Comment on "Solution of the Schrödinger equation for the time-dependent linear potential"

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It is shown that the wave function recently found by Guedes [Phys. Rev. A **63**, 034102 (2001)] is a special case of the well-known Volkov solution to the time-dependent Schrödinger equation describing a nonrelativistic charged particle moving in an electromagnetic field.

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In a recent paper [1], Guedes solved the time-dependent Schrödinger equation for the linear potential of the form $V(x,t)=q\varepsilon_0 x+q\varepsilon\ x\cos\omega t$. The author of Ref. [1] used the so-called Lewis and Riesenfeld (LR) invariant method [2] to obtain analytically the wave function [Eq. (18) of Ref. [1]]. The main purpose of the present Comment is to explain that this wave function is simply a special case of the well-known Volkov solution. The latter one can be parametrized by a wave vector k. We show that the Guedes' result is the Volkov solution with $k=0$. Moreover, it is shown that the LR invariant method may be easily extended to get the more general Volkov solution with any real k.

One of the quantum-mechanical problems, whose exact analytical solution to the time-dependent Schrödinger equation exists, is a problem of charged particle interacting only with a plane-wave electromagnetic field. This solution is known as the Volkov or Gordon-Volkov wave function [3,4]. In applications it is assumed that the field can be treated in the so-called dipole approximation, i.e., the electric-field strength is only a function of time, but does not depend on coordinates. This is the case where the wavelength of electromagnetic radiation is much larger than the spatial size of interest (for example, in the process of ionization of atoms by a laser beam). For a nonrelativistic particle, the Schrödinger equation and its solution (in a one-dimensional version) can be written as follows:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=\frac{1}{2m}\left(\hat{p}+\frac{q}{c}A(t)\right)^2\Psi(x,t), \tag{1a}$$

$$\Psi_k^V(x,t)=N\exp\left[ikx-\frac{i}{2m\hbar}\int_0^t\left(\hbar k+\frac{q}{c}A(\tau)\right)^2 d\tau\right]. \tag{1b}$$

For convenience, we use the same units and notation as in Ref. [1]. $\Psi_k^V(x,t)$ is just the Volkov solution. It depends on a real parameter–the wave vector k (or the canonical momentum $\hbar k$) of the particle, which has an electric charge $-q$, and a mass m. $\hat{p}=-i\hbar\partial/\partial x$ is the momentum operator, c is the speed of light, and N is the normalization constant. The vector potential of the electromagnetic field $A(t)$ is given here in the Coulomb gauge. One can easily verify, that the wave function (1b) is a solution of Eq. (1a) for any time dependence of $A(t)$. The vector potential describes the interaction between the charged particle and the electromagnetic field in the so-called velocity form (or velocity gauge). But within the dipole approximation one can use another equivalent description, which is called the length form (or length gauge). In this case, Eqs. (1) have the following counterparts:

$$i\hbar\frac{\partial}{\partial t}\Psi'(x,t)=\left[\frac{\hat{p}^2}{2m}+qF(t)x\right]\Psi'(x,t), \tag{2a}$$

$$\Psi_k'^V(x,t)=N\exp\left[ikx+\frac{iq}{\hbar c}A(t)x-\frac{i}{2m\hbar}\int_0^t\left(\hbar k+\frac{q}{c}A(\tau)\right)^2 d\tau\right], \tag{2b}$$

where the electric-field strength vector is $F(t)=-1/c\,\partial A/\partial t$. One can obtain Eqs. (2) from Eqs. (1) by the Zienau-Power transformation [5,6]. This is a unitary gauge transformation $\hat{U}$ of the wave function and operators ($\Psi'=\hat{U}\Psi$, and $\hat{O}'=\hat{U}\hat{O}\hat{U}^{-1}$), where

$$\hat{U}=\exp\left[\frac{iq}{\hbar c}A(t)x\right]. \tag{3}$$

Then the operators in Eq. (1a) become transformed as follows:

$$\left(\frac{\partial}{\partial t}\right)'=\frac{\partial}{\partial t}-\frac{iq}{\hbar c}\frac{\partial A}{\partial t}x,\quad (\hat{p})'=\hat{p}-\frac{q}{c}A(t),\quad (x)'=x. \tag{4}$$

In the Ref. [1] the author says, "... it seems that no one had reported the solution of the Schrödinger equation for a particle in a general time-dependent linear potential, $V(x,t)=f(t)x$." However, we think that this solution is very well known to the community of atomic physicists, particularly to those who investigate interactions of intense laser fields with atoms. The solution for this time-dependent linear potential is given here in Eq. (2b). A general solution of Eq. (2a) is a linear combination of the Volkov wave functions corresponding to different wave vectors k.

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Substituting $A(t)=-c(\varepsilon_0 t+\varepsilon/\omega \sin \omega t)$ in Eq. (2b) for the vector potential of electromagnetic field, we obtain

$$\Psi_k'^V(x,t)=N \exp\Bigg\{ ikx-\frac{iqx}{\hbar\omega}(\varepsilon_0\omega t+\varepsilon \sin \omega t)+ -\frac{ikq}{2m\omega^2}\left(\frac{\hbar k\omega^2 t}{q}-\varepsilon_0(\omega t)^2-2\varepsilon(1-\cos \omega t)\right)+ -\frac{iq^2}{2m\hbar\omega^3}\left[\frac{\varepsilon_0^2(\omega t)^3}{3}+2\varepsilon\varepsilon_0(\sin \omega t-\omega t \cos \omega t) +\varepsilon^2\left(\frac{\omega t}{2}-\frac{\sin 2\omega t}{4}\right)\right]\Bigg\}. \tag{5}$$

This substitution just corresponds to the interaction potential (in the length form) from Ref. [1], namely $qF(t)x=q\varepsilon_0 x+q\varepsilon\ x \cos \omega t$. When $k=0$, the Volkov wave function given in Eq. (5) exactly reduces to Eq. (18) of Ref. [1]. [By the way, we note two tiny misprints in the powers of ε_0 in this equation, and a similar one in Eq. (17b).] Perhaps, due to the nonvanishing constant component of the electric-field strength ε_0, it is more difficult than usual, to recognize the Volkov solution in Eq. (5).

But the LR invariant method [1,2] is also able to reproduce the Volkov wave function with any wave vector k. The function of time $\eta(t)$ from Eq. (16) of Ref. [1] [which is given there explicitly in Eq. (17a)] satisfies the additional condition $\eta(0)=0$. This condition seems to have no justification. If one releases this constraint assuming $\eta(0)=ik$, and recalculates the wave function, the final result is just the Volkov solution given in Eq. (5) of this Comment. Of course, there is no reason to assume that k is a real number, except the fact that it has a clear physical interpretation of the wave vector.

In conclusion, we have proved that the wave function from Eq. (5) can be derived in two different ways. Both methods and their relationship deserve further investigations. The LR invariant method has been already used in a few problems (see the references in [1]), and probably will find some other applications. Also, the method of unitary transformations of the time-dependent Schrödinger equation has been useful in finding an approximate Coulomb-corrected Volkov-type solution in a certain specific case [7].

Note added in proof. Recently M. Feng [8] showed a yet more general analytical solution to the Schrödinger equation with the time-dependent linear potential. In fact, our Eq. (5) is a special case of his Eq. (7). However, the Volkov solution was not mentioned by M. Feng in this context.

[1] I. Guedes, Phys. Rev. A **63**, 034102 (2000).
[2] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[3] W. Gordon, Z. Phys. **40**, 117 (1926).
[4] D. M. Volkov, Z. Phys. **94**, 250 (1935).
[5] E. A. Power and S. Zienau, Philos. Trans. R. Soc. London, Ser. A **251**, 427 (1959).
[6] F. H. M. Faisal, *Theory of Multiphoton Processes* (Plenum Press, New York, 1987).
[7] J. Bauer, J. Phys. B **34**, 1343 (2001).
[8] M. Feng, Phys. Rev. A **64**, 034101 (2001).