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Testing Quantum Nonlocality in Phase Space

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We propose an experimental scheme for testing nonlocality of a correlated two-mode quantum state of light. We show that the correlation functions violating the Bell inequalities in the proposed experiment are equal to the joint two-mode Q function and the Wigner function. This assigns a novel operational meaning to these two quasidistribution functions in tests of quantum nonlocality and also establishes a direct relationship between two intriguing aspects of quantum mechanics: the nonlocality of entangled states and the noncommutativity of quantum observables, which underlies the nonclassical structure of the phase-space quasidistribution functions. [S0031-9007(99)08691-3]

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A fundamental step providing a bridge between classical and quantum physics has been given by Wigner in the form of a quantum mechanical phase-space distribution: the Wigner function [1]. From the pioneering work of Weyl, Wigner, and Moyal, it follows that the noncommutativity of quantum observables leads to a real abundance of different-in-form quantum mechanical phase-space quasi-distributions. This provided a milestone step towards a c-number formulation of quantum effects in phase space and led to the development of efficient theoretical tools in various fields of modern physics [2].

Because of Einstein, Podolsky, and Rosen [3], followed by the seminal contribution of Bell [4], the meaning of quantum reality and quantum nonlocality has become a central issue of the modern interpretation and understanding of quantum phenomena [5]. Concepts such as entanglement and quantum nonlocality have generated a real flood of theoretical work devoted to various connections of the quantum description with different views or representations of the quantum formalism.

Despite all of these theoretical works a direct link between various phase-space distributions and the nonlocality of quantum mechanics has been missing. In several works [6,7] the quantum phase space has been treated as a model for a hidden variable theory, and the incompatibility of quantum mechanics with local theories has been discussed in connection with the nonpositive character of the Wigner function. However, no direct link between these two aspects of quantum theory has been found, and it has been argued that these two issues are, in fact, rather loosely connected [7].

It is the purpose of this Letter to assign a direct role to phase-space quasidistribution functions in demonstrating quantum nonlocality. We propose an experimental test of nonlocal effects in phase space. The quantum entanglement will be represented by an arbitrary correlated state of light, which refers to two spatially separated modes of the electromagnetic field. We show that the proposed experiment establishes a direct relationship between quantum nonlocality and the positive phase-space Q function, as well as the nonpositive Wigner function. We demonstrate that for a certain class of experiments these two quasi-probability distributions *are* nonlocal correlation functions violating Bell's inequalities. This result assigns a novel operational meaning to these quasidistribution functions.

In this Letter we propose a photon counting experiment which leads directly to a measurement that is described by the phase-space Q function or the Wigner function. We show that these functions are given by joint photon count correlations and as such can be used to test local realism in the form of Bell's inequalities. Our approach is different from all the previous discussions involving the relation of quantum nonlocality and the phase-space quasiprobability distributions. To the best of our knowledge, no such direct relation between various phase-space quasidistributions and the nonlocality of quantum correlations has ever been

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satisfactorily established. The general character of the scheme proposed in this Letter allows one to test an arbitrary entangled state of light. Moreover, the measured photon count correlations revealing the nonlocality have a natural theoretical description in terms of phase-space quasidistribution functions.

The link of quantum nonlocality to the Q function is a rather striking result, since this particular distribution function is positive everywhere, which has been considered as a loss of quantum properties due to simultaneous measurement of canonically conjugated observables.

The setup to demonstrate quantum nonlocality in the phase space is presented in Fig. 1. For concreteness, we will take the source of the correlated state of light to be a single photon impinging onto a 50:50 beam splitter. We label the outgoing modes a and b. From the following discussion it will be obvious that the same scheme can be used to test the nonlocal character of any correlated state of modes a and b and that the corresponding Wigner and Q functions will play the same operational role of nonlocal correlations. The quantum state of our exemplary source, written in terms of the outgoing modes, is of the form analogous to the singlet state of two spin-1/2 particles [8]:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b - |0\rangle_a|1\rangle_b). \tag{1}$$

We will now demonstrate how nonlocality of this state is revealed by the Wigner and the Q functions.

Each of the measuring apparatuses in our setup consists of a photon counting detector preceded by a beam splitter with the power transmission T. The second input port of the beam splitter is fed with a highly excited coherent state $|\gamma\rangle$. As is known [9], in the limit $T \to 1$ and $\gamma \to \infty$, the effect of the beam splitter is described by the displacement operator $\hat{D}(\sqrt{1-T}\,\gamma)$ with the parameter equal to the amplitude of the reflected part of the coherent state. In the following, we will assume that this limit describes sufficiently well the measuring apparatuses.

The first type of the measurement we will consider is the test for the presence of photons. This is a more realistic case, as the most efficient detectors available currently for single-photon level light, namely, the avalanche photodiodes operating in the Geiger mode, are not capable of resolving the number of photons that triggered the output signal. This type of measurement is described by a pair of two orthogonal projection operators depending on the coherent displacement $\alpha = \sqrt{1-T}\,\gamma$:

$$\begin{aligned} \hat{Q}(\alpha) &= \hat{D}(\alpha)|0\rangle\langle 0|\hat{D}^\dagger(\alpha), \\ \hat{P}(\alpha) &= \hat{D}(\alpha)\sum_{n=1}^{\infty}|n\rangle\langle n|\hat{D}^\dagger(\alpha), \end{aligned} \tag{2}$$

which satisfy the completeness relation:

$$\hat{Q}(\alpha) + \hat{P}(\alpha) = \hat{\mathbb{1}}. \tag{3}$$

In the following, we will use the indices a and b to refer to the two apparatuses.

In contrast to the standard approach, we will be interested in events when *no photons* were registered. Let us assign 1 to no-count events and 0 otherwise. This establishes a strict analogy with two-particle coincidence experiments, where each of the spatially separated analyzers provides a binary outcome. The role of adjustable parameters of the analyzers is now played by coherent displacements α and β. Consequently, all Bell inequalities derived for a measurement of local realities bounded by 0 and 1 can be applied to test the nonlocal character of correlations obtained in our setup.

The joint quantum mechanical probability of no-count events simultaneously in both the detectors is

$$\begin{aligned} Q_{ab}(\alpha,\beta) &= \langle\Psi|\hat{Q}_a(\alpha)\otimes\hat{Q}_b(\beta)|\Psi\rangle \\ &= \frac{1}{2}|\alpha - \beta|^2 e^{-|\alpha|^2-|\beta|^2}, \end{aligned} \tag{4}$$

where α and β are coherent displacements for the modes a and b, respectively. The probabilities on single detectors are

$$\begin{aligned} Q_a(\alpha) &= \langle\Psi|\hat{Q}_a(\alpha)\otimes\hat{\mathbb{1}}_b|\Psi\rangle = \frac{1}{2}(|\alpha|^2+1)e^{-|\alpha|^2}, \\ Q_b(\beta) &= \langle\Psi|\hat{\mathbb{1}}_a\otimes\hat{Q}_b(\beta)|\Psi\rangle = \frac{1}{2}(|\beta|^2+1)e^{-|\beta|^2}. \end{aligned} \tag{5}$$

The measurement is now performed for two settings of the coherent displacement in each of the apparatuses: zero

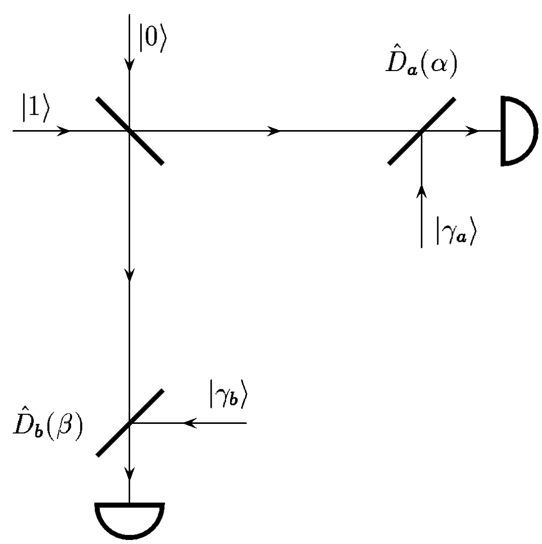

FIG. 1. The optical setup proposed to demonstrate quantum nonlocality in phase space. The exemplary source of nonclassical correlated radiation is a single photon incident on a 50:50 beam splitter, which generates a quantum singletlike state. The measuring devices are photon counting detectors preceded by beam splitters. The beam splitters have the transmission coefficient close to one and strong coherent states injected into the auxiliary ports. In this limit, they effectively perform coherent displacements $\hat{D}_a(\alpha)$ and $\hat{D}_b(\beta)$ on the two modes of the input field.

or α for mode a and zero or β for mode b. From the resulting four different correlation functions we build the Clauser-Horne combination [10]:

$$\mathcal{CH} = Q_{ab}(0,0) + Q_{ab}(\alpha,0) + Q_{ab}(0,\beta) - Q_{ab}(\alpha,\beta) - Q_a(0) - Q_b(0), \qquad (6)$$

which for local theories satisfies the inequality $-1 \leq \mathcal{CH} \leq 0$. We will take the coherent displacements to have equal magnitudes $|\alpha|^2 = |\beta|^2 = \mathcal{J}$ and an arbitrary phase difference $\beta = e^{2i\varphi}\alpha$. For these values we obtain

$$\mathcal{CH} = -1 + \mathcal{J}e^{-\mathcal{J}} - 2\mathcal{J}e^{-2\mathcal{J}}\sin^2\varphi. \qquad (7)$$

As depicted in Fig. 2, this result violates the lower bound imposed by local theories. The violation is most significant for the phase φ which minimizes the last term in Eq. (7). This takes place when the coherent displacements have opposite phases $\beta = -\alpha$.

The only measurement that is required to demonstrate the nonlocality of this state requires single and joint registration of *no photons*. When the state is not shifted, this measurement is described by the projection on the vacuum state $|0\rangle$. Furthermore, application of a coherent displacement $\hat{D}(\alpha)$ is equivalent to the projection on a coherent state $|\alpha\rangle$. And here comes the most striking link of the quantum nonlocality with the phase-space quasidistribution: $Q_{ab}(\alpha,\beta)$ is consequently equal, up to a multiplicative constant $1/\pi^2$, to the joint Q function of the state $|\Psi\rangle$. The operator $\hat{Q}(\alpha)$, defined above, represents a projection on a coherent state $|\alpha\rangle$, and the correlation function is

$$Q_{ab}(\alpha,\beta) = |\langle\alpha,\beta\,|\,\Psi\rangle|^2, \qquad (8)$$

where $|\alpha,\beta\rangle = |\alpha\rangle_a \otimes |\beta\rangle_b$. The probabilities of no-count events on single detectors are given by marginal Q functions:

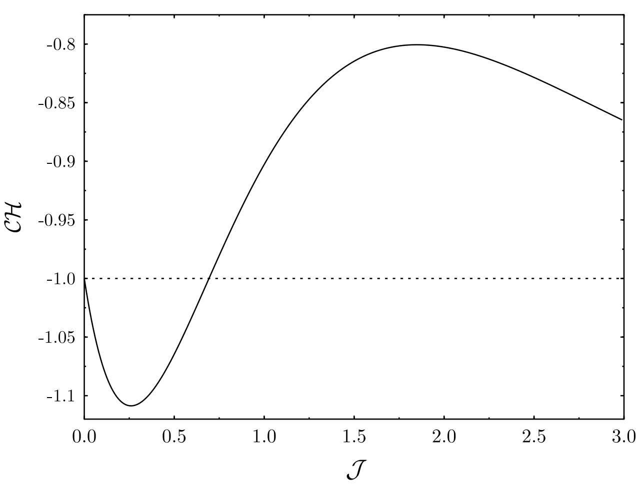

FIG. 2. The plot of the Clauser-Horne combination defined in Eq. (6) as a function of the intensity of coherent displacements $\mathcal{J} = |\alpha|^2 = |\beta|^2$, for opposite phases $\beta = -\alpha$. The dotted line indicates the lower bound imposed by local theories.

$$Q_a(\alpha) = \langle\alpha|\mathrm{Tr}_b(|\Psi\rangle\langle\Psi|)\,|\alpha\rangle_a,$$
$$Q_b(\beta) = \langle\beta|\mathrm{Tr}_a(|\Psi\rangle\langle\Psi|)\,|\beta\rangle_b. \qquad (9)$$

Thus, we now clearly see that the Q function contains direct information on nonlocal quantum correlations. If a four-point combination of the type given in Eq. (6) violates the inequality $0 \leq \mathcal{CH} \leq 1$, this immediately certifies the nonlocal properties of the quantum state. This definition has an obvious operational meaning, as we have discussed an experiment in which the nonlocal character of the Q function can be tested [11].

In order to give an operational meaning to the Wigner function, we will now consider the case when the detectors are capable of resolving the number of absorbed photons. Let us assign to each event $+1$ or -1, depending on whether an even or an odd number of photons has been registered. This measurement is described by a pair of projection operators:

$$\hat{\Pi}^{(+)}(\alpha) = \hat{D}(\alpha)\sum_{k=0}^{\infty}|2k\rangle\langle 2k|\hat{D}^\dagger(\alpha), \qquad (10)$$

$$\hat{\Pi}^{(-)}(\alpha) = \hat{D}(\alpha)\sum_{k=0}^{\infty}|2k+1\rangle\langle 2k+1|\hat{D}^\dagger(\alpha). \qquad (11)$$

Using these projections, we construct the correlation function between the outcomes of the apparatuses a and b. It has a clear analogy to spin or to photon polarization joint measurements: the spin value is replaced by the parity of the registered number of photons, and the coherent displacements correspond to the orientations of the polarizers. The correlation function measured in our scheme is given by the expectation value of the operator:

$$\hat{\Pi}_{ab}(\alpha,\beta) = [\hat{\Pi}_a^{(+)}(\alpha) - \hat{\Pi}_a^{(-)}(\alpha)] \otimes [\hat{\Pi}_b^{(+)}(\beta) - \hat{\Pi}_b^{(-)}(\beta)], \qquad (12)$$

and, as we will show, it is proportional to the joint two-mode Wigner function of the state $|\Psi\rangle$. This link becomes obvious if we rewrite $\hat{\Pi}_{ab}(\alpha,\beta)$ to the form

$$\hat{\Pi}_{ab}(\alpha,\beta) = \hat{D}_a(\alpha)\hat{D}_b(\beta)(-1)^{\hat{n}_a+\hat{n}_b}\hat{D}_a^\dagger(\alpha)\hat{D}_b^\dagger(\beta), \qquad (13)$$

showing that the correlation function is given by the displaced parity operator $(-1)^{\hat{n}_a+\hat{n}_b}$, which is one of equivalent definitions of the Wigner function [12]. It is a striking result that the nonlocality in a dichotomous correlation measurement in our setup is given directly by the phase-space Wigner function of the state $|\Psi\rangle$.

An easy calculation yields the expectation value of the operator $\hat{\Pi}_{ab}(\alpha,\beta)$ over the state $|\Psi\rangle$:

$$\Pi_{ab}(\alpha,\beta) = \langle\Psi|\hat{\Pi}_{ab}(\alpha,\beta)|\Psi\rangle = (2|\alpha-\beta|^2 - 1)e^{-2|\alpha|^2-2|\beta|^2}. \qquad (14)$$

Now we consider the combination [13]:

$$\mathcal{B} = \Pi_{ab}(0,0) + \Pi_{ab}(\alpha,0) + \Pi_{ab}(0,\beta) - \Pi_{ab}(\alpha,\beta) \qquad (15)$$

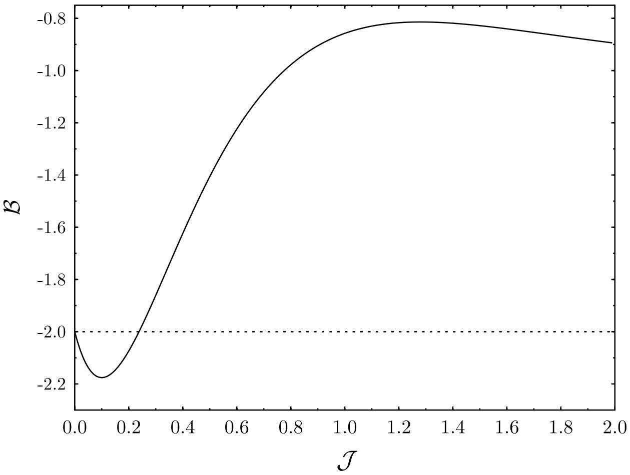

FIG. 3. The plot of the combination defined in Eq. (15) as a function of the magnitude of coherent displacements parametrized with $\mathcal{J} = |\alpha|^2 = |\beta|^2$, for $\beta = -\alpha$. The dotted line indicates the lower bound imposed by local theories.

for which local theories impose the bound $-2 \leq \mathcal{B} \leq 2$. Again we will take equal magnitudes of the coherent displacements $|\alpha|^2 = |\beta|^2 = \mathcal{J}$ and a certain phase difference between them $\beta = e^{2i\varphi}\alpha$. Then the combination $\mathcal{B}$ takes the form

$$\mathcal{B} = -1 + (4\mathcal{J} - 2)e^{-2\mathcal{J}} - (8\mathcal{J}\sin^2\varphi - 1)e^{-4\mathcal{J}}, \tag{16}$$

which, as shown in Fig. 3, for sufficiently small intensities $\mathcal{J}$ violates the lower bound of the inequality imposed by local theories. As before, the strongest violation is obtained for $\varphi = \pi/2$, i.e., when the coherent displacements have opposite phases.

It is now an interesting question whether the nonlocality of the Wigner function exhibited in the proposed experiment is connected to its nonpositivity. The Wigner function of the state $|\Psi\rangle$, containing only one photon, is not positive definite and exhibits the nonlocal character of quantum correlations. The nonlocal character of this phase-space function is directly measured in an experiment involving a detection that resolves the number of absorbed photons. However, it should be pointed out that the above measurement for an incoherent mixture of the two components forming the state $|\Psi\rangle$ leads to a joint correlation equal to $(2|\alpha|^2 + 2|\beta|^2 - 1)e^{-2|\alpha|^2-2|\beta|^2}$. Note that this joint correlation is the Wigner function of the incoherent mixture. This function is not positive definite, but it does not exhibit any quantum interference effects and as a result the Bell inequality is not violated in this case. This shows that the nonpositivity of the Wigner function does not automatically guarantee violation of local realism [14].

In conclusion, we have demonstrated that phase-space quasidistribution functions, the Wigner function and the Q function, carry explicit information on nonlocality of entangled quantum states. This is due to the fact that these two quasiprobability distributions directly correspond to nonlocal correlation functions which can be measured in a class of photon counting experiments involving application of coherent displacements. In addition, the discussed setup provides a new method for measuring directly the two-mode quasidistribution functions.

In this Letter attention was focused on the principle linking quasidistribution functions with quantum nonlocality, which provides a novel operational meaning of the former. A realistic analysis of a photon counting experiment should take into account detector inefficiencies and dark counts [15]. On the other hand, it should be possible to improve the performance of the experiment by optimizing the controllable parameters such as coherent displacements and by selecting carefully the two-mode source of nonclassical radiation. A complete discussion of all experimental aspects would require much more space and will be presented elsewhere.

Finally, let us recall that the past several years have witnessed fascinating advances in the field of quantum state reconstruction, which, in particular, provided feasible experimental schemes for measuring quantum mechanical quasidistribution functions [16]. The results presented in this Letter suggest an exciting route of applying these novel methods in the studies of quantum entanglement exhibited by optical systems.

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[1] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[2] For a review, see M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984); W. P. Schleich, E. Mayr, and D. Krämer, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 1998).

[3] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[4] J. S. Bell, Physics **1**, 195 (1965).

[5] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993).

[6] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987), Chap. 21.

[7] A. M. Cetto, L. De La Pena, and E. Santos, Phys. Lett. **113A**, 304 (1985); L. M. Johansen, Phys. Lett. A **236**, 173 (1997); O. Cohen, Phys. Rev. A **56**, 3484 (1997).

[8] S. M. Tan, D. F. Walls, and M. J. Collett, Phys. Rev. Lett. **66**, 252 (1991); L. Hardy, Phys. Rev. Lett. **73**, 2279 (1994).

[9] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. **70**, 548 (1993); K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996); S. Wallentowitz and W. Vogel, Phys. Rev. A **53**, 4528 (1996).

[10] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).

[11] Following a more traditional approach, the combination $\mathcal{CH}$ defined in Eq. (6) can be also related to the probabilities of registering photons by the detectors. In this case, 1 is assigned to count events and 0 otherwise. A simple calculation shows that $\mathcal{CH}$ can also be expressed as

$$\begin{aligned}\mathcal{CH} &= P_{ab}(0,0) + P_{ab}(\alpha,0) + P_{ab}(0,\beta) \\ &\quad - P_{ab}(\alpha,\beta) - P_a(0) - P_b(0),\end{aligned}$$

where $P_a(\alpha)$, $P_b(\beta)$, and $P_{ab}(\alpha,\beta)$ are given by the expectation values over the state $|\Psi\rangle$ of the operators: $\hat{P}_a(\alpha)$, $\hat{P}_b(\beta)$, and $\hat{P}_a(\alpha)\otimes\hat{P}_b(\beta)$, respectively. For local theories, this combination of count probabilities satisfies the same inequality as before; i.e., $-1 \leq \mathcal{CH} \leq 0$.

[12] A. Royer, Phys. Rev. A **15**, 449 (1977); H. Moya-Cessa and P. L. Knight, Phys. Rev. A **48**, 2479 (1993).

[13] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969); J. S. Bell, in *Foundations of Quantum Mechanics*, edited by B. d'Espagnat (Academic, New York, 1971).

[14] An example of a quantum state described by a positive definite Wigner function, which exhibits nonlocality, is given by K. Banaszek and K. Wódkiewicz, Phys. Rev. A **58**, 4345 (1998).

[15] H.-J. Briegel, B.-G. Englert, N. Sterpi, and H. Walther, Phys. Rev. A **49**, 2962 (1994).

[16] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993). For a review of recent advances, see Special issue on Quantum State Preparation and Measurement, edited by M. G. Raymer and W. P. Schleich [J. Mod. Opt. **44**, Nos. 11/12 (1997).