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12 March 1998

PHYSICS LETTERS B

Physics Letters B 422 (1998) 247–256

# QCD at finite baryon density: nucleon droplets and color superconductivity

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Received 2 December 1997

Editor: H. Georgi

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## Abstract

We use a variational procedure to study finite density QCD in an approximation in which the interaction between quarks is modelled by that induced by instantons. We find that uniform states with conventional chiral symmetry breaking have negative pressure with respect to empty space at all but the lowest densities, and are therefore unstable. This is a precisely defined phenomenon which motivates the basic picture of hadrons assumed in the MIT bag model, with nucleons as droplets of chiral symmetry restored phase. At all densities high enough that the chirally symmetric phase fills space, we find that color symmetry is broken by the formation of a  $\langle qq \rangle$  condensate of quark Cooper pairs. A plausible ordering scheme leads to a substantial gap in a Lorentz scalar channel involving quarks of two colors, and a much smaller gap in an axial vector channel involving quarks of the third color. © 1998 Elsevier Science B.V.

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## 1. Introduction

The behavior of QCD at high density is of fundamental interest and has potential applications to cosmology, to the astrophysics of neutron stars, and to heavy ion collisions. One can make a heuristic, but we think extremely plausible, case that essentially new forms of ordering will emerge in this regime.

Motivated by asymptotic freedom, let us suppose as a starting point that at high density quarks behave nearly freely and form large Fermi surfaces [1]. As we turn on the interactions, we notice that most of the important interquark scattering processes allowed by the conservation laws and Fermi statistics involve large momentum transfer and are therefore weak at asymptotically high density. No matter how weak the

attraction, pairing of the BCS type can be expected if there is an attractive channel [2]. Pairs of quarks cannot be color singlets, and so a  $\langle qq \rangle$  condensate inevitably breaks color symmetry. This breaking is analogous to the breaking of electromagnetic gauge invariance in superconductivity, and might be called ‘color superconductivity’. In this phase, the Higgs mechanism operates and (some) gluons become massive. The proposed symmetry breaking in diquark channels is of course quite different from chiral symmetry breaking in QCD at zero density, which occurs in color singlet quark-antiquark channels.

Our goal here is to explore such new forms of ordering in a context that is definite, qualitatively reasonable, and yet sufficiently tractable that likely patterns of symmetry breaking and rough magnitudes

of their effects can be identified concretely. In the course of looking for new patterns we will need to discuss the fate of the old one, and here a surprise emerges: we find that the uniform phase with broken chiral symmetry is unstable at any nonzero density. At all but the lowest densities, this instability is signalled by negative pressure, which presumably triggers the break-up of the uniform state into regions of high density separated by empty space.

## 2. Model

Several of the methods that are used to good effect, either quantitatively or qualitatively, in analyzing nonperturbative QCD at zero density, do not seem well adapted to finite density. Lattice gauge theory simulations have given very limited results, fundamentally because all known methods having acceptable efficiency rely on importance sampling by Monte Carlo techniques which require positivity of the action configuration by configuration, and this positivity fails at nonzero (real) chemical potential. Extrapolation from supersymmetric models seems hopeless, simply because the nuclear world would be a very different place if one had bosonic quarks (or even baryons). Large  $N_c$  methods are suspect, first because one expects even-odd effects depending on the baryon statistics and furthermore because we shall see concrete effects that depend critically on the precise value  $N_c = 3$ .

We already briefly alluded to the fact that asymptotic freedom suggests qualitatively new types of order at very high density. This regime has been studied [3] by approximating the interquark interactions by one gluon exchange, which is in fact attractive in the color antitriplet channel. Perturbative treatments cannot, by their nature, do full justice to a problem whose main physical interest arises at moderate densities. To get more insight into the phenomena, and in particular to make quantitative estimates, it seems appropriate to analyze a tractable, physically motivated model.

In this letter we present results obtained from a variational treatment of a two-parameter class of models having two flavors and three colors of massless quarks. We leave details of the calculations to a longer paper in preparation. The kinetic part of the

Hamiltonian is that for free quarks, while the interaction Hamiltonian is a slight idealization of the instanton vertex [4] from QCD, explicitly:

$$H_I = -K \int d^3x \bar{\psi}_{R1\alpha} \psi_{Lk}^\gamma \bar{\psi}_{R2\beta} \psi_{Ll}^\delta \varepsilon^{kl} \times (3\delta_\gamma^\alpha \delta_\delta^\beta - \delta_\delta^\alpha \delta_\gamma^\beta) + \text{h.c.}, \quad (2.1)$$

where  $1,2,k,l$  are flavor indices,  $\alpha,\beta,\gamma,\delta$  are color indices, repeated indices are summed, and the spinor indices are not shown. The overall sign is chosen negative for later convenience, so that  $K > 0$  results in chiral symmetry breaking.  $H_I$  is not yet a good representation of the instanton interaction in QCD: in order to mimic the effects of asymptotic freedom, we must modify it in such a way that the interaction decreases with increasing momentum. We write  $H_I$  as a mode expansion in momentum space involving creation and annihilation operators and spinors, and multiply the result by a product of form factors each of the form

$$F(p) = \left( \frac{\Lambda^2}{p^2 + \Lambda^2} \right)^\nu, \quad (2.2)$$

one for each of the momenta of the four fermions. This factorized form is taken for later convenience, and is an idealization.  $\Lambda$ , of course, is some effective QCD cutoff scale, which one might anticipate should be in the range 300–1000 MeV.  $\nu$  parametrizes the shape of the form factor; we consider  $\nu = 1/2$  and  $\nu = 1$ . Since the interaction we have chosen is not necessarily an accurate rendering of QCD,<sup>1</sup> we will have faith only in conclusions that are robust with respect to the parameter choices.

The color, flavor, and Lorentz structure of our interaction has been taken over directly from the instanton vertex for two-flavor QCD. For our purposes it is very important that this interaction properly reflect the chiral symmetry of QCD: axial baryon number symmetry is broken, while chiral  $SU(2) \times SU(2)$  is respected. Color (but *not* axial color) is realized as a global symmetry. There are other four-fermion interactions in addition to  $H_I$  which respect the unbroken symmetries of QCD; using  $H_I$  alone is

<sup>1</sup> For example, one could use a four-fermion interaction based on one gluon exchange [5]

the simplest way of breaking all symmetries broken by QCD, and is therefore a good starting point. This model of fermions interacting via a four-fermion interaction is one in the long line of such models inspired by the work of BCS as adapted to particle physics, starting with Vaks and Larkin [6] and Nambu and Jona-Lasinio [7] and studied subsequently by many others [8]. There is also a tradition, going back more than twenty years and flourishing now perhaps as never before, to model the low-energy dynamics of QCD, and specifically the dynamics of chiral symmetry breaking, with instanton interactions among quarks derived semi-microscopically [9]. This approach cannot explain the other main qualitative aspect of low-energy QCD dynamics, that is strict confinement of quarks, but it is adequate for many purposes and is generating an impressive phenomenology both of real and of numerical experiments.

### 3. Chiral symmetry breaking and restoration

We will first consider symmetry breaking in the familiar pattern known for QCD at zero density. Working first at zero density, we choose a variational wave function of the form

$$\begin{aligned}
 |\psi\rangle = & \prod_{p,i,\alpha} (\cos(\theta^L(p)) \\
 & + e^{i\xi^L(p)} \sin(\theta^L(p)) a_{L i \alpha}^\dagger(p) b_{R i \alpha}^\dagger(-p)) \\
 & \times (\cos(\theta^R(p)) + e^{i\xi^R(p)} \\
 & \times \sin(\theta^R(p)) a_{R i \alpha}^\dagger(p) b_{L i \alpha}^\dagger(-p)) |0\rangle, \quad (3.1)
 \end{aligned}$$

with variational parameters  $\theta$  and  $\xi$  depending on the modes.  $a_{i\alpha}^\dagger$  and  $b_{i\alpha}^\dagger$  create particles and antiparticles respectively, with flavor  $i$  and color  $\alpha$ . This standard pairing form preserves the normalization of the wave function. The pairing occurs between particles and antiparticles with the same flavor and color but opposite helicity and opposite 3-momentum. It is in a Lorentz scalar, isospin singlet component of the chiral (2,2) representation, *i.e.* has the  $\sigma$ -field quantum numbers standard in this context. Following a

well-trodden path we find that the energy is minimized when

$$\begin{aligned}
 \tan(2\theta^L(p)) = \tan(2\theta^R(p)) = \frac{F^2(p)\Delta}{p}; \\
 \xi^L(p) + \xi^R(p) = \pi. \quad (3.2)
 \end{aligned}$$

The full specification of  $\xi^R$  and  $\xi^L$  depends on spinor conventions. The gap parameter  $\Delta$  is momentum independent and is defined by

$$\Delta \equiv 16K \int_0^\infty \frac{p^2 dp}{2\pi^2} F(p) \sin(\theta(p)) \cos(\theta(p)). \quad (3.3)$$

(3.2) and (3.3) are consistent only if  $\Delta = 0$  or if  $\Delta$  satisfies the gap equation

$$1 = 8K \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{F^4(p)}{\sqrt{F^4(p)\Delta^2 + p^2}}, \quad (3.4)$$

Note that, unlike for standard pairing at a Fermi surface, this equation does not have solutions for arbitrarily weak coupling, but only for couplings above a certain threshold value. Also note that although it is common to refer to  $\Delta$  as the gap parameter, it is best thought of as inducing an effective quark mass, which takes the form  $\Delta F^2(p)$ . In the interests of simplicity of presentation, we quote all results in this paper for  $\Delta = 400$  MeV; we have verified that the picture we present is qualitatively unchanged for  $300 \text{ MeV} < \Delta < 500 \text{ MeV}$ . Fixing  $\Delta$  fixes the magnitude (and sign) of the coupling  $K$  for each  $\Lambda$  and  $\nu$ .

We now generalize this calculation to nonzero quark number density  $n$ . It is most favorable energetically to fill the particle states up to some Fermi momentum  $p_F$ , while leaving the corresponding antiparticle states empty. That is, we replace  $|0\rangle$  on the right hand side of (3.1) by  $|p_F\rangle$ , in which all particle states with  $|\mathbf{p}| < p_F$  are occupied. The quark number density,

$$n = \frac{2}{\pi^2} p_F^3, \quad (3.5)$$

is determined by  $p_F$  but as we shall see  $p_F$  is not equal to the chemical potential  $\mu$ . The creation operators in (3.1) for modes below the Fermi surface annihilate  $|p_F\rangle$ , and so for these modes effectively

$\theta(\mathbf{p}) = 0$ . On the other hand for the unoccupied states, with  $|\mathbf{p}| > p_F$ , the variational scheme is unmodified. Note that the condensate does not affect  $n$ . Thus we arrive at a very simple modification of the gap equation: the lower limit of the integral in (3.4) is  $p_F$ , rather than 0. Since the interaction is ever more effectively quenched as  $p_F$  increases, the gap parameter  $\Delta(p_F)$  arising as the solution of (3.4) will shrink monotonically and eventually vanish as  $p_F$  increases. Let us define  $n_c$  to be the critical density at which  $\Delta$  vanishes.

Having obtained a definite wave function, we can evaluate the energy density in terms of the gap parameter. Relative to the energy of the naive vacuum state  $|0\rangle$ , the energy density  $\varepsilon(n)$  is given by

$$\begin{aligned} \frac{1}{24} \varepsilon(n) = & \frac{p_F^4}{16\pi^2} \\ & + \int_{p_F}^{\infty} \frac{p^2 dp}{2\pi^2} \frac{p}{2} \left( 1 - \frac{p}{\sqrt{\Delta^2 F^4(p) + p^2}} \right) \\ & - \frac{\Delta^2}{32K}, \end{aligned} \quad (3.6)$$

where  $p_F(n)$  is to be obtained from (3.5). The first and second terms are the kinetic energies of the modes respectively below and above the nominal Fermi surface, while the third term is the interaction energy. It is startling, perhaps, that in the first term the bare energy occurs, with no effective mass, but this is a direct reflection of the situation discussed in the previous paragraph. For a proper interpretation, however, it is important to note that adding a particle at the Fermi surface modifies both the kinetic and the interaction term, and that these conspire to give a gap or effective mass  $F(p_F)^2 \Delta$  for the physical excitations there. The chemical potential  $\mu$ , the minimum energy required to add one more quark to the state, is given by  $\mu = \partial\varepsilon/\partial n$  at fixed volume. We have verified that  $\mu^2 = p_F^2 + \Delta^2 F^4(p_F)$ . Note that  $\varepsilon(0) < 0$ , reflecting the fact that the physical vacuum state with its chiral condensate has a lower energy than the state  $|0\rangle$ . Measured relative to the physical vacuum, the energy density at nonzero  $n$  is  $e(n) \equiv \varepsilon(n) - \varepsilon(0)$ .

The equation for  $\Delta(p_F)$  can be solved numeri-

cally, and the resulting energy evaluated. Great physical interest attaches to the pressure

$$P(n) = n \frac{\partial e}{\partial n} - e = n^2 \frac{\partial}{\partial n} \left( \frac{e}{n} \right) \quad (3.7)$$

of a region with density  $n$ , where the pressure of the physical vacuum is by definition zero. For all values of the parameters that we consider reasonable, we find that after a tiny interval of very low densities at which the pressure is positive, *the pressure becomes negative, and continues to decrease until the critical density  $n_c$  at which chiral symmetry is restored*. At that point we switch over to an essentially free quark phase, in which the energy density relative to the physical vacuum is  $3p_F^4/(2\pi^2) - \varepsilon(0)$ , and the pressure is given by  $p_F^4/(2\pi^2) + \varepsilon(0)$ . At  $n = n_c$ , where these phases join, the pressure function is continuous, with a finite negative value, but it has a cusp. As density increases in the free-quark phase, the pressure increases monotonically, and passes through zero again at some density  $n_0 > n_c$ . At  $n = n_0$ , the energy per quark  $e/n$  is minimized and this phase is stable at zero pressure. Representative results are recorded in Table 1, and  $\Delta$  and  $P$  are shown for one set of parameters in Fig. 1.

Evidently, at all but the lowest densities (which we discuss below) in the presence of a chiral condensate the negative pressure associated with increasing vacuum energy overcompensates the increasing Fermi pressure. This negative pressure signals mechanical instability of the uniform chiral symmetry broken phase. There is an attractive physi-

Table 1

Comparison of the Fermi momenta corresponding to the density  $n_c$  at which the chiral gap vanishes, the density  $n_0$  at which there is a stable zero pressure phase, and the energy density  $\varepsilon(0)$  of the vacuum phase, for six choices of parameters. In each case the coupling is fixed by requiring  $\Delta = 0.4$  GeV at  $n = 0$ . All numbers are in GeV

$\nu$	$\Lambda$	$p_F(n_c)$	$p_F(n_0)$	$\varepsilon(0)$
1	0.6	0.212	0.268	$-(0.127)^4$
1	0.8	0.246	0.298	$-(0.141)^4$
1	1.0	0.274	0.321	$-(0.152)^4$
0.5	0.6	0.262	0.310	$-(0.147)^4$
0.5	0.8	0.298	0.340	$-(0.161)^4$
0.5	1.0	0.328	0.362	$-(0.172)^4$

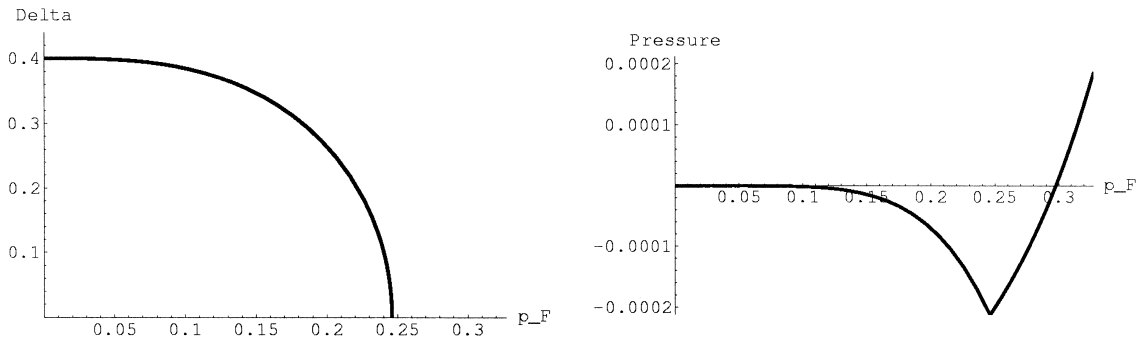


Fig. 1. Chiral gap (in GeV) and the pressure (in  $\text{GeV}^4$ ) as a function of  $p_F = (n\pi^2/2)^{1/3}$  in GeV. The pressure is positive at very small  $p_F$ ; it becomes negative at a  $p_F$  which is less than 0.001. At  $n = n_c$ , the gap vanishes and the pressure, which is still negative, has a cusp. At  $n = n_0 > n_c$ , the pressure crosses zero and becomes positive.

cal interpretation of this phenomenon. The uniform nonzero density phase will break up into stable droplets of high density  $n = n_0$  in which the pressure is zero and chiral symmetry is restored, surrounded by empty space with  $n = P = 0$ . There are preliminary indications of this behavior in numerical simulations of a model similar to ours [11]. Although our simple calculations do not allow us to follow the evolution and eventual stabilization of the original quark cloud, it is hard to avoid identifying the droplets of chiral symmetric phase into which it condenses with physical nucleons. Nothing within the model tells us that the stable droplets have quark number 3; nucleons are simply the only candidates in nature which can be identified with droplets within which the quark density is nonzero and the chiral condensate is zero. If correct, this identification is very reminiscent of the MIT bag philosophy, here arising in the description of a sharply defined physical phenomenon.<sup>2</sup> It seems quite different, at least superficially, from the Skyrme model, where the chiral symmetry order parameter changes in direction but not in magnitude within the nucleon.

What, then, of the positive pressure phase at very low density? Without external pressure, this dilute gas of quarks with mass  $\Delta(0)$  would expand and dissipate. Even if some external pressure prevents

expansion, however, this phase is only metastable: its energy per quark  $e/n \sim \Delta(0)$  is greater than that in the stable phase at  $n = n_0$ , which satisfies

$$\frac{e(n_0)}{n_0} = p_F(n_0) \quad (3.8)$$

and can therefore be read off Table 1. When fluctuations in this dilute gas increase the density in a region enough that the pressure becomes negative, this region can collapse to density  $n_0$ . In this way, the metastable phase converts to regions of the stable phase at  $n = n_0$ , surrounded by vacuum. (Note that at the density at which the pressure first becomes negative,  $e/n$  is at a local maximum and so this zero pressure phase is unstable, unlike that at  $n = n_0$ .) We see that the fate of the low density positive pressure phase is the same as that of the negative pressure phase. Any uniform phase with chiral symmetry broken evolves into an inhomogeneous mixture of droplets within which  $n = n_0$  and chiral symmetry is restored, surrounded by regions of vacuum.

The satisfying picture just discussed is not obtained for all parameter values, however. For example, for  $\nu = 1$  and  $2.2 < \Lambda < 3.2$  the phase at  $n = n_0$  has higher  $e/n$  than that of a dilute gas of quarks with mass  $\Delta(0)$ . For  $\Lambda > 3.2$  GeV, the pressure is positive for all  $n$ . In these (fortunately, unreasonable) parameter ranges, the model, without further modification, has no reasonable physical interpretation.

At a quantitative level, a naive implementation of our proposed identification of droplets of  $n = n_0$  matter with nucleons works surprisingly well. One

<sup>2</sup> Considerations similar to those we describe also lead Buballa [10] to conclude that in a Nambu Jona-Lasinio model with an interaction which differs from the one we use, matter with broken chiral symmetry is unstable and nucleons can therefore only be viewed as bags within which chiral symmetry is restored.

might want to identify  $n_0$  with the quark density at the center of baryons. Taking this to be three times nuclear matter density yields  $p_F(n_0) \sim 0.39$  GeV. On the other hand, requiring  $e(n_0)/n_0$  to be one third the nucleon mass yields  $p_F(n_0) \sim 0.31$  GeV. Our toy model treatment cannot meet both criteria simultaneously, which is not surprising, but we see from Table 1 that the magnitude of  $p_F(n_0)$  is very reasonable. The vacuum energy, which becomes the bag constant, is also of the correct order of magnitude. Adding further interactions to  $H_I$  would obviously make a quantitative difference, but there is no reason to expect the qualitative picture to change.

In any case, the physical picture suggested here has significant implications for the phase transition, as a function of density, to restored chiral symmetry. Since the nucleons are regions where the symmetry is already restored, the transition should occur by a mechanism analogous to percolation as nucleons merge [12]. This transition should be complete once a density characteristic of the center of nucleons is achieved. The fact that some external pressure must be imposed in order to induce the nucleons to merge (e.g. the fact that in nuclear matter at zero pressure the nucleon droplets remain unmerged) must reflect interactions between droplets, which we have not treated here. The mechanism of chiral symmetry restoration at finite density but zero temperature is quite different from the one we expect at finite temperature and zero density: it occurs by percolation among pre-formed bags of symmetric phase. Of course, this is no contradiction, because the finite temperature transition occurs at such a low temperature that few baryons are present.

#### 4. Color superconductivity

At high density, pairing of particles near the Fermi surface as in the original BCS scheme [2] becomes more favorable. Our Hamiltonian supports condensation in quark-quark channels. The condensation is now between identical fermions with the same helicity, and the Hamiltonian selects chiral isosinglets — that is, antisymmetry in flavor. One can therefore have spin 0 — antisymmetric in spin and therefore in color, forming a  $\bar{\mathbf{3}}$ , or spin 1 —

symmetric in spin and therefore in color, forming a  $\mathbf{6}$ .

We first consider the former. A suitable trial wave function is

$$|\psi\rangle = G_L^\dagger G_R^\dagger |p_F\rangle \quad (4.1)$$

where

$$\begin{aligned} G_L^\dagger = & \prod_{\alpha, \beta, \mathbf{p}} \left( \cos(\theta_A^L(\mathbf{p})) + \varepsilon^{\alpha\beta 3} e^{i\xi_A^L(\mathbf{p})} \right) \\ & \times \sin(\theta_A^L(\mathbf{p})) a_{L1\alpha}^\dagger(\mathbf{p}) a_{L2\beta}^\dagger(-\mathbf{p}) \\ & \times \left( \cos(\theta_B^R(\mathbf{p})) + \varepsilon^{\alpha\beta 3} e^{i\xi_B^R(\mathbf{p})} \right) \\ & \times \sin(\theta_B^R(\mathbf{p})) b_{R1\alpha}^\dagger(\mathbf{p}) b_{R2\beta}^\dagger(-\mathbf{p}) \\ & \times \left( \cos(\theta_C^R(\mathbf{p})) + \varepsilon^{\alpha\beta 3} e^{i\xi_C^R(\mathbf{p})} \right) \\ & \times \sin(\theta_C^R(\mathbf{p})) a_{R1\alpha}(\mathbf{p}) a_{R2\beta}(-\mathbf{p}) \end{aligned}$$

$$G_R^\dagger = \text{same, with } R \leftrightarrow L. \quad (4.2)$$

Here,  $\alpha$  and  $\beta$  are color indices, and we have chosen to pair quarks of the first two colors. 1 and 2 are flavor indices. The first term in (4.2) creates particles above the Fermi surface; the second creates antiparticles; the third creates holes below the Fermi surface. In this state, the Lorentz scalar  $\langle q^{i\alpha} C \gamma^5 q^{j\beta} \varepsilon_{ij} \varepsilon_{\alpha\beta 3} \rangle$  is nonzero. This singles out a preferred direction in color space and breaks color  $SU(3) \rightarrow SU(2)$ . The  $U(1)$  of electromagnetism is spontaneously broken but there is a linear combination of electric charge and color hypercharge under which the condensate is neutral, and which therefore generates an unbroken  $U(1)$  gauge symmetry. No flavor symmetries, not even the chiral ones, are broken.

Note that  $n$  is now not given by (3.5) because the operators in (4.2) can change particle number. Varying the expectation value of  $H - \mu N$  in this state with respect to  $p_F$  yields  $p_F = \mu$ , unlike in the case of the chiral condensate. This difference reflects the fact that a gap in a  $\langle qq \rangle$  channel does not act as an effective mass term in the way that a chiral gap does. Upon adding a quark, the condensate can adjust in such a way that the energy cost is only  $p_F$ .  $\Delta$  is, however, a true gap in the sense of condensed matter physics: the energy cost of making a particle-hole excitation is  $2\Delta$  at minimum. Varying with respect

to all the other variational parameters yields  $\xi_{A,B,C}^R + \xi_{A,B,C}^L = \pi$ ,  $\theta_{A,B,C}^R = \theta_{A,B,C}^L$ , and

$$\begin{aligned} \tan(2\theta_A^L(p)) &= \frac{F^2(p)\Delta}{p-\mu}, \\ \tan(2\theta_B^L(p)) &= \frac{F^2(p)\Delta}{p+\mu}, \\ \tan(2\theta_C^L(p)) &= \frac{F^2(p)\Delta}{\mu-p}. \end{aligned} \quad (4.3)$$

Here, the gap  $\Delta$  satisfies a self-consistency equation of the form

$$\begin{aligned} 1 = 2K \left\{ \int_{\mu}^{\infty} \frac{p^2 dp}{2\pi^2} \frac{F^4(p)}{\sqrt{F^4(p)\Delta^2 + (p-\mu)^2}} \right. \\ + \int_0^{\infty} \frac{p^2 dp}{2\pi^2} \frac{F^4(p)}{\sqrt{F^4(p)\Delta^2 + (p+\mu)^2}} \\ \left. + \int_0^{\mu} \frac{p^2 dp}{2\pi^2} \frac{F^4(p)}{\sqrt{F^4(p)\Delta^2 + (\mu-p)^2}} \right\}. \end{aligned} \quad (4.4)$$

The three terms in this equation arise respectively from particles above the Fermi surface, antiparticles, and particles below the Fermi surface. For  $\mu > 0$  the particle and hole integrals diverge logarithmically at the Fermi surface as  $\Delta \rightarrow 0$ , which signals the possibility of condensation for arbitrarily weak attraction.

The numerical coefficient in this color superconducting gap equation is smaller than the corresponding coefficient in the chiral symmetry breaking case. The exact factor follows from the precise form of the Hamiltonian, but part of the explanation is simple and robust: chiral condensation makes good use of all three colors coherently, but the color superconducting condensation, which breaks color symmetry, cannot.

One can form reasonable qualitative expectations for the solution of the gap equation without detailed calculations. Because the numerical coefficient in the gap equation is smaller than the threshold value at which one would have a nonzero  $\Delta$  at  $\mu = 0$ ,  $\Delta$  would be zero were it not for the logarithmically divergent contribution to the integral from the region near  $\mu$ . This means that at small  $\mu$ , the gap must be small because the density of states at the Fermi

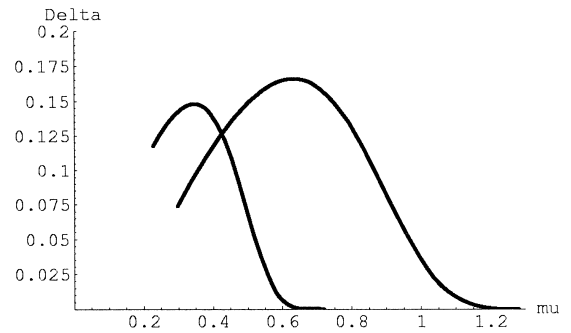


Fig. 2. Gap created by the Lorentz scalar color superconductor condensate, as a function of  $\mu = p_F$  for  $\nu = 1$  and (from left to right)  $\Lambda = 0.4, 0.8$  GeV. Each curve begins where  $n$  is given by the appropriate  $n_0$ .

surface is small. This has only formal significance, because the only densities of physical relevance are  $n = 0$  and  $n \geq n_0$ . At intermediate densities, matter is in an inhomogeneous mixture of the  $n = 0$  and  $n = n_0$  phases. (We are assuming that the color breaking condensate does not significantly affect  $n_0$ ; this will be discussed below.) As  $\mu$  increases, the density of states at the Fermi surface increases and the gap parameter grows. Finally, at large  $\mu$  the effect of the form factor  $F$  is felt, the effective coupling decreases, and the gap parameter goes back down. For the parameter ranges we have examined the gap parameter is quite substantial:  $\sim 50$ – $150$  MeV at  $n_0$ , and peaking at  $100$ – $200$  MeV at a density somewhat higher. We plot  $\Delta$  for two sets of parameters in Fig. 2. The density at which the gap peaks depends on  $\Lambda$ ; the shape of the curve depends on  $\nu$ ; the height of the curve is almost independent of both.

As in the previous section, one can obtain expressions for the energy and the density, and thus derive the equation of state. We find that the equation of state is hardly modified from the free-quark values — the pressures, at equal density, are equal to within a few per cent. This makes it very plausible that, as we assume, the color condensation makes only a small change in the density  $n_0$  at which a stable phase exists at zero pressure. To make the argument rigorous, we must do a calculation in which we consider chiral and color condensation simultaneously; we should form a trial wave function that allows for both possibilities and allows them to

compete. We have begun this calculation, but will not report on it here other than to note that since the two condensates compete for the same quarks the bigger of the two tends to suppress the smaller. This is further evidence that the potential for the formation of a color breaking gap does not affect the result that  $n_0 > n_c$ . For practical purposes, it appears to be a very good approximation to treat the condensations separately, as we do here, because where one is large the other is small, even before they compete.

The third color has so far been left out in the cold, but we can gain energy by allowing it too to condense. The available channel is the color **6**. We find that there is attraction in this channel, for a pairing that is a spatial axial vector. Thus we predict that not only color, but also rotation invariance, is spontaneously broken by QCD at high density. Axial vector quantum numbers, of course, are characteristic of spin alignment. The appropriate trial wave function for this condensation is obtained by acting on  $|p_F\rangle$  by a product of operators analogous to the product in (4.2) except that they now involve only the third color and depend on the direction in momentum space. The particle pair creation operator, for example, can be written

$$\frac{1}{\sqrt{1 + [\theta_L^A(|\mathbf{p}|) p_z / |\mathbf{p}|]^2}} \times \left( 1 + \theta_A^L(|\mathbf{p}|) \frac{p_z}{|\mathbf{p}|} e^{i\xi_L^t(p)} a_{L13}^\dagger(\mathbf{p}) a_{L23}^\dagger(-\mathbf{p}) \right). \quad (4.5)$$

In this state, the nonzero condensate is  $\langle q^{i3} C \sigma_{03} q^{j3} \epsilon_{ij} \rangle$ , where  $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$ . This is an axial vector pointing in the  $z$ -direction, manifestly breaking rotation invariance. It does not change sign under spatial inversion. It breaks the gauged  $U(1)$  symmetry left unbroken by the scalar superconductor condensate, but does not affect the unbroken  $SU(2)$  subgroup of color  $SU(3)$ .

Finding the self-consistent solution to the variational problem for the axial condensate is somewhat involved, because of the expressions which arise when the angular integrals in  $\langle H \rangle$  are performed. The expression analogous to  $\tan(2\theta)$  of (3.2) cannot explicitly be inverted, so it is not possible to reduce the two equations analogous to (3.2) and (3.3) to a

single gap equation. The results are nevertheless easily described. We define a momentum independent parameter  $\Delta$  which can be viewed as the average of the gap over the Fermi surface, and which satisfies self-consistency relations. The effective coupling is smaller than in the scalar channel for two reasons. First, only one color participates. Second, most regions of the Fermi surface do not participate fully. We find that the gap must be pushed to very small values. Because of the logarithmic singularity in the integrals, the resulting gap is exponentially sensitive to parameter choices. We find  $\Delta$  of order a few keV at most, but this should not be considered a robust result. It is worth exploring whether plausible interactions can be added to  $H_I$  which have the effect of strengthening the axial vector condensate, particularly as such a condensate could lead to signatures in heavy ion collisions. The existence of a preferred direction for spins could be observable, if it were reasonably efficiently handed down to  $\Lambda$  baryons, as it would lead to correlation between the polarization of different  $\Lambda$ 's in a single event. Without modification, our model suggests that this axial vector condensate is *much* smaller than the scalar color breaking condensate. A gap this small is surely irrelevant in heavy ion collisions, but has both formal consequences and implications for neutron star physics. It is striking that a single interaction generates coexisting condensates with scales which differ by five orders of magnitude.

In our model as it stands, color is realized as a global symmetry. Breaking of this symmetry generates Nambu-Goldstone bosons, formally. However, in reality color is of course a gauge symmetry, and the true spectrum does not contain massless scalars, but rather massive vectors. Aside from a node along the equator of the Fermi surface for one color, there is a gap everywhere on the quark Fermi surfaces. To this point, we have described the color superconducting phase as a Higgs phase. One expects, however, that there is a complementary description in which this is a confining phase, albeit one with two vastly different confinement lengths, neither of which is related to the confinement length at zero density. As a formal matter, it is of some interest that the color superconducting phase can be considered a realization of *confinement without chiral symmetry breaking*.



In looking for signatures of color superconductivity in heavy ion physics and in neutron stars, it is unfortunate that the equation of state is almost equal to that for a deconfined phase with no diquark condensate. Superconducting condensates do modify the gauge interactions and this may have implications in heavy ion collisions. The scalar condensate carries electric as well as color charge. It is neutral under a certain combination of electrodynamic and color hypercharge, so taken by itself it would leave a modified massless photon. If densities above  $n_0$  are achieved at low enough temperatures that the scalar condensate forms, there will be a mixture of the ordinary photon and the color hypercharge gauge boson which is massless. (This modified photon would acquire a small mass from the axial vector condensate if temperatures were low enough for this condensate to be present.) There is also a residual  $SU(2)$  gauge symmetry, presumably deconfined, and there are five gluons whose mass is set by the scalar condensate. Either the modification of the photon or the loss of massless gluons could have consequences, but dramatic effects do not seem apparent.

Turning to neutron star phenomenology, there is some indication, from the slowness of observed neutron star cooling rates, that a gap in the excitation spectrum for quark matter might be welcome [13], as this suppresses neutrino emission via weak interaction processes involving single thermally excited  $u$  and  $d$  quarks by  $\exp(-\Delta/T)$ . A 400 keV gap has dramatic consequences [14]; the scalar gap is therefore enormous in this context, and the axial vector gap plays a role too, shutting down these direct neutrino emission processes completely once the core cools to temperatures at which the axial condensate forms. It would also be worthwhile to explore the effects of the presence of macroscopic regions in which an axial vector condensate is ordered.

## 5. Discussion

Many things were ignored in this analysis. Most important, perhaps, is the strange quark. In the spirit of the analysis, we should consider the modified instanton vertex including the strange quark as well. This adds an incoming left-handed and an outgoing right-handed leg. If the mass of this quark were

large, we could connect these legs with a large coefficient, and reduce to the previous case, perhaps with an additional four-fermion vertex involving all three flavors modelled on one-gluon exchange. Whatever the interaction(s), color superconductivity in a three flavor theory necessarily introduces the new feature of flavor symmetry breaking. Both the condensates considered in this paper are flavor singlets; this is impossible for a  $\langle qq \rangle$  condensate in a three flavor theory. One particularly attractive possibility is condensation in the  $\langle q_i^\alpha C \gamma^5 q_j^\beta \varepsilon^{ijA} \varepsilon_{\alpha\beta A} \rangle$  channel, with summation over  $A$ . This breaks flavor and color in a coordinated fashion, leaving unbroken the diagonal subgroup of  $SU(3)_{\text{color}} \times SU(3)_{\text{flavor}}$ .

Another question concerns the postulated Hamiltonian. While there are good reasons to take an effective interaction of the instanton type as a starting point, there could well be significant corrections affecting the more delicate consequences such as axial vector  $\mathbf{6}$  condensation. A specific, important example is to compare the effective interaction derived from one-gluon exchange. It turns out that this interaction has a similar pattern, for our purposes, to the instanton: it is very attractive in the  $\sigma$  channel, attractive in the color antitriplet scalar, and neither attractive nor repulsive in the color sextet axial vector. More generally, it would be desirable to use a renormalization group treatment to find the interactions which are most relevant near the Fermi surface.

The qualitative model we have treated suggests a compelling picture both for the chiral restoration transition and for the color superconductivity which sets in at densities just beyond. It points toward future work in many directions: the percolation transition must be characterized; consequences in neutron star and heavy ion physics remain to be elucidated; the superconducting ordering patterns may hold further surprises, particularly as flavor becomes important. The whole subject needs more work; the microscopic phenomenon is so remarkable, that we suspect our imaginations have failed adequately to grasp its implications.

## Acknowledgements

Related work on color superconductivity has been done independently by R. Rapp, T. Schaefer, E. V.

Shuryak and M. Velkovsky. We wish to thank them and M. Gyulassy, R. L. Jaffe, M. Prakash, H. Satz, M. Stephanov and I. Zahed for helpful discussions. We thank the RIKEN-BNL Center, where this work was completed, for its hospitality.

The research of MA and FW is supported in part by DOE grant DE-FG02-90ER40542; that of KR is supported in part by DOE cooperative research agreement DE-FC02-94ER40818.

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