapping ϕ was introduced by Szpilrajn (Marczewski) in [264] and is usually called the characteristic function of $\{X_n : n < \omega\}$.)

Show by a straightforward verification that

(a) ϕ is an injection;

(b) ϕ is a ν-measurable mapping from X into the Cantor space 2^ω; moreover, the equality

$$\{\phi^{-1}(Z) : Z \in \mathcal{B}(2^\omega)\} = \mathcal{S}_X$$

holds true.

Therefore, it is possible to define a Borel diffused probability measure μ on 2^ω by putting

$$\mu(Z) = \nu(\phi^{-1}(Z)) \qquad (Z \in \mathcal{B}(2^\omega)).$$

Accordingly, one gets

$$\mu^*(\phi(X)) = 1 > 0.$$

Now, identifying 2^ω with $[0,1]$ by a Borel isomorphism, one obtains that $\phi(X)$ is not a universal measure zero subset of $[0,1]$. Hence, in view of Theorem 8 from Chapter 13, there exists a homeomorphic image of $\phi(X)$ lying on $[0,1]$ and having strictly positive outer Lebesgue measure. But the latter fact yields a contradiction with the relation

$$\mathrm{card}(\phi(X)) = \mathrm{card}(X) < \kappa.$$

This contradiction gives the desired result.

7*. Let κ be as in Exercise 6.

Demonstrate that there exists a countably generated σ-algebra of subsets of κ, separating the points in κ and not admitting a nonzero σ-finite diffused measure.

Argue as follows. First of all, take a set $E \subset \mathbf{R}$ such that

$$\mathrm{card}(E) = \kappa, \qquad 0 < \lambda^*(E) < +\infty.$$

Further, equip E with the induced topology and denote by μ the Borel diffused measure on E produced by λ. Evidently, $0 < \mu(E) < +\infty$. Also, fix a countable base $\{V_n : n < \omega\}$ of the topology of E. Let $\{e_\xi : \xi < \kappa\}$ be an injective enumeration of all points of E (where, as usual, κ is identified with the least ordinal number of cardinality κ). In view of the definition of κ, for each ordinal $\alpha < \kappa$ one has

$$\mu^*(\{e_\xi : \xi < \alpha\}) = 0.$$

This implies that there exists an open set $U_\alpha \subset E$ satisfying the relations

$$\{e_\xi : \xi < \alpha\} \subset U_\alpha, \qquad \mu(U_\alpha) \leq (1/2)\mu(E).$$

Take now the product set $\kappa \times E$ and consider its subset

$$G = \cup\{\{\alpha\} \times U_\alpha : \alpha < \kappa\}.$$

In addition to this, for each $n < \omega$, introduce the set

$$A_n = \{\alpha < \kappa : V_n \subset U_\alpha\}$$

and verify that the equality

$$G = \cup\{A_n \times V_n : n < \omega\}$$

is true.

For this purpose, pick any pair $(\alpha, e_\beta) \in G$. Then $e_\beta \in U_\alpha$. Since the family $\{V_n : n < \omega\}$ forms a base of the space E, there exists a natural number m such that $e_\beta \in V_m \subset U_\alpha$. This circumstance implies the relation $(\alpha, e_\beta) \in A_m \times V_m$ and, therefore,

$$(\alpha, e_\beta) \in \cup\{A_n \times V_n : n < \omega\}.$$

Thus, one comes to the inclusion

$$G \subset \cup\{A_n \times V_n : n < \omega\}.$$

In order to show the reverse inclusion, take any pair

$$(\alpha, e_\beta) \in \cup\{A_n \times V_n : n < \omega\}.$$

Then, for some natural number m, one has $(\alpha, e_\beta) \in A_m \times V_m$, whence it follows that

$$\alpha \in A_m, \quad e_\beta \in V_m, \quad V_m \subset U_\alpha, \quad (\alpha, e_\beta) \in \{\alpha\} \times U_\alpha.$$

Consequently,

$$\cup\{A_n \times V_n : n < \omega\} \subset G.$$

In this way, one gets the required equality

$$G = \cup\{A_n \times V_n : n < \omega\}.$$

Now, since $\kappa = \mathrm{card}(E)$ and E is a subset of $\mathbf{R}$, there exist many countably generated σ-algebras of subsets of κ, separating the points of κ. Denote by $\mathcal{S}'$ one of such σ-algebras and let $\mathcal{S}$ be the σ-algebra of subsets of κ, generated by the family $\mathcal{S}' \cup \{A_n : n < \omega\}$. Obviously, $\mathcal{S}$ is also countably generated and separates the points of κ.

It is time to demonstrate that $\mathcal{S}$ does not admit a nonzero σ-finite diffused measure. Suppose otherwise, and let ν be a probability diffused measure with $\mathrm{dom}(\nu) = \mathcal{S}$. Then one may consider the product measure $\nu \otimes \mu$ on $\kappa \times E$. Evidently,

$$(\nu \otimes \mu)(\kappa \times E) = \nu(\kappa)\mu(E) = \mu(E).$$

As shown above, the set

$$G = \cup\{\{\alpha\} \times U_\alpha : \alpha < \kappa\}$$

belongs to the domain of $\nu \otimes \mu$. Now, try to apply the Fubini theorem to G. Fix any $e_\beta \in E$ and consider the horizontal section $G^{-1}(e_\beta)$ of G corresponding to e_β. Observe that, for all $\alpha \in \ ]\beta, \kappa[$, one has

$$e_\beta \in U_\alpha, \quad (\alpha, e_\beta) \in \{\alpha\} \times U_\alpha \subset G, \quad \alpha \in G^{-1}(e_\beta),$$

from which it follows (in view of Exercise 6) that

$$\nu(G^{-1}(e_\beta)) = \nu(\kappa) = 1.$$

In other words, all horizontal sections of G turn out to be of full measure (with respect to ν). Therefore, by the Fubini theorem, the equalities

$$(\nu \otimes \mu)(G) = (\nu \otimes \mu)(\kappa \times E) = \mu(E)$$

must be valid. On the other hand, if $\alpha < \kappa$, then the vertical section $G(\alpha)$ of G, corresponding to α, coincides with the set U_α, and, by the definition of U_α, one has $\mu(U_\alpha) \leq (1/2)\mu(E)$. Taking this fact into account and utilizing Fubini's theorem once more, one comes to the relation

$$(\nu \times \mu)(G) \leq (1/2)\mu(E)\nu(\kappa) = (1/2)\mu(E)$$

and, finally, $\mu(E) \leq (1/2)\mu(E)$, which is impossible because of the inequalities

$$0 < \mu(E) < +\infty.$$

The contradiction obtained gives the required result.

8*. Let κ be as in Exercises 6 and 7.

Work in **ZFC** theory and prove Grzegorek's result [84] stating that there exists an absolute null subset of **R** whose cardinality is equal to κ (so there exists an uncountable absolute null subset of **R**).

Argue step by step as follows.

The argument used in Exercise 7 yields the existence of a countably generated σ-algebra $\mathcal{S}$ of subsets of κ, which separates the points of κ and does not admit a nonzero σ-finite diffused measure.

Let $\{D_n : n < \omega\}$ be a countable family of subsets of κ, generating $\mathcal{S}$ and separating the points of κ. Further, let $\phi : \kappa \to 2^\omega$ be Marczewski's characteristic function of the family $\{D_n : n < \omega\}$. As known, the mapping ϕ is injective and

$$\{\phi^{-1}(Z) : Z \in \mathcal{B}(2^\omega)\} = \mathcal{S}.$$

Consider the set $\phi(\kappa)$ and check that this set is a universal measure zero subspace of the Cantor space 2^ω.

For this purpose, suppose to the contrary that there exists a Borel diffused probability measure on $\phi(\kappa)$. Then it is clear that there exists a Borel diffused probability measure μ on 2^ω such that

$$\mu^*(\phi(\kappa)) = \mu(2^\omega) = 1 > 0.$$

Now, put

$$\nu(\phi^{-1}(Z)) = \mu(Z) \qquad (Z \in \mathcal{B}(2^\omega)).$$

Verify that the functional ν is well-defined and turns out to be a diffused probability measure on the σ-algebra $\mathcal{S}$.

Since such a measure cannot exist on $\mathcal{S}$, conclude that the set $\phi(\kappa)$ must be an absolute null subspace of 2^ω with cardinality equal to κ. The latter fact directly leads to the existence of an absolute null subspace of $\mathbf{R}$ with the same cardinality κ.

Remark 3. Without appealing to any additional set-theoretical assumptions, the delicate result of Exercise 8 shows, in particular, that there exist uncountable absolute null subsets of $\mathbf{R}$. There are also several other clever constructions of uncountable absolute null subsets of $\mathbf{R}$ within **ZFC** theory (see, e.g., [149], [169], [184], [207], [283]). Of course, under **CH** or **MA**, Luzin sets or generalized Luzin sets in $\mathbf{R}$ give us analogous examples, and the constructions of such sets are much easier than that presented in Exercise 8. However, the constructions of Luzin sets and of generalized Luzin sets are necessarily based on certain extra axioms and, therefore, are less preferable.

9. Demonstrate that

(a) the union of a countable family of strong measure zero subsets of $\mathbf{R}^2$ is also a strong measure zero set in $\mathbf{R}^2$;

(b) the class of all strong measure zero subsets of $\mathbf{R}^2$ is preserved under the family of all continuous mappings of $\mathbf{R}^2$ into itself;

(c) if $Z \subset \mathbf{R}^2$ is a strong measure zero set and l is an arbitrary straight line in $\mathbf{R}^2$, then $Z \cap l$ is a strong measure zero subset of l.

Appendix 1
Luzin's theorem on the existence of primitives

It was noted in Chapter 2 of this book that any derivative is a function of first Baire class, hence is Borel and measurable in the Lebesgue sense. This simple fact admits a partial converse assertion. Namely, it will be shown below that, for every Lebesgue measurable function f acting from $\mathbf{R}$ into $\mathbf{R}$, there exists at least one primitive of f, i.e., there exists a continuous function $F : \mathbf{R} \to \mathbf{R}$ such that $F'(x) = f(x)$ for almost all (with respect to the Lebesgue measure λ) points $x \in \mathbf{R}$. This important statement was first established by Luzin (see his fundamental work [168] where an application of the above-mentioned result to the generalized Dirichlet boundary problem is also given).

In this context, it should be indicated that Luzin's method allows one to construct some universal real-valued functions in the sense of certain path derivatives. Those functions may be treated as strange or extraordinary from the analytic viewpoint.

To present Luzin's theorem on primitives (which is remarkable in various respects), let us begin with the following auxiliary proposition.

Lemma 1. *Let $[a, b]$ be a segment of $\mathbf{R}$ and let $g : [a, b] \to \mathbf{R}$ be a function integrable in the Lebesgue sense. Then, for every real $\varepsilon > 0$, there exists a function $G : [a, b] \to \mathbf{R}$ such that*

(1) G is continuous on $[a, b]$;

(2) $G'(x) = g(x)$ for almost all points $x \in [a, b]$;

(3) $G(a) = G(b) = 0$;

(4) $|G(x)| \leq \varepsilon$ for all points $x \in [a, b]$.

Proof. First of all, we introduce a function H defined by the formula

$$H(x) = \int_a^x g(t)dt \qquad (x \in [a, b]).$$

Since H is continuous (even absolutely continuous), we can choose some points

$$a = a_0 < a_1 < ... < a_{n-1} < a_n = b$$

on $\mathbf{R}$ satisfying the relations

$$|H(x) - H(y)| \leq \varepsilon/2$$

for all x and y belonging to the interval $[a_i, a_{i+1}]$ $(i = 0, 1, ..., n-1)$. Let $F_i : [a_i, a_{i+1}] \to \mathbf{R}$ be a continuous monotone function which satisfies the equalities

$$F_i(a_i) = H(a_i), \qquad F_i(a_{i+1}) = H(a_{i+1})$$

and whose derivative vanishes almost everywhere on $[a_i, a_{i+1}]$. The existence of such a function F_i readily follows from the existence of singular increasing (or decreasing) continuous functions on $[a_i, a_{i+1}]$ (see Chapter 4). Now, denote by $F : [a, b] \to \mathbf{R}$ the common extension of all F_i and put

$$G(x) = H(x) - F(x) \qquad (x \in [a, b]).$$

It can easily be verified that the function $G : [a, b] \to \mathbf{R}$ is as required.

Lemma 2. *Let $[a, b]$ be a segment of $\mathbf{R}$, let $g : [a, b] \to \mathbf{R}$ be a Lebesgue integrable function, and let P be a closed subset of $[a, b]$. Then, for any real $\varepsilon > 0$, there exists a function $G : [a, b] \to \mathbf{R}$ satisfying these three relations:*

(1) G is continuous on $[a, b]$ and $G'(x) = g(x)$ for almost all points $x \in [a, b] \setminus P$;

(2) $G(x) = 0$ for all points $x \in P$ and $G'(x) = 0$ at all points $x \in P$ except countably many of them;

(3) $|G(x + h)| \leq \varepsilon|h|$ for any $x \in P$ and $h \in \mathbf{R}$ such that $x + h \in [a, b]$.

Proof. Consider the open set $]a, b[\ \setminus P$ and represent this set in the form

$$]a, b[\ \setminus P = \cup\{]a_k, b_k[\ :\ k \in \mathbf{N}\},$$

where $\{]a_k, b_k[\ :\ k \in \mathbf{N}\}$ is a family of pairwise disjoint open subintervals of $\mathbf{R}$. Further, in each nonempty interval $]a_k, b_k[$ fix a countable family of points

$$... < c_k^{(i-1)} < c_k^{(i)} < c_k^{(i+1)} < ... \qquad (i \in \mathbf{Z})$$

such that

$$\lim_{i \to -\infty} c_k^{(i)} = a_k, \qquad \lim_{i \to +\infty} c_k^{(i)} = b_k,$$

and denote

$$\varepsilon(k, i) = (1/2)\min\left(\frac{\varepsilon(c_k^{(i)} - a_k)}{k + |i| + 1}, \frac{\varepsilon(b_k - c_k^{(i+1)})}{k + |i| + 1}\right) \qquad (i \in \mathbf{Z}).$$

Now, repeatedly applying Lemma 1 to each segment $[c_k^{(i)}, c_k^{(i+1)}]$, where i ranges over $\mathbf{Z}$, we get a continuous function $G_k :]a_k, b_k[\to \mathbf{R}$ satisfying the following conditions:

(a) $G_k'(x) = g(x)$ for almost all points $x \in [a_k, b_k]$;

(b) $G_k(c_k^{(i)}) = 0$ for all indices $i \in \mathbf{Z}$;

(c) $|G_k(x)| \leq \varepsilon(k, i)$ for any point x from $[c_k^{(i)}, c_k^{(i+1)}]$.

Finally, putting $G(x) = G_k(x)$ if x belongs to some $]a_k, b_k[$, and $G(x) = 0$ if x belongs to $P \cup \{a, b\}$, we obtain the function $G : [a, b] \to \mathbf{R}$. It is not difficult to check that G is the required one.

Now, we are able to prove Luzin's classical theorem (see [168], [225]).

Theorem 1. *Let $[a, b]$ be a segment in $\mathbf{R}$ and let $f : [a, b] \to \mathbf{R}$ be a Lebesgue measurable function. Then there exists a continuous function $F : [a, b] \to \mathbf{R}$ such that $F'(x) = f(x)$ for almost all points $x \in [a, b]$.*

Proof. We shall define by recursion a sequence $\{G_n : n \in \mathbf{N}\}$ of continuous functions acting from $[a, b]$ into $\mathbf{R}$ and differentiable almost everywhere on $[a, b]$, and a sequence $\{P_n : n \in \mathbf{N}\}$ of closed subsets of $[a, b]$ satisfying the relations presented below. To formulate these relations, we first denote

$$Q_n = P_0 \cup P_1 \cup \ldots \cup P_n,$$

$$F_n = G_0 + G_1 + \ldots + G_n.$$

After introducing this notation, we wish the following four relations would be valid for all natural numbers $n \geq 1$:

(a) at almost every point $x \in Q_n$ there exists $F_n'(x)$ equal to $f(x)$;

(b) if $x \in Q_{n-1}$, then $G_m(x) = 0$ for each natural number $m \geq n$;

(c) if $x \in Q_{n-1}$ and $x + h \in [a, b]$, then $|G_m(x + h)| \leq h/2^m$ for each natural number $m \geq n$;

(d) $\lambda([a, b] \setminus Q_n) < 1/n$, where λ is the Lebesgue measure on $\mathbf{R}$.

Let us put $P_0 = \emptyset$ and $G_0 = 0$. Suppose that, for a natural number r, the partial families

$$G_0,\ G_1,\ \ldots,\ G_r, \qquad P_0,\ P_1,\ \ldots,\ P_r$$

have already been defined fulfilling (a), (b), (c), and (d). Since f is Lebesgue measurable and F_r is differentiable almost everywhere on $[a, b]$, there exists a closed set $P_{r+1} \subset [0, 1] \setminus Q_r$ such that

$$\lambda([a, b] \setminus (P_{r+1} \cup Q_r)) < 1/(r + 1),$$

the derivative $F'_r(x)$ exists at each point $x \in P_{r+1}$, and, for some constant $M > 0$, we have

$$|F'_r(x)| \leq M, \qquad |f(x)| \leq M \qquad (x \in P_{r+1}).$$

We extend $(f - F'_r)|P_{r+1}$ to a bounded λ-measurable function on $[a, b]$ and preserve the same notation $f - F'_r$ for the extended function. Applying Lemma 2 to the set Q_r and to the function $f - F'_r$, we can define a continuous function $G_{r+1} : [a, b] \to \mathbf{R}$ such that

(i) G_{r+1} is differentiable almost everywhere on $[a, b]$;

(ii) $G'_{r+1}(x) = f(x) - F'_r(x)$ for almost all points $x \in P_{r+1}$;

(iii) $G_{r+1}(x) = 0$ for all points $x \in Q_r$ and $G'_{r+1}(x) = 0$ for almost all points $x \in Q_r$;

(iv) $|G_{r+1}(x + h)| \leq h/2^{r+1}$ for any $x \in Q_r$ and $h \in [a, b] - x$.

Notice that conditions (a), (b), (c), and (d) are satisfied for $n = r + 1$. Finally, we put

$$F = \lim_{n\to+\infty} F_n = \sum_{n\in\mathbf{N}} G_n,$$

$$Q = \cup\{P_n : n \in \mathbf{N}\}.$$

Clearly, the relation $\lambda(Q) = \lambda([a, b]) = b - a$ holds true. Also, in view of the uniform convergence of the series $\sum_{n\in\mathbf{N}} G_n$, the function F is continuous.

Now, take a point $x_0 \in Q$. Then, for all sufficiently large natural numbers n, we have $x_0 \in Q_n$, and we may suppose that there exists $F'_n(x_0)$ equal to $f(x_0)$ (see (a)). Further, we can write

$$\frac{F(x_0 + h) - F(x_0)}{h} = \frac{F_n(x_0 + h) - F_n(x_0)}{h} + \sum_{k=n+1}^{\infty} \frac{G_k(x_0 + h) - G_k(x_0)}{h}.$$

Keeping in mind (b) and (c), we get

$$|(F(x_0 + h) - F(x_0))/h - f(x_0)| \leq |(F_n(x_0 + h) - F_n(x_0))/h - f(x_0)| + \frac{1}{2^n}.$$

In addition, by virtue of (a), we have

$$\lim_{h\to 0, h\neq 0} \frac{F_n(x_0 + h) - F_n(x_0)}{h} = F'_n(x_0) = f(x_0),$$

whence it follows

$$\limsup_{h\to 0, h\neq 0} |(F(x_0 + h) - F(x_0))/h - f(x_0)| \leq \frac{1}{2^n}.$$

Since n can be chosen arbitrarily large, we conclude that

$$F'(x_0) = f(x_0)$$

for almost all $x_0 \in Q$, which ends the proof of Luzin's theorem.

Remark 1. Obviously, Theorem 1 can be formulated in a slightly more general form. Namely, remembering that

$$\mathbf{R}^* = \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$$

denotes the extended real line, suppose that $f : [a, b] \to \mathbf{R}^*$ is a partial Lebesgue measurable function defined at almost all points of the segment $[a, b]$ and finite almost everywhere on $[a, b]$. Then there exists a continuous function $F : [a, b] \to \mathbf{R}$ such that $F'(x) = f(x)$ for almost all points x from $[a, b]$. The proof remains the same.

Remark 2. A substantial extension of Luzin's theorem is obtained by his disciple Bari in the context of convergence of trigonometric series. Also, a deep generalization of Luzin's theorem to the case of all λ_n-measurable functions acting from $\mathbf{R}^n$ into $\mathbf{R}^n$ is given in [188] (here λ_n denotes the standard n-dimensional Lebesgue measure on $\mathbf{R}^n$). A certain analogue of Luzin's theorem in terms of the Baire property is formulated and proved by Landis (see [168]).

Using a method very similar to the proof of Theorem 1, Marcinkiewicz established in [175] the following interesting and somewhat unexpected result.

Theorem 2. *There exists a continuous function $F : \mathbf{R} \to \mathbf{R}$ such that, for any Lebesgue measurable and finite almost everywhere function*

$$f : \mathbf{R} \to \mathbf{R}^*,$$

there is a sequence $\{h_n : n \in \mathbf{N}\}$ of nonzero real numbers which has the property that $\lim_{n\to+\infty} h_n = 0$ and

$$\lim_{n\to+\infty} \frac{F(x + h_n) - F(x)}{h_n} = f(x)$$

for almost all points $x \in \mathbf{R}$.

Remark 3. In connection with Theorem 2, it should be mentioned that the existence of a universal sequence $\{h_n : n \in \mathbf{N}\} \subset \mathbf{R}$ for all Lebesgue measurable almost everywhere finite functions $f : \mathbf{R} \to \mathbf{R}^*$ is impossible.

The natural question arises whether the finiteness of f (at least almost everywhere) is essential in Luzin's theorem. Luzin himself showed that this assumption is indeed necessary. To prove his result, let us first recall some notions concerning Dini's derived numbers (see [194], [225] or Chapters 4 and 5 of the present book).

Let $F : \mathbf{R} \to \mathbf{R}$ be a function and let $x \in \mathbf{R}$. Recall that a real number t is said to be a Dini derived number of F at a point x if there exists a sequence $\{x_n : n \in \mathbf{N}\} \subset \mathbf{R}$ of points distinct from x and tending to x such that

$$\lim_{n\to+\infty} \frac{F(x_n) - F(x)}{x_n - x} = t.$$

We can denote by $F'_D(x)$ the set of all Dini's derived numbers of F at x. Then the relation $t \in F'_D(x)$ means that t is a Dini derived number of F at x. In other words, we have a set-valued mapping

$$F'_D : \mathbf{R} \to \mathcal{P}(\mathbf{R})$$

which may be treated as a Dini mapping associated with F. The properties of this mapping were extensively investigated (cf. [225]).

For our further purposes, it is convenient to introduce the following notation:

$$(D_F(x))^+ = \limsup_{y\to x+} \frac{F(y) - F(x)}{y - x},$$

$$(D_F(x))_+ = \liminf_{y\to x+} \frac{F(y) - F(x)}{y - x},$$

$$(D_F(x))^- = \limsup_{y\to x-} \frac{F(y) - F(x)}{y - x},$$

$$(D_F(x))_- = \liminf_{y\to x-} \frac{F(y) - F(x)}{y - x}.$$

Observe that

$$(D_F(x))^+ = -(D_{-F}(x))_+ \qquad (D_F(x))_+ = -(D_{-F}(x))^+.$$

We need the next auxiliary proposition.

Lemma 3. *For an arbitrary function $F : \mathbf{R} \to \mathbf{R}$, the sets*

$$\{x \in \mathbf{R} : (D_F(x))^+ = -\infty\}, \qquad \{x \in \mathbf{R} : (D_F(x))_+ = +\infty\}$$

are of Lebesgue measure zero.

Proof. The argument is quite similar in both cases. Let us denote

$$T = \{x \in \mathbf{R} : (D_F(x))^+ = -\infty\}$$

and let us show that $\lambda(T) = 0$. Suppose otherwise, i.e., $\lambda^*(T) > 0$.

For each point $x \in T$, there exists a natural number $m > 0$ such that

$$(\forall y)((x < y < x + 1/m) \Rightarrow (F(y) < F(x))).$$

The set of all those $x \in T$ which satisfy the preceding relation will be denoted by T_m. Obviously, we have the equality

$$T = \cup\{T_m : m \in \mathbf{N} \setminus \{0\}\}.$$

Consequently, $\lambda^*(T_n) > 0$ for some natural number $n > 0$. Choose any two points a and b from T_n satisfying the conditions

$$a < b, \qquad b - a < 1/n, \qquad \lambda^*(T_n \cap\ ]a, b[) > 0.$$

Let X be the set of all those points from $T_n \cap\ ]a, b[$ which are density points of a fixed measurable hull of $T_n \cap\ ]a, b[$. According to the Lebesgue theorem on density points (see Chapter 0),

$$\lambda^*(X) = \lambda^*(T_n \cap\ ]a, b[) > 0.$$

Take an arbitrary real $r > 0$. For each point $x \in X$, there exists a strictly decreasing sequence $\{x_k : k \in \mathbf{N}\} \subset X$ tending to x and such that

$$F(x_k) - F(x) < -r(x_k - x) \qquad (k \in \mathbf{N}).$$

The system $\mathcal{V}$ of nondegenerate segments $\{[x, x_k] : x \in X, k \in \mathbf{N}\}$ covers the set X in the sense of Vitali. By virtue of the Vitali theorem (see again Chapter 0), we can find a finite disjoint subsystem

$$\{[a_1, b_1],\ [a_2, b_2],\ \ldots, [a_p, b_p]\}$$

of $\mathcal{V}$ such that

$$[a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_p, b_p] \subset\ ]a, b[,$$

$$\lambda([a_1 b_1] \cup [a_2, b_2] \cup \ldots \cup [a_p, b_p]) > (1/2)\lambda^*(X).$$

Thus, the open interval $]a, b[$ can be represented as a union of the segments $[a_1, b_1],\ [a_2, b_2],\ \ldots, [a_p, b_p]$ and their conjugate open intervals

$$]u_1, v_1[,\ ]u_2, v_2[,\ \ldots,\ ]u_{p+1}, v_{p+1}[\ .$$

Now, we can write

$$F(b_i) - F(a_i) < -r(b_i - a_i) \qquad (i = 1, 2, ..., p),$$

$$F(v_i) - F(u_i) < 0 \qquad (i = 1, 2, ..., p+1).$$

It immediately follows from these inequalities that

$$F(b) - F(a) = \sum_{1 \le i \le p} (F(b_i) - F(a_i)) + \sum_{1 \le i \le p+1} (F(v_i) - F(u_i)) <$$

$$-r\lambda([a_1, b_1] \cup [a_2, b_2] \cup ... \cup [a_p, b_p]) < -(r/2)\lambda^*(X).$$

Since $r > 0$ can be arbitrarily large, we finally obtain

$$F(b) - F(a) = -\infty,$$

which is impossible. The obtained contradiction finishes the proof of the lemma.

The following result is due to Luzin [168].

Theorem 3. *Let $F : \mathbf{R} \to \mathbf{R}$ be a function and let $R_\infty(F)$ denote the set of all those points of $\mathbf{R}$ at which the right derivative of F is infinite. Then $R_\infty(F)$ is of Lebesgue measure zero.*

The same is true for the set $L_\infty(F)$ of all those points of $\mathbf{R}$ at which the left derivative of F is infinite.

Proof. It suffices to establish only the first part of this theorem. Obviously, we have the inclusion

$$R_\infty(F) \subset \{x \in \mathbf{R} : (D_F(x))^+ = -\infty\} \cup \{x \in \mathbf{R} : (D_F(x))_+ = +\infty\}.$$

In view of Lemma 3, both sets of the right-hand side of the above inclusion are of Lebesgue measure zero. Consequently, we directly come to the required equality $\lambda(R_\infty(F)) = 0$, which completes the proof.

Appendix 2
Banach limits on the real line

In Chapter 12 we repeatedly stressed that the problem of summation of various divergent series is closely connected with uncountable forms of the Axiom of Choice (**AC**) and that there are linear functionals defined on the family of all bounded real sequences which substantially extend the operation of taking limits of convergent real sequences.

In this appendix we want to give a certain version of Banach's classical construction of functionals of such a type (cf. [15]).

We begin with recalling the fundamental notion of a filter in a base (ground) set E (see, e.g., [27]).

Let $\mathcal{P}(E)$ denote, as usual, the power set of E.

A family $\Phi \subset \mathcal{P}(E)$ is a filter in E if the following three conditions are satisfied:

(i) $E \in \Phi$ and $\emptyset \notin \Phi$;

(ii) if $X \in \Phi$ and $X \subset Y \subset E$, then $Y \in \Phi$;

(iii) if $X \in \Phi$ and $Y \in \Phi$, then $X \cap Y \in \Phi$.

The idea of this notion is to reflect (in an appropriate way) the main properties of "big" subsets of E.

Indeed, E itself must be big, a set containing a big set also must be big, and the intersection of two big sets must be big, too.

It directly follows from condition (i) that if $E = \emptyset$, then there exists no filter in E.

At present, it is well known that filters are a quite powerful technical tool in various fields of contemporary mathematics. Moreover, by using filters it becomes possible to define the general concept of convergence in abstract topological spaces (see, e.g., [27], [64], [107]). Especially, filters are frequently met in those topics of general topology which are concerned with quasi-compact spaces and other spaces similar to quasi-compact ones (see, in particular, [27], [64]).

Obviously, the family of all filters in E is partially ordered by the standard inclusion relation $\subset$. If $E \neq \emptyset$, then the smallest element in this partially ordered set is $\{E\}$.

Example 1. Suppose that E is a finite ground set and Φ is some filter in E. It is not difficult to verify that there exists a unique nonempty set $X_0 \subset E$ such that

$$\Phi = \{X \subset E : X_0 \subset X\}.$$

In fact, the set X_0 is defined by the equality

$$X_0 = \cap\Phi,$$

so X_0 is the least (by inclusion) element of Φ.

Example 2. Let $E = \omega$ and take the family of all co-finite subsets of E as Φ. It can easily be seen that Φ is a filter in E which is usually called the Frechet filter in ω. Clearly, the Frechet filter does not have minimal elements (by the inclusion relation).

Let Φ be a filter in a ground set E and let X be a subset of E.

One can readily show that, for the existence of a filter Φ' in E containing Φ and satisfying $X \in \Phi'$, it is necessary and sufficient that $Z \cap X \neq \emptyset$ for all sets $Z \in \Phi$. Indeed, supposing that this condition holds, one may put

$$\Phi' = \{Y \subset E : (\exists Z \in \Phi)(X \cap Z \subset Y)\}.$$

Observe that Φ' defined in this manner is a filter in E. This new filter properly contains Φ if and only if $X \notin \Phi$.

A filter Φ in E is called an ultrafilter if Φ is maximal with respect to the inclusion relation (i.e., there is no filter Φ^* in E properly containing Φ).

The following simple statement gives a characterization of ultrafilters within **ZF** set theory.

Theorem 1. *Let Φ be a filter in a base (ground) set E. These three assertions are equivalent for Φ:*

(1) Φ is an ultrafilter;

(2) for any set $X \subset E$, either $X \in \Phi$ or $E \setminus X \in \Phi$;

(3) if $\{X_1, X_2, ..., X_k\}$ is a finite family of subsets of E such that

$$X_1 \cup X_2 \cup ... \cup X_k \in \Phi,$$

then there exists a natural index $i \in [1, k]$ such that $X_i \in \Phi$.

Proof. Obviously, the implications

$$(3) \Rightarrow (2) \Rightarrow (1)$$

are valid. So it only remains to demonstrate the validity of the implication (1) ⇒ (3). A trivial inductive argument shows that it suffices to consider the case $k = 2$.

Let (1) hold true and let $X \cup Y \in \Phi$. Suppose to the contrary that

$$X \notin \Phi \;\;\&\;\; Y \notin \Phi.$$

Then, according to the remarks made before the formulation of Theorem 1, there exist two sets $X' \in \Phi$ and $Y' \in \Phi$ such that

$$X' \cap X = Y' \cap Y = \emptyset.$$

It follows from these equalities that

$$(X' \cap Y') \cap (X \cup Y) = \emptyset,$$

which is impossible in view of the relations

$$X' \cap Y' \in \Phi, \qquad X \cup Y \in \Phi.$$

The obtained contradiction finishes the proof of Theorem 1.

The next statement is a direct consequence of the Zorn lemma, so we omit its proof here.

Theorem 2. *Let E be a ground set and let Φ be a filter in E. There exists an ultrafilter Φ^* in E containing Φ.*

Example 3. According to Theorem 2, there exists an ultrafilter Φ^* in ω containing the Frechet filter Φ (see Example 2). However, the existence of such a Φ^* cannot be established within **ZF** & **DC** theory. Indeed, as is well known (see Chapter 10), if one assumes the existence of Φ^*, then one obtains in the same theory the existence of a Lebesgue nonmeasurable set on $\mathbf{R}$, which is impossible by virtue of Solovay's result [258].

In our further considerations we fix some ultrafilter Φ^* in ω containing the Frechet filter.

Let $\{x_n : n < \omega\}$ be any bounded sequence of real numbers. By using the ultrafilter Φ^*, we associate with this sequence some point $x^* \in \mathbf{R}$. For this purpose take any segment $[a, b] \subset \mathbf{R}$ such that

$$\{x_n : n < \omega\} \subset [a, b]$$

and introduce the following two sets of natural numbers:

$$N_1 = \{n < \omega : x_n \in [a, (a + b)/2]\}, \qquad N_2 = \{n < \omega : x_n \in [(a + b)/2, b]\}.$$

Evidently, we have the equality

$$\omega = N_1 \cup N_2.$$

According to Theorem 1, at least one of the sets N_1 and N_2 belongs to Φ^*.

If $N_1 \in \Phi^*$, then we put $\Delta_1 = [a, (a+b)/2]$.

If $N_1 \notin \Phi^*$, then we put $\Delta_1 = [(a+b)/2, b]$ (clearly, in this case $N_2 \in \Phi^*$).

Further, repeat the same procedure for Δ_1 instead of $[a, b]$, and so forth. Proceeding in such a manner, we come to the sequence

$$\Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n \supset \ldots$$

of line segments whose lengths tend to zero. There exists a unique point x^* belonging to all of these segments.

The construction just described allows us to assert that x^* has the following characteristic property:

(*) For any neighborhood $U(x^*)$ of x^*, the set $\{n < \omega : x_n \in U(x^*)\}$ belongs to Φ^*.

This property also shows that the point x^* is uniquely determined by Φ^* (so the choice of an initial segment $[a, b]$ does not play any role here). We thus obtain a functional Lim on the family of all bounded real sequences, defined by the formula

$$\mathrm{Lim}(\{x_n : n < \omega\}) = x^*.$$

Theorem 3. *The functional* Lim *satisfies the following relations:*

(1) $\mathrm{liminf}\{x_n : n < \omega\} \le \mathrm{Lim}(\{x_n : n < \omega\}) \le \mathrm{limsup}\{x_n : n < \omega\}$ *and, in particular,* Lim *is an extension of the standard functional* lim *defined on the family of all convergent real sequences;*

(2) Lim *is a positive linear functional on the family of all bounded real sequences;*

(3) $|\mathrm{Lim}(\{x_n : n < \omega\})| \le \sup\{|x_n| : n < \omega\}$.

Proof. First, let us show the linearity of Lim. Indeed, taking any two bounded real sequences $\{x_n : n < \omega\}$ and $\{y_n : n < \omega\}$ and remembering that

$$\mathrm{Lim}(\{x_n : n < \omega\}) = x^*, \qquad \mathrm{Lim}(\{y_n : n < \omega\}) = y^*,$$

one can readily deduce from the characteristic property (*) that the point $x^* + y^*$ corresponds to the sequence $\{x_n + y_n : n < \omega\}$ and hence coincides with $\mathrm{Lim}(\{x_n + y_n : n < \omega\})$.

Also, it is almost trivial that

$$\mathrm{Lim}(\{tx_n : n < \omega\}) = t\mathrm{Lim}(\{x_n : n < \omega\})$$

for every real number t.

Moreover, $\mathrm{Lim}(\{x_n : n < \omega\}) \geq 0$ whenever there exists a natural index n such that $x_m \geq 0$ for all natural numbers $m > n$.

Further, denoting

$$r = \mathrm{liminf}\{x_n : n < \omega\}, \qquad s = \mathrm{limsup}\{x_n : n < \omega\},$$

take any real $\varepsilon > 0$ and consider the sequence

$$\{z_n : n < \omega\} = \{r - \varepsilon, r - \varepsilon, ..., r - \varepsilon, ...\}.$$

Then we may write

$$r - \varepsilon = \mathrm{Lim}(\{z_n : n < \omega\}) \leq \mathrm{Lim}(\{x_n : n < \omega\}).$$

Since ε can be chosen arbitrarily small, we get

$$\mathrm{liminf}\{x_n : n < \omega\} = r \leq \mathrm{Lim}(\{x_n : n < \omega\}).$$

The analogous argument yields the inequality

$$\mathrm{Lim}(\{x_n : n < \omega\}) \leq s = \mathrm{limsup}\{x_n : n < \omega\}.$$

Finally, observe that relation (3) is a straightforward consequence of relation (1). This completes the proof of Theorem 3.

The considerations just presented show that the functionals of type Lim are naturally associated with ultrafilters in ω.

Functionals of such a type are usually called Banach limits on the real line $\mathbf{R}$.

Among Banach limits, one may encounter those functionals which have one important additional property, namely, the property of shift invariance.

The shift operation is standard for the family of all real sequences. Actually, this operation transforms any sequence $\{x_n : n < \omega\}$ to the sequence $\{y_n : n < \omega\}$, where

$$y_n = x_{n+1} \qquad (n < \omega).$$

The next statement is due to Banach (see his classical monograph [15]).

Theorem 4. *There exists a Banach limit* Lim' *on* $\mathbf{R}$ *which is shift invariant, i.e., for every bounded real sequence* $\{x_n : n < \omega\}$*, one has*

$$\mathrm{Lim}'(\{x_n : n < \omega\}) = \mathrm{Lim}'(\{y_n : n < \omega\}),$$

where $y_n = x_{n+1}$ *for all natural numbers* n.

Proof. We start with an arbitrary Banach limit Lim on $\mathbf{R}$. Consider any bounded real sequence $\{x_n : n < \omega\}$ and apply to it Cesáro's transform (cf. Chapter 12), i.e., take the sequence of arithmetical means

$$s_0 = x_0, \quad s_1 = (x_0 + x_1)/2, \quad \ldots, \quad s_n = (x_0 + x_1 + \ldots + x_n)/(n+1), \quad \ldots,$$

which is bounded, too. So we may define Lim' as follows:

$$\mathrm{Lim}'(\{x_n : n < \omega\}) = \mathrm{Lim}(\{s_n : n < \omega\}).$$

The reader can easily check that the functional Lim' is also a Banach limit and, in addition to this, Lim' is shift invariant. Theorem 4 has thus been proved.

It is reasonable to underline once again that the existence of at least one Banach limit on $\mathbf{R}$ implies the existence of subsets of $\mathbf{R}$ with pathological descriptive properties. In fact, any Banach limit Lim canonically produces a normalized finitely additive measure μ on $\mathcal{P}(\omega)$ which vanishes at all singletons in ω. To see this circumstance, it suffices to take an arbitrary set $X \subset \omega$ and stipulate

$$\mu(X) = \mathrm{Lim}(\chi_X),$$

where χ_X denotes, as usual, the characteristic function of X.

Even the existence of such a μ leads to certain pathological subsets of $\mathbf{R}$. More precisely, in **ZF** & **DC** theory the existence of μ yields the existence of a subset of $\mathbf{R}$ lacking the Baire property. In order to establish this fact, we need two simple auxiliary statements.

Lemma 1. *Let* ν *be a normalized finitely additive measure on some algebra* $\mathcal{A}$ *of subsets of a ground set* E *and let* $\{X_i : i \in I\}$ *be a* ν*-almost disjoint family of sets from* $\mathcal{A}$*, i.e., for any two distinct indices* $i \in I$ *and* $j \in I$*, the equality* $\nu(X_i \cap X_j) = 0$ *holds true.*

If I *is uncountable, then there exists* $i \in I$ *such that* $\nu(X_i) = 0$ *(actually, there are uncountably many indices* $j \in I$ *such that* $\nu(X_j) = 0$*).*

Proof. Suppose to the contrary that $\nu(X_i) > 0$ for all $i \in I$. Then, for some natural number $n > 0$, one can find an uncountable subset I' of I satisfying the relation

$$(\forall i \in I')(\nu(X_i) > 1/n).$$

Let $i_1, i_2, \ldots, i_n$ be any pairwise distinct indices from I'. Keeping in mind the almost disjointness of the sets $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$, we may write

$$\nu(E) \geq \nu(X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_n}) > n \cdot (1/n) = 1,$$

which contradicts the assumption $\nu(E) = 1$. The contradiction obtained finishes the proof.

Lemma 2. *Let E be a Baire topological space and let G be a group of homeomorphisms of E acting almost transitively in E, i.e., for any point $x \in E$, the orbit $G(x)$ is everywhere dense in E. Further, let X be a subset of E having the Baire property and almost invariant with respect to G, i.e., for each $g \in G$, the set $g(X)\triangle X$ is of first category in E.*

Then either X is of first category in E or $E \setminus X$ is of first category in E.

Proof. Suppose to the contrary that both sets X and $E \setminus X$ are of second category in E. This means that

$$X = (U \cup Y) \setminus Y', \quad E \setminus X = (V \cup Z) \setminus Z',$$

where U and V are two nonempty open sets in E and Y, Y', Z, Z' are some first category subsets of E. By virtue of the almost transitivity of G, there exists a transformation $g \in G$ such that $g(U) \cap V \neq \emptyset$. Since $g(U) \cap V$ is a nonempty open set in the Baire space E, we infer that $g(U) \cap V$ is of second category in E. Also, it is not difficult to check that

$$g(X) \setminus X \supset (g(U) \cap V) \setminus (g(Y') \cup Z'),$$

so $g(X)\triangle X$ is of second category, which yields a contradiction and finishes the proof.

Remark 1. By taking into account Banach's statement on first category open sets (see Exercise 30 from Chapter 0), it is not hard to show that the analogue of Lemma 2 remains true for any topological space E. However, such a generalization of Lemma 2 needs an uncountable form of the Axiom of Choice.

Now, we are able to prove the following statement due to Pincus (cf. [273]).

Theorem 5. *In* **ZF** & **DC** *theory the existence of a normalized finitely additive measure μ defined on $\mathcal{P}(\omega)$ and vanishing on all singletons of ω implies the existence of a subset of* **R** *without the Baire property.*

Proof. In our further considerations it will be convenient to identify the subsets of ω with the points of Cantor's space $C = \{0, 1\}^{\omega}$. Clearly, such an identification can be realized by a canonical one-to-one correspondence between the subsets of ω and their characteristic functions. Also, we will treat C as a compact topological group with respect to the addition operation + modulo 2. As is well known, in **ZF** & **DC** theory there exists a Borel isomorphism

$$\psi : C \to [0,1]$$

which preserves the category of sets (cf. Exercise 18 from Chapter 10). This ψ preserves the Baire property of sets, too. So, in the same theory, the existence of a subset of C without the Baire property is equivalent to the existence of a subset of $[0,1]$ without the Baire property.

Now, let μ be as in the formulation of Theorem 5, and suppose to the contrary that all subsets of C possess the Baire property. Let us denote

$$\mathcal{I} = \{X \subset \omega : \mu(X) = 0\}.$$

The ideal $\mathcal{I}$ is a certain subset of C having the Baire property in C and satisfying the following two relations:

(i) $\mathcal{I}$ is invariant under the subgroup $(G, +)$ of $(C, +)$ defined by

$$G = \{t \in C : (\exists n \in \omega)(\forall m > n)(t(m) = 0)\};$$

(ii) $\mathcal{I}$ is of first category in C.

The validity of (i) follows from the circumstance that μ vanishes on all singletons of ω.

To show the validity of (ii), it suffices to apply Lemma 2 to the Baire space C, to the everywhere dense subgroup G of C, and to the subset $\mathcal{I}$ of C. Indeed, according to Lemma 2, either $\mathcal{I}$ is meager in C or $C \setminus \mathcal{I}$ is meager in C. Assuming for a moment that $C \setminus \mathcal{I}$ is meager, we get that $\mathcal{I}$ is co-meager. Consequently, the set

$$\mathcal{Z} = \mathcal{I} + (1, 1, 1, ..., 1, ...)$$

must be co-meager in C, too. But it can readily be observed that

$$\mathcal{Z} \cap \mathcal{I} = \emptyset,$$

which is impossible for the two co-meager sets $\mathcal{Z}$ and $\mathcal{I}$. The obtained contradiction yields (ii).

We thus come to the equality $\mathcal{I} = \cup\{X_n : n < \omega\}$, where all sets X_n $(n < \omega)$ are nowhere dense in C.

Notice now that if X is a nowhere dense subset of C and n is a natural number (identified with $\{0, 1, ..., n-1\}$), then for every mapping

$$s : n \to \{0, 1\}$$

there exist a natural number $m > n$ and a mapping

$$s' : m \to \{0, 1\}$$

such that $s \subset s'$ and

$$(\forall t \in C)(s' \subset t \Rightarrow t \notin X).$$

Keeping in mind the above circumstance, it is not difficult to construct recursively a sequence $\{k(n) : n < \omega\}$ of natural numbers and a family

$$\{s_{i_1 i_2 \ldots i_n} \ : \ n < \omega, \ i_1 \in \{0,1\}, \ i_2 \in \{0,1\}, \ . \ . \ . \ , \ i_n \in \{0,1\}\}$$

satisfying the following five conditions:

(a) $k(n) < k(n+1)$ for each $n < \omega$;

(b) if $n < \omega$ and $i_1, i_2, \ . \ . \ . \ , i_n$ are natural numbers belonging to $\{0, 1\}$, then $s_{i_1 i_2 \ldots i_n}$ is a mapping acting from $\{0, 1, ..., k(n)\}$ into $\{0, 1\}$;

(c) if $n < \omega$ and $i_1, i_2, \ . \ . \ . \ , i_n$ are natural numbers belonging to $\{0, 1\}$, then $s_{i_1 i_2 \ldots i_n 0}$ and $s_{i_1 i_2 \ldots i_n 1}$ are two distinct extensions of $s_{i_1 i_2 \ldots i_n}$;

(d) if $n < \omega$, $t \in C$, and there exists $(i_1, i_2, ..., i_n) \in \{0, 1\}^{\{1,2,...,n\}}$ such that

$$s_{i_1 i_2 \ldots i_n} \subset t,$$

then t does not belong to the set $X_0 \cup X_1 \cup ... \cup X_n$;

(e) if $n < \omega$ and $j \in \{0, 1, ..., k(n)\}$, then in the finite family

$$\{s_{i_1 i_2 \ldots i_n}(j) : (i_1, i_2, ..., i_n) \in \{0, 1\}^{\{1,2,...,n\}}\}$$

at most one term is equal to 1.

Further, take any mapping $\sigma : \omega \to \{0, 1\}$. This σ can be identified with the sequence of its restrictions $\{\sigma|n : n < \omega\}$. Obviously, for the corresponding family $\{s_{\sigma|n} : n < \omega\}$, we get

$$s_{\sigma|0} \subset s_{\sigma|1} \subset \ . \ . \ . \ \subset \ s_{\sigma|n} \ \subset \ . \ . \ . \ ,$$

so there exists a unique $s_\sigma \in C$ extending all $s_{\sigma|n}$ $(n < \omega)$.

It immediately follows from the described construction that

(f) $s_\sigma \cap s_{\sigma'}$ is finite for any two distinct mappings

$$\sigma : \omega \to \{0, 1\}, \quad \sigma' : \omega \to \{0, 1\};$$

(g) for every $\sigma : \omega \to \{0, 1\}$, the mapping s_σ does not belong to $\mathcal{I}$, so one has $\mu(s_\sigma) > 0$.

But the relations (f) and (g) contradict Lemma 1, which completes the proof of Theorem 5.

Remark 2. Let E be a ground set, $\mathcal{A}$ be a σ-algebra of subsets of E, and let ν be a normalized finitely additive measure with $\mathrm{dom}(\nu) = \mathcal{A}$. Working in **ZF** & **DC** theory, suppose that ν is not countably additive. This means that there exists a disjoint family $\{Y_n : n < \omega\}$ of members from $\mathcal{A}$ such that

$$\nu(\cup\{Y_n : n < \omega\}) > \sum\{\nu(Y_n) : n < \omega\}.$$

Let us denote

$$r = \nu(\cup\{Y_n : n < \omega\}) - \sum\{\nu(Y_n) : n < \omega\}$$

and, for every set $K \subset \omega$, let us put

$$\mu(K) = (1/r)(\nu(\cup\{Y_n : n \in K\}) - \sum\{\nu(Y_n) : n \in K\}).$$

A straightforward verification shows that μ is a normalized finitely additive measure defined on the power set of ω and vanishing on all singletons of ω. So, applying Theorem 5, we conclude that in **ZF** & **DC** theory the existence of ν (which is not countably additive) also implies the existence of a subset of $\mathbf{R}$ lacking the Baire property.

Bibliography

[1] A. Abian, An example of a nonmeasurable set, *Boll. Un. Mat. Ital.*, v. 1 (4), 1968, pp. 366–368.

[2] S. I. Adian, P. S. Novikov, On a semicontinuous function, *Uchen. Zap. Moskov. Gos. Ped. Inst.*, v. 138, n. 3, 1958, pp. 3–10 (in Russian).

[3] A. Andretta, A. Marcone, Ordinary differential equations and descriptive set theory: Uniqueness and globality of solutions of Cauchy problems in one dimension, *Fund. Math.*, v. 153, n. 2, 1997, pp. 157–190.

[4] T. W. Apostol, *Mathematical Analysis*, Addison Wesley Longman, New York, 1974.

[5] J. Appell, P. P. Zabrejko. *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1990.

[6] R. Baire, Sur les fonctions de variables réelles, *Ann. di Math.*, v. 3, n. 3, 1899.

[7] R. Baire, Sur la représentation des fonctions discontinues, *Acta Math.*, v. 30, 1905.

[8] R. Baire, *Leçons sur les Fonctions Discontinues*, Gauthier-Villars, Paris, 1905.

[9] M. Balcerzak, Some remarks on sup-measurability, *Real Analysis Exchange*, v. 17, n. 2, 1991–1992, pp. 597–607.

[10] M. Balcerzak, Another nonmeasurable set with property (s^0), *Real Analysis Exchange*, v. 17, n. 2, 1991–1992, pp. 781–784.

[11] M. Balcerzak, Typical properties of continuous functions via the Vietoris topology, *Real Analysis Exchange*, v. 18, n. 2, 1992–1993, pp. 532–536.

[12] S. Baldwin, Martin's axiom implies a stronger version of Blumberg's theorem, *Real Analysis Exchange*, v. 16, 1990–1991, pp. 67–73.

[13] S. Banach, Sur l'équation fonctionnelle $f(x+y) = f(x) + f(y)$, *Fund. Math.*, v. 1, 1920, p. 123.

[14] S. Banach, Über die Baire'sche Kategorie gewisser Funktionenmengen, *Studia Math.*, v. 3, 1931, pp. 174–179.

[15] S. Banach, *Théorie des Opérations Linéaires*, 2nd edition, Chelsea Publ. Comp., New York, 1978.

[16] S. Banach, C. Kuratowski, Sur une généralisation du probléme de la mesure, *Fund. Math.*, v. 14, 1929, pp. 127–131.

[17] S. Banach, H. Steinhaus, Sur le principe de la condensation de singularités, *Fund. Math.*, v. 9, 1927, pp. 50–61.

[18] J. Barwise (editor), *Handbook of Mathematical Logic*, North-Holland Publishing Comp., Amsterdam, 1977.

[19] F. Bernstein, Zur Theorie der trigonometrischen Reihen, *Sitzungsber. Sächs. Akad. Wiss. Leipzig. Math.-Natur. Kl.*, 60, 1908, pp. 325–338.

[20] A. Blass, A partition theorem for perfect sets, *Proc. Amer. Math. Soc.*, v. 82, 1981, pp. 271–277.

[21] J. Blazek, E. Borák, J. Malý, On Köpcke and Pompeiu functions, *Ćas. pro pest. mat.*, v. 103, 1978, pp. 53–61.

[22] H. Blumberg, New properties of all real functions, *Trans. Amer. Math. Soc.*, v. 24, 1922, pp. 113–128.

[23] V. Bogachev, *Measure Theory*, Springer-Verlag, Berlin-Heidelberg, 2007.

[24] V. G. Boltjanskii, *Hilbert's Third Problem*, Izd. Nauka, Moscow, 1977 (in Russian).

[25] J. Boos, *Classical and Modern Methods in Summability*, Clarendon Press, Oxford, 2000.

[26] N. Bourbaki, *Theory of Sets*, Hermann et Cie, Paris, 1968.

[27] N. Bourbaki, *General Topology*, Springer-Verlag, Berlin, 1989.

[28] N. Bourbaki, *Integration*, v. I, v. II, Springer-Verlag, Berlin, 2004.

[29] J. B. Brown, Variations on Blumberg's theorem, *Real Analysis Exchange*, v. 9, 1983–1984, pp. 123–137.

[30] J. B. Brown, Restriction theorems in real analysis, *Real Analysis Exchange*, v. 20, n. 2, 1994–1995, pp. 510–526.

[31] J. B. Brown, G. V. Cox, Classical theory of totally imperfect spaces, *Real Analysis Exchange*, v. 7, 1982, pp. 1–39.

[32] J. B. Brown, K. Prikry, Variations on Lusin's theorem, *Trans. Amer. Math. Soc.*, v. 302, 1987, pp. 77–86.

[33] A. Bruckner, *Differentiation of Real Functions*, Springer, Berlin, 1978.

[34] A. Bruckner, J. Haussermann, Strong porosity features of typical continuous functions, *Acta Math. Hung.*, v. 45, n. 1–2, 1985, pp. 7–13.

[35] A. Bruckner, J. L. Leonard, Derivatives, *Amer. Math. Monthly*, v. 73, n. 4, 1966, pp. 24–56.

[36] A. M. Bruckner, R. J. O'Malley, B. S. Thomson, Path derivatives: A unified view of certain generalized derivatives, *Trans. Amer. Math. Soc.*, v. 283, n. 1, 1984, pp. 97–125.

[37] L. Bukovský, *The Structure of the Real Line*, Birkhäuser, Basel, 2011.

[38] L. Bukovský, N. N. Kholshchevnikova, M. Repický, Thin sets of harmonic analysis and infinite combinatorics, *Real Analysis Exchange*, v. 20, n. 2, 1994–1995, pp. 454–509.

[39] V. V. Buldygin, A. B. Kharazishvili, *Geometric Aspects of Probability Theory and Mathematical Statistics*, Kluwer Academic Publishers, Dordrecht, 2000.

[40] C. Burstin, Die Spaltung des Kontinuums in c im L. Sinne nichtmessbare Mengen, *Wien. Ber.*, v. 125, 1916, pp. 209–217.

[41] J. Cichoń, A. Kharazishvili, B. Węglorz, On selectors associated with some subgroups of the real line, *Bull. Acad. Sci. of Georgian SSR*, v. 144, n. 2, 1991, pp. 237–239.

[42] J. Cichoń, A. Kharazishvili, B. Węglorz, On sets of Vitali's type, *Proc. Amer. Math. Soc.*, v. 118, n. 4, 1993, pp. 1243–1249.

[43] J. Cichoń, A. Kharazishvili, B. Węglorz, *Subsets of the Real Line*, Łódź University Press, Łódź, 1995.

[44] J. Cichoń, M. Morayne, On differentiability of Peano type functions, III, *Proc. Amer. Math. Soc.*, v. 92, n. 3, 1984, pp. 432–438.

[45] J. Cichoń, M. Morayne, Universal functions and generalized classes of functions, *Proc. Amer. Math. Soc.*, v. 102, n. 1, 1988, pp. 83–89.

[46] J. Cichoń, M. Morayne, J. Pawlikowski, S. Solecki, Decomposing Baire functions, *The Journal of Symbolic Logic*, v. 56, n. 4, 1991, pp. 1273–1283.

[47] K. Ciesielski, Set-theoretic real analysis, *Journal of Applied Analysis*, v. 3, n. 2, 1997, pp. 143–190.

[48] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts, v. 39, Cambridge University Press, Cambridge, 1997.

[49] K. Ciesielski, L. Larson, K. Ostaszewski, Differentiability and density continuity, *Real Analysis Exchange*, v. 15, n. 1, 1989–1990, pp. 239–247.

[50] K. Ciesielski, L. Larson, K. Ostaszewski, I-density continuous functions, *Memoirs of the Amer. Math. Soc.*, v. 107, n. 515, 1994, pp. 1–132.

[51] K. Ciesielski, S. Shelah, Category analogue of sup-measurability problem, *Journal of Applied Analysis*, v. 6, n. 2, 2000, pp. 159–172.

[52] P. M. Cohn, *Universal Algebra*, Harper & Row, London, 1965.

[53] P. Corazza, Ramsey sets, the Ramsey ideal, and other classes over $\mathbf{R}$, *The Journal of Symbolic Logic*, v. 57, 1992, pp. 1441–1468.

[54] U. B. Darji, Decomposition of functions, *Real Analysis Exchange*, v. 21, n. 1, 1995–1996, pp. 19–25.

[55] U. B. Darji, Countable decomposition of derivatives and Baire 1 functions, *Journal of Applied Analysis*, v. 2, n. 2, 1996, pp. 119–124.

[56] M. Dehn, Über den Rauminhalt, *Math. Ann.*, v. 55, 1902, pp. 465–478.

[57] C. Dellacherie, *Capacités et Processus Stochastiques*, Springer-Verlag, Berlin, 1972.

[58] A. Denjoy, Sur les fonctions dérivées sommables, *Bull. Soc. Math. France*, v. 43, 1915, pp. 161–248.

[59] J. Dieudonne, *The Foundations of Modern Analysis*, Academic Press, New York, 1960.

[60] J. L. Doob, *Stochastic Processes*, Wiley and Sons, Inc., New York, 1953.

[61] N. Dunford, J. T. Schwartz, *Linear Operators (General Theory)*, Part 1, Wiley-Interscience, New York, 1988.

[62] D. Egoroff (D. Egorov), Sur les suites de fonctions mesurables, *C. R. Acad. Sci. Paris*, v. CLII, 1911, pp. 244–246.

[63] I. Ekeland, *Elements d'Economie Mathematique*, Hermann, Paris, 1979.

[64] R. Engelking, *General Topology*, PWN, Warszawa, 1985.

[65] P. Erdös, S. Kakutani, On nondenumerable graphs, *Bull. Amer. Math. Soc.*, v. 49, 1943, pp. 457–461.

[66] P. Erdös, K. Kunen, R. D. Mauldin, Some additive properties of sets of real numbers, *Fund. Math.*, v. CXIII, 1981, pp. 187–199.

[67] P. Erdös, R. D. Mauldin, The nonexistence of certain invariant measures, *Proc. Amer. Math. Soc.*, vol. 59, 1976, pp. 321–322.

[68] P. Erdös, A. H. Stone, On the sum of two Borel sets, *Proc. Amer. Math. Soc.*, v. 25, n. 2, 1970, pp. 304–306.

[69] M. P. Ershov, Extension of measures and stochastic equations, *Probability Theory and Its Applications*, v. 19, n. 3, 1974, pp. 457–471 (in Russian).

[70] M. Evans, On continuous functions and the approximate symmetric derivatives, *Colloq. Math.*, v. 31, 1974, pp. 129–136.

[71] G. Fichtenholz, Sur une fonction de deux variables sans intégrale double, *Fund. Math.*, v. 6, 1924, pp. 30–36.

[72] G. M. Fichtenholz, *Course in Differential and Integral Calculus*, v. 2, Moscow: Izdat. Nauka, 1966 (in Russian).

[73] C. Freiling, Axioms of symmetry: Throwing darts at the real number line, *The Journal of Symbolic Logic*, v. 51, 1986, pp. 190–200.

[74] C. Freiling, A converse to a theorem of Sierpiński on almost symmetric sets, *Real Analysis Exchange*, v. 15, n. 2, 1989–1990, pp. 760–767.

[75] D. H. Fremlin, Measure additive coverings and measurable selectors, *Dissertationes Mathematicae*, v. 260, 1987, pp. 1–116.

[76] H. Friedman, A consistent Fubini-Tonelli theorem for non-measurable functions, *Illinois Journal of Math.*, v. 24, 1980, pp. 390–395.

[77] S. Fuchino, Sz. Plewik, On a theorem of E. Helly, *Proc. Amer. Math. Soc.*, v. 127, n. 2, 1999, pp. 491–497.

[78] B. R. Gelbaum, J. M. H. Olmsted, *Counterexamples in Analysis*, Holden-Day, San Francisco, 1964.

[79] C. Goffman, Everywhere differentiable functions and the density topology, *Proc. Amer. Math. Soc.*, v. 51, 1975, pp. 250–251.

[80] C. Goffman, C. J. Neugebauer, T. Nishiura, Density topology and approximate continuity, *Duke Mathem. Journal*, v. 28, 1961, pp. 497–505.

[81] R. L. Graham, B. Rothschild, J. H. Spencer, *Ramsey Theory*, Wiley and Sons, New York, 1990.

[82] Z. Grande, J. Lipiński, Un exemple d'une fonction sup-mesurable qui n'est pas mesurable, *Colloq. Math.*, v. 39, 1978, pp. 77–79.

[83] E. Grzegorek, Remarks on σ-fields without continuous measures, *Colloq. Math.*, v. 39, 1978, pp. 73–75.

[84] E. Grzegorek, Solution of a problem of Banach on σ-fields without continuous measures, *Bull. Acad. Polon. Sci., Ser. Sci. Math.*, v. 28, 1980, pp. 7–10.

[85] P. R. Halmos, *Measure Theory*, D. Van Nostrand, New York, 1950.

[86] P. R. Halmos, *Naive Set Theory*, Van Nostrand Reinhold Co., New York, 1960.

[87] G. Hamel, Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$, *Math. Ann.*, v. 60, 1905, pp. 459–462.

[88] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[89] H. Herrlich, *Axiom of Choice*, Springer-Verlag, Berlin, 2006.

[90] E. Hewitt, K. Ross, *Abstract Harmonic Analysis*, v. 1, Springer-Verlag, Berlin, 1963.

[91] E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.

[92] E. W. Hobson, *Theory of Functions of a Real Variable*, II, Dover, New York, 1957.

[93] D. H. Hyers, S. M. Ulam, Approximately convex functions, *Proc. Amer. Math. Soc.*, v. 3, 1952, pp. 821–828.

[94] S. Jackson, R. D. Mauldin, Some complexity results in topology and analysis, *Fund. Math.*, v. 141, 1992, pp. 75–83.

[95] V. Jarník, Sur la dérivabilité des fonctions continues, *Spisy Privodov, Fak. Univ. Karlovy*, v. 129, 1934, pp. 3–9.

[96] J. Jasiński, I. Reclaw, Restrictions to continuous and pointwise discontinuous functions, *Real Analysis Exchange*, v. 23, n. 1, 1997–1998, pp. 161–174.

[97] T. Jech, *Set Theory*, Springer-Verlag, Berlin, 2003.

[98] B. Jessen, The algebra of polyhedra and the Dehn-Sydler theorem, *Math. Scand.*, v. 22, 1968, pp. 241–256.

[99] S. Kaczmarz, H. Steinhaus, *Theorie der Orthogonalreihen*, Monogr. Matem., Warszawa–Lwow, 1935.

[100] S. Kakutani, J. C. Oxtoby, Construction of a non-separable invariant extension of the Lebesgue measure space, *Ann. Math.*, v. 52, 1950, pp. 580–590.

[101] A. Kanamori, The emergence of descriptive set theory, Synthese, v. 251, 1995, pp. 241–262.

[102] A. Kanamori, The mathematical development of set theory from Cantor to Cohen, *The Bulletin of Symbolic Logic*, v. 2, n. 1, 1996, pp. 1–71.

[103] A. Kanamori, *The Higher Infinite*, Springer-Verlag, Berlin-Heidelberg, 2003.

[104] Y. Katznelson, K. Stromberg, Everywhere differentiable, nowhere monotone functions, *Amer. Math. Monthly*, v. 81, 1974, pp. 349–354.

[105] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[106] J. L. Kelley, The Tychonoff product theorem implies the Axiom of Choice, *Fund. Math.*, v. 37, 1950, pp. 75–76.

[107] J. L. Kelley, *General Topology*, Springer-Verlag, New York, 1991.

[108] A. B. Kharazishvili, Certain types of invariant measures, *Dokl. Akad. Nauk SSSR*, v. 222, n. 3, 1975, pp. 538–540 (in Russian).

[109] A. B. Kharazishvili, Some applications of Hamel bases, *Bull. Acad. Sci. Georgian SSR*, v. 85, n. 1, 1977, pp. 17–20 (in Russian).

[110] A. B. Kharazishvili, *Some Questions of Set Theory and Measure Theory*, Izd. Tbil. Gos. Univ., Tbilisi, 1978 (in Russian).

[111] A. B. Kharazishvili, Some questions of the theory of invariant measures, *Bull. Acad. Sci. of Georgian SSR*, v. 100, n. 3, 1980, pp. 533–536 (in Russian).

[112] A. B. Kharazishvili, *Invariant Extensions of Lebesgue Measure*, Izd. Tbil. Gos. Univ., Tbilisi, 1983 (in Russian).

[113] A. B. Kharazishvili, *Applications of Set Theory*, Izd. Tbil. Gos. Univ., Tbilisi, 1989 (in Russian).

[114] A. B. Kharazishvili, Martin's axiom and Γ-selectors, *Bull. Acad. Sci. of Georgian SSR*, v. 137, n. 2, 1990, pp. 245–248 (in Russian).

[115] A. B. Kharazishvili, *Vitali's Theorem and Its Generalizations*, Izd. Tbil. Gos. Univ., Tbilisi, 1991 (in Russian).

[116] A. B. Kharazishvili, Baire property and its applications, *Proc. I. Vekua Inst. of Appl. Math.*, v. 43, 1992, pp. 5–113 (in Russian).

[117] A. B. Kharazishvili, On sections of analytic sets, *Bull. Acad. Sci. of Georgia*, v. 147, n. 2, 1993, pp. 223–226 (in Russian).

[118] A. B. Kharazishvili, Some remarks on density points and the uniqueness property for invariant extensions of the Lebesgue measure, *Acta Universitatis Carolinae - Mathematica et Physica*, v. 35, n. 2, 1994, pp. 33–39.

[119] A. B. Kharazishvili, Some questions concerning invariant extensions of Lebesgue measure, *Real Analysis Exchange*, v. 20, n. 2, 1994–1995, pp. 580–592.

[120] A. B. Kharazishvili, Some remarks on the property (N) of Luzin, *Annales Mathematicae Silesianae*, v. 9, 1995, pp. 33–42.

[121] A. B. Kharazishvili, On translations of sets and functions, *Journal of Applied Analysis*, v. 1, n. 2, 1995, pp. 145–158.

[122] A. B. Kharazishvili, *Selected Topics of Point Set Theory*, Łódź University Press, Łódź, 1996.

[123] A. B. Kharazishvili, Sup-measurable and weakly sup-measurable mappings in the theory of ordinary differential equations, *Journal of Applied Analysis*, v. 3, n. 2, 1997, pp. 211–224.

[124] A. B. Kharazishvili, On countably generated invariant σ-algebras that do not admit measure type functionals, *Real Analysis Exchange*, v. 23, n. 1, 1997–1998, pp. 287–294.

[125] A. B. Kharazishvili, *Transformation Groups and Invariant Measures: Set-theoretical Aspects*, World Scientific Publ. Co., Singapore, 1998.

[126] A. B. Kharazishvili, On measurability properties connected with the superposition operator, *Real Analysis Exchange*, v. 28, n. 1, 2002–2003, pp. 205–214,

[127] A. B. Kharazishvili, On absolutely nonmeasurable additive functions, *Georgian Mathematical Journal*, v. 11, n. 2, 2004, pp. 301–306.

[128] A. B. Kharazishvili, On generalized step-functions and superposition operators, *Georgian Mathematical Journal*, v. 11, n. 4, 2004, pp. 753–758.

[129] A. B. Kharazishvili, *Nonmeasurable Sets and Functions*, Elsevier, Amsterdam, 2004.

[130] A. B. Kharazishvili, On a bad descriptive structure of Minkowski sum of certain small sets in a topological vector space, *Theory of Stochastic Processes*, v. 14 (30), n. 2, 2008, pp. 35–41.

[131] A. B. Kharazishvili, *Topics in Measure Theory and Real Analysis*, Atlantis Press, Amsterdam-Paris, 2009.

[132] A. B. Kharazishvili, *Set Theoretical Aspects of Real Analysis*, Chapman and Hall/CRC, Boca Raton, 2015.

[133] A. B. Kharazishvili, Strangely convergent sequences, *Inference: International Review of Science*, v. 1, issue 4, 2015, pp. 1–13.

[134] A. B. Kharazishvili, A.P. Kirtadze, On the measurability of functions with respect to certain classes of measures, *Georgian Mathematical Journal*, v. 11, n. 3, 2004, pp. 489–494.

[135] B. King, Some remarks on difference sets of Bernstein sets, *Real Analysis Exchange*, v. 19, n. 2, 1993–1994, pp. 478–490.

[136] E. M. Kleinberg, *Infinitary Combinatorics and the Axiom of Determinateness*, Springer-Verlag, Berlin, 1977.

[137] K. Knopp, *Theory and Application of Infinite Series*, Dover Publications, New York, 1990.

[138] K. Kodaira, S. Kakutani, A non-separable translation-invariant extension of the Lebesgue measure space, *Ann. Math.*, v. 52, 1950, pp. 574–579.

[139] A. N. Kolmogoroff (A. N. Kolmogorov), Sur la possibilité de la définition générale de la dérivée, de l'intégrale et de la sommation des séries divergentes, *C. R. Acad. Sci. Paris*, v. 180, 1925, pp. 362–364.

[140] P. Komjáth, A note on set mappings with meager images, *Studia Scientiarum Mathematicarum Hungarica*, v. 30, 1995, pp. 461–467.

[141] P. Komjáth, V. Totik, Ultrafilters, *Amer. Math. Monthly*, v. 115, n. 1, 2008, pp. 33–44.

[142] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, GITTL, Moscow, 1956 (in Russian).

[143] M. A. Krasnoselskii, A. V. Pokrovskii, On a discontinuous superposition operator, *Uspekhi Mat. Nauk*, v. 32, n. 1, 1977, pp. 169–170 (in Russian).

[144] M. A. Krasnoselskii, A. V. Pokrovskii, *Systems with Hysteresis*, Springer-Verlag, Berlin, 1988.

[145] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, PWN, Katowice, 1985.

[146] K. Kunen, *Set Theory*, North-Holland Publishing Comp., Amsterdam, 1980.

[147] K. Kunen, Random and Cohen reals, In: *Handbook of Set-Theoretic Topology*, North-Holland Publishing Comp., Amsterdam, 1984, pp. 887–911.

[148] K. Kuratowski, Une méthode d'elimination des nombres transfinis des raisonnements mathématiques, *Fund. Math.*, v. 5, 1922, pp. 76–108.

[149] K. Kuratowski, *Topology*, v. 1, Academic Press, New York, 1966.

[150] K. Kuratowski, *Topology*, v. 2, Academic Press, New York, 1968.

[151] K. Kuratowski, On the concept of strongly transitive systems in topology, *Annali di Matematica Pura ed Applicata*, v. 98, no. 4, 1974, pp. 357–363.

[152] K. Kuratowski, A theorem on ideals and its application to the Baire property in Polish spaces, *Uspekhi Mat. Nauk*, v. 31, 5 (191), 1976, pp. 124–127 (in Russian).

[153] K. Kuratowski, A. Mostowski, *Set Theory*, North-Holland Publishing Comp., Amsterdam, 1967.

[154] K. Kuratowski, C. Ryll–Nardzewski, A general theorem on selectors, *Bull. Acad. Polon. Sci., Ser. Math.*, v. 13, n. 6, 1965, pp. 397–402.

[155] K. Kuratowski, S. Ulam, Quelques propriétés topologiques du produit combinatoire, *Fund. Math.*, v. 19, 1932, pp. 248–251.

[156] M. Laczkovich, G. Petruska, On the transformers of derivatives, *Fund. Math.*, v. 100, 1978, pp. 179–199.

[157] R. Laver, On the consistency of Borel's conjecture, *Acta Math.*, v. 137, 1976, pp. 151–169.

[158] M. A. Lavrentieff (M. A. Lavrentiev), Contribution á la théorie des ensembles homéomorphes, *Fund. Math.*, v. 6, 1924, pp. 149–160.

[159] M. A. Lavrentieff (M. A. Lavrentiev), Sur une équation différentielle du premiér ordre, *Math. Zeitschrift*, v. 23, 1925, pp. 197–209.

[160] H. Lebesgue, Sur les fonctions représentables analytiquement, *C. R. Acad. Sci. Paris*, v. 139, 1904, pp. 29–31.

[161] H. Lebesgue, Sur les fonctions représentables analytiquement, *Journal de Math. Pures et Appl.*, v. 1, Ser. 6, 1905, pp. 139–216.

[162] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1965.

[163] A. Liapunoff, Sur quelques propriétés des cribles rectilignes et des ensembles C, *C.r. Soc. sci. Varsovie*, v. 29, 1936, pp. 1–8.

[164] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Math.*, v. 80, 1948, pp. 167–190.

[165] G. G. Lorentz, K. Zeller, Series rearrangements and analytic sets, *Acta Math.*, v. 100, 1958, pp. 149–169.

[166] N. Lusin (N. Luzin), Sur une probléme de M. Baire, *C. R. Acad. Sci. Paris*, v. 158, 1914, p. 1259.

[167] N. Lusin (N. Luzin), *Leçons sur les Ensembles Analytiques et leurs Applications*, Gauthier-Villars, Paris, 1930.

[168] N. N. Lusin (N. N. Luzin), *The Integral and Trigonometric Series*, Izd. GITTL, Moscow, 1956 (in Russian).

[169] N. N. Lusin (N. N. Luzin), *Collected Works*, v. 2, Izd. Akad. Nauk SSSR, Moscow, 1958 (in Russian).

[170] N. Lusin (N. Luzin), W. Sierpiński, Sur une décomposition du continu, *C. R. Acad. Sci. Paris*, v. 175, 1922, pp. 357–359.

[171] Yu. I. Lyubich, Axiomatic theory of divergent series and cohomological equations, *Fund. Math.*, v. 198, n. 3, 2008, pp. 263–282.

[172] D. Maharam, On a theorem of von Neumann, *Proc. Amer. Math. Soc.*, v. 9, 1958, pp. 987–994.

[173] D. Maharam, A. H. Stone, Expressing measurable functions by one-one ones, *Advances in Mathematics*, v. 46, n. 2, 1982, pp. 151–161.

[174] J. Malý, The Peano curve and the density topology, *Real Analysis Exchange*, v. 5, 1979–1980, pp. 326–329.

[175] J. Marcinkiewicz, Sur les nombres dérivés, *Fund. Math.*, v. 24, 1935, pp. 305–308.

[176] E. Marczewski, C. Ryll-Nardzewski, Sur la mesurabilité des fonctions de plusieurs variables, *Annales de la Societé Polonaise de Mathématique*, v. 25, 1952, pp. 145–155.

[177] R. D. Mauldin, The set of continuous nowhere differentiable functions, *Pacific Journal of Mathematics*, v. 83, n. 1, 1979, pp. 199–205; v. 121, n. 1, 1986, pp. 119–120.

[178] S. Mazurkiewicz, Sur les fonctions non-derivables, *Studia Math.*, v. 3, 1931, pp. 92–94.

[179] S. Mazurkiewicz, Sur les suites de fonctions continues, *Fund. Math.*, v. 18, 1932, pp. 114–117.

[180] E. Michael, Continuous selections, I, *Ann. Math.*, v. 63, n. 2, 1956, pp. 361–382.

[181] E. Michael, Selected selection theorems, *Amer. Math. Monthly*, v. 63, 1956, pp. 233–238.

[182] J. van Mill, *Infinite-dimensional Topology: Prerequisites and Introduction*, North-Holland Publishing Comp., Amsterdam, 1989.

[183] J. van Mill, R. Pol, Baire 1 functions which are not countable unions of continuous functions, *Acta Math. Hungar.*, v. 66, no. 4, 1995, pp. 289–300.

[184] A. W. Miller, Special subsets of the real line, In: *Handbook of Set-Theoretic Topology*, North-Holland Publishing Comp., Amsterdam, 1984, pp. 201–233.

[185] A. W. Miller, Infinite combinatorics and definability, *Annals of Pure and Applied Logic*, v. 41, 1989, pp. 179–203.

[186] A. W. Miller, *Descriptive Set Theory and Forcing*, Lecture Notes in Logic, Springer-Verlag, Berlin, 1995.

[187] G. Mokobodzki, Ensembles á coupes dénombrables et capacités dominées par une mesure, *Lecture Notes in Mathematics*, v. 649, 1978, pp. 491–508.

[188] L. Moonens, W. F. Pfeffer, The multi-dimensional Luzin theorem, *J. Math. Anal. Appl.*, v. 339, n. 1, 2008, pp. 746–752.

[189] G. H. Moore, *Zermelo's Axiom of Choice*, Springer-Verlag, New York, 1982.

[190] M. Morayne, On differentiability of Peano type functions, *Colloq. Math.*, v. 53, n. 1, 1987, pp. 129–132.

[191] M. Morayne, On differentiability of Peano type functions, II, *Colloq. Math.*, v. 53, n. 1, 1987, pp. 133–135.

[192] J. C. Morgan II, *Point Set Theory*, Marcel Dekker, Inc., New York, 1990.

[193] J. R. Munkres, *Topology*, 2nd edition, MIT, Prentice Hall, NJ, 2000.

[194] I. P. Natanson, *The Theory of Functions of a Real Variable*, Gos. Izd. Techn.-Teor. Lit., Moscow, 1957 (in Russian).

[195] T. Natkaniec, *Almost Continuity*, WSP, Bydgoszcz, 1992.

[196] J. Neveu, *Bases Mathématiques du Calcul des Probabilités*, Masson et Cie, Paris, 1964.

[197] P. S. Novikov, Separation of C-sets, *Izv. Akad. Nauk SSSR, Ser. Mat.*, v. 1, n. 2, 1937, pp. 253–264 (in Russian).

[198] P. S. Novikov, *Selected Works*, Izd. Nauka, Moscow, 1979 (in Russian).

[199] W. Orlicz, Zur Theorie der Differentialgleichung $y' = f(x, y)$, *Polska Akademia Umiejetnosci*, Krakow, 1932, pp. 221–228.

[200] W. Orlicz, *Collected Papers*, v. 1, v. 2, PWN, Warszawa, 1988.

[201] J. C. Oxtoby, Cartesian products of Baire spaces, *Fund. Math.*, v. XLIX, 1961, pp. 157–166.

[202] J. C. Oxtoby, *Measure and Category*, Springer-Verlag, Berlin, 1971.

[203] G. R. Pantsulaia, Density points and invariant extensions of the Lebesgue measure, *Bull. Acad. Sci. of Georgia*, v. 151, n. 2, 1995, pp. 216–219 (in Russian).

[204] J. Pawlikowski, The Hahn-Banach theorem implies the Banach-Tarski paradox, *Fund. Math.*, v. 138, 1991, pp. 21–22.

[205] A. Pelc, K. Prikry, On a problem of Banach, *Proc. Amer. Math. Soc.*, v. 89, n. 4, 1983, pp. 608–610.

[206] I. G. Petrovskii, *Lectures on the Theory of Ordinary Differential Equations*, GITTL, Moscow, 1949 (in Russian).

[207] W. F. Pfeffer, K. Prikry, Small spaces, *Proc. London Math. Soc.*, v. 58, n. 3, 1989, pp. 417–438.

[208] Z. Piotrowski, Separate and joint continuity, *Real Analysis Exchange*, v. 11, 1985–1986, pp. 293–322.

[209] Yu. V. Prokhorov, Convergence of random processes and limit theorems in probability theory, *Teor. Veroyatnost. i Primenen.*, v. 1, 1956, pp. 177–238 (in Russian).

[210] J. Raisonnier, A mathematical proof of S. Shelah's theorem on the measure problem and related results, *Israel Journal of Mathematics*, v. 48, n. 1, 1984, pp. 48–56.

[211] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.*, ser. 2, v. 30, 1930, pp. 264–286.

[212] M. M. Rao, *Stochastic Processes and Integration*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1979.

[213] I. Reclaw, Restrictions to continuous functions and boolean algebras, *Proc. Amer. Math. Soc.*, v. 118, n. 3, 1993, pp. 791–796.

[214] D. Repovs, P. V. Semenov, E. Michael's theory of continuous selections; development and applications, *Uspekhi Mat. Nauk*, v. 49, n. 6 (300), 1994, pp. 151–190 (in Russian).

[215] L. Rodriguez-Piazza, Every separable Banach space is isometric to a space of continuous nowhere differentiable functions, *Proc. Amer. Math. Soc.*, v. 123, n. 12, 1995, pp. 3649–3654.

[216] C. A. Rogers, A linear Borel set whose difference set is not a Borel set, *Bull. London Math. Soc.*, v. 2, 1970, pp. 41–42.

[217] A. Rosental, On the continuity of functions of several variables, *Math. Zeitschrift*, v. 63, no. 1, 1955, pp. 31–38.

[218] A. Roslanowski, S. Shelah, Measured Creatures, *Israel Journal of Mathematics*, v. 151, n. 1, 2006, pp. 61–110.

[219] K. A. Ross, *Elementary Analysis: The Theory of Calculus*, Springer, New York, 1980.

[220] F. Rothberger, Eine Äquivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinschen und Sierpińskischen Mengen, *Fund. Math.*, v. 30, 1938, pp. 215–217.

[221] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1976.

[222] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, New York, 1987.

[223] S. Ruziewicz, Sur une proprieté de la base hamelienne, *Fund. Math.*, v. 26, 1936, pp. 56–58.

[224] C. Ryll-Nardzewski, R. Telgarský, The nonexistence of universal invariant measures, *Proc. Amer. Math. Soc.*, v. 69, 1978, pp. 240–242.

[225] S. Saks, *Theory of the Integral*, Warszawa-Lwów, 1937.

[226] E. Schechter, *Handbook of Analysis and its Foundations*, Academic Press, New York, 1997.

[227] E. A. Selivanovskii, On a class of effective sets, *Mat. Sborn.*, v. 35, 1928, pp. 379–413, (in Russian).

[228] S. Shelah, Can you take Solovay's inaccessible away?, *Israel Journal of Mathematics*, v. 48, n. 1, 1984, pp. 1–47.

[229] S. Shelah, Possibly every real function is continuous on a non-meagre set, *Publ. Inst. Mat. Beograd (N.S.)*, v. 57, n. 71, 1995, pp. 47–60.

[230] J. Shinoda, Some consequences of Martin's axiom and the negation of the continuum hypothesis, *Nagoya Math. Journal*, v. 49, 1973, pp. 117–125.

[231] J. Shipman, Cardinal conditions for strong Fubini theorems, *Trans. Amer. Math. Soc.*, v. 321, 1990, pp. 465–481.

[232] J. R. Shoenfield, Measurable cardinals, In: *Logic Colloquium'69*, North-Holland Publishing Comp., Amsterdam, 1970, pp. 19–49.

[233] J. W. Shragin, Conditions for measurability of superpositions, *Dokl. Akad. Nauk SSSR*, v. 197, 1971, pp. 295–298 (in Russian).

[234] J. W. Shragin, On representation of a locally defined operator in the form of the Nemytskii operator, *Functional Differential Equations*, v. 3, n. 3–4, 1996, pp. 447–452.

[235] W. Sierpiński, L'axiome de M.Zermelo et son role dans la théorie des ensembles et l'analyse, *Bull. Intern. Acad. Sci. Cracovie, Ser. A*, 1918, pp. 97–152.

[236] W. Sierpiński, Sur un théoréme équivalent á l'hypothése du continu, *Bull. Internat. Acad. Sci. Cracovie, Ser. A*, 1919, pp. 1–3.

[237] W. Sierpiński, Sur la question de la mesurabilité de la base de M. Hamel, *Fund. Math.*, v. 1, 1920, pp. 105–111.

[238] W. Sierpiński, Sur l'équation fonctionnelle $f(x + y) = f(x) + f(y)$, *Fund. Math.*, v. 1, 1920, pp. 116–122.

[239] W. Sierpiński, Sur les fonctions convexes mesurables, *Fund. Math.*, v. 1, 1920, pp. 125–128.

[240] W. Sierpiński, Sur l'ensemble de distances entre les points d'un ensemble, *Fund. Math.*, v. 7, 1925, pp. 144–148.

[241] W. Sierpiński, Sur le produit combinatoire de deux ensembles jouissant de la propriété C, *Fund. Math.*, v. 24, 1935, pp. 48–50.

[242] W. Sierpiński, Sur une fonction non mesurable partout presque symetrique, *Acta Litt. Scient. Szeged*, v. 8, 1936, pp. 1–6.

[243] W. Sierpiński, Sur un probleme concernant les fonctions semi-continues, *Fund. Math.*, v. 28, 1937, pp. 1–6.

[244] W. Sierpiński, Les correspondances multivoques et l'axiome du choix, *Fund. Math.*, v. 34, 1947, pp. 39–44.

[245] W. Sierpiński, *Introduction to Set Theory and Topology*, PZWS, Warszawa, 1947 (in Polish).

[246] W. Sierpiński, *Hypothése du Continu*, 2-nd Edition, Chelsea Publ. Co., New York, 1956.

[247] W. Sierpiński, *Cardinal and Ordinal Numbers*, PWN, Warszawa, 1958.

[248] W. Sierpiński, *Oeuvres Choisies*, Volumes I, II, III, PWN, Warszawa, 1972-1976.

[249] W. Sierpiński, A. Zygmund, Sur une fonction qui est discontinue sur tout ensemble de puissance du continu, *Fund. Math.*, v. 4, 1923, pp. 316–318.

[250] L. L. Silverman, On the definition of the sum of a divergent series, *University of Missouri Studies, Math. Series 1*, 1913, pp. 1-96.

[251] J. C. Simms, Sierpiński's theorem, *Simon Stevin*, v. 65, n. 1–2, 1991, pp. 69–163.

[252] S. G. Simpson, Which set existence axioms are needed to prove the Cauchy–Peano theorem for ordinary differential equations?, *The Journal of Symbolic Logic*, v. 49, 1984, pp. 783–802.

[253] A. V. Skorokhod, *Integration in Hilbert Space*, Izd. Nauka, Moscow, 1975 (in Russian).

[254] B. S. Sodnomov, On arithmetical sums of sets, *Dokl. Akad. Nauk SSSR*, v. 80, n. 2, 1951, pp. 173–175 (in Russian).

[255] B. S. Sodnomov, An example of two G_δ-sets whose arithmetical sum is not Borel measurable, *Dokl. Akad. Nauk SSSR*, v. 99, 1954, pp. 507–510 (in Russian).

[256] S. Solecki, On sets nonmeasurable with respect to invariant measures, *Proc. Amer. Math. Soc.*, v. 119, n. 1, 1993, pp. 115–124.

[257] S. Solecki, Measurability properties of sets of Vitali's type, *Proc. Amer. Math. Soc.*, v. 119, n. 3, 1993, pp. 897–902.

[258] R. M. Solovay, A model of set theory in which every set of reals is Lebesgue measurable, *Ann. Math.*, v. 92, 1970, pp. 1–56.

[259] R. M. Solovay, Real-valued measurable cardinals, *Proceedings of Symposia in Pure Mathematics*, v. XII, 1971, pp. 397–428.

[260] J. Stallings, Fixed point theorems for connectivity maps, *Fund. Math.*, v. 47, 1959, pp. 249–263.

[261] H. Steinhaus, Sur les distances des points de mesure positive, *Fund. Math.*, v. 1, 1920, pp. 93–104.

[262] J.-P. Sydler, Conditions nécessaires et suffisantes pour l'équivalence des polyédres de l'espace euclidien á trois dimensions, *Comment. Math. Helv.*, v. 40, 1965, pp. 43–80.

[263] E. Szpilrajn (E. Marczewski), Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles, *Fund. Math.*, v. 24, 1935, pp. 17–34.

[264] E. Szpilrajn (E. Marczewski), The characteristic function of a sequence of sets and some of its applications, *Fund. Math.*, v. 31, 1938, pp. 207–223.

[265] M. Talagrand, Sommes vectorielles d'ensembles de mesure nulle, *C. R. Acad. Sci. Paris*, v. 280, 1975, pp. 853–855.

[266] F. Tall, The density topology, *Pacific Journal of Mathematics*, v. 62, 1976, pp. 275–284.

[267] G. P. Tolstov, A note on D. F. Egorov's theorem, *Dokl. Akad. Nauk SSSR*, v. 22, 1939, pp. 309–311 (in Russian).

[268] G. P. Tolstov, On partial derivatives, *Izv. Akad. Nauk SSSR, Ser. Mat.*, v. 13, n. 5, 1949, pp. 425–446 (in Russian).

[269] T. Traynor, An elementary proof of the lifting theorem, *Pacific Journal of Mathematics*, v. 53, 1974, pp. 267–272.

[270] S. Ulam, Zur Masstheorie in der allgemeinen Mengenlehre, *Fund. Math.*, v. 16, 1930, pp. 140–150.

[271] S. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

[272] G. Vitali, Sul problema della misura dei gruppi di punti di una retta, Tip. Gamberini e Parmeggiani, Bologna, 1905.

[273] S. Wagon, *The Banach–Tarski Paradox*, Cambridge University Press, Cambridge, 1985.

[274] J. T. Walsh, Marczewski sets, measure and the Baire property II, *Proc. Amer. Math. Soc.*, v. 106, n. 4, 1989, pp. 1027–1030.

[275] C. Weil, On nowhere monotone functions, *Proc. Amer. Math. Soc.*, v. 56, 1976, pp. 388–389.

[276] H. E. White Jr., Topological spaces in which Blumberg's theorem holds, *Proc. Amer. Math. Soc.*, v. 44, 1974, pp. 454–462.

[277] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, v. 1, v. 2, Cambridge University Press, Cambridge, 1990.

[278] W. Wilczyński, A category analogue of the density topology, approximate continuity and the approximate derivative, *Real Analysis Exchange*, v. 10, n. 2, 1984–1985, pp. 241–265.

[279] W. Wilczyński, A. Kharazishvili, On translations of measurable sets and of sets with the Baire property, *Bull. Acad. Sci. of Georgia*, v. 145, n. 1, 1992, pp. 43–46 (in Russian).

[280] Z. Zahorski, Sur la premiére dérivée, *Trans. Amer. Math. Soc.*, v. 69, 1950, pp. 1–54.

[281] P. Zakrzewski, Extensions of isometrically invariant measures on Euclidean spaces, *Proc. Amer. Math. Soc.*, v. 110, 1990, pp. 325–331.

[282] P. Zakrzewski, Measures on algebraic-topological structures, In: *Handbook of Measure Theory*, North-Holland Publ. Co., Amsterdam, 2002, pp. 1091-1130.

[283] P. Zakrzewski, On a construction of universally small sets, *Real Analysis Exchange*, v. 28, n. 1, 2002–2003, pp. 221–226.

[284] O. Zindulka, Dimension zero vs. measure zero, *Proc. Amer. Math. Soc.*, v. 128, n. 6, 2000, pp. 1769–1778.

[285] A. Zygmund, *Trigonometric Series*, v. 1, Cambridge University Press, Cambridge, 2002.

Index