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NONLINEAR PHENOMENA

Operator of Pair Electron–Ion Collisions in Alternating Electromagnetic Fields

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Abstract—Collisions of electrons with ions in the presence of an alternating electromagnetic field are considered. Based on the first principles (the Liouville equations for N particles), a general expression for the collisional operator in the approximation of pair collisions at an arbitrary scattering potential, including that depending periodically on time, is derived. The problem of collisions in plasma in the presence of an electromagnetic field can be reduced to this case by introducing drift coordinates. It is shown that the method of test particles can be applied to the problem of particle collisions in an alternating electromagnetic field.

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1. INTRODUCTION

Collisions play a fundamental role in plasma processes. They determine the shape and evolution of the distribution function and, consequently, different types of instabilities, plasma radiation, and plasma heating. The importance of collisions is difficult to overestimate. In this context, the question naturally arises of the form of the collisional operator under different conditions.

The operator of pair collisions in plasma in weak electric and magnetic fields is well known [1–3]. This is the Landau collisional operator, to which far particle collisions resulting in small-angle scattering make the main contribution. In this case, the particle trajectories are nearly straight and the form of the collisional operator can easily be calculated with logarithmic accuracy. The logarithmic factor (the so-called Coulomb logarithm) takes into account limitations with respect to the minimum impact parameter (corresponding to large scattering angles or quantum effects) and the maximum one (determined by collective effects or the adiabaticity of particle motion in the wave field and corresponding to small scattering angles). Evidently, when the logarithmic factor is small, the condition under which particle trajectories are nearly straight is violated and a more complicated problem of exact variations in the particle momentum during scattering must be considered [4, 5].

Unfortunately, some difficulties also arise in considering electron–ion collisions in strong electromagnetic (EM) fields [6, 7]. In this case, the logarithmic factor is formally large and it would seem that collisions can be regarded as small-angle. However, at sufficiently low particle velocities, $v \ll v_{\rm osc} = eE/m\omega$, a situation arises in which, due to the attracting character of electron–ion interaction, the trajectories of scattered particles cannot be considered straight. Consequently, the applicability conditions of the Landau collisional operator are also violated. Indeed, if the field is so strong that the amplitude of electron oscillations $r_{\rm osc}$ is much larger than the characteristic scattering length (the Rutherford radius estimated from the oscillation velocity),

$$b_{\rm osc} = e^2 Z/m v_{\rm osc}^2 \ll r_{\rm osc} = eE/m\omega^2, \qquad (1)$$

the scattered electron performs multiple oscillations near the ion. Each time it passes near the ion, it is scattered by a small angle; however, during the time period $T = 2\pi/\omega \gg t_{\rm coll} \sim b_{\rm osc}/v_{\rm osc}$, it appreciably approaches the ion. As a result, each subsequent scattering event is stronger than the previous one (a cumulative effect). In this case, the electron–ion collision frequency turns out to be much greater [6] than that evaluated in the approximation of straight drift trajectories. This leads to the generation of fast particles and coherent radiation.

In the present paper, basic expressions for the collisional operator in plasma in the presence of alternating EM fields are derived. Further, we plan to use them to derive a collisional operator in strong fields (see Eq. (1)). However, these expressions have their own value. They confirm the applicability of the test particle method to calculating pair collisions in arbitrary alternating EM fields. Moreover, they allow one to refine the Landau collisional operator in the presence of alternating fields and to more correctly determine its applicability conditions.

Since the problem of deriving the collisional operator in the presence of alternating EM fields is rather complicated, we start from the Liouville equation and the Bogolyubov chain for s-particle distribution functions [3, 8] by generalizing it to the case of arbitrary Hamiltonian scattering systems. As a result, we obtain a general collisional operator in the Boltzmann form (see Sections 2, 3) but for scattering in a time-dependent potential (field). When deriving this operator, we use the Hamiltonian formalism, which allows us to simplify calculations and generalize them to arbitrary Hamiltonian systems. The formulas obtained confirm the applicability of the test particle method to calculating collision characteristics in alternating EM fields. In Section 4, these formulas are used, as an example, to derive the collisional operator in weak fields, in fact repeating the classical results [1–3]. Simultaneously, the applicability conditions of this collisional operator are determined and the part responsible for the transport characteristics of scattering is separated out from it. In the Conclusions, the possibility of further generalization and application of the formulas derived in this paper is discussed.

2. KINETIC EQUATION IN A CANONICALLY INVARIANT FORM

The most complete description of plasma consisting of N particles is provided by the N-particle distribution function $D_N(t, \varsigma_1, \ldots, \varsigma_N)$. Here, $\varsigma_i = \{\mathbf{r}_i, \mathbf{p}_i\}$ are the radius-vector and momentum of the ith particle. The physical meaning of the function D_N is as follows: the quantity $D_N d\varsigma_1 \ldots d\varsigma_N$ is the probability of the coordinates and momenta of particles ($i = 1, \ldots, N$) being within the interval from ς_i to $\varsigma_i + d\varsigma_i$. As a normalization condition for D_N in a closed system, the equality $\int D_N d\varsigma_1 \ldots d\varsigma_N = 1$ can be used. The function D_N obeys the Liouville equation

$$\partial_t D_N + [\mathcal{H}_N^{\Sigma}, D_N] = 0, \tag{2}$$

which reflects the fact that the particle distribution probability in a given phase volume can change only if a particle passes through the volume boundary. Here, $\partial_t f = \partial f/\partial t$ and the second summand is Poisson's brackets,

$$[f, g] = \sum_i \frac{\partial f}{\partial \mathbf{r}_r}\frac{\partial g}{\partial \mathbf{p}_i} - \frac{\partial f}{\partial \mathbf{p}_i}\frac{\partial g}{\partial \mathbf{r}_i}.$$

Assuming that only pair interactions between particles take place,[1] the Hamiltonian $\mathcal{H}_N^{\Sigma}$ of the system of N particles can be represented in the form

$$\mathcal{H}_N^{\Sigma} = \sum_i \mathcal{H}_i + \sum_{j>i} U_{ij}, \tag{3}$$

where $\mathcal{H}_i$ is the Hamiltonian of a free particle and $U_{ij}(\mathbf{r}_i - \mathbf{r}_j, t)$ is the potential of interaction between nonrelativistic particles.

[1] This is certainly valid for a classical plasma (see also [3], Section 16).

Equation (2) is too complicated because it determines the time evolution of the function of $6N$ variables (where $N \sim 10^{23}$ is the total number of plasma particles). On the other hand, since the average interaction energy is smaller than the kinetic energy of plasma particles, the quantities characterizing individual particles should play a greater role in plasma kinetics. To describe them, s-particle distribution functions are introduced:

$$D_s = V^s \int D_N d\varsigma_{s+1} \ldots d\varsigma_N, \tag{4}$$

where V is the plasma volume. The equations for each of these functions are called the Bogolyubov chains, and the sources for them are higher order distribution functions:

$$\partial_t D_1 + [\mathcal{H}_1, D_1] = -\int \frac{d\varsigma_2}{V}[U_{12}, D_2], \tag{5}$$

$$\begin{aligned} &\partial_t D_2 + [\mathcal{H}_1 + \mathcal{H}_2 + U_{12}, D_2] \\ &= -\int \frac{d\varsigma_3}{V}([U_{13}, D_3] + [U_{23}, D_3]) \end{aligned} \tag{6}$$

and so on. Note that, in these equations, EM fields are determined by external sources in which plasma self-fields are not included.

In the absence of correlations (interactions), we have $D_n = \prod_{i=1}^{n} D_1(\varsigma_i)$; therefore, in the functions D_2, D_3, …, it is convenient to single out the terms responsible for correlations,

$$D_1(\varsigma_a) = \frac{V}{N_a} f_a, \quad D_2(\varsigma_a, \varsigma_b) = \frac{V^2}{N_a N_b}(f_a f_b + g_{ab}),$$

$$\begin{aligned} D_3(\varsigma_a, \varsigma_b, \varsigma_c) &= \frac{V^3}{N_a N_b N_c} \\ &\times (f_a f_b f_c + f_a g_{bc} + f_b g_{ac} + f_c g_{ab} + d_{abc}). \end{aligned} \tag{7}$$

Hereafter, the subscripts indicate the dependence of the functions on the coordinates (e.g., $g_{ab} \equiv g_{ab}(\varsigma_a, \varsigma_b)$).

Substituting definitions (7) into Eq. (5), we obtain the kinetic equation for the distribution function f_a,

$$\partial_t f_a + [H_a, f_a] = St_{ab}[f_a] \equiv -\int d\varsigma_b [U_{ab}, g_{ab}]. \tag{8}$$

Here, the Hamiltonian on the left-hand side contains the self-consistent (acting) field in plasma,

$$H_a = \mathcal{H}_a + \sum_b \int f_b U_{ab} d\varsigma_b.$$

The right-hand side of Eq. (8), $St_{ab}[f_a]$, is called the collisional operator for particles of species b. To calculate it, it is necessary to know the pair correlation function g_{ab}. The equation for the function g_{ab} is derived from Eq. (6) by means of simple algebra with account of definitions (7),

$$\partial_t g_{ab} + [H_a + H_b + U_{ab}, g_{ab}] = -[U_{ab}, f_a f_b]. \quad (9)$$

Here, the terms $\int d\varsigma_c ([U_{ac}, f_a g_{bc}] + [U_{bc}, f_b g_{ac}])$, which are responsible for the dynamic polarization of plasma and plays an important role at distances from the scattering particle greater than the Debye radius, $r \geq r_D = \sqrt{T/8\pi e^2 n_e}$, are omitted. In Eq. (9), the term d_{abc}, describing three-particle correlations is also omitted, because the probability of simultaneous collision of three particles is assumed to be negligible.

Equation (9) is an equation of a hyperbolic type, and its solution can easily be found taking into account the smoothness of the distribution functions on the characteristic collision scale-length $\partial_t f_a f_b + [H_a + H_b, f_a f_b] \ll [U_{ab}, f_a f_b]$, as well as the absence of correlations before the interaction, $\lim_{t \to -\infty} g_{ab} = 0$ (the condition proposed by Bogolyubov [9]). As a result, we obtain

$$g_{ab} = -f_a f_b|_{\text{tr}} \equiv -f_a(\varsigma_{0a}(\varsigma_a, \varsigma_b)) f_b(\varsigma_{0b}(\varsigma_a, \varsigma_b)). \quad (10)$$

The expression $f_a f_b|_{\text{tr}}$ should be understood as a dependence on the running coordinates and momenta ς along the trajectories of test particles, which, in turn, depend on the initial coordinates and momenta ς_0. By the test particles we mean particles moving in a system with the Hamiltonian $H_\Sigma = H_a + H_b + U_{ab}$.

Substituting expression (10) into Eq. (8), we obtain the collisional operator in the form

$$St_{ab}[f_a] = \int d\varsigma_b [U_{ab}, f_a f_b|_{\text{tr}}]. \quad (11)$$

This expression is easy to transform into a classical integral operator with the kernel $w_{ab}(\varsigma_a, \varsigma_a^0)$,

$$St_{ab}[f_a] = \int f(\varsigma_{a0}) w_{ab}(\varsigma_a, \varsigma_{a0}) d\varsigma_{a0}. \quad (12)$$

Taking into account that the system is Hamiltonian, we obtain a very simple expression for w_{ab},

$$w_{ab} = \int f_b(\varsigma_b) \frac{d}{dt} \delta(\tilde{\varsigma}_a(\varsigma_{a0}, \varsigma_{b0}, t) - \varsigma_a) d\varsigma_b. \quad (13)$$

This expression has a simple physical meaning: the change in the distribution function during a collision event is determined only by the motion along test particle trajectories. Therefore, the distribution over momenta is determined by the total change of the test particle momenta along their trajectories. Thus, to calculate the collisional operator, it is necessary and sufficient to find the trajectories particle trajectories of scattered particles, after which the distribution over their momentum variations should be integrated over all possible initial coordinates.

In fact, these formulas create a basis for the method of calculating the characteristics of pair collisions by using test particles. The essence of the method is to mentally gather all the plasma ions of the same species into one point (e.g., into the point with the coordinates {0, 0, 0}. In this case, collisions of the entire ensemble of electrons with plasma ions are reduced to successive collisions of noninteracting (test) particles with one ion located at the point {0, 0, 0}. In fact, space averaging (over collisions with different ions) is substituted by time averaging (over successive collisions with one ion). In computer simulations, a sufficiently large number of test particles are used (10^7–10^8 for typical plasma parameters). For each particle, random initial parameters $\varsigma_0 = \{\mathbf{r}_0, \mathbf{p}_0\}$ are chosen and its trajectory during a collision event with an ion is calculated. Then, the results of all collisions are summed up by analogy with the Monte Carlo method. Thus, the above formulas give grounds for the applicability of the test particle method in an arbitrary scattering potential, including a time-dependent one.

3. KERNEL OF THE COLLISIONAL OPERATOR

One can see that Eq. (13) has a canonically invariant form, because it takes into account the shifts of particles in both momentum and coordinate spaces. Unfortunately, in spite of a very simple form of Eq. (13), its application presents great difficulties. In what follows, in view of the smallness of collision scales, we will ignore variations in the spatial coordinates of particles (assuming $\mathbf{r}_{a0} = \mathbf{r}_a$) and will focus on the dependence on the momentum. Under the assumption that the particle coordinates remain unchanged after scattering, collisional operators (12) and (13) lose their canonic invariance, because information on collisions is coarsened. As a result, the collisional operator acquires a diffusion character typical of a Boltzmann collisional operator.

Above, we considered the collisional operator for arbitrarily colliding particles. Let us now consider the important particular case of nonrelativistic electron–ion collisions, because, in the general case, calculations are very cumbersome. All the formulas for relativistic collisions are derived in a similar way.[2]

The Hamiltonian for two charged (test) nonrelativistic particles is well-known,

$$H_\Sigma = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + \frac{e_a e_b}{|\mathbf{r}_a - \mathbf{r}_b|} + \mathbf{E}(t) \cdot (e_a \mathbf{r}_a + e_b \mathbf{r}_b).$$

[2] At some points, we would have to make some remarks and expand formulas to include particles with relativistic velocities. The matter is that, in the relativistic case, the scattering potential in the drift coordinates becomes dependent of the particle momentum [10].

However, for practical application, it is more convenient to write it in the drift coordinates $\mathbf{r}^{\text{drift}}_{[a,b]} = \mathbf{r}^{\text{lab}}_{[a,b]} - \mathbf{r}_{\text{osc},[a,b]}$ and $\mathbf{p}^{\text{drift}}_{[a,b]} = \mathbf{p}^{\text{lab}}_{[a,b]} - \mathbf{p}_{\text{osc},[a,b]}$,

$$H_\Sigma = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + U_{ab},$$
$$U_{ab} = \frac{e_a e_b}{|\mathbf{r}_a + \mathbf{r}_{\text{osc},a}(t) - \mathbf{r}_b - \mathbf{r}_{\text{osc},b}(t)|}. \quad (14)$$

Here, $\mathbf{p}_{\text{osc},[a,b]} = \int e_{[a,b]}\mathbf{E}(t)dt$ is the oscillatory momentum and $\mathbf{r}_{\text{osc},[a,b]} = \int \mathbf{p}_{\text{osc},[a,b]}/m_{[a,b]}dt$ is the oscillation radius. Such a representation is convenient because the scattering potential U_{ab} is spatially localized. However, this transformation results in an explicit dependence of the potential U_{ab} on time. Note that, when one particle (electron) is much lighter than another (ion), we can ignore the oscillations $\mathbf{r}_{\text{osc},b}(t)$ of the heavy particle in the interaction potential in view of the smallness of the parameter m_a/m_b.

Hamiltonian (14) depends only on the relative distances $\mathbf{r} = \mathbf{r}_a - \mathbf{r}_b$. Moreover, we will assume that variations in the ion velocities during electron–ion collisions are small and ignore the change in the coordinate part of the distribution functions f_a and f_b due to variations in the particle positions. As a result, kernel (13) can be represented in the form

$$w_{ab} = n_b(\mathbf{r}_a)\delta(\mathbf{r}_a - \mathbf{r}_{a0})\int \frac{d}{dt}\delta(\tilde{p}_a(\mathbf{r}, \mathbf{p}_{a0}) - \mathbf{p}_a)d^3r, \quad (15)$$

where $n_b = \int f_b\, d^3p_b$ is the density of b-species particles at the point r_a. Further on, we omit $\delta(\mathbf{r}_a - \mathbf{r}_{a0})$ in Eq. (15), assuming that all the collisions are local.

Since H_a and U_{ab} are periodic functions of time with the period $T = 2\pi/\omega_0$ (e.g., for collisions in the field of a plane EM wave), the collision function and, accordingly, the collisional operator must be periodic functions of time. Therefore, it is of interest to find the mean value and the amplitudes of harmonics of the collision function. Let us first find the mean of the collision kernel,

$$\langle w_{ab}(\mathbf{p}, \mathbf{p}_0)\rangle = \frac{n_b}{T}\int_t^{t+T} w(\mathbf{p}, \mathbf{p}_0)dt. \quad (16)$$

Hereafter, the indices a by the momenta are omitted.

Let us switch to Cartesian coordinates ξ, ρ_1, and ρ_2 (with the origin at the ion position) oriented in such a way that the ξ axis is directed along the initial momentum $\mathbf{p}_0$. Since the Hamiltonian is a periodic function of time with the period T, the relation between particle momenta at different instants at $\xi \longrightarrow -\infty$ can be written as

$$\mathbf{p}(t;\xi) = \mathbf{p}(t-T;\xi-\zeta), \quad \zeta = |\mathbf{v}_0|T. \quad (17)$$

Here, the initial velocity $\mathbf{v}_0$ is related to the initial momentum by $\mathbf{v}_0 = \mathbf{p}_0/m$; i.e., the expression for the function w in Eq. (16) can be rewritten as

$$\langle w_{ab}(\mathbf{p}, \mathbf{p}_0)\rangle = \frac{n_b}{T}\sum_{n=-\infty}^{\infty}\int_n^{(n+1)\zeta}\int_\zeta (\delta(\tilde{\mathbf{p}}(t+T) - \mathbf{p})$$
$$- \delta(\tilde{\mathbf{p}}(t) - \mathbf{p}))d\xi d^2\rho_-$$
$$= \frac{n_b}{T}\lim_{\Xi\to-\infty}\iint_\Xi^{\Xi+\zeta}\sum_{n=-\infty}^{\infty}(\delta(\tilde{\mathbf{p}}(t+(n+1)T) - \mathbf{p})$$
$$- \delta(\tilde{\mathbf{p}}(t+nT) - \mathbf{p}))d\xi d^2\rho_-$$
$$= \frac{n_b}{T}\lim_{\Xi\to-\infty}\iint_\Xi^{\Xi+\zeta}(\delta(\tilde{\mathbf{p}}(+\infty) - \mathbf{p}) - \delta(\tilde{\mathbf{p}}(-\infty) - \mathbf{p}))d\xi d^2\rho_-.$$

Taking into account that the initial value of the momentum, $\tilde{\mathbf{p}}(-\infty)$, is identically equal to $\mathbf{p}_0$, we obtain

$$\langle w_{ab}(\mathbf{p}, \mathbf{p}_0)\rangle = \frac{n_b}{T}\lim_{\Xi\to-\infty}\iint_\Xi^{\Xi+\zeta}(\delta(\tilde{\mathbf{p}}(+\infty) - \mathbf{p})$$
$$- \delta(\mathbf{p}_0 - \mathbf{p}))d\xi d^2\rho_-. \quad (18)$$

Thus, the physical meaning of the collision kernel averaged over the oscillation period is as follows: it is equal to the particle velocity distribution function in the problem of scattering of a monoenergetic beam.

Using formula (12), we obtain the expression for the collisional operator,

$$\langle St_{ab}[f]\rangle = \frac{n_b}{T}\lim_{\Xi\to-\infty}\iiint_\Xi^{\Xi+\zeta} f(\mathbf{p}_0)$$
$$\times(\delta(\tilde{\mathbf{p}}(+\infty) - \mathbf{p}) - \delta(\mathbf{p}_0 - \mathbf{p}))d^3p_0 d\xi d^2\rho. \quad (19)$$

This expression generalizes the Boltzmann collisional operator to the case of scattering in a potential depending periodically on time. The physical meaning of formulas (18) and (19) is as follows: variations in the distribution function during collisions are determined only by the motion along particle trajectories. Accordingly, the momentum distribution is calculated using variations in the momenta of particles along their trajectories.

The expression for the amplitudes of harmonics of the collision kernel is derived in a similar way,

$$w_k^{ab} = ik\omega_0 \lim_{\Xi \to -\infty} \iint\limits_{\Xi}^{\Xi+\zeta} \int\limits_{-\infty}^{+\infty} \delta(\tilde{\mathbf{p}}(\tau) - \mathbf{p}) \times e^{i2\pi\tau k/T} d\tau d\xi d^2\rho. \tag{20}$$

Let us note another important and more or less obvious property of Eqs. (18)–(20). Operator (19) allows one to exactly calculate the evolution of the time-averaged distribution function in quasi-monochromatic fields, while expression (20) allows one to find the spectrum of its harmonics. Indeed, on the left-hand side of kinetic equation (8), only the term $\int \partial_t f dt = f(t+T) - f(t)$ is left after averaging over the period, while on the right-hand side, only collisional operator (19) remains. This allows one to develop numerical methods (such as the particles-in-cell method) that accurately and quickly calculate plasma evolution with a sufficiently large time step equal to the period of the EM field.

Collisional operator (12) and expressions (18) and (20) for the function ω allow one to easily determine the collision characteristics (moments). Thus, for Joule plasma heating, we have

$$\begin{aligned} \frac{dT_e}{dt} &\equiv \frac{d}{dt}\int \frac{p^2}{2m} f(\mathbf{p}) d^3 p \\ &= \frac{m v_{\text{osc}}^2}{2} \int \nu_{ei}(\mathbf{p}_0) f(\mathbf{p}) d^3 p_0, \end{aligned} \tag{21}$$

where the effective collision frequency

$$\nu_{ei}(\mathbf{p}_0) = \frac{2m}{\mathbf{p}_{\text{osc}}^2} \frac{n_i}{2m} \int p^2 \langle w_{ei}(\mathbf{p}, \mathbf{p}_0) \rangle d^3 p, \tag{22}$$

is introduced as the ratio of the mean power deposited in the plasma electron component to the oscillation energy of electrons. Using formula (18), it is easy to obtain the expression for the collision frequency,

$$\nu_{ei}(\mathbf{p}_0) = \frac{n_i}{T\mathbf{p}_{\text{osc}}^2} \lim_{\Xi \to -\infty} \iint\limits_{\Xi}^{\Xi+\zeta} (p_+^2 - p_0^2) d\xi d^2\rho_0, \tag{23}$$

which shows that, in order to calculate energy variations during collisions, it is sufficient to find variations in the energy of test particles along their characteristics within a layer of width ζ over ξ from $t = -\infty$ to ∞.

Using the variable $\varphi = \zeta\xi/(2\pi) = p_0 T\xi/2\pi m$, formula (23) can be rewritten in a more convenient form

$$\begin{aligned} \nu_{ei}(\mathbf{p}_0) &= n_i v_0 \sigma_{\text{eff}}, \\ \sigma_{\text{eff}}(\mathbf{p}_0) &= \lim_{\Xi \to -\infty} \iint\limits_{0}^{2\pi} \frac{p_+^2 - p_0^2}{p_{\text{osc}}^2} d\varphi d^2\rho. \end{aligned} \tag{24}$$

The effective cross section σ_{eff} characterizes the effective area from which the scattered particles change their energy by $p_{\text{osc}}^2/2m$. It is this quantity that was used in [6] to calculate Joule plasma heating.

In a similar way, we can find the electron beam current induced by collisions,

$$\begin{aligned} \frac{d\mathbf{j}_{ei}}{dt} &\equiv e\frac{d}{dt}\int \mathbf{p}\delta(\mathbf{p} - \mathbf{p}_0) d^3 p \\ &= en_i \int \mathbf{p} w_{ei}(\mathbf{p}, \mathbf{p}_0, t) d^3 p. \end{aligned} \tag{25}$$

Substituting into Eq. (25) expression (20) for the function w_{ei}, we obtain the Fourier-spectrum of the current $\mathbf{j}_{ei}$,

$$\begin{aligned} \mathbf{j}_\omega &= \int \mathbf{j}_{ei} e^{i\omega t} dt \\ &= en_i \lim_{\Xi \to -\infty} \int \int\limits_{\Xi}^{\Xi+\delta} \mathbf{p}_\omega \cdot \left(\sum_{n=-\infty}^{\infty} \exp\left(i\omega n \frac{2\pi}{\omega_0} \right) \right) d\xi d^2\rho. \end{aligned} \tag{26}$$

The sum in this expression is a sum of δ functions,

$$\mathbf{j}_\omega = e \lim_{\Xi \to -\infty} \int \int\limits_{\Xi}^{\Xi+\delta} \mathbf{p}_\omega d\xi d^2\rho \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0). \tag{27}$$

The physical meaning of the δ functions in this expression is that the response of the system (in particular, plasma) perturbed periodically at the frequency ω_0 should be periodic with frequencies multiple to ω_0.

Note that, for an isotropic distribution function $f(|\mathbf{p}|)$,[3] in view of the symmetry at $\xi \longrightarrow -\infty$, we have

$$\mathbf{p}(t;\, \xi, \mathbf{p}_0) = \mathbf{p}(t + T/2;\, -\xi, -\mathbf{p}_0) \tag{28}$$

i.e., only odd harmonics of the frequency ω_0 remain in the sum.

In a similar way, we can find expressions for other quantities, such as the transport cross section, characterizing the area from which the particles scatter at an angle on the order of $\pi/2$,

$$\sigma_{\text{tr}}(\mathbf{p}_0) = \lim_{\Xi \to -\infty} \iint\limits_{0}^{2\pi} \left(1 - \frac{\mathbf{p}_+ \mathbf{p}_0}{\mathbf{p}_+ p_0} \right) d\varphi d^2\rho, \tag{29}$$

and the cross section for incoherent radiation, having the dimensionality of area per energy,

$$\chi_{\text{inc}}(\mathbf{p}_0) = \frac{4}{3c^3} \lim_{\Xi \to -\infty} \iint\limits_{0}^{2\pi} p_\omega^2 d\varphi d^2\rho. \tag{30}$$

[3] Since the external field does not affect electron–electron collisions and the electron–electron collision frequency ν_{ee} is higher than the electron–ion one, any anisotropy in the distribution function should get smeared over the time $1/\nu_{ee}$; i.e., in analyzing electron–ion collisions, the distribution function should be considered isotropic if $\nu_{ei} \ll \nu_{ee}$, which usually holds.

Note that, in the cross section for incoherent radiation (in contrast to collision current (27)), the powers of radiating sources are summed without allowance for their possible correlation. This is a commonly adopted method for evaluating the intensity of incoherent radiation [11].

4. PERTURBATION METHOD

Unfortunately, it is rather difficult to find an exact analytic solution to the equations of motion of a test particle. However, in some cases, perturbation theory can be developed for the case in which the actual trajectory only slightly deviates from a straight line. Such approximation is called "straight-line approach" [1, 2, 12–15].[4] In fact, the approach consists in ignoring the term

$$[U_{ab}, g_{ab}] \ll [U_{ab}, f_a f_b] \tag{31}$$

in Eq. (9) for the correlation function. As a result, the equation transformed into

$$\partial_t g_{ab} + [H_a + H_b, g_{ab}] = -[U_{ab}, f_a f_b]. \tag{32}$$

Note that inequality (31) implies that particle trajectories remain straight during collisions. This is quite justified for high-energy electrons, $v \gg v_{\rm osc}$, which collide with an ion only once and never return to it or for a repulsive interaction potential, when each subsequent collision is weaker than the previous one. However, for slower electrons, $v \le v_{\rm osc}$, the situation is not so clear, because the electron can return to the same ion after one oscillation period and undergo a stronger collision. In particular, in [6, 7, 16], it was shown that taking into account the return of electrons to the ion drastically changes the pattern of scattering.

Thus, let us concentrate on high-energy particles, $v \gg v_{\rm osc}$. In this case, the solution can be easily written if we take into account that Poisson's brackets $[H_a, \mathbf{p}_a]$ are equal to zero (i.e., that the drift momentum is constant along the particle trajectory),

$$g_{ab} = -f_b \int [U_{ab}, f_a]\big|_{\rm tr} dt = f_b \frac{\partial f_a}{\partial \mathbf{p}} \int \frac{\partial U_{ab}}{\partial \mathbf{r}}\bigg|_{\mathbf{r} \to \mathbf{r} + \mathbf{v}t} dt.$$

Substituting this expression into Eq. (12), we obtain the collisional operator expressed in quadratures,

$$\mathrm{St}[f_a] = n_b \int d^3 r \left[U_{ab}, \frac{\partial f_a}{\partial \mathbf{p}} \int \frac{\partial U_{ab}}{\partial \mathbf{r}}\bigg|_{\mathbf{r} \to \mathbf{r} + \mathbf{v}t} dt \right]. \tag{33}$$

Note that the square dependence on the Coulomb potential (or on the ion charge) appears here only due to condition (31), but is not a consequence (criterion) of the pair collision approach. Moreover, a more rigorous solution of Eq. (9) could yield any power-law dependence on the ion charge, because the trajectories of test particles depend on it transcendentally.

In the nonrelativistic case, when the interaction potential does not depend on momenta (see Eq. (14)), the integral in Eq. (33) can be taken analytically. Indeed, it is easy to see that the collisional operator in this case has the form

$$\mathrm{St}_{ab}[f_a] = \frac{\partial}{\partial p_j} B_{ij} \frac{\partial f(\mathbf{p})}{\partial p_i}, \tag{34}$$

where the tensor B_{ij} is described by the expression

$$B_{ij} = \iint \frac{\partial U_{ab}}{\partial r_j}\bigg|_{\mathbf{r} \to \mathbf{r} + \mathbf{v}t} dt \frac{\partial U_{ab}}{\partial r_i} d^3 r. \tag{35}$$

Applying the Fourier transform to the Coulomb potential, $1/r = 4\pi \int \frac{e^{i\mathbf{kr}}}{k^2} \frac{d^3 k}{(2\pi)^3}$, we obtain

$$B_{ij}(t) = \frac{2Z^2 e^4}{\pi} \int \frac{d^3 k}{k^4} k_i k_j e^{-i\mathbf{k}\mathbf{r}_{\rm osc}(t)} \int e^{i\mathbf{kv}t' - i\mathbf{k}\mathbf{r}_{\rm osc}(t')} dt'. \tag{36}$$

In the laboratory frame, this corresponds to the collisional operator derived by Silin [1],

$$B_{ij}(t) = \frac{2Z^2 e^4}{\pi} \iint \frac{d^3 k dt}{k^4} k_i k_j e^{i\mathbf{k}(\mathbf{v} + \mathbf{v}_{\rm osc}(t))t}. \tag{37}$$

Collisional operator (37) can be simplified to within logarithmic accuracy. Note that, for large k (which contribute significantly to the effective collision frequency), the total particle velocity at the instant of scattering in expression (37) can be ignored in view of the large value of the parameter $kv/\omega \equiv kr_a \gg 1$. As a result, we obtain

$$\begin{aligned} B_{ij}(t) &= 2Z^2 e^4 \int \frac{d^3 k}{k^4} k_i k_j \delta(\mathbf{k}(\mathbf{v} + \mathbf{v}_{\rm osc})) \\ &= 2\pi Z^2 e^4 \frac{\delta_{ij} V^2 - V_i V_j}{V^3} \ln \frac{r_a}{b_v}, \end{aligned} \tag{38}$$

where $b_v = e^2 Z/mv^2$ is the radius at which a particle is scattered at an angle on the order of $\pi/2$ and $\mathbf{V}(t) = \mathbf{v} + \mathbf{v}_{\rm osc}(t)$ is the total particle velocity at the instant of collision. In the range of small k ($kr_{\rm osc} \ll kr_a \le 1$), we can ignore the term $\mathbf{kr}_{\rm osc}(t)$ in expression (36). This approximation implies that the effect of the external field on far collisions is negligible. As a result, for the tensor B_{ij}, we have

$$\begin{aligned} B_{ij} &= 2Z^2 e^4 \int \frac{d^3 k}{k^4} k_i k_j \delta(\mathbf{kv}) \\ &= 2\pi Z^2 e^4 \frac{\delta_{ij} v^2 - v_i v_j}{v^3} \ln \frac{r_D}{r_a}. \end{aligned} \tag{39}$$

[4] In the Russian literature, the term "small-angle approximation" is usually used, which, in my opinion, is not quite correct.

Finally, for the collisional operator in the domain of applicability of the straight-line approach, we obtain

$$B_{ij} = 2\pi Z^2 e^4 \frac{\delta_{ij} V^2 - V_i V_j}{V^3} \ln\frac{r_a}{b_v} + 2\pi Z^2 e^4 \frac{\delta_{ij} v^2 - v_i v_j}{v^3} \ln\frac{r_D}{r_a}. \tag{40}$$

The first summand is responsible, first of all, for the change in the plasma energy during the interaction with the external EM field. The second summand does not lead to the change in the plasma energy and, as a rule, is not taken into account. It is responsible only for the transport characteristics of scattering (by analogy with the Landau collisional operator for electron–electron collisions).

Expression (40) for the tensor B_{ij} has a fairly general form. In some specific cases, it can be simplified and reduced to the well-known expressions or allows one to draw a conclusion regarding the particle dynamics in these regimes. Let us consider them.

First, when the external field is switched off ($v_{\text{osc}} \longrightarrow 0$), the total velocity becomes equal to the particle drift velocity $\mathbf{V}(t) \longrightarrow \mathbf{v}$. Consequently, the tensor B_{ij} transforms into the Landau tensor B_{ij}^0 (see Eq. (39)), which enters into the Landau collisional operator,

$$B_{ij} \longrightarrow 2\pi Z^2 e^4 \frac{\delta_{ij} v^2 - v_i v_j}{v^3} \left[\ln\frac{r_a}{b_v} + \ln\frac{r_D}{r_a}\right] = 2\pi Z^2 e^4 \frac{\delta_{ij} v^2 - v_i v_j}{v^3} \ln\frac{r_D}{b_v}. \tag{41}$$

Note that the conventional form of the collisional operator in the presence of an EM field [1] does not allow such a transition.

For a nontransparent plasma, $\omega < \omega_p \Leftrightarrow r_a > r_D$, the tensor B_{ij} has the form of the Landau tensor, in which the drift velocity is replaced with the total particle velocity,

$$B_{ij} = 2\pi Z^2 e^4 \frac{\delta_{ij} V^2 - V_i V_j}{V^3} \ln\frac{r_D}{b_v}. \tag{42}$$

This tensor is traditionally used to determine the permittivity of a nontransparent plasma [3].

In the opposite limiting case, $r_a < b_v$, the tensor B_{ij} is identically equal to Landau tensor (39) in the absence of a field,

$$B_{ij} = 2\pi Z^2 e^4 \frac{\delta_{ij} v^2 - v_i v_j}{v^3} \ln\frac{r_D}{b_v}. \tag{43}$$

Note that the same form of the tensor B_{ij} can also be obtained from Eq. (36) when the oscillation radius is smaller than the Rutherford radius, $r_{\text{osc}} \ll b_v$, irrespective of the ratio between the drift and oscillation velocities. In both cases (r_a, $r_{\text{osc}} \ll b_v$), only the part responsible for the change in the particle momentum direction (rather than for variations in the particle energy) remains in the tensor B_{ij}. In particular, this means that, to within a logarithmic factor, the transport cross section remains the Rutherford one in these ranges. That the energy in expression (43) remains unchanged means that it is necessary to take into account large-angle scattering [4, 5] in order to determine the effective frequency responsible for the change in the plasma energy in these ranges. However, it does not follow from this that there is no energy exchange in these ranges, as was stated, e.g., in [13].

Finally, in a transparent plasma, for

$$v \gg v_{\text{osc}}, \quad b_v \ll r_a \ll r_D \tag{44}$$

the tensor B_{ij} contains both summands. The summand responsible for energy variations (the first term in Eq. (40)) has the form of a tensor proposed for the first time by Silin [1] and traditionally used for such plasmas.

A further generalization of Silin's collisional operator (40) to the range of low velocities should be performed with allowance for the curvature of the characteristics of the equation for the correlation function; i.e., it should go beyond the applicability range of the straight-line approach.

5. CONCLUSIONS

In the present paper, we have applied the apparatus of Hamiltonian formalism to generalize the method for deriving the collisional operator to the case of an arbitrary scattering potential, including a time-dependent one. The latter case is of particular importance, because it can be reduced to the problem of particle scattering by the Coulomb potential in the presence of an arbitrarily large EM pump wave. In fact, passing from laboratory coordinates to the drift ones, we can exclude the effect of the external field on the free particle motion. In this case, the scattering potential becomes time-dependent.

Formulas (18)–(20) for the collisional operator have a universal character. They confirm the applicability of the test particle method underlying numerical simulations of collisional processes in plasma in the presence of an alternating EM field. In particular, the main characteristics of collisions, such as effective collision frequency (23), collision current (27), etc., have been calculated by integrating the corresponding characteristics of test particles over their initial conditions.

As an example, the collisional operator in the Landau form (in Silin's form for alternating EM fields) has been derived in the straight-line approach. Assuming that an electron collides with an ion only once, which is quite justified for the high-energy particles ($v_T > v_{\text{osc}}$), one can easily find the trajectory of test particles and take the resulting integrals. The applicability conditions for such an approach have been determined.

In the case of multiple electron–ion collisions (for $v_T \ll v_{osc}$), it is necessary to use the complete (unreduced) expression for the collisional operator (see Eqs. (18)–(20)). Since the corresponding calculations are rather cumbersome, this will be done in a separate paper.

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