Relativistic theory of the above-threshold multiphoton ionization of hydrogenlike atoms in ultrastrong laser fields

H. K. Avetissian,* A. G. Markossian, and G. F. Mkrtchian
Department of Theoretical Physics, Plasma Physics Laboratory, Yerevan State University, 1, A. Manukian St., 375049 Yerevan, Armenia
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The relativistic theory of above-threshold ionization (ATI) of hydrogenlike atoms in ultrastrong radiation fields, taking into account the photoelectron-induced rescattering in the continuum spectrum, is developed. It is shown that the contribution of the latter in the multiphoton ionization probability even in the Born approximation by the Coulomb field is of the order of ATI probability in the scope of the Keldysh-Faisal-Reiss ansatz.

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I. INTRODUCTION

The increasing interest in the process of multiphoton above-threshold ionization (ATI) of atoms in superintense laser fields is due in large part to the problem of high harmonic generation and short-wave coherent radiation implementation via multiphoton bound-free transitions through a free continuum spectrum (one of the possible version of a free-electron x-ray laser). During the last two decades numerous investigations have been carried out to study ATI of atoms both theoretically and experimentally and many review papers (see, e.g., [1–10]) and monographs [11–15] are devoted to this problem.

The main ansatz in the nonrelativistic theory of multiphoton ionization of atoms in strong electromagnetic (EM) radiation fields that of Keldysh [16] (also called the Keldysh-Faisal-Reiss ansatz [17,18]). The advantage of this approach is that it leads in a very simple way to reveal some of the main qualitative features of the photoelectron energy spectrum in ATI experiments [19–21]. Within the scope of this ansatz, the photoelectron rescattering in the field of atomic remainder is neglected. In order to bridge this gap, attempts have been made to describe the photoelectron final state by a "Coulomb-Volkov" wave function that is a product of the Coulomb wave function of elastic scattering and a wave function of electrons in the EM wave field [22–28]. This wave function results in the factorization of the probability of multiphoton ionization and restricts both the frequency (low-frequency approximation) and intensity of the wave. The use of another ansatz for the definition of multiphoton ionization probabilities [29] should also be noted.

The description of the photoelectron final state, taking into account the stimulated bremsstrahlung (SB) at the photoelectron scattering on the electrostatic potential of the ionized atom in the presence of a strong EM radiation field (induced free-free transitions), still remains one of the main problems in the ATI process. Moreover, the definition of the dynamic wave function of an electron in the SB process is already problematic, therefore the main results concerning multiphoton SB probabilities have been found through the S matrix formalism for "free-free" (over the electrostatic potential) transitions in the Born approximation between Volkov states in an EM wave [30]. However, in many cases when the condition of the Born approximation is broken, the scattering process is described in low-frequency [31,32] or eikonal [33] approximations. Though the Born and low-frequency approximations in the SB process are applicable for describing free-free transitions in high-intensity radiation fields, they do not take into account the mutual influence of the scattering and the radiation fields (i.e., the probability of SB is factorized by elastic scattering and photon emission or absorption processes). What concerns the eikonal approximation in the SB process it is not applicable beyond the interaction region ($z \ll pa^2/\hbar$, where z is the coordinate along the direction of initial momentum $\vec{p}$ of the particle, a is the range of the interaction region, and $\hbar$ is the Planck constant). A description of the electron eigenstates in the SB process beyond of the scope of these approximations has been made in [34], developing a generalized eikonal approximation (GEA). The obtained GEA wave function enables us to leave the framework of the ordinary eikonal approximation and to be free from the restriction $z \ll pa^2/\hbar$. Besides, such a wave function simultaneously takes into account the influence of both the scattering and radiation fields on the particle state. Therefore, to determine the multiphoton probabilities of above-threshold ionization of an atom, one should know the wave function of the ejected photoelectron in the SB process with more accuracy. On the other hand, in the current superintense laser fields, the state of the electron becomes relativistic already at the distances $l \ll \lambda$ (λ is the wavelength of a laser radiation), independent of its initial state. Hence, the problem of ATI of atoms with the photodetached electron SB process requires a relativistic consideration.

A relativistic generalization of multiphoton SB in the first Born and eikonal approximations has been made in the papers [35], [36], and [37], respectively. In [38], on the basis of the solution of the Dirac equation, the GEA [34] has been developed for relativistic scattering theory in the arbitrary electrostatic and plane EM wave fields, including both the Born and eikonal approximations in corresponding limits and describing the spin interaction as well. Such a wave function allows us to describe the final state of the photoelectron with more accuracy in the ATI process of atoms.

A relativistic description of multiphoton ATI of hydrogen-

*Electronic address: avetissian@ysu.am

like atoms for high-intensity laser fields taking into account the spin interaction has been developed analytically in the papers [39–41] with an approximation in which the stimulated bremsstrahlung of the emergent electron is neglected. The relativistic consideration of ATI is important as it is generally assumed that the problem of stabilization of atoms in ultraintense laser fields must be solved within the framework of relativistic theory [42]. From this point of view, some attempts have been made to solve analytically the Klein-Gordon equation [43,44] or numerically the Dirac equation [42,45] in the fields of a static potential and monochromatic EM wave (using various model potentials of one or two dimensions and various approximations). In the papers [46–48] relativistic corrections to the nonrelativistic results have been given.

Note that at present, analytic formulas for these probabilities, taking into account the photoelectron rescattering, are unknown even in the first Born approximation for the Coulomb scattering field. Therefore, in the present paper the relativistic probabilities of multiphoton ATI in the limit of the Born approximation for the photoelectron rescattering are calculated. Moreover, it is shown that the neglect of the photoelectron rescattering in the relativistic domain especially [39,41] is invalid, since the contribution of the electron rescattering process in the matrix elements of transitions has the same order by a scattering potential in the Born approximation as the matrix elements of bound-free transitions for the ATI process.

The organization of the paper is as follows. In Sec. II, we present the multiphoton cross sections of the above-threshold ionization of hydrogenlike atoms in an ultraintense laser field (with the help of the GEA wave function), taking into account the induced free-free transitions of the ejected photoelectron in the continuum spectrum. Because of the many complicated expressions in the GEA, the spin interaction is neglected and the ultimate analytic results for the multiphoton probabilities are performed in the limit of the first Born approximation by the ion (atomic remainder) potential, which we present in Sec. III. In Sec. IV, we treat the dependence of ATI probability on the polarization of an EM wave and we consider the differences between circular and linear polarizations of an electromagnetic wave.

II. THE IONIZATION PROBABILITY FROM THE RELATIVISTIC GEA SOLUTION OF A WAVE EQUATION OF AN ELECTRON

The problem has been reduced to an investigation of the relativistic exploration of the transition S matrix formalism utilizing the relativistic GEA wave function [38] as a wave function of the final state of a photodetached electron (it has been neglected with the spin interaction in the relativistic GEA wave function). Following the relativistic S matrix formalism, the bound-free transition amplitude can be written in this integral form (in natural units $\hbar=c=1$),

$$T_{i\to f}=-i\int_{-\infty}^{\infty}\Psi^{(-)\dagger}(x)\hat{V}\Phi(x)d^4x, \tag{1}$$

where $x=(t,\vec{r})$ is the four-component radius-vector x^{μ}, $\Phi(x)$ is the initial unperturbed bound state of the atomic system, and $\Psi^{(-)}(x)$ is the final out-state of an electron in the potential of atomic remainder and in the field of a plane EM wave ($K^{\dagger}$ denotes the complex conjugation of K). We assume the EM wave to be quasimonochromatic and of an arbitrary polarization with the vector potential,

$$\vec{A}(\varphi)=A_0(\varphi)(\vec{e}_1\cos\varphi+\vec{e}_2\zeta\sin\varphi),\quad \varphi=k\cdot x=\omega t-\vec{k}\cdot\vec{r}, \tag{2}$$

where $k=(\omega,\vec{k})$ is the four-wave vector, $A_0(\varphi)$ is the slowly varying amplitude of the vector potential of a plane wave, $\vec{e}_1$ and $\vec{e}_2$ are unit vectors ($\vec{e}_1\perp\vec{e}_2\perp\vec{k}$), and $\arctan\zeta$ is the polarization angle.

According to the Klein-Gordon equation, the interaction operator is

$$\hat{V}=-2e\vec{A}(\varphi)(-i\vec{\nabla})+e^2\vec{A}^2(\varphi), \tag{3}$$

where e is the electron charge.

The wave function of the final state of the photodetached electron in the relativistic GEA approximation has the following form [38]:

$$\Psi^{(-)\dagger}(x)=\frac{1}{\sqrt{2\Pi_0}}F^{\dagger}(x)\exp[-iS_V(x)]. \tag{4}$$

The $S_V(x)$ is the action of a photoelectron in the field (2),

$$S_V(x)=\vec{\Pi}\cdot\vec{r}-\Pi_0 t+\alpha\left(\frac{\vec{p}}{k\cdot p}\right)\sin[\varphi-\theta(\vec{p})]-\frac{Z}{2}(1-\zeta^2)\sin 2\varphi. \tag{5}$$

Here $\Pi=(\Pi_0,\vec{\Pi})$ is the average four-kinetic momentum or ''quasimomentum'' of the electron in the plane EM wave field, which is defined via the free-electron four-momentum $p=(\varepsilon_0,\vec{p})$ and the relative parameter of the wave intensity Z by the following equation:

$$\Pi=p+kZ(1+\zeta^2),\quad Z=\frac{e^2\overline{A_0^2}}{4k\cdot p}, \tag{6}$$

where $\overline{A_0}$ is the averaged value of the amplitude $A_0(\varphi)$. The wave function (4) is normalized for the one particle in the unit volume $V=1$.

Included in Eq. (5), the quantity $\alpha(\vec{p}/(k\cdot p))$ is the intensity-dependent amplitude of the electron-wave interaction and as a function of any three-vector $\vec{b}$ it has the following definition:

$$\alpha(\vec{b})=e\overline{A_0}\sqrt{(\vec{b}\cdot\vec{e}_1)^2+\zeta^2(\vec{b}\cdot\vec{e}_2)^2}, \tag{7}$$

with the phase angle

$$\theta(\vec{p})=\arctan\left(\frac{\vec{p}\cdot\vec{e}_2}{\vec{p}\cdot\vec{e}_1}\zeta\right). \tag{8}$$

The function $F^{\dagger}(x)$ in Eq. (4), describing the impact of both the scattering and EM radiation fields on the photoelectron state simultaneously, has the following form [38]:

$$F^{\dagger}(x)=\exp\Bigg[\frac{1}{4\pi^3}\sum_{n=-\infty}^{\infty}e^{in\varphi}\int\frac{\left\{\omega\left[\alpha\left(\frac{\vec{p}}{k\cdot p}\right)D^{\dagger}_{1,n}(\theta_1(\vec{q})-\theta(\vec{p}))-Z(1-\zeta^2)D^{\dagger}_{2,n}\right]-\Pi_0D^{\dagger}_n\right\}}{\vec{q}^2+2\vec{\Pi}\cdot\vec{q}-2n(k\cdot p-\vec{k}\cdot\vec{q})+i0}\tilde{U}(\vec{q})$$
$$\times\exp(-i\{\vec{q}\cdot\vec{r}+\alpha_1(\vec{q})\sin[\varphi-\theta_1(\vec{q})]-\alpha_2(\vec{q})\sin 2\varphi+\theta_1(\vec{q})n\})d\vec{q}\Bigg], \tag{9}$$

where

$$\tilde{U}(\vec{q})=\int U(\vec{r})\exp(-i\vec{q}\cdot\vec{r})d\vec{r} \tag{10}$$

is the Fourier transform of the potential of the atomic remainder, $\alpha_1(\vec{q})$, $\alpha_2(\vec{q})$ are dynamic parameters of the interaction defined by the expression

$$\alpha_1(\vec{q})=\alpha((\vec{k}\cdot\vec{q})\vec{p}/k\cdot p+\vec{q}),$$
$$\alpha_2(\vec{q})=\frac{\vec{k}\cdot\vec{q}}{2(k\cdot p-\vec{k}\cdot\vec{q})}Z(1-\zeta^2), \tag{11}$$

and $\theta_1(\vec{q})$ is the phase angle

$$\theta_1(\vec{q})=\theta((\vec{k}\cdot\vec{q})\vec{p}/k\cdot p+\vec{q}). \tag{12}$$

The functions $J_n(u,v,\triangle)$, D_n, $D_{1,n}(\theta_1(\vec{q})-\theta(\vec{p}))$, and $D_{2,n}$ are defined by the expressions (also see Ref. [38])

$$D_n=J_n(\alpha_1(\vec{q}),-\alpha_2(\vec{q}),\theta_1(\vec{q})), \tag{13}$$

$$D_{1,n}(\theta_1(\vec{q})-\theta(\vec{p}))=\frac{1}{2}[J_{n-1}(\alpha_1(\vec{q}),-\alpha_2(\vec{q}),\theta_1(\vec{q}))e^{-i(\theta_1(\vec{q})-\theta(\vec{p}))}+J_{n+1}(\alpha_1(\vec{q}),-\alpha_2(\vec{q}),\theta_1(\vec{q}))e^{i(\theta_1(\vec{q})-\theta(\vec{p}))}], \tag{14}$$

and

$$D_{2,n}=\frac{1}{2}[J_{n-2}(\alpha_1(\vec{q}),-\alpha_2(\vec{q}),\theta_1(\vec{q}))e^{-i2\theta_1(\vec{q})}+J_{n+2}(\alpha_1(\vec{q}),-\alpha_2(\vec{q}),\theta_1(\vec{q}))e^{i2\theta_1(\vec{q})}. \tag{15}$$

In the denominator of the integral in expression (9), $+i0$ is an imaginary infinitesimal, which shows how the path around the pole in the integrand should be chosen to obtain a certain asymptotic behavior of the wave function, i.e., the ingoing spherical wave [to determine that one must be passed to the limit of the Born approximation at $\vec{A}(\varphi)=\vec{0}$].

Since we consider the ATI problem for hydrogenlike atoms ($Z_a\ll 137$), the initial velocities of atomic electrons are nonrelativistic, and as an initial-state wave function Φ in the transition amplitude, Eq. (1) will be taken as a stationary wave function of the hydrogenlike atom bound state in the nonrelativistic limit,

$$\Phi(\vec{r},t)=\frac{1}{\sqrt{2m}}\Phi_0(\vec{r})\exp(-i\varepsilon_0 t), \quad \varepsilon_0=m-E_B, \tag{16}$$

where $E_B>0$ is the binding energy of the valence electron in the atom,

$$2mE_B=a^{-2}. \tag{17}$$

Concerning the relativism of the photoelectron final state in a strong EM field, it should be mentioned that at the wave intensities already $\xi\sim 10^{-1}$, where

$$\xi=\frac{e\overline{A_0}}{m} \tag{18}$$

is the relativistic invariant parameter of the wave intensity, relativistic effects become observable, and the final state of the photoelectron should be described in the context of relativistic theory. Moreover, at the currently available laser intensities $\xi>1$ (even $\xi\gg 1$), a free electron becomes essentially relativistic already at distances smaller than one wavelength. On the other hand, in such fields we see the production of electron-positron pairs from an intense photon field on the electrostatic potential of atomic remainder through multiphoton channels. However, we can calculate separately the ATI probability in superstrong laser fields without restricting intensities by the threshold value of multiphoton pair production ($\xi\simeq 2$; see [49] and [50]) since those are independent processes.

Since $\hat{V}$ is a Hermitian operator, the transition amplitude (1) can be written in the form

$$T_{i\to f}=-i\int_{-\infty}^{\infty}\Phi(x)\hat{V}^{\dagger}(x)\Psi^{(-)\dagger}(x)d^4x. \tag{19}$$

To integrate this expression, it is convenient to turn from the variables $t,\vec{r}$ to $\vec{\eta}\equiv\vec{r},\varphi$ [see Eq. (2)],

$$T_{i\to f}=-\frac{i}{\omega}\int_{-\infty}^{\infty}\Phi(\varphi,\vec{\eta})\hat{V}^{\dagger}(\varphi,\vec{\eta})\Psi^{(-)\dagger}(\varphi,\vec{\eta})d\varphi\, d\vec{\eta}, \tag{20}$$

and make a Fourier transformation of the function $F^{\dagger}(x)$ over the variable φ,

$$F^{\dagger}(\varphi,\vec{\eta})=\sum_{l=-\infty}^{\infty}\tilde{F}_l(\vec{\eta})\exp(-il\varphi), \tag{21}$$

$$\tilde{F}_l(\vec{\eta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\varphi,\vec{\eta})\exp(il\varphi)d\varphi. \tag{22}$$

Then with the help of Eqs. (2)–(16) [using Eq. (A3) as well] and taking into account the Lorentz condition for the plane–wave field $\vec{k}\cdot\vec{A}(\varphi)=0$, we can accomplish the integration over the variable φ in Eq. (20). After a simple transformation with the help of the formula (A5), we obtain the following expression for the transition amplitude:

$$T_{i\to f}=\frac{i2\pi(k\cdot p)}{\omega\sqrt{m\Pi_0}}\sum_{L,l=-\infty}^{\infty}\Bigg\{[L-Z(1+\zeta^2)]\tilde{\Phi}_l(\vec{g})J_L\left(\alpha\left[\frac{\vec{p}}{k\cdot p}\right],-\frac{Z}{2}(1-\zeta^2),\theta(\vec{p})\right)e^{iL\theta(\vec{p})}\times\delta\left(\frac{\Pi_0-\varepsilon_0}{\omega}-L-l\right)$$
$$+2\sum_{n=-\infty}^{\infty}\int\frac{d\vec{q}}{(2\pi)^3}\tilde{\Phi}_l(\vec{g}+\vec{q})\tilde{U}(\vec{q})\times\alpha\left(\frac{\vec{q}}{k\cdot p}\right)C_{1,L}^{\dagger}[\theta(\vec{p}+\vec{q})-\theta(\vec{q})]e^{-in\theta_1(\vec{q})+iL\theta(\vec{p}+\vec{q})}$$
$$\times\frac{\left\{\omega\left[\alpha\left(\frac{\vec{p}}{k\cdot p}\right)D_{1,n}^{\dagger}[\theta_1(\vec{q})-\theta(\vec{p})]-Z(1-\zeta^2)D_{2,n}^{\dagger}\right]-\Pi_0D_n^{\dagger}\right\}}{\vec{q}^2+2\vec{\Pi}\cdot\vec{q}-2n(k\cdot p-\vec{k}\cdot\vec{q})+i0}\times\delta\left(\frac{\Pi_0-\varepsilon_0}{\omega}-L-l+n\right)\Bigg\}, \tag{23}$$

where $\vec{g}$ is the three-vector,

$$\vec{g}=\vec{p}-\frac{(\varepsilon-\varepsilon_0)\vec{k}}{\omega}, \tag{24}$$

and the function $\tilde{\Phi}_l(\vec{b})$ is the Fourier transform of $\Phi_l(\vec{\eta})\equiv\Phi(\vec{\eta})\tilde{F}_l(\vec{\eta})$, and as a function of any three-vector $\vec{b}$ is defined by the analogous formula (10)

$$C_{1,n}[\theta(\vec{p}+\vec{q})-\theta(\vec{q})]=\frac{1}{2}\Bigg[J_{n-1}\left(\alpha_1(\vec{p}+\vec{q}),-\frac{Z_1}{2}(1-\zeta^2),\theta(\vec{p}+\vec{q})\right)e^{-i[\theta(\vec{p}+\vec{q})-\theta(\vec{q})]}$$
$$+J_{n+1}\left(\alpha_1(\vec{q}),-\frac{Z_1}{2}(1-\zeta^2),\theta(\vec{p}+\vec{q})\right)e^{i[\theta(\vec{p}+\vec{q})-\theta(\vec{q})]}\Bigg], \tag{25}$$

where the parameters $\alpha((\vec{p}+\vec{q})/(k\cdot p-\vec{k}\cdot\vec{q}))$, $\theta(\vec{p}+\vec{q})$ are determined by the expressions (7) and (8), and

$$Z_1=\frac{e^2\overline{A_0^2}}{4(k\cdot p-\vec{k}\cdot\vec{q})}. \tag{26}$$

Using the general conservation law, the probability amplitude of the above-threshold ionization in final form can be presented in this ultimate form,

$$T_{i\to f}=\frac{i2\pi(k\cdot p)}{\sqrt{m\Pi_0}}\sum_{N,l=-\infty}^{\infty}\Bigg\{[N+l-Z(1+\zeta^2)]\tilde{\Phi}_l(\vec{g})\times J_{N-l}\left(\alpha\left(\frac{\vec{p}}{k\cdot p}\right),-\frac{Z}{2}(1-\zeta^2),\theta(\vec{p})\right)e^{i(N-l)\theta(\vec{p})}$$

$$+2\sum_{n=-\infty}^{\infty}\int\frac{d\vec{q}}{(2\pi)^3}\tilde{\Phi}_l(\vec{g}+\vec{q})\tilde{U}(\vec{q})\times\alpha\left(\frac{\vec{q}}{k\cdot p}\right)C^{\dagger}_{1,N-l+n}[\theta(\vec{p}+\vec{q})-\theta(\vec{q})]e^{-in\theta_1(\vec{q})+i(N-l+n)\theta(\vec{p}+\vec{q})}$$

$$\times\frac{\left\{\omega\left[\alpha\left(\frac{\vec{p}}{k\cdot p}\right)D^{\dagger}_{1,n}[\theta_1(\vec{q})-\theta(\vec{p})]-Z(1-\zeta^2)D^{\dagger}_{2,n}\right]-\Pi_0D^{\dagger}_n\right\}}{\vec{q}^2+2\vec{\Pi}\cdot\vec{q}-2n(k\cdot p-\vec{k}\cdot\vec{q})+i0}\Bigg\}\times\delta(\Pi_0-\varepsilon_0-\omega N). \tag{27}$$

The differential probability of the ATI process per unit time in the phase space $d\vec{\Pi}/(2\pi)^3$ (space volume $V=1$ in accordance with normalization of the electron wave function) taking into account all the final states of a photoelectron with quasimomenta in the interval $\vec{\Pi}$, $\vec{\Pi}+d\vec{\Pi}$ is

$$dW_{i\to f}=\frac{|T_{i\to f}|^2}{\tau}\frac{d\vec{\Pi}}{(2\pi)^3}=\frac{|T_{i\to f}|^2}{\tau}\sqrt{\Pi_0^2-m_*^2}\,\Pi_0 d\Pi_0\frac{d\Omega}{(2\pi)^3}, \tag{28}$$

where τ is the interaction time, $d\Omega$ is the differential solid angle, and

$$m_*=\sqrt{\Pi_0^2-\vec{\Pi}^2}=\sqrt{m^2+e^2\overline{A_0^2}\frac{(1+\zeta^2)}{2}} \tag{29}$$

is the "effective mass" of the relativistic electron in the EM wave field.

As follows from Eq. (27) and formulas

$$2\pi\delta(\Pi_0-\varepsilon_0-\omega N)\delta(\Pi_0-\varepsilon_0-\omega N')=\begin{bmatrix}0 \quad \text{if} \quad N\neq N'\\ \tau\delta(\Pi_0-\varepsilon_0-\omega N) \quad \text{if} \quad N=N',\end{bmatrix} \tag{30}$$

the differential probability of the ATI process $dW_{i\to f}$ (28) per unit time does not depend on interaction time.

III. THE RELATIVISTIC BORN APPROXIMATION FROM THE POTENTIAL OF ATOMIC REMAINDER FOR HYDROGENLIKE ATOM IONIZATION

The impact of the rescattering effect on the ATI process is more transparent in the limit of the Born approximation by the scattering potential. The latter takes place if the corresponding part of the action in the GEA wave function, describing the impact of both the scattering and EM radiation fields on the photoelectron state simultaneously, is enough small (see Ref. [38]).

Expanding Eq. (27) into the series and keeping only the terms to the first order over $U(\vec{r})$, after a simple transformation, utilizing Eqs. (A2), (A3), and (A5), we obtain

$$T_{i\to f}=\frac{i2\pi}{\sqrt{m\Pi_0}}\sum_{N=-\infty}^{\infty}\Bigg\{[N-Z(1+\zeta^2)](k\cdot p)\tilde{\Phi}(\vec{g})e^{iN\theta(\vec{p})}J_N\left(\alpha\left(\frac{\vec{p}}{k\cdot p}\right),-\frac{Z}{2}(1-\zeta^2),\theta(\vec{p})\right)$$

$$+2\sum_{n=-\infty}^{\infty}\int\frac{d\vec{q}}{(2\pi)^3}[N+n-Z_1(1+\zeta^2)](k\cdot p-\vec{k}\cdot\vec{q})\tilde{\Phi}(\vec{g}+\vec{q})\tilde{U}(\vec{q})e^{-in\theta_1(\vec{q})+i(N+n)\theta(\vec{p}+\vec{q})}$$

$$\times\left[\omega\left\{\alpha\left(\frac{\vec{p}}{k\cdot p}\right)D^{\dagger}_{1,n}[\theta_1(\vec{q})-\theta(\vec{p})]-Z(1-\zeta^2)D^{\dagger}_{2,n}\right\}-\Pi_0D^{\dagger}_n\right]\times\frac{J_{(N+n)}\left(\alpha\left(\frac{\vec{p}+\vec{q}}{k\cdot p-\vec{k}\cdot\vec{q}}\right),-\frac{Z_1}{2}(1-\zeta^2),\theta(\vec{p}+\vec{q})\right)}{\vec{q}^2+2\vec{\Pi}\cdot\vec{q}-2n(k\cdot p-\vec{k}\cdot\vec{q})+i0}\Bigg\}$$

$$\times\delta(\Pi_0-\varepsilon_0-\omega N). \tag{31}$$

For hydrogenlike atoms with the charge number Z_a, the condition of the Born approximation for the photoelectron scattering (in the Coulomb field),

$$\frac{Z_ae^2}{\hbar v}\ll 1, \tag{32}$$

requires electron velocities $v \gg Z_a \alpha$, where $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant (it is assumed that $Z_a \ll 137$; $\hbar$ and c are restored for clarity). The photoelectron acquires such velocities in the EM wave field at the intensities

$$\xi \gg \frac{Z_a}{137}. \tag{33}$$

As will be shown below, Eq. (33) is the condition of the Born approximation in the ATI process of hydrogenlike atoms taking into account the photoelectron rescattering.

The initial bound state enters into Eq. (27) through its momentum space wave function $\tilde{\Phi}(\vec{b})$. For hydrogenlike atoms, the bound state wave function is

$$\Phi(\vec{\eta}) = \frac{\exp(-\eta/a)}{\sqrt{\pi a^3}}, \tag{34}$$

where $a = a_0/Z_a$ ($a_0 = 1/me^2$ is the Bohr radius) and the corresponding momentum space wave function has the following form:

$$\tilde{\Phi}(\vec{b}) = \frac{2^3(\pi a^3)^{1/2}}{\vec{b}^4 a^4}. \tag{35}$$

Note that in Eq. (35) it has been taken into account that $|\vec{b}|a \gg 1$ in accordance with the Born approximation. Then the function $\tilde{\Phi}(\vec{g}+\vec{q})$ in the second term in curly brackets of Eq. (31) can be replaced by the quantity $\delta(\vec{g}+\vec{q})/\sqrt{\pi a^3}$ because of the small contributions of the other terms in an expansion of $T_{i\rightarrow f}$ over the parameter $\vec{g}^2 a^2$ (see, e.g., [51]), which will be shown below. Such a δ function can be used to accomplish the integration over $\vec{q}$ in the second term of the sum in the large curly brackets of Eq. (31).

For the scattering of a charged particle in the Coulomb field for which the Fourier transform is

$$\tilde{U}(\vec{g}) = \frac{4\pi}{am\vec{g}^2}, \tag{36}$$

we have the following expression for the transition amplitude in the field of arbitrary polarization of an EM wave:

$$T_{i\rightarrow f} = \frac{i2^4(\pi a)^{3/2}}{\sqrt{m\Pi_0}} \frac{(k\cdot p)}{\vec{g}^4 a^4} \sum_{N=-\infty}^{\infty} \Bigg\{ [N - Z(1+\zeta^2)] e^{iN\theta(\vec{p})} J_N\left(\alpha\left(\frac{\vec{p}}{k\cdot p}\right), -\frac{Z}{2}(1-\zeta^2), \theta(\vec{p})\right) - \frac{\omega\varepsilon_0 \vec{g}^2}{m(k\cdot p)}$$
$$\times \sum_{n=-\infty}^{\infty} [2n - \alpha'(1+\zeta^2)] e^{-i(2n-N)\theta(\vec{p})} \frac{\{[\omega(2n-N)+\Pi_0] C^{\dagger}_{N-2n} + \omega\alpha'(1-\zeta^2) C^{\dagger}_{2,N-2n}\} J_n\left(\frac{-\alpha'(1-\zeta^2)}{2}\right)}{m_*^2 + \varepsilon_0^2 - 2\varepsilon_0[\Pi_0 + \omega(2n-N)]} \Bigg\}$$
$$\times \delta(\Pi_0 - \varepsilon_0 - \omega N), \tag{37}$$

where α' is defined by Eq. (26) at $\vec{q} = -\vec{g}$ and $\alpha' = e^2\overline{A_0^2}/4\omega\varepsilon_0$; then $J_n([-\alpha'(1-\zeta^2)]/2)$ is the ordinary Bessel function $[J_{2n}(0,x,0) = J_n(x)$, Eq. (A7)], and C_s and $C_{2,s}$ are defined by the expressions

$$C_s = J_s(\alpha(\vec{p}/kp), (Z-\alpha')(1-\zeta^2)/2, \theta(\vec{p})) \tag{38}$$

and

$$C_{2,s} = \frac{1}{2}[J_{s-2}(\alpha(\vec{p}/kp), (Z-\alpha')(1-\zeta^2)/2, \theta(\vec{p})) e^{-i2\theta(\vec{p})} + J_{s+2}(\alpha(\vec{p}/kp), (Z-\alpha')(1-\zeta^2)/2, \theta(\vec{p})) e^{i2\theta(\vec{p})}]. \tag{39}$$

Integrating the expression (28) over Π_0, taking into account Eqs. (37) and (30), for differential probability of the ATI we obtain the formulas

$$\frac{dW_{i\rightarrow f}}{d\Omega} = \frac{2^4}{\pi m a^5} \sum_{N=N_0}^{\infty} \frac{[N - Z(1+\zeta^2)]^2 (k\cdot\Pi)^2 |\vec{\Pi}|}{\vec{g}^8} \Bigg| \Bigg\{ e^{iN\theta(\vec{\Pi})} J_N\left(\alpha\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right), -\frac{Z}{2}(1-\zeta^2), \theta(\vec{\Pi})\right) + \frac{\vec{g}^2}{2m[N - Z(1+\zeta^2)](k\cdot\Pi)}$$
$$\times \sum_{n=-\infty}^{\infty} e^{-i(2n-N)\theta(\vec{\Pi})} J_n\left(\frac{-\alpha'(1-\zeta^2)}{2}\right) \times [(\varepsilon_0 + 2n\omega) C^{\dagger}_{N-2n} + \omega\alpha'(1-\zeta^2) C^{\dagger}_{2,N-2n}] \Bigg\} \Bigg|^2, \tag{40}$$

where $\vec{g}$ is the three-vector,

$$\vec{g} = \vec{\Pi} - N\vec{k}, \quad |\vec{\Pi}| = \sqrt{(\varepsilon_0 + \omega N)^2 - m_*^2}. \tag{41}$$

The number N_0 from which we carry out the summation in Eq. (40) is defined from the energy conservation law of the ATI process: $N_0 = \langle (m_* - \varepsilon_0)/\omega \rangle$.

The first term in the curly brackets of Eq. (40) corresponds to the result of the KFR approximation, and the second term shows the dependence of the ATI probability on the ejected photoelectron stimulated bremsstrahlung (SB) probability, i.e., it takes into account the rescattering process.

IV. PROBABILITY OF THE ATI PROCESS FOR THE CIRCULAR AND LINEAR POLARIZATION OF AN EM WAVE

The state of a photoelectron in the field of a strong EM wave and consequently the ionization probability essentially depends on the polarization of the wave (the nonlinear effect of intensity conditioned by the impact of a strong magnetic field). Thus, for circular polarization the relativistic parameter of the wave intensity $\xi^2 = \text{const} = \xi_0^2$ and the longitudinal velocity of the electron in the wave $v_{II} = \text{const}$ (eliminating this inertial motion—in the framework of the electron—we have the uniform rotation in the polarization plane with the wave frequency ω), meanwhile for the linear one $\xi^2 = \xi_0^2 \cos^2\varphi$ and v_{II} oscillates with the frequencies of all wave harmonics $n\omega$ corresponding to strongly unharmonic oscillatory motion of a photoelectron. The latter leads principally to different behavior of the ionization process and corresponding formulas depending on the polarization of a strong wave. Therefore, we shall consider the cases of circular and linear polarizations of an EM wave field separately.

From Eq. (40), for the circularly polarized wave ($\zeta = 1$) in the first Born approximation by the ionized atom potential, we obtain the following formula for the differential probability of the ATI process:

$$\frac{dW_{i\to f}}{d\Omega} = \frac{2^4}{\pi m a^5} \sum_{N=N_0}^{\infty} \frac{(N-2Z)^2 (k\cdot\Pi)^2 |\vec{\Pi}|}{\vec{g}^8} J_N^2\left(\alpha\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right)\right) \times \left\{1 + \frac{\vec{g}^2}{2(N-2Z)(k\cdot\Pi)}\right\}^2 . \tag{42}$$

As is seen from this formula, in contrast to the case of another polarization, the differential probability of the ATI process is defined by the ordinary Bessel function instead of the function $J_n(u,v,\triangle)$ and the sum over n vanishes. The latter corresponds to the above-mentioned fact that for the circular polarization, the parameter of the intensity of the wave $\xi^2 = \text{const}$, and the effect of the intensity of a strong wave appears in the form of constant renormalization of the characteristic parameters of the interacting system.

Let us estimate the contribution of photoelectron rescattering in the probability of the ATI process that is the second term in the curly brackets in Eq. (42). The latter is

$$\frac{\vec{g}^2}{2(N-2Z)(k\cdot\Pi)} \simeq 1 \tag{43}$$

for the most probable number of absorbed photons at which the Bessel function has the maximum value. So, the rescattering effect has the same order as the probability of the direct transition in the SFA. It should be noted that the derivations relying upon the SFA (e.g., [41]) are expected to become more accurate at a high-intensity EM field. However, the prediction of the SFA with regard to the rescattering effect in a high-intensity EM field, i.e., for a relativistic photoelectron (according to which the rescattering will be negligible in the relativistic domain with increasing the wave intensity [41]), is not true, especially for the Coulomb field, as we have obtained a significant contribution even in the Born approximation when the impact of the scattering potential is the smallest. Indeed, beyond the context of the Born approximation, the contribution of the rescattering effect in the ATI process will be more considerable (for instant, in the above-considered GEA for the scattering potential).

In the context of the current approximation $\xi \gg Z_a/137$, the explicit analytic formulas for the total ionization rate can be obtained utilizing the properties of the Bessel function. With the condition (33), the argument of the Bessel function $X(N) \gg 1$ and always $X < N$. Therefore, the terms with $N \gg 1$ and $N \sim X$ give the main contribution in the sum (42). Besides, in this limit one can replace the summation over N with integration and approximate the Bessel function by the Airy one,

$$J_N(x) \simeq \left(\frac{2}{N}\right)^{1/3} \text{Ai}\left[\left(\frac{N}{2}\right)^{2/3}\left(1 - \frac{x^2}{N^2}\right)\right]. \tag{44}$$

Turning to spherical coordinates, we carry out the integration over the φ since there is azimuthal symmetry with respect to the direction $\vec{k}$ (the OZ axis), and for the ionization rate we have

$$W_{i\to f} = \frac{2^5}{ma^5}\int_0^{\pi} \sin\theta d\theta \int_{N=N_0}^{\infty} dN \left(\frac{2}{N}\right)^{2/3} \times \frac{(N-2Z)^2 (k\cdot\Pi)^2 |\vec{\Pi}|}{\vec{g}^8} \text{Ai}^2[y(N,\theta)] \times \left\{1 + \frac{\vec{g}^2}{2(N-2Z)(k\cdot\Pi)}\right\}^2, \tag{45}$$

where

$$y(N,\theta) = \left(\frac{N}{2}\right)^{2/3}\left[1 - \frac{\alpha^2\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right)}{N^2}\right]. \tag{46}$$

The $y(N,\theta)$ has a minimum as a function of N and θ, and since the Airy function decreases exponentially with increasing argument, one can use the Laplace method (the method of steepest descent) in order to carry out the integration over N as well as over θ. The extremum points of the function $y(N,\theta)$, i.e., the most probable values of N and θ, are

$$N_m = \frac{m_*^2 - \varepsilon_0^2}{\varepsilon_0 \omega} \simeq \frac{m}{\omega}\xi^2, \quad \cos\theta_m = \frac{|\vec{\Pi}(N_m)|}{\Pi_0(N_m)}, \tag{47}$$

and

$$y_m = y(N_m, \theta_m) = \frac{2^{1/3}E_B}{N_m^{1/3}\omega} = \left(\frac{F_{\rm at}}{2F_0}\right)^{2/3}, \tag{48}$$

where F_0 and $F_{\rm at} = Z_a^3 m^2 e^5$ are wave and atomic electric-field strengths. At $N = N_m$ and $\theta = \theta_m$, we have a peak for angular and energetic distribution. Let us note that the contribution of the rescattering effect to the angular distribution of the photoelectrons is nonessential.

For $y_m \ll 1$, when the wave electric-field strength greatly exceeds the atomic one ($F_0 \gg F_{\rm at}$) the main contribution in the integral gives

$$\delta\theta \simeq (N_m/2)^{-1/3}/\sqrt{1+\xi^2} \quad \text{and} \quad \delta N \simeq 2(N_m/2)^{2/3} \tag{49}$$

(angular and energetic widths of the peak) and for ionization rate we have an explicit formula that expresses directly the dependence upon the wave intensity,

$$W_{i\to f} = \frac{2^{7/3}}{3^{4/3}\Gamma^2(2/3)} \pi\omega \left(\frac{\omega}{E_B}\right)^3 \left(\frac{F_{\rm at}}{F_0}\right)^{11/3}. \tag{50}$$

For $y_m \gg 1$ or $F_0 \ll F_{\rm at}$ (the so-called tunneling regime of ionization), we shall use the following asymptotic formula for the Airy function:

$$\mathrm{Ai}(x) \simeq \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(-\frac{2x^{3/2}}{3}\right), \tag{51}$$

and applying the Laplace method we have

$$W_{i\to f} = 2\omega \left(\frac{\omega}{E_B}\right)^3 \left(\frac{F_{\rm at}}{F_0}\right)^3 \exp\left\{-\frac{2}{3}\frac{F_{\rm at}}{F_0}\right\}. \tag{52}$$

Let us revert back to the Born condition (32) to substantiate the condition (33). As is shown above, we have a peak for angular and energetic distribution (42) at θ_m and N_m (47), and the electron mean velocity will be defined by these values,

$$v = \frac{|\vec{\Pi}(N_m)|}{\Pi_0(N_m)} \simeq \frac{\xi}{\sqrt{1+\xi^2}}. \tag{53}$$

Substituting Eq. (53) into Eq. (32), we have the condition of the Born approximation in ATI process of hydrogenlike atoms [Eqs. (33)].

Using the explicit analytic formulas for the total ionization rate, we can conclude that at $N = N_m$ and $\theta = \theta_m$ we have peaks for angular and energetic distribution that are given by Eqs. (47) and (49) with the angular and energetic widths of the peaks $\delta\theta$ and δN, respectively.

In the case of linear polarization of the wave from Eq. (40), we have

$$\frac{dW_{i\to f}}{d\Omega} = \frac{2^4}{\pi m a^5} \sum_{N=N_0}^{\infty} \frac{(N-Z)^2 (k\cdot\Pi)^2 |\vec{\Pi}|}{\vec{g}^8} \times \left\{ J_N\left(\alpha\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right), -\frac{Z}{2}\right) + \frac{\vec{g}^2}{2m(N-Z)(k\cdot\Pi)} \times \sum_{n=-\infty}^{\infty} J_n(-\alpha'/2) \left\{ (\varepsilon_0 + 2n\omega) J_{N-2n} \times \left(\alpha\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right), (Z-\alpha')/2\right) + \frac{\omega\alpha'}{2} \times \left[J_{N-2n-2}\left(\alpha\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right), (Z-\alpha')/2\right) + J_{N-2n+2}\left(\alpha\left(\frac{\vec{\Pi}}{k\cdot\Pi}\right), (Z-\alpha')/2\right)\right]\right\}\right\}^2, \tag{54}$$

where $J_n(u,v)$ is the real *generalized* Bessel function (e.g., see [18]). As is seen from the formula (54), in this case the total probability of the ATI process includes all intermediate transitions of a photoelectron through the virtual vacuum states as well, corresponding to the emission and absorption of wave photons of number $-\infty < n < \infty$ (the sum over n) in accordance with the above-mentioned behavior of the wave intensity effect at linear polarization (strongly unharmonic oscillatory motion of a photoelectron).

Let us now consider the ATI process with the rescattering effect in the nonrelativistic limit since the theoretical treatments of this problem—mainly the Keldysh-Faisal-Reiss (KFR) ansatz [16–18]—in general have been carried out for a nonrelativistic photoelectron when the rescattering effect is neglected. In the pioneering result of Keldysh [16], the rescattering of a photoelectron from the potential of atomic remainder has been approximately estimated and put in the form of a coefficient in the ultimate formula for the ionization probability (for wave fields much smaller than atomic ones). Furthermore, the same approach has been taken in [53] for relatively large wave fields up to the atomic ones. Besides, in the existing nonrelativistic theory of ATI, the gauge problem for the description of interaction with the wave field and different views concerning the role of wave intensity in the dipole approximation have arisen. For a discussion of these problems, a special paper has been devoted [52]. Moreover, in the context of the same Keldysh-Faisal-Reiss ansatz, the existence of a stabilization effect depends on the gauge of the wave field [9]. Therefore, we shall consider the results of the present paper in the nonrelativistic limit taking into account the photoelectron rescattering.

From the formula (42), for the differential probability of the ATI transition rate in the case of circular polarization of an EM wave in the nonrelativistic limit we have

$$\frac{dW_{i\to f}^{\rm nrel}}{d\Omega} = \frac{8\omega}{\pi}\left(\frac{E_B}{\omega}\right)^{5/2} \sum_{N=N_0}^{\infty} \frac{(N-2z-E_B/\omega)^{1/2}}{(N-2z)^2} J_N^2(\vartheta) \times \left[1 + \frac{N-2z-E_B/\omega}{N-2z}\right]^2, \tag{55}$$

where

$$\vartheta = \frac{e\overline{A}_0}{\omega m}\sqrt{(\vec{p}\cdot\hat{\vec{e}}_1)^2+(\vec{p}\cdot\hat{\vec{e}}_2)^2}, \tag{56}$$

$z = Z = Z_1 = e^2\overline{A_0^2}/4m\omega$, and $N_0 = \langle(\vec{p}^2/2m - E_B)/\omega + z\rangle$.

The corresponding condition of the Born approximation [Eq. (33)] in the nonrelativistic limit is

$$1 \gg \xi \gg \frac{Z_a}{137}. \tag{57}$$

The first term in the quadratic brackets of Eq. (55) coincides with the above-threshold ionization differential probability obtained in the SFA for the nonrelativistic photoelectron [18] without the rescattering effect. According to [41], the SFA is expected to become valid when the ponderomotive potential $U_p = e^2\overline{A_0^2}/2m$ due to an EM radiation field larger than the ionization potential of the atom, $U_p \gg E_B$, and consequently $\vec{p}^2/2m \gg E_B$, which is the condition of the Born approximation. Then taking into account the scattering potential by perturbation theory, we obtain [Eq. (55)] that the contribution of the photoelectron rescattering in the ATI probability (in the first order of the Born approximation over the Coulomb potential) is of the order of the main results of the KFR ansatz. Therefore, neglecting the SB process for the photoelectron in the Coulomb field of an atomic remainder is incorrect.

In the case of a linear polarized EM wave from the formula (54) we have the differential probability of the ATI process in the nonrelativistic domain,

$$\frac{dW_{i\to f}^{\mathrm{nrel}}}{d\Omega} = \frac{8\omega}{\pi}\left(\frac{E_B}{\omega}\right)^{5/2}\sum_{N=N_0}^{\infty}\frac{(N-z-E_B/\omega)^{1/2}}{(N-z)^2}J_N^2\left(u,-\frac{z}{2}\right)\times\left\{1+\frac{(N-z-E_B/\omega)}{(N-z)}\right\}, \tag{58}$$

where

$$u = z^{1/2}\chi, \quad \chi = 8^{1/2}\left(N - z - \frac{E_B}{\omega}\right)^{1/2}\cos\theta \tag{59}$$

and θ is the angle between the velocity vector of the emitted photoelectron and the wave polarization vector.

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APPENDIX: DEFINITION OF THE FUNCTION $J_n(u,v,\triangle)$

A function $J_n(u,v,\triangle)$ may be defined by

$$J_n(u,v,\triangle) = (2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\exp\{i[u\sin(\theta+\triangle) + v\sin(2\theta) - n(\theta+\triangle)]\} \tag{A1}$$

or by an infinite series representation

$$J_n(u,v,\triangle) = \sum_{k=-\infty}^{\infty}e^{-i2k\triangle}J_{n-2k}(u)J_k(v). \tag{A2}$$

We perform two important theorems, which can be proved from Eq. (A1):

$$\sum_{n=-\infty}^{\infty}e^{in(\varphi+\triangle)}J_n(u,v,\triangle) = \exp\{i[u\sin(\varphi+\triangle)+v\sin 2\varphi]\} \tag{A3}$$

and

$$\sum_{k=-\infty}^{\infty}J_{n\mp k}(u,v,\triangle)J_k(u',v',\pm\triangle) = J_n(u\pm u',v\pm v',\triangle). \tag{A4}$$

An integration by parts in Eq. (A1) yields the relation

$$2nJ_n(u,v,\triangle) = u[J_{n-1}(u,v,\triangle)+J_{n+1}(u,v,\triangle)] + 2v[e^{-i2\triangle}J_{n-2}(u,v,\triangle) + e^{i2\triangle}J_{n+2}(u,v,\triangle)]. \tag{A5}$$

From either Eq. (A1) or Eq. (A2) it follows that

$$J_n(u,0,\triangle) = J_n(u) \tag{A6}$$

and

$$J_n(0,v,\triangle) = \begin{cases} e^{-i\triangle n}J_{n/2}(v), & n \text{ even} \\ 0, & n \text{ odd}. \end{cases} \tag{A7}$$

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