Dynamics of a charged particle in a circularly polarized traveling electromagnetic wave

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The relativistic motion of a charged particle in a transverse circularly or almost circularly polarized homogeneous electromagnetic wave is studied using the Hamiltonian formalism. First, the case of a circularly and almost circularly polarized traveling wave propagating in a nonmagnetized space is studied. In the case of an almost circularly polarized wave, it is shown that the charged particle has an average velocity along the propagation direction of the wave. The same result is derived considering a cold electron plasma. The case of a traveling wave propagating along a constant homogeneous magnetic field is then considered. Using canonical transformations, it is shown that the equations of motion can be derived from an autonomous Hamiltonian which has two degrees of freedom and a first integral. As a consequence, the system is completely integrable. An equation is found for the particle energy when it is initially resonant. This equation is solved exactly, and the asymptotic solution is obtained. The expression for the energy allows a solution for the system in terms of quadratures, and in consequence the asymptotic solution for all the variables. The case of an almost circularly polarized wave propagating along a constant homogeneous magnetic field is also studied. Finally, a magnetic field gradient is considered, and new acceleration mechanisms are found.

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I. INTRODUCTION

The dynamics of a charged particle in a transverse circularly polarized electromagnetic wave is studied in this paper by using the Hamiltonian formalism. Several important points in Hamiltonian dynamics must first be recalled. An autonomous Hamiltonian system is called completely integrable if it possesses *n* independent, time-independent invariants in involution [1–4]. This is often called the Liouville theorem [1,2]. When it is completely integrable, the solution for the motion can be expressed in terms of canonical action-angle variables [1–4]. Kozlov and Kolesnikov showed that the solution of a time-dependent Hamiltonian system with *n* degrees of freedom possessing *n* independent, possibly time-dependent invariants in involution can be found by quadratures [1,5]. It was shown that in this case no chaos can take place [6]. We shall call such a system integrable. This is an extension of the first definition of integrability. In this sense, Liouville's theorem on integrability still holds in the case of time-dependent Hamiltonian systems.

By considering the problem of a charged particle in a purely circularly polarized wave, we demonstrate simply how invariants can be found and some of their properties. Similar invariants with the same properties exist in many of the more complex problems studied further in this paper. Complete integrability is shown. Then the dynamics of one particle in an almost circularly polarized wave is considered. Complete integrability is also proved in this case. It is shown that the wave generates a constant electron current along its propagation direction. The existence of such an effect is confirmed for a cold electron plasma approach by using the wave equations first derived by Akhiezer and Polovin [7] (also see Ref. [8]).

The dynamics of a charged particle in a circularly polarized electromagnetic wave propagating along a constant magnetic field is studied next. First, the magnetic field is assumed to be homogeneous, this problem has already been explored by Roberts and Buchsbaum [9]. We present elegant derivations for some results previously derived by them. When the index of refraction is unity, Roberts and Buchsbaum considered a charged particle starting from rest in the field of a circularly polarized plane wave, whose frequency is equal to the rest mass cyclotron frequency (eB_0/m). They found a ''synchronous'' solution in which the particle gains energy indefinitely. This solution occurs because the particle gains energy parallel to, as well as perpendicular to, the propagation direction of the circularly polarized plane wave. The increase in perpendicular energy lowers the cyclotron frequency of the charged particle, while the increase in parallel energy changes the velocity of the particle, resulting in a Doppler shift to a lower frequency as ''seen'' by the particle. In this case, the Doppler shift to the lower frequency equals the reduction in the cyclotron frequency, and the particle remains ''synchronously'' in cyclotron-resonance condition. This problem is studied again in this paper by using the Hamiltonian formalism, which provides a way to prove the integrability of the system. The system has three degrees of freedom and three independent constants in involution, one obtained by using Noether's theorem [1,2,10–12]. Alternatively, the system can be reduced to a two-dimensional problem. Canonical transformations [1–4,11,12] permit the

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use of two invariants as two conjugate variables, and, consequently, the system can be reduced to a time-dependent Hamiltonian system with two degrees of freedom possessing two independent invariants in involution. This problem is therefore integrable in the new ''Liouville sense.'' A formal solution is given in terms of quadratures. It is also shown that a canonical transformation can change this system into an autonomous one with two degrees of freedom. In this way, the problem is shown to be completely integrable. The resonance condition is identical to one of the invariants when expressed in terms of resonant initial conditions [13]. This gives a mathematical explanation for the synchronous case already discussed above. The approach by quadratures shows that the solution can be written in terms of the energy of the particle, and that the energy is a solution of an integrable differential equation in the general case. We only study the resonant case, and asymptotic solutions for the energy and the different variables are derived. The asymptotic solution for the energy is used to show that the acceleration mechanism described above is more efficient than a linear accelerator with the same field strength for a short distance in the direction of the wave propagation only. It is shown numerically that, for a short distance, the particle gains more transverse energy than parallel energy. The situation where the wave is almost circularly polarized is also studied, and it is shown numerically that the synchronous solution still exists.

Finally, the situation where a charged particle is first accelerated over a short distance (the distance where the longitudinal energy catches up with the transverse energy) using the synchronous resonance, then is introduced to a region where the magnetic field is no longer homogeneous, is considered, and new acceleration mechanisms are described. When the magnetic field grows linearly very rapidly, the final energy reached by the particle can be higher than when the magnetic field is homogeneous. When one has a decreasing magnetic field i.e., if a charged particle is initially resonant and at rest and then immersed in a linear magnetic field gradient, or initially resonant and at rest in the middle of Helmoltz coils, an interesting phenomenon takes place. when the magnitude of the electric field is higher than some value the particle is accelerated just as if the magnetic field were homogeneous, over a distance which is roughly five times the magnetic field gradient length.

II. DYNAMICS OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC CIRCULARLY AND ALMOST CIRCULARLY POLARIZED TRAVELING WAVE

A. Dynamics of a charged particle in an electromagnetic circularly polarized wave

A circularly polarized traveling wave propagating along the z direction (wave vector k_0 parallel to the z direction) is considered. The fields are given by

$$E_x = E_0 \sin(\omega_0 t - k_0 z), \quad E_y = -E_0 \cos(\omega_0 t - k_0 z),$$

$$E_z = 0,$$

$$B_x = \frac{k_0 E_0}{\omega_0}\cos(\omega_0 t - k_0 z), \quad B_y = \frac{k_0 E_0}{\omega_0}\sin(\omega_0 t - k_0 z),$$

$$B_z = 0, \tag{1}$$

where E_0, ω_0, and k_0 are constants. The following vector potential is chosen:

$$\mathbf{A} = \frac{E_0}{\omega_0}\cos(\omega_0 t - k_0 z)\hat{\mathbf{e}}_x + \frac{E_0}{\omega_0}\sin(\omega_0 t - k_0 z)\hat{\mathbf{e}}_y . \tag{2}$$

The relativistic Hamiltonian of a charged particle submitted to this wave is, in MKS units, given by

$$H = \left[\left[P_x + \frac{eE_0}{\omega_0}\cos(\omega_0 t - k_0 z)\right]^2 c^2 + \left[P_y + \frac{eE_0}{\omega_0}\sin(\omega_0 t - k_0 z)\right]^2 c^2 + P_z^2 c^2 + m^2 c^4\right]^{1/2}, \tag{3}$$

where $-e$ and m are the charge and the rest mass of the particle. This is a time-dependent Hamiltonian with one degree of freedom. The constants that we are going to derive now, and their properties, are not always necessary to study this system. The fact similar constants with the same properties exist in most of the more complex systems studied next justify this presentation. As x and y are two cyclic variables the system has the two following constants of motion:

$$A_1 = P_x ,$$

$$A_2 = P_y . \tag{4}$$

These two constants are obviously independent and in involution. Another constant can be found simply by deriving with respect to time the quantity

$$A_3 = yP_x - xP_y + H/\omega_0 , \tag{5}$$

with the help of Hamilton's equations. This invariant can also be evidenced by using Noether's theorem as follows [10,12]. The relations

$$[A_1, A_3] = A_2 , \quad [A_2, A_3] = -A_1 , \tag{6}$$

are satisfied, where $[A,B]$ stands for the Poisson bracket of A with B. Another first integral is given by

$$A_4 = A_1^2 + A_2^2. \tag{7}$$

It satisfies the relation

$$[A_4, A_3] = 2A_1[A_1, A_3] + 2A_2[A_2, A_3] = 0. \tag{8}$$

The Poisson theorem does not allow one to find new constants of motion when considering the four constants of motion that we have found [2]. One can notice that

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}=-\frac{\omega_0}{k_0}\frac{\partial H}{\partial z}=\frac{\omega_0}{k_0}\dot{P}_z. \tag{9}$$

One obtains a fifth constant of motion by integrating this equation

$$A_5=H-\frac{\omega_0}{k_0}P_z. \tag{10}$$

This constant satisfies

$$[A_1,A_5]=0,\quad [A_2,A_5]=0,\quad [A_3,A_5]=0 \tag{11}$$

and

$$[A_4,A_5]=2A_1[A_1,A_5]+2A_2[A_2,A_5]=0. \tag{12}$$

We have: $[A_3,A_4]=0$, $[A_4,A_5]=0$, and $[A_3,A_5]=0$. Only one of these constants is enough to prove that the system is integrable according to the definition given in Refs. [6]. This is important from a practical point of view, as it means that trajectories are not sensitive to their initial conditions.

Let us now introduce the following dimensionless variables and parameters:

$$\hat{z}=k_0z,\hat{P}_{x,y,z}=\frac{P_{x,y,z}}{mc},\quad \hat{t}=\omega_0 t,\hat{H}=\gamma=\frac{H}{mc^2},\quad a=\frac{eE_0}{mc\omega_0}. \tag{13}$$

The new Hamiltonian of the charged particle, expressed in terms of these normalized quantities, is

$$\hat{H}=[[\hat{P}_x+a\cos(\hat{t}-\hat{z})]^2+[\hat{P}_y+a\sin(\hat{t}-\hat{z})]^2+\hat{P}_z^2+1]^{1/2}. \tag{14}$$

A canonical transformation is introduced $(\hat{x},\hat{P}_x,\hat{y},\hat{P}_y,\hat{z},\hat{P}_z)\rightarrow(\hat{x},\hat{P}_x,\hat{y},\hat{P}_y,\phi,\hat{P}_z)$, given the type-2 generating function [1–4,6]

$$F_2(\hat{x},\hat{y},\hat{z},\hat{P}_x,\hat{P}_y,\hat{P}_z,\hat{t})=\hat{x}\hat{P}_x+\hat{y}\hat{P}_y+\hat{P}_z(\hat{z}-\hat{t}). \tag{15}$$

When one has a type-2 generating function $F_2(q_i,\overline{P_i},\hat{t})$, where $\overline{P_i}$ and the $\overline{Q_i}$ are the new coordinates, and P_i and the q_i the old ones, the canonical transformations are given by

$$p_i=\frac{\partial F_2(q_i,\overline{P_i},\hat{t})}{\partial q_i},$$

$$\overline{Q_i}=\frac{\partial F_2(q_i,\overline{P_i},\hat{t})}{\partial \overline{P_i}}. \tag{16}$$

Consequently, the generating function defined by Eq. (15) yields the canonical transformation

$$\phi=\hat{z}-\hat{t}. \tag{17}$$

The Hamiltonian expressed in terms of the new variables is

$$\hat{H}(\overline{Q_i},\overline{P_i})=\hat{H}(q_i,p_i)+\frac{\partial F_2}{\partial \hat{t}} \tag{18}$$

or

$$\hat{H}=[(\hat{P}_x+a\cos\phi)^2+(\hat{P}_y-a\sin\phi)^2+\hat{P}_z^2+1]^{1/2}-\hat{P}_z. \tag{19}$$

Since this Hamiltonian is time independent, it is a constant of motion. One can notice that $\hat{H}$ is the constant $\hat{A}_5$ ($\hat{A}_5=A_5/mc^2$) exponent expressed in terms of the new variables. Thus $\hat{P}_x$ and $\hat{P}_y$ and the Hamiltonian are three constants. One of these constants is sufficient to prove this time that the system is completely integrable. As shown in Refs. [6], both integrable and completely integrable systems are nonchaotic. When the constants $\hat{P}_x$ and $\hat{P}_y$ equal zero, the fact that A_3 and A_5 are constants implies that γ, $\hat{P}_z$, and consequently $\dot{\phi}(\dot{g}=\partial g/\partial\hat{t})$ are constant. In this case, it is very simple to show that trajectories are circles when projected onto the $(\hat{x},\hat{y})$ plane. One can assume $\hat{P}_x=0$, $\hat{P}_y=0$ and $\hat{P}_z=0$, which implies that the electron is at rest on average in the $(\hat{x},\hat{y})$ plane, and is at rest along the $\hat{z}$ axis, as is assumed when considering the propagation of circularly polarized wave in a plasma. One can then predict that purely transverse circularly polarized waves can propagate in a cold electron plasma.

B. Dynamics of a charged particle in an almost circularly polarized wave

1. Dynamics of one particle only

The fields are given by

$$E_x=E_0\sin(\omega_0 t-k_0 z),$$

$$E_y=-E_0(1+\delta)\cos(\omega_0 t-k_0 z),\quad E_z=0, \tag{20}$$

$$B_x=\frac{k_0E_0}{\omega_0}(1+\delta)\cos(\omega_0 t-k_0 z),$$

$$B_y=\frac{k_0E_0}{\omega_0}\sin(\omega_0 t-k_0 z),\quad B_z=0,$$

where δ is a quantity such that $|\delta|\ll 1$. The following gauge was chosen:

$$\mathbf{A}=\frac{E_0}{\omega_0}\cos(\omega_0 t-k_0 z)\hat{\mathbf{e}}_x+\frac{E_0}{\omega_0}(1+\delta)\sin(\omega_0 t-k_0 z)\hat{\mathbf{e}}_y. \tag{21}$$

The relativistic Hamiltonian of a charged particle in this wave is

$$H=\left[\left[P_x+\frac{eE_0}{\omega_0}\cos(\omega_0 t-k_0 z)\right]^2 c^2 +\left[P_y+\frac{eE_0}{\omega_0}(1+\delta)\sin(\omega_0 t-k_0 z)\right]^2 \times c^2+P_z^2c^2+m^2c^4\right]^{1/2}. \tag{22}$$

This system still has only one degree of freedom. On the one hand, P_x, P_y (A_1 and A_2), and A_5 are still constants. On the other hand, A_3 is no longer a first integral. The system is of course integrable.

The normalized Hamiltonian is obtained by introducing the same dimensionless parameters and variables as in the previous case:

$$\hat{H}=[[\hat{P}_x+a\cos(\hat{t}-\hat{z})]^2 +[\hat{P}_y+a(1+\delta)\sin(\hat{t}-\hat{z})]^2+\hat{P}_z^2+1]^{1/2}. \tag{23}$$

Performing the canonical transformation defined by the type-2 generating function given by Eq. (15), the following autonomous normalized Hamiltonian is obtained:

$$\hat{H}=[[\hat{P}_x+a\cos\phi]^2 +[\hat{P}_y-a(1+\delta)\sin\phi]^2+\hat{P}_z^2+1]^{1/2}-\hat{P}_z. \tag{24}$$

$\hat{P}_x$, $\hat{P}_y$, and $\hat{H}$ are three independent constants in involution. One of these constants is sufficient to show that this system is completely integrable.

Assuming $|\delta|<1$ and neglecting terms in δ^2, Eq. (24) leads to

$$\hat{P}_z\approx\frac{1}{2\hat{A}_5}[(1+a^2+\hat{P}_x^2+\hat{P}_y^2-\hat{A}_5^2) +2\hat{P}_x a\cos\phi-2\hat{P}_y a(1+\delta)\sin\phi+2a^2\delta\sin^2\phi]. \tag{25}$$

When considering that the constants $\hat{P}_x$ and $\hat{P}_y$ equal to zero, and $\hat{P}_z=0$ when $\phi=0$, the constant term between parentheses on the right hand of Eq. (25) is zero, and this equation becomes

$$\hat{P}_z\approx\frac{a^2\delta\sin^2\phi}{\sqrt{1+a^2}}. \tag{26}$$

With these hypotheses, the velocity along the z axis of the charged particle is given by

$$\hat{v}_z=\frac{v_z}{c}=\frac{\hat{P}_z}{\gamma}\approx\frac{a^2\delta\sin^2\phi}{1+a^2}. \tag{27}$$

By averaging over ϕ, one finds that the charged particle has an average velocity along the z axis:

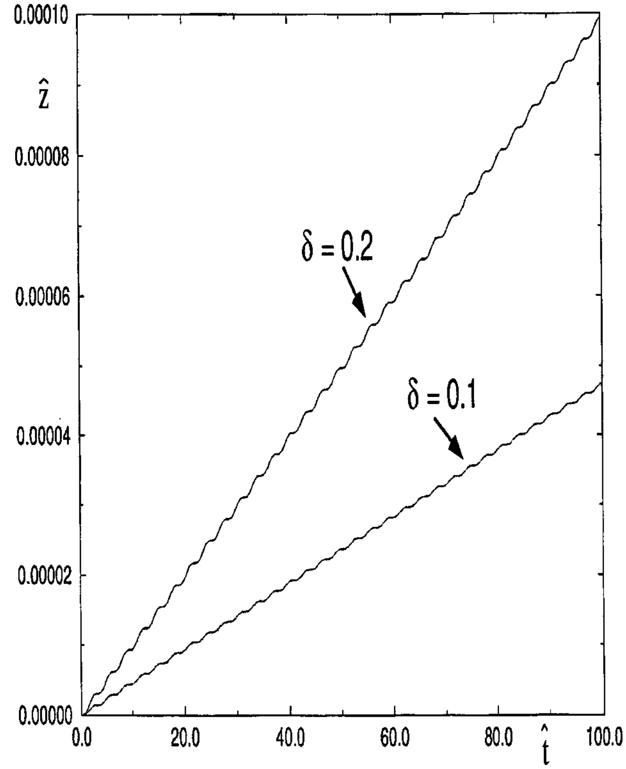

FIG. 1. $\hat{z}$ component of the charged particle for different values of δ. $a=3\times10^{-3}$.

$$\langle\hat{v}_z\rangle\approx\frac{1}{2}\frac{a^2}{1+a^2}\delta. \tag{28}$$

We have verified, by numerically solving the exact equation of motion, that the average velocity is indeed proportional to δ (Fig. 1). This result shows that the propagation of an almost circularly polarized traveling wave can produce a constant electron current in a plasma when its density is very low or when the wave has a relativistic intensity.

2. Cold electron plasma approach

We now show that the propagation of a strong electromagnetic wave in a cold electron plasma generates a constant current along the propagation direction of the wave when its phase velocity is very close to the speed of light in vacuum.

To describe the propagation of a relativistically strong wave in a cold electron plasma, we start from the Maxwell and Lorentz equations

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},$$

$$\nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial E}{\partial t}-\mu_0 ne\mathbf{v}, \tag{29}$$

$$\nabla\cdot\mathbf{E}=-\frac{e}{\varepsilon_0}(n-N_0),$$

$$\frac{\partial \mathbf{p}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}=-e\mathbf{E}-e\mathbf{v}\times\mathbf{B},$$

where $\mathbf{v}$ and $\mathbf{p}$ are respectively the velocity and momentum of the electrons, respectively, n is their density and N_0 is the one of ions.

All the variables entering into Eqs. (29) are assumed not to be functions of space and time separately, but only of the combination $\mathbf{i}\cdot\mathbf{r}-Vt$, where $\mathbf{i}$ is a constant unit vector, and V a constant. This means that we look for plane wave solutions travelling in the direction $\mathbf{i}$ with speed V. Introducing the variables

$$\hat{\mathbf{p}}=\frac{p}{mc},\quad \hat{\mathbf{v}}=\frac{v}{c},\quad \tau=t-\frac{\mathbf{i}\cdot\mathbf{r}}{V},\quad \beta=\frac{V}{c},\quad \omega_p^2=\frac{e^2N_0}{\varepsilon_0 m},\tag{30}$$

after some algebra we find the following wave equations:

$$\frac{d^2\hat{p}_x}{d\tau^2}+\omega_p^2\frac{\beta^2}{\beta^2-1}\,\frac{\beta\hat{p}_x}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z}=0,\tag{31a}$$

$$\frac{d^2\hat{p}_y}{d\tau^2}+\omega_p^2\frac{\beta^2}{\beta^2-1}\,\frac{\beta\hat{p}_y}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z}=0,\tag{31b}$$

$$\frac{d^2}{d\tau^2}(\beta\hat{p}_z-\sqrt{1+\hat{p}^2})+\omega_p^2\frac{\beta^2\hat{p}_z}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z}=0.\tag{31c}$$

While the electron density is given by

$$n=\frac{N_0V}{V-\mathbf{i}\cdot\mathbf{v}}.\tag{32}$$

Letting $\theta=\omega_p(\beta^2-1)^{-1/2}\tau$, the propagation equations become

$$\frac{d^2\hat{p}_x}{d\theta^2}+\frac{\beta^3\hat{p}_x}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z}=0,\tag{33a}$$

$$\frac{d^2\hat{p}_y}{d\theta^2}+\frac{\beta^3\hat{p}_y}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z}=0,\tag{33b}$$

$$\frac{d^2}{d\theta^2}(\beta\hat{p}_z-\sqrt{1+\hat{p}^2})+\frac{\beta^2(\beta^2-1)\hat{p}_z}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z}=0.\tag{33c}$$

Assuming that the phase velocity is close to the velocity of light in vacuum, i.e., $\beta\approx 1$, Eq. (33c) leads to the fact that the following quantity is a constant:

$$\hat{A}_5=\alpha^2=\sqrt{1+\hat{p}^2}-\hat{p}_z\,.\tag{34}$$

This invariant is the same as the one found in the case of one particle in an almost circularly polarized wave. It follows from Eqs. (33a) and (33b) that

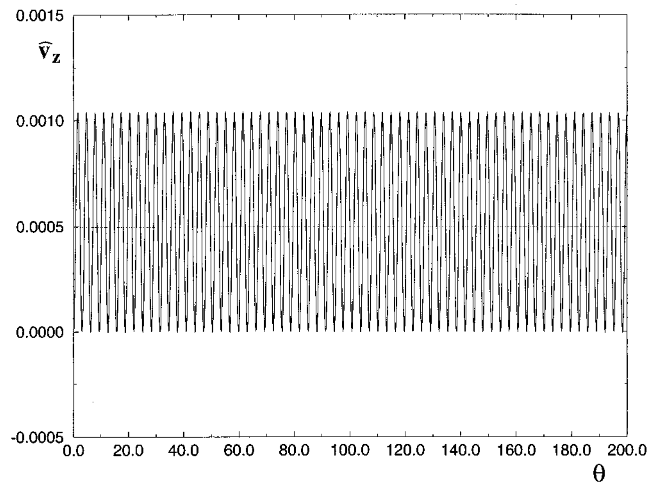

FIG. 2. Velocity of the particle along the z axis vs θ, compared to the average velocity given by Eq. (38) (horizontal solid line). $\delta=0.1$, $a=0.1$, and $\beta=1+10^{-7}$.

$$\hat{p}_x=\hat{p}_{0x}\cos(\theta/\alpha),$$

$$\hat{p}_y=\hat{p}_{0y}\sin(\theta/\alpha),\tag{35}$$

where $\hat{p}_{0x}$ and $\hat{p}_{0y}$ are constants.

Considering that the wave is almost circularly polarized, we let $\hat{p}_{0y}=\hat{p}_{0x}(1+\delta)(\delta\ll 1)$. It is assumed that $\hat{p}_z$ is small compared to $\hat{p}_{0x}$ and $\hat{p}_z=0$ when $\theta=0$. The electric field can be calculated by using the expression [14]

$$e\mathbf{E}=-\frac{\partial \mathbf{p}}{\partial t}-\mathrm{grad}\, cp_0\,,\tag{36}$$

with $p_0=\sqrt{p^2+m^2c^2}$. Then, it is consistent to take $\hat{p}_{0x}=a$. Equation (34) leads to

$$\hat{p}_z\approx\frac{a^2\delta}{\sqrt{1+a^2}}\sin^2(\theta/\alpha);\tag{37}$$

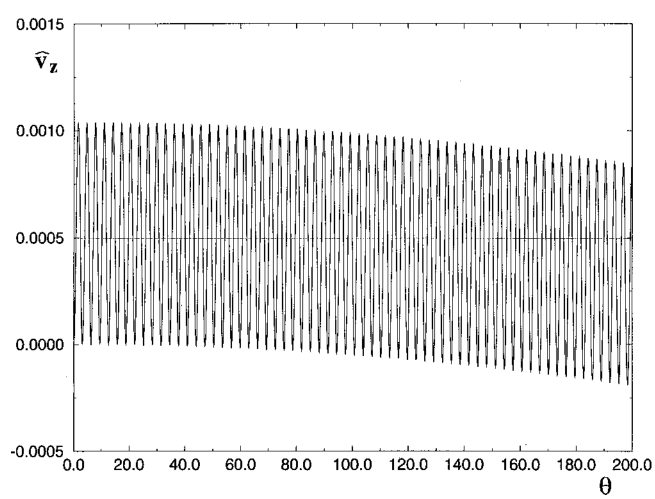

FIG. 3. Velocity of the particle along the z axis vs θ, compared to the average velocity given by Eq. (38) (horizontal solid line). $\delta=0.1$, $a=0.1$ and $\beta=1+10^{-5}$.

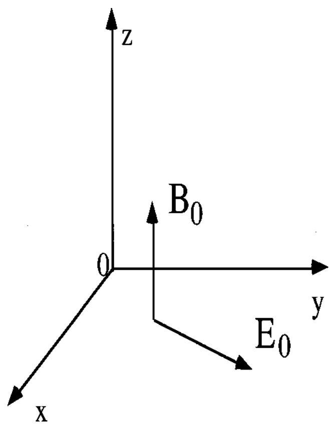

FIG. 4. Coordinate system.

as a consequence, the average velocity along the z axis is again [8]

$$\langle \hat{v}_z \rangle \approx \frac{1}{2} \frac{a^2}{1+a^2} \delta. \tag{38}$$

This is the same result as in the case of one particle.

Equations (33) are solved numerically considering very low density plasmas (it has been assumed that $\beta = 1 + \varepsilon'$ with $\varepsilon' \ll 1$). When the phase velocity of the wave is close to the speed of light in vacuum, Fig. 2 shows the good agreement between the numerical solution of Eqs. (33) and the average value of $\hat{v}_z$ determined analytically [Eq. (38)]. Figure 3 shows for a slightly higher phase speed velocity that the agreement exists only over a very short distance.

According to Eq. (32) the electron density is given by

$$n \approx N_0 \left[1 + \frac{a^2}{1+a^2} \delta \sin^2(\theta/\alpha) \right], \tag{39}$$

the following average density increase is created:

$$\langle \Delta n \rangle \approx \frac{1}{2} \frac{a^2}{1+a^2} \delta N_0. \tag{40}$$

The average electron current density is

$$\langle j_z \rangle = \langle n v_z \rangle \approx \frac{c}{2} \frac{N_0 a^2}{1+a^2} \delta. \tag{41}$$

III. MOTION OF A CHARGED PARTICLE IN A CIRCULARLY POLARIZED ELECTROMAGNETIC TRAVELING WAVE PROPAGATING ALONG A CONSTANT MAGNETIC FIELD

A. Homogeneous constant magnetic field case

1. The wave is purely circularly polarized

(a) Description of the system. Numerical solution of Hamilton's equations. The constant magnetic field $\mathbf{B}_0$ is assumed to be along the z axis (Fig. 4). The traveling wave is

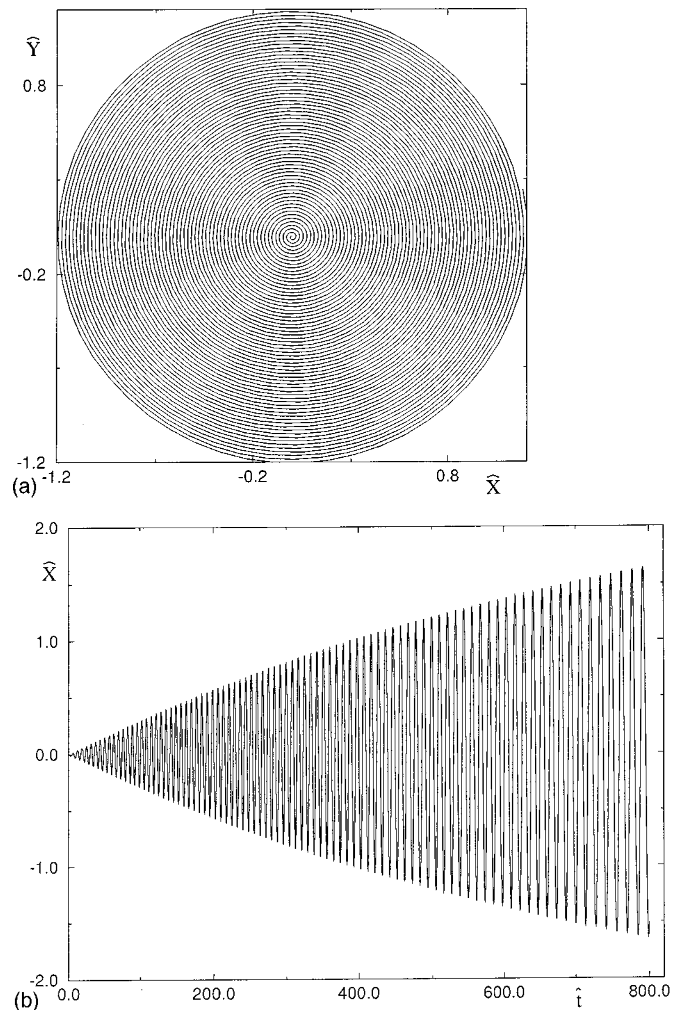

FIG. 5. (a) Trajectory of a charged particle initially resonant and at rest ($\gamma_0 = \Omega_0 = 1$) at $\hat{x}_0 = \hat{y}_0 = 0$ (initial values of $\hat{x}$ and $\hat{y}$) in the $\hat{x}-\hat{y}$ plane. $a = 3 \times 10^{-3}$. (b) $\hat{x}$ component of the charged particle in the same conditions as those of (a).

circularly polarized and has a propagation vector $\mathbf{k}_0$ parallel to $\mathbf{B}_0$. The fields are given by

$$E_x = E_0 \sin(\omega_0 t - k_0 z), \quad E_y = -E_0 \cos(\omega_0 t - k_0 z),$$

$$E_z = 0,$$

$$B_x = \frac{k_0 E_0}{\omega_0} \cos(\omega_0 t - k_0 z), \quad B_y = \frac{k_0 E_0}{\omega_0} \sin(\omega_0 t - k_0 z),$$

$$B_z = B_0. \tag{42}$$

where E_0, B_0, ω_0, and k_0 are constants. The following vector potential is chosen for the electromagnetic field:

$$\mathbf{A} = \left(-\frac{B_0}{2} y + \frac{E_0}{\omega_0} \cos(\omega_0 t - k_0 z) \right) \hat{e}_x + \left(\frac{B_0}{2} x + \frac{E_0}{\omega_0} \sin(\omega_0 t - k_0 z) \right) \hat{\mathbf{e}}_y. \tag{43}$$

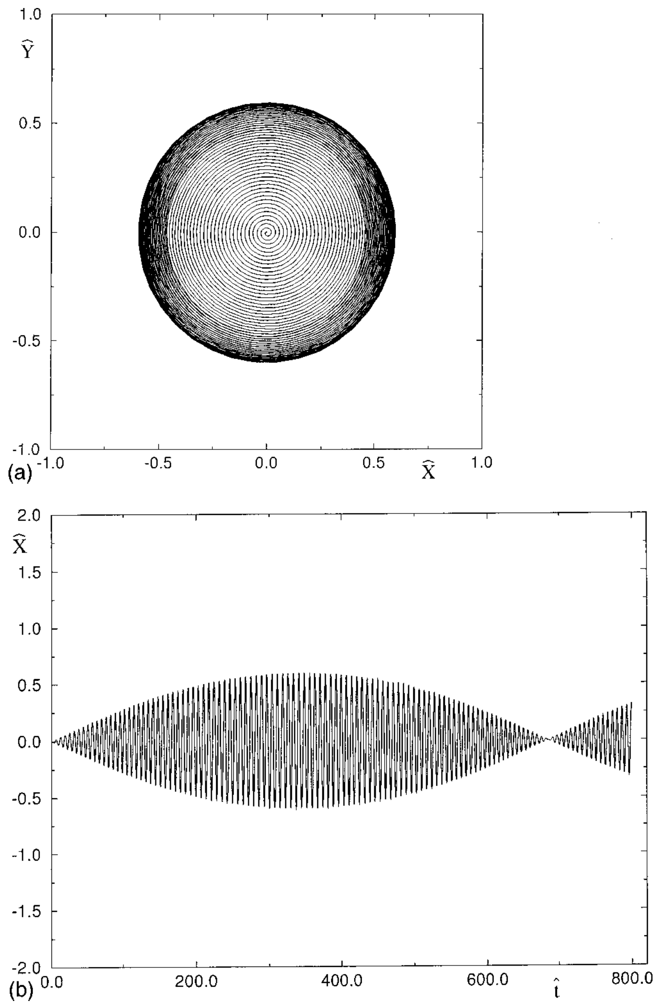

FIG. 6. (a) Trajectory of a charged particle that is initially nonresonant and at rest ($\gamma_0=1$, $\Omega_0=1{,}01$) at $\hat{x}_0=\hat{y}_0=0$ in the $\hat{x}-\hat{y}$ plane. $a=3\times10^{-3}$. (b) $\hat{x}$ component of the charged particle in the same conditions as those of (a).

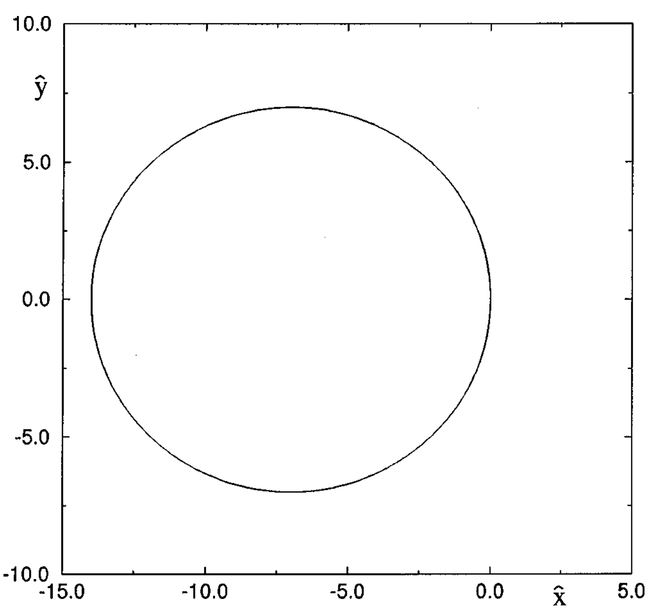

FIG. 7. Trajectory of a charged particle with a high initial energy in the $\hat{x}-\hat{y}$ plane ($\gamma_0=7.07$). $\Omega_0=1$, and $a=3\times10^{-3}$.

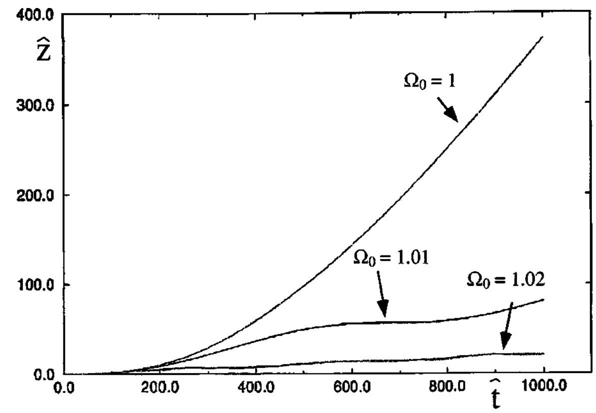

FIG. 8. $\hat{z}$ component of the particle vs $\hat{t}$ for three values of Ω_0. $\gamma_0=1$ and $a=3\times10^{-3}$.

The relativistic Hamiltonian for the motion expressed in MKS units is

$$H=\Bigg[\left(P_x+\frac{eE_0}{\omega_0}\cos(\omega_0 t-k_0 z)-\frac{eB_0}{2}y\right)^2c^2$$
$$+\left(P_y+\frac{eE_0}{\omega_0}\sin(\omega_0 t-k_0 z)+\frac{eB_0}{2}x\right)^2$$
$$\times c^2+P_z^2c^2+m^2c^4\Bigg]^{1/2}. \tag{44}$$

This is a time-dependent Hamiltonian with three degrees of freedom. Two new dimensionless variables and a new dimensionless parameter are now introduced:

$$\hat{x}=x\frac{\omega_0}{c},\quad \hat{y}=y\frac{\omega_0}{c},\quad \Omega_0=\frac{eB_0}{m\omega_0}.$$

As it is assumed that the electromagnetic wave propagates in vacuum ($k_0c/\omega_0=1$), the normalized Hamiltonian is expressed in terms of the dimensionless variables and parameters defined in the following way:

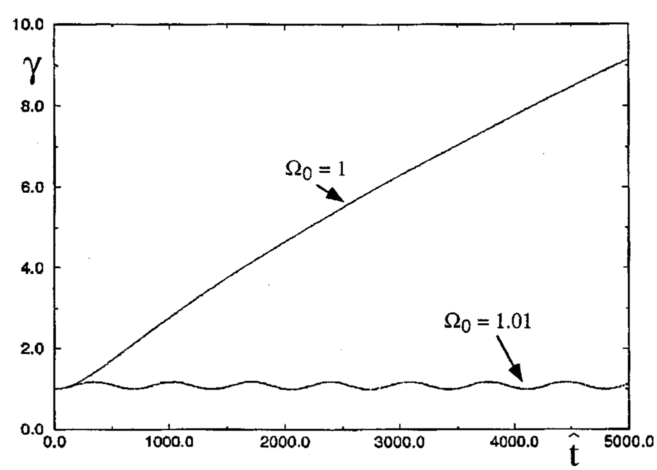

FIG. 9. γ vs time for two values of Ω_0. $\gamma_0=1$ and $a=3\times10^{-3}$.

$$\hat{H}=\left[\left(\hat{P}_x+a\cos(\hat{t}-\hat{z})-\frac{\Omega_0}{2}\hat{y}\right)^2+\left(\hat{P}_y+a\sin(\hat{t}-\hat{z})+\frac{\Omega_0}{2}\hat{x}\right)^2+\hat{P}_z^2+1\right]^{1/2}. \quad (45)$$

The canonical equations are solved numerically using a fourth order Runge-Kutta method. Figures 5–7 show different types of trajectories. When the particle is initially at rest and resonant, it spirals outward in the plane perpendicular to the z axis (Fig. 5). When the particle is initially nonresonant and at rest, it spirals outward and inward (Fig. 6). When the initial energy of the particle is high, its trajectory is a circle (Fig. 7). The z component of the particle strongly depends on the initial mismatch with the resonance condition; for a given finite time the final energy reached by the particle is greater in the resonant case than in any nonresonant case (Fig. 8). Figure 9 compares the Lorentz factor variation of the particle versus time in a resonant case to a nonresonant one.

(b) First demonstration of the integrability of the problem. The Hamilton equations allow us to readily find two constants of motion [12]:

$$C_1=P_x+\frac{eB_0}{2}y,$$

$$C_2=P_y-\frac{eB_0}{2}x. \quad (46)$$

A third constant of motion can be obtained by using Noether's theorem [10,12]. If the Lagrangian is invariant under the infinitesimal transformation

$$t\to t+\varepsilon g(t,\mathbf{r}),$$

$$\mathbf{r}\to\mathbf{r}+\varepsilon\mathbf{u}(t,\mathbf{r}), \quad (47)$$

where ε is an infinitesimal, then a constant of motion is

$$\frac{\partial L}{\partial\mathbf{v}}\cdot\mathbf{u}+\left(L-\mathbf{v}\cdot\frac{\partial L}{\partial\mathbf{v}}\right)g, \quad (48)$$

where $L=-mc^2\sqrt{1-v^2/c^2}-e\mathbf{A}\cdot\mathbf{v}$ is the relativistic Lagrangian, and $\mathbf{v}$ the velocity of the charged particle. It is simple to show that the Lagrangian of the system is invariant under the following transformations:

$$t\to t-\varepsilon/\omega_0,$$

$$x\to x+\varepsilon y,$$

$$y\to y-\varepsilon x, \quad (49)$$

$$z\to z.$$

Therefore, a third constant of motion is

$$C_3=yP_x-xP_y+H/\omega_0. \quad (50)$$

It can be noted that the first two constants (C_1 and C_2/eB_0) are canonically conjugate:

$$\left[C_1,\frac{C_2}{eB_0}\right]=1. \quad (51)$$

Among these three constants of motion, one cannot find two constants in involution:

$$[C_1,C_3]=C_2,\quad [C_2,C_3]=-C_1. \quad (52)$$

Another constant of motion is given by

$$C_4=C_1^2+C_2^2. \quad (53)$$

This satisfies the relation

$$[C_4,C_3]=2C_1[C_1,C_3]+2C_2[C_2,C_3]=0. \quad (54)$$

A constant of motion can be found very simply, as above, by integrating $dH/dt=(\omega_0/k_0)\dot{P}_z$, it gives

$$C_5=H-\frac{\omega_0}{k_0}P_z. \quad (55)$$

It satisfies the relations

$$[C_1,C_5]=0,\quad [C_2,C_5]=0, \quad (56)$$

and consequently

$$[C_4,C_5]=0. \quad (57)$$

It also satisfies

$$[C_3,C_5]=0. \quad (58)$$

As C_3, C_4, and C_5 are three independent constants in involution, the system is integrable according to the definition given in Sec. I and in Refs. [6].

(c) Reduction to a two-dimensional problem, second demonstration of the integrability of the problem. In the normalized variables, the constants of motion corresponding to C_3 and C_5 are

$$\hat{C}_3=\hat{y}\hat{P}_x-\hat{x}\hat{P}_y+\hat{H},$$

$$\hat{C}_5=\hat{H}-\hat{P}_z. \quad (59)$$

Those corresponding to C_1 and C_2 are

$$\hat{C}_1=\hat{P}_x+\frac{\Omega_0}{2}\hat{y},$$

$$\hat{C}_2=\hat{P}_y-\frac{\Omega_0}{2}\hat{x}; \tag{60}$$

they satisfy

$$[\hat{C}_1,\hat{C}_2]=\Omega_0. \tag{61}$$

By using this property, we can show that the system can be described by a time-dependent Hamiltonian with two degrees of freedom. To do so, let us choose the two constants $\hat{C}_1$ and $\hat{C}_2$ (one must be normalized by Ω_0) as new momentum and coordinate conjugate [12]. We choose a first canonical transformation, $(\hat{x},\hat{y},\hat{P}_x,\hat{P}_y)\to(\tilde{x},\tilde{y},\tilde{P}_x,\tilde{P}_y)$, defined by the following type-2 generating function:

$$F_2=\left(\tilde{P}_x-\frac{\Omega_0}{2}\hat{y}\right)\hat{x}+\tilde{P}_y\hat{y}. \tag{62}$$

This yields the canonical transformations

$$\hat{x}=\tilde{x},$$

$$\hat{y}=\tilde{y},$$

$$\hat{P}_x=\tilde{P}_x-\frac{\Omega_0}{2}\tilde{y}, \tag{63}$$

$$\hat{P}_y=\tilde{P}_y-\frac{\Omega_0}{2}\tilde{x}.$$

In these variables, $\hat{C}_1$ and $\hat{C}_2$ become

$$\tilde{C}_1=\tilde{P}_x,$$

$$\tilde{C}_2=\tilde{P}_y-\Omega_0\tilde{x}. \tag{64}$$

Then we introduce a second canonical transformation, $(\tilde{x},\tilde{y},\tilde{P}_x,\tilde{P}_y)\to(Q_1,Q_2,P_1,P_2)$, generated by

$$F_2=(P_2+\Omega_0\tilde{x})\tilde{y}+P_1\left(\tilde{x}+\frac{P_2}{\Omega_0}\right), \tag{65}$$

and yielding

$$\tilde{x}=Q_1-\frac{P_2}{\Omega_0},$$

$$\tilde{y}=Q_2-\frac{P_1}{\Omega_0},$$

$$\tilde{P}_x=\Omega_0 Q_2, \tag{66}$$

$$\tilde{P}_y=\Omega_0 Q_1.$$

The resulting transformation, which is the product of the two transformations, is given by

$$\hat{x}=Q_1-\frac{P_2}{\Omega_0},$$

$$\hat{y}=Q_2-\frac{P_1}{\Omega_0},$$

$$\hat{P}_x=\frac{1}{2}(\Omega_0 Q_2+P_1), \tag{67}$$

$$\hat{P}_y=\frac{1}{2}(\Omega_0 Q_1+P_2).$$

In terms of these variables, one has

$$Q_2=\frac{\hat{C}_1}{\Omega_0},\quad P_2=\hat{C}_2, \tag{68}$$

and the new Hamiltonian is

$$\bar{H}=\{[(P_1+a\cos(\hat{t}-\hat{z}))]^2+[\Omega_0 Q_1+a\sin(\hat{t}-\hat{z})]^2+\hat{P}_z^2+1\}^{1/2}. \tag{69}$$

The constants of motion corresponding to C_3 and C_5 are now

$$\bar{K}=\bar{H}-\frac{P_1^2}{2\Omega_0}-\frac{\Omega_0}{2}Q_1^2 \tag{70}$$

and

$$\bar{\Delta}=\bar{H}-\hat{P}_z. \tag{71}$$

As $[\bar{K},\bar{\Delta}]=0$, the system is integrable. Then the canonical transformation $(Q_1,P_1,\hat{z},\hat{P}_z)\to(Q_1,P_1,\phi,\hat{P}_z)$, generated by

$$F_2(Q_1,\hat{z},P_1,\hat{P}_z,\hat{t})=Q_1P_1+\hat{P}_z(\hat{z}-\hat{t}), \tag{72}$$

is performed, yielding

$$\phi=\hat{z}-\hat{t}. \tag{73}$$

The Hamiltonian becomes

$$\bar{\bar{H}}=[(P_1+a\cos\phi)^2+(\Omega_0 Q_1-a\sin\phi)^2+\hat{P}_z^2+1]^{1/2}-\hat{P}_z. \tag{74}$$

One must point out that this is the expression of $\bar{\Delta}$ expressed in terms of the new variables. The Hamiltonian is now time-independent and the system has a constant of motion. The invariant obtained by using Noether's theorem written in these variables is

$$K=\bar{\bar{H}}+\hat{P}_z-\frac{P_1^2}{2\Omega_0}-\frac{\Omega_0}{2}Q_1^2, \tag{75}$$

which is in involution with $\bar{\bar{H}}$. As a consequence, the system is completely integrable.

(d) Equation for the energy of the particle and another way to show the system can be solved by quadratures. The equations of Hamilton derived from Eq. (74) are

$$\dot{P}_1=-\frac{\Omega_0}{\gamma}(\Omega_0 Q_1-a\sin\phi),$$

$$\dot{Q}_1=\frac{1}{\gamma}(P_1+a\cos\phi),$$

$$\dot{\hat{P}}_z=\frac{a}{\gamma}(P_1\sin\phi+\Omega_0 Q_1\cos\phi), \tag{76}$$

$$\dot{\phi}=\frac{\hat{P}_z}{\gamma}-1.$$

Introducing the variables

$$\overline{Q_1}=Q_1-a/\Omega_0\sin\phi,$$

$$\overline{P_1}=P_1+a\cos\phi, \tag{77}$$

and the complex quantity $Z=\overline{P_1}+i\Omega_0\overline{Q_1}$, the two first equations of Hamilton [Eqs. (76)] are equivalent to the following equation

$$\dot{Z}=\frac{i\Omega_0 Z}{\gamma}-ia\dot{\phi}\exp(-i\phi), \tag{78}$$

which is the equation of a nonlinear oscillator under the action of an external force. Formally, the solution of this equation can be written

$$Z=A_0\exp i[\sigma(\hat{t})+\delta]-ia\int_0^{\hat{t}}\dot{\phi}(\tau)\exp i[\sigma(\hat{t})-\sigma(\tau)-\phi(\tau)]d\tau, \tag{79}$$

where A_0 and δ are real constants, and

$$\sigma(\hat{t})=\Omega_0\int_0^{\hat{t}}d\tau\,\gamma^{-1}(\tau). \tag{80}$$

Then

$$P_1=A_0\cos[\sigma(\hat{t})+\delta]-a\cos\phi+a\int_0^{\hat{t}}\dot{\phi}(\tau)\sin[\sigma(\hat{t})-\sigma(\tau)-\phi(\tau)]d\tau,$$

$$Q_1=\frac{A_0}{\Omega_0}\sin[\sigma(\hat{t})+\delta]+\frac{a}{\Omega_0}\sin\phi-\frac{a}{\Omega_0}\int_0^{\hat{t}}\dot{\phi}(\tau)\cos[\sigma(\hat{t})-\sigma(\tau)-\phi(\tau)]d\tau. \tag{81}$$

The quantities A_0 and δ are determined, so that, at $\hat{t}=0$, $A_0^2=\gamma_0^2-\hat{p}_{z0}^2-1=\hat{p}_{x0}^2+\hat{p}_{y0}^2$ and $\tan\delta=\hat{p}_{y0}/\hat{p}_{x0}$ ($\hat{p}=p/mc$, and p is the momentum of the particle). The subscript 0 appended to variables γ and p refers to their initial values. This formal solution shows that when the particle has an initial normalized energy γ_0 large compared to a ($A_0\gg a$), the trajectory of the particule is an ellipse in the Q_1-P_1 plane which is transformed into a circle in the $\hat{x}-\hat{y}$ plane according to Eqs. (67). Equations (81) are now substituted into the equation for $\dot{\hat{P}}_z$ [Eqs. (76)], to obtain a nonlinear integro differential equation

$$\dot{\hat{P}}_z=\frac{a}{\gamma}A_0\sin[\sigma(\hat{t})+\phi(\hat{t})+\delta]-\frac{a^2}{\gamma}\int_0^{\hat{t}}\dot{\phi}(\tau)\cos\{[\sigma(\hat{t})+\phi(\hat{t})]-[\sigma(\tau)+\phi(\tau)]\}d\tau. \tag{82}$$

Let us now consider the synchronous case only. In terms of the original coordinates the condition for resonance is

$$\omega_0-k_0\dot{z}-\frac{eB_0}{m\gamma}=0. \tag{83}$$

This condition implies that the invariant defined by Eq. (55) is such as $C_5=eB_0c^2/\omega_0$ [13]. This means that if the particule is initially resonant, it remains resonant all the time. In terms of the new coordinates, this condition becomes

$$\dot{\phi}+\frac{\Omega_0}{\gamma}=0. \tag{84}$$

Integrating this expression from 0 to $\hat{t}$ gives

$$\phi(\hat{t})+\sigma(\hat{t})-\phi_0=0, \tag{85}$$

where ϕ_0 is the value of ϕ at $\hat{t}=0$. Then Eq. (82) becomes

$$\gamma\dot{\gamma}=aA_0\sin\theta_0+a^2\sigma(\hat{t}), \tag{86}$$

with $\theta_0=\phi_0+\delta$. Equation (86) can be divided by γ and integrated between 0 and $\hat{t}$ to give

$$\gamma-\gamma_0=\frac{aA_0}{\Omega_0}\sin\theta_0\sigma(\hat{t})+\frac{a^2}{2\Omega_0}[\sigma(\hat{t})]^2. \tag{87}$$

Then $\sigma(\hat{t})$ is calculated from Eq. (86) and substituted in Eq. (87). We obtain

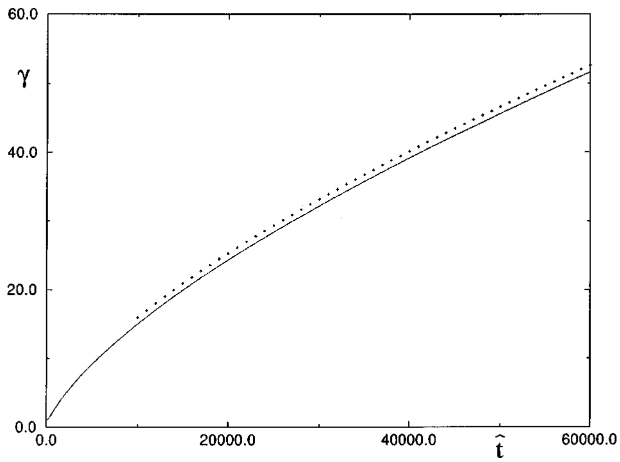

FIG. 10. Comparison of the evolution of γ when the particle is initially resonant obtained with the exact equations [Eqs. (76)] or Eq. (94) and the asymptotic solution [Eq. (95)]. $\gamma_0=\Omega_0=1$, and $a=3\times10^{-3}$.

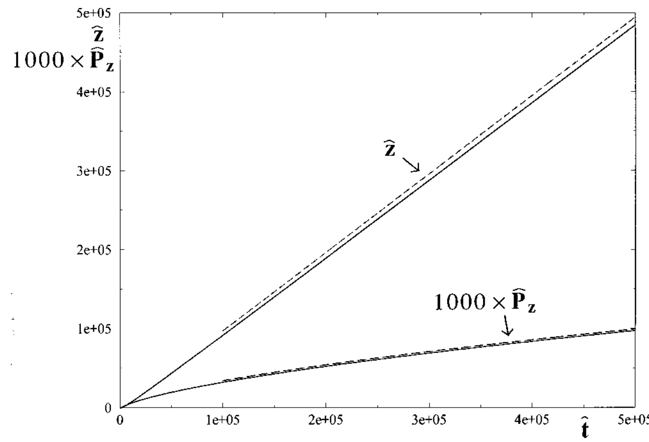

FIG. 11. Comparison between two numerical solutions derived through the exact equations of motion (solid lines) to their corresponding asymptotic solution (long dashed lines). $E_0=10^5$ V/m and $f=10$ GHz ($a\neq 9.337\times10^{-4}$).

$$(\dot{\gamma})^2+V(\gamma)=0, \tag{88}$$

where

$$V(\gamma)=-\frac{2\Omega_0 a^2}{\gamma}-\frac{[(aA_0\sin\theta_0)^2-2\Omega_0\gamma_0 a^2]}{\gamma^2}. \tag{89}$$

Equation (88) describes motion in the one-dimensional well defined by $V(\gamma)$. Letting $\alpha=2\Omega_0 a^2$, $\beta=(aA_0\sin\theta_0)^2-2\Omega_0\gamma_0 a^2$ and $\zeta=\sqrt{\alpha\gamma+\beta}$, it reads [15]

$$2(\zeta^2-\beta)d\zeta=\varepsilon_1\alpha^2 d\hat{t}, \tag{90}$$

where $\varepsilon_1=\pm1$. Considering that ζ_0 is the value of ζ at $\hat{t}=\hat{t}_0$, this equation is trivially integrated to give

$$\frac{2}{3}\zeta^3-2\beta\zeta=\frac{2}{3}\zeta_0^3-2\beta\zeta_0+\varepsilon_1\alpha^2(\hat{t}-\hat{t}_0). \tag{91}$$

This expression can be expressed in terms of $\zeta=\lambda T$. As β is always negative when the charged particle is resonant, one can consider that $-3(\beta/\lambda^2)=1$; that is to say, $\lambda=\varepsilon_2\sqrt{3|\beta|}$, with $\varepsilon_2=\pm1$. Then Eq. (91) takes the canonical form

$$T^3+T+A=0, \tag{92}$$

where

$$A=-\varepsilon_2(-3\beta)^{-3/2}\left[\zeta_0^3-3\beta\zeta_0+\frac{3\varepsilon_1\alpha^2}{2}(\hat{t}-\hat{t}_0)\right]. \tag{93}$$

Equation (92) can be easily solved. Hence the exact expression for the normalized energy of the particle is

$$\gamma=-\frac{\beta}{\alpha}\left\{3\times\left(\frac{A}{2}\right)^{2/3}\left[\left(\left(1+\frac{4}{27A^2}\right)^{1/2}-1\right)^{2/3}+\left(\left(1+\frac{4}{27A^2}\right)^{1/2}+1\right)^{2/3}-\frac{2}{3}\left(\frac{2}{A}\right)^{2/3}\right]+1\right\}. \tag{94}$$

This expression, substituted into Eqs. (81) and (82), shows that, at least in the resonant case, the solution can be expressed in terms of quadratures according to Kozlov and Kolesnikov. As A goes to ∞ in the limit when $\hat{t}\to\infty$, the asymptotic form of the solution [Eq. (94)] is

$$\gamma\approx(\tfrac{9}{2}\Omega_0 a^2)^{1/3}\hat{t}^{2/3}. \tag{95}$$

The comparison between this solution and the exact one is performed in Fig. 10.

Equations (84) and (95) lead to the following asymptotic expression for $\hat{z}$:

$$\hat{z}=\hat{t}-(\tfrac{9}{2}\Omega_0 a^2)^{-1/3}\hat{t}^{1/3}+\hat{z}_0, \tag{96}$$

where $\hat{z}_0$ is the value of $\hat{z}$ at $\hat{t}=0$.

Equations (81) and (85) provide the following asymptotic expressions for P_1 and Q_1:

$$P_1=A_0\cos[\phi_0-\phi(\hat{t})+\delta)]-a\{\cos\phi+[\phi(\hat{t})-\phi_0]\sin\phi(\hat{t})\},$$

$$Q_1=\frac{A_0}{\Omega_0}\sin[\phi_0-\phi(\hat{t})+\delta)]+\frac{a}{\Omega_0}\{\sin\phi-[\phi(\hat{t})-\phi_0]\cos\phi(\hat{t})\}. \tag{97}$$

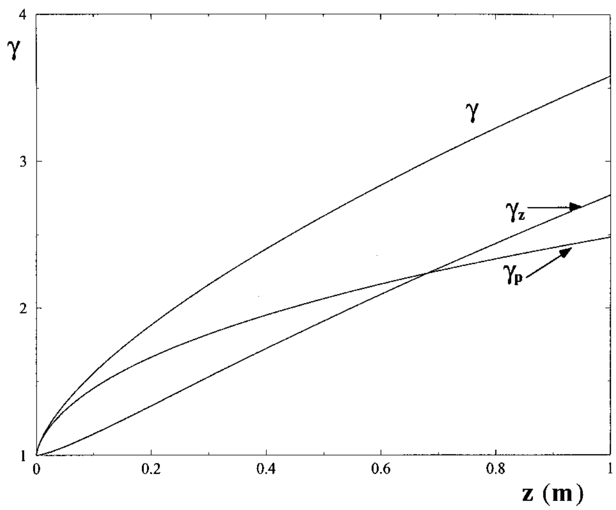

FIG. 12. Evolution of the total energy γ, compared to the longitudinal energy γ_z and the transverse energy γ_p. ($\gamma_0=\Omega_0=1$). $E_0=10^6$ V/m and $f=2.5$ GHz ($a\approx 3.735\times 10^{-2}$).

$\phi(\hat{t})$ is given by Eqs. (80), (85), and (95):

$$\phi(\hat{t})=\phi_0-\left(\frac{1}{6}\frac{a^2}{\Omega_0^2}\right)^{-1/3}\hat{t}^{1/3}. \tag{98}$$

Taking into account the resonance condition, the equation for $\hat{P}_z$ [Eq. (82)] becomes

$$\hat{P}_z=\frac{a}{\gamma}A_0\sin(\phi_0+\delta)-\frac{a^2}{\gamma}[\phi(\hat{t})-\phi_0], \tag{99}$$

which leads to

$$\hat{P}_z=A_0\left(\frac{6a}{\Omega_0}\right)^{1/3}\sin(\phi_0+\delta)\hat{t}^{1/3}+\left(\frac{9}{2}\Omega_0 a^2\right)^{1/3}\hat{t}^{2/3}+\hat{P}_{z0}, \tag{100}$$

where $\hat{P}_{z0}$ is the value of $\hat{P}_z$ at $\hat{t}=0$. The asymptotic trajectory in the physical space is given by Eqs. (67).

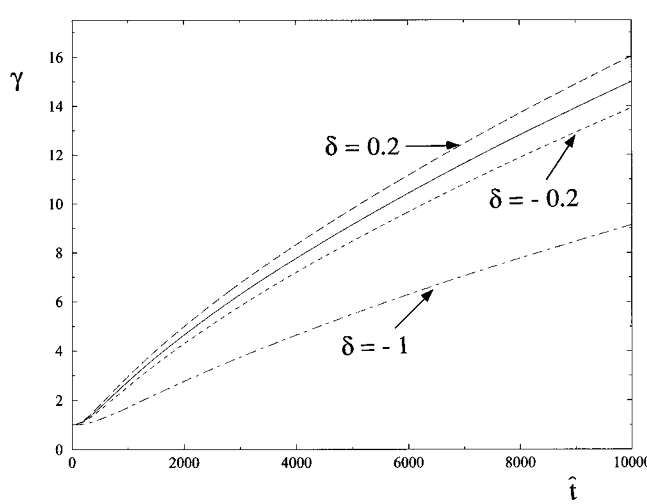

FIG. 13. Evolution of γ for different values of δ. The solid line is for $\delta=0$. $\gamma_0=\Omega_0=1$ and $a=3\times 10^{-3}$.

In Fig. 11, the asymptotic evolutions of $\hat{z}$ and $\hat{P}_z$ are compared to those when calculated numerically through the exact equations of motion. Assuming that the charged particle has a velocity close to the speed of light in vacuum, we have

$$z\approx ct. \tag{101}$$

If the charged particle is initially at rest, the Lorentz factor can be expressed as a function of z:

$$\gamma=\left(\frac{3}{\sqrt{2}}\frac{eE_0}{mc^2}z\right)^{2/3}. \tag{102}$$

The energy gained by a particle in a linear accelerator is the product of axial electric field and distance

$$\gamma_L=\frac{eE_0}{mc^2}z. \tag{103}$$

At the same field strength, the efficiency of the linear accel-

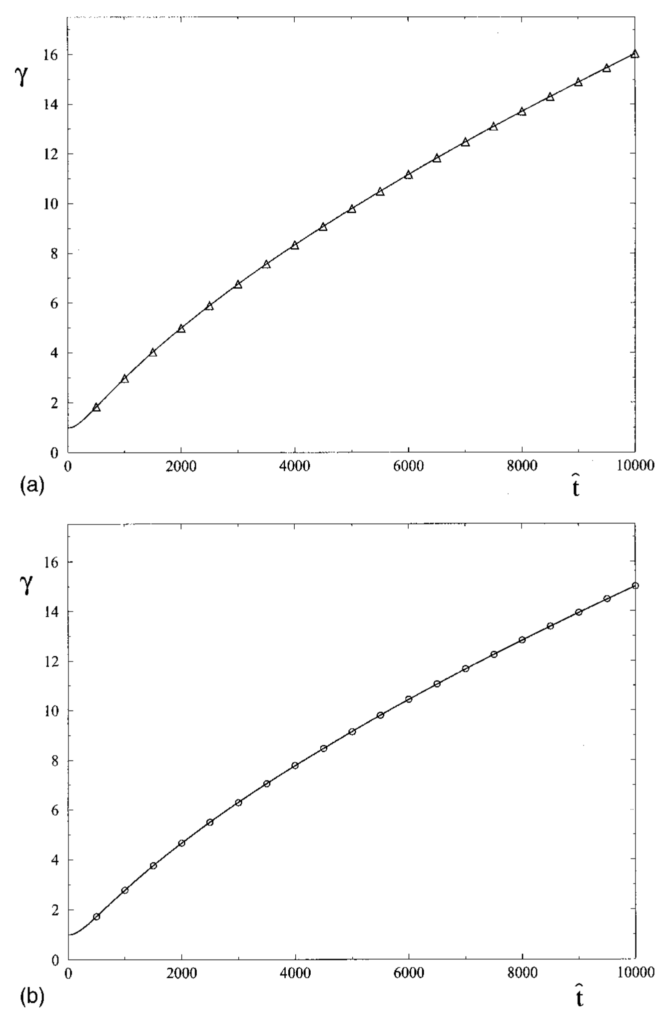

FIG. 14. γ vs time: (a) The solid line corresponds to $a=3.3\times 10^{-3}$ and $\delta=0$, and the triangles correspond to $a=3\times 10^{-3}$ and $\delta=0.2$. (b) The solid line corresponds to $a=3\times 10^{-3}$ and $\delta=0$, and the circles correspond to $a=6\times 10^{-3}$ and $\delta=-1$.

erator for an electron is the same as the one associated with the synchronous solution ($\gamma = \gamma_L$), when [16]

$$z = z_L = \frac{2.3\times 10^6}{E_0}. \tag{104}$$

For example, $z_L = 2.3$ m for $E_0 = 10^6$ V/m. This shows that the cyclotron resonance accelerator is competitive only if a very short acceleration distance is requested. Moreover, one must note that, over a short distance, the transverse acceleration in the x-y plane is more efficient than the longitudinal one (Fig. 12).

2. The wave is almost circularly polarized

The fields are given by

$$E_x = E_0 \sin(\omega_0 t - k_0 z),$$

$$E_y = -E_0(1+\delta)\cos(\omega_0 t - k_0 z), \quad E_z = 0,$$

$$B_x = \frac{k_0 E_0}{\omega_0}(1+\delta)\cos(\omega_0 t - k_0 z),$$

$$B_y = \frac{k_0 E_0}{\omega_0}\sin(\omega_0 t - k_0 z), \quad B_z = B_0, \tag{105}$$

where δ is defined in the same way as for Eqs. (20).

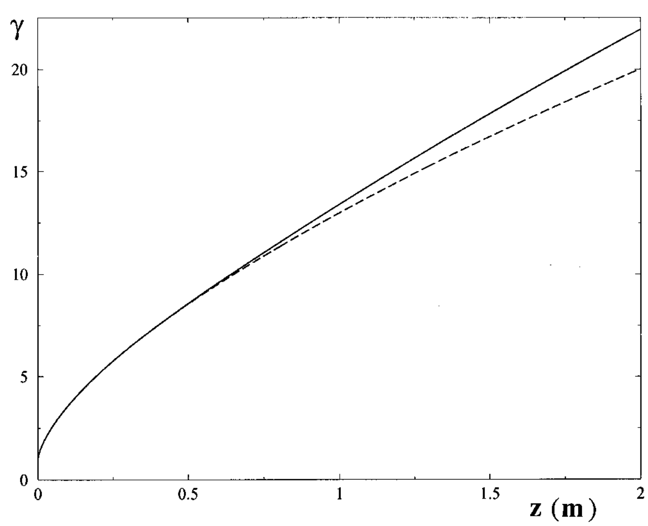

FIG. 15. γ vs z when the particle is initially resonant and at rest. $E_0 = 10^7$ V/m and $f = 25$ GHz ($a \approx 3.735\times 10^{-2}$) when the magnetic field is linearly increasing ($L^{-1} = 0.5\,\mathrm{m}^{-1}$) after $z_c = 0.4$ m (solid line), compared to the case when the magnetic field is homogeneous (long dashed line).

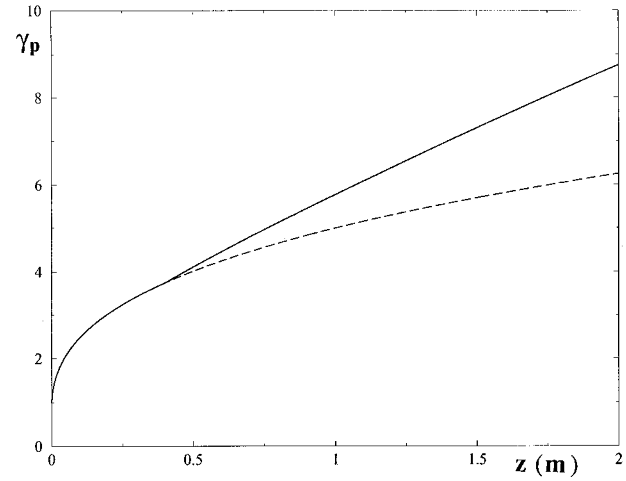

FIG. 16. γ_p vs z when the particle is initially resonant and at rest. $E_0 = 10^7$ V/m and $f = 25$ GHz ($a \approx 3.735\times 10^{-2}$), when the magnetic field is linearly increasing ($L^{-1} = 0.5\,\mathrm{m}^{-1}$) after $z_c = 0.4$ m (solid line), compared to the case when the magnetic field is homogeneous (long dashed line).

The normalized Hamiltonian is obtained by introducing the same dimensionless parameters and variables as above

$$\hat{H} = \left[\left[\hat{P}_x + a\cos(\hat{t}-\hat{z}) - \frac{\Omega_0}{2}\hat{y}\right]^2 + \left[\hat{P}_y + a(1+\delta)\sin(\hat{t}-\hat{z}) + \frac{\Omega_0}{2}\hat{x}\right]^2 + \hat{P}_z^2 + 1\right]^{1/2}. \tag{106}$$

It can be easily checked that C_1, C_2, and C_5 are still constants of the motion.

The canonical transformation defined by Eqs. (67) is used again in order to reduce the number of degrees of freedom of the system. The new Hamiltonian is

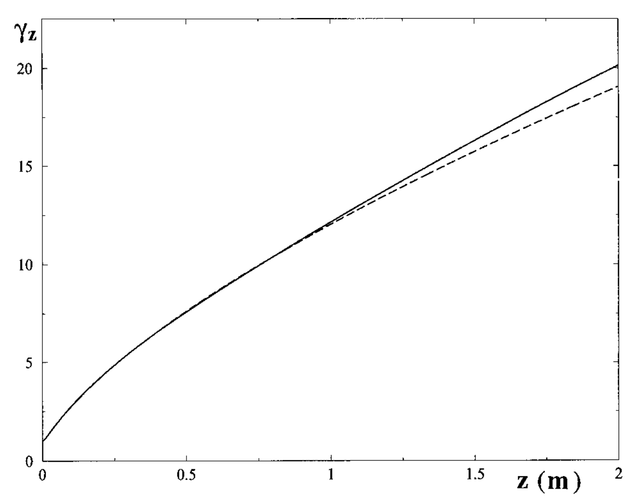

FIG. 17. γ_z vs z when the particle is initially resonant and at rest. $E_0 = 10^7$ V/m and $f = 25$ GHz ($a \approx 3.735\times 10^{-2}$), when the magnetic field is linearly increasing ($L^{-1} = 0.5\,\mathrm{m}^{-1}$) after $z_c = 0.4$ m (solid line), compared to the case when the magnetic field is homogeneous (long dashed line).

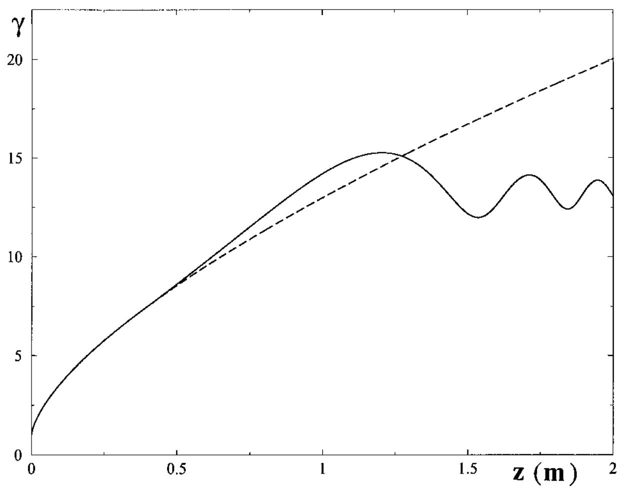

FIG. 18. γ vs z when the particle is initially resonant and at rest. $E_0 = 10^7$ V/m and $f = 25$ GHz ($a \approx 3.735\times10^{-2}$) when the magnetic field is linearly increasing ($L^{-1} = 2\ \mathrm{m}^{-1}$) after $z_c = 0.4$ m (solid line) compared to the case when the magnetic field is homogeneous (long dashed line).

$$\bar{H} = \{[(P_1 + a\cos(\hat{t}-\hat{z}))]^2 + [\Omega_0 Q_1 + a(1+\delta)\sin(\hat{t}-\hat{z})]^2 + \hat{P}_z^2 + 1\}^{1/2}. \tag{107}$$

Then the canonical transformation defined by Eq. (73) is performed. The Hamiltonian becomes

$$\bar{\bar{H}} = \{(P_1 + a\cos\phi)^2 + [\Omega_0 Q_1 - a(1+\delta)\sin\phi]^2 + \hat{P}_z^2 + 1\}^{1/2} - \hat{P}_z. \tag{108}$$

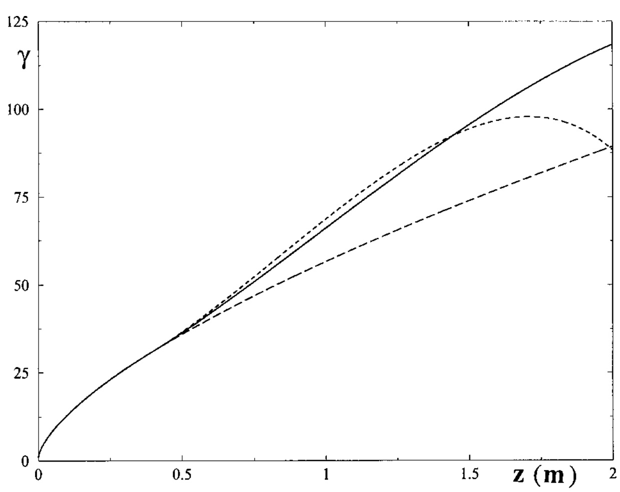

FIG. 19. γ vs z when the particle is initially resonant and at rest. $E_0 = 10^8$ V/m and $f = 25$ GHz ($a \approx 3.735\times10^{-1}$) when the magnetic field is linearly increasing with $L^{-1} = 3.5\ \mathrm{m}^{-1}$ after $z_c = 0.4$ m (solid line), compared to the case when $L^{-1} = 5\ \mathrm{m}^{-1}$ and $z_c = 0.4$ (dashed line), and when the magnetic field is homogeneous (long dashed line).

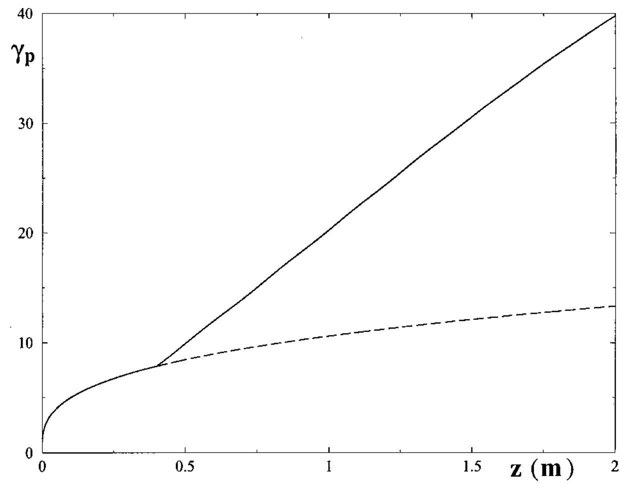

FIG. 20. γ_p vs z when the particle is initially resonant and at rest. $E_0 = 10^8$ V/m and $f = 25$ GHz ($a \approx 3.735\times10^{-1}$), when the magnetic field is linearly increasing ($L^{-1} = 3.5\ \mathrm{m}^{-1}$) after $z_c = 0.4$ m (solid line), compared to the case when the magnetic field is homogeneous (long dashed line).

This Hamiltonian is a constant, as it does not explicitly depend on time. The system now has two degrees of freedom, and a set of only four coupled equations has to be solved numerically. It has been checked numerically that, even when $\delta \neq 0$, the synchronous solution is still possible (Fig. 13). The energy of the particle versus time is compared to the one when the wave is purely circularly polarized and has the same Poynting vector [Figs. 14(a) and 14(b)]. It has been shown numerically that when one has $a_{ac}(1+\delta/2) = a_c$, where a_{ac} and a_c are the parameter a in the almost circularly polarized case and in the circularly polarized one, respectively the evolution of the charged particle's energy is the same. Consequently, at the same power, an almost circularly polarized wave is as effective to accelerate a particle as a purely circularly one.

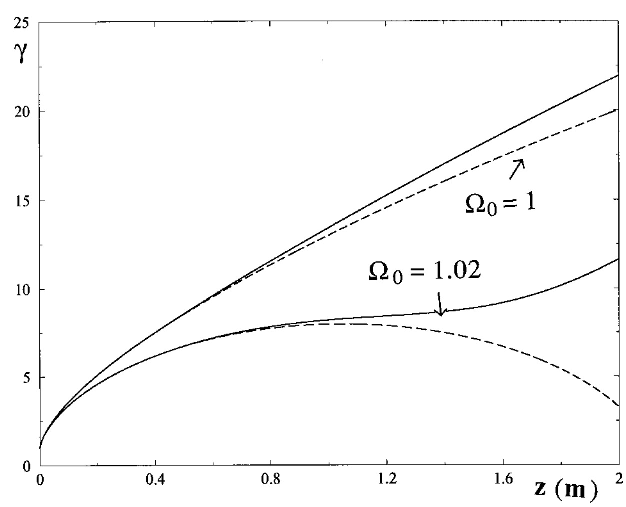

FIG. 21. γ vs z when the particle is initially at rest. $E_0 = 10^7$ V/m and $f = 25$ GHz ($a \approx 3.735\times10^{-2}$) when the magnetic field is linearly increasing ($L^{-1} = 0.5\ \mathrm{m}^{-1}$) after $z_c = 0.4$ m (solid line), compared to the case when the magnetic field is homogeneous (long dashed line). Cases when the particle is initially resonant ($\Omega_0 = 1$) and nonresonant ($\Omega_0 = 1.02$) are considered.

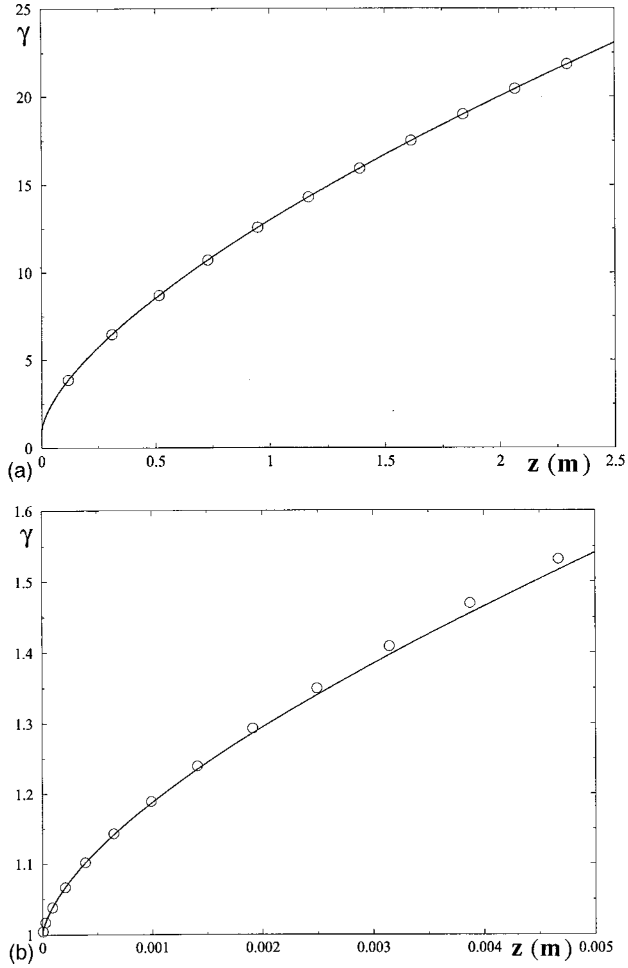

FIG. 22. (a) γ vs z when the particle is initially resonant and at rest. $E_0=10^7$ V/m and $f=2\times10^7$ Hz ($a\approx46.68$) when the magnetic field is linearly decreasing ($L=0.5$ m) after $z_c=0.4$ m (solid line) compared to the case when the magnetic field is homogeneous (circles). (b) γ vs z when the particle is initially resonant and at rest. $E_0=2\times10^7$ V/m and $f=10^9$ Hz ($a\approx1.867$) when the magnetic field is linearly decreasing ($L=10^{-3}$ m) after $z_c=10^{-4}$ m (solid line), compared to the case when the magnetic field is homogeneous (circles).

B. Circularly polarized wave traveling along a non-homogeneous constant magnetic field

Let us take advantage of the fact that the synchronous resonance is an efficient acceleration mechanism over a short distance only, as was shown above. More precisely, by short distances we mean distances such that the transverse energy of the particle is about the same as the longitudinal one. The idea is to use this mechanism first, so that the particle may

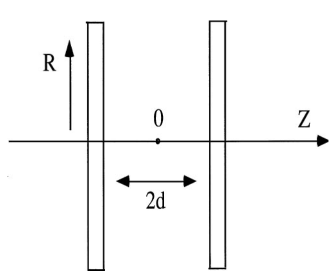

FIG. 23. Helmoltz coils.

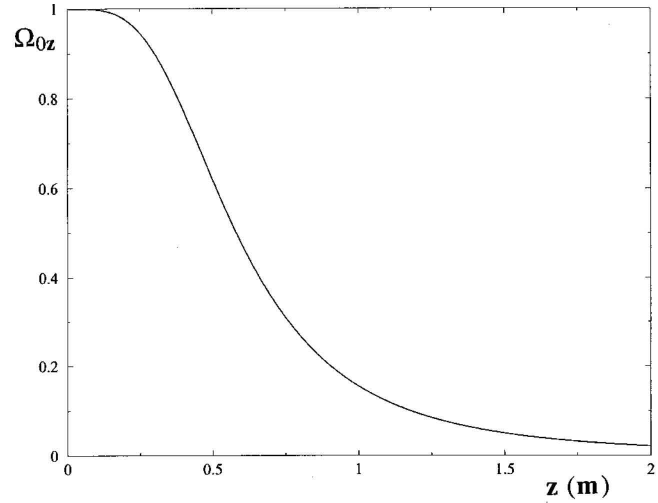

FIG. 24. Normalized z component of the magnetic field vs z in the case of some magnetic field gradient. $R=0.5$ m and $d=0.25$ m.

have a high velocity component along the z axis and in the plane perpendicular to this axis at the same time. The other point is that over such a short distance the final energy reached is not very sensitive to inaccuracies in the initial conditions. Then the particle is immersed in a magnetic field gradient in order to obtain a very high energy on a short distance.

Let us first consider such a constant nonhomogeneous magnetic field

$$\mathbf{B}_0(\mathbf{r})=B_0\left[1+\varepsilon Y(z-z_c)\frac{(z-z_c)}{L}\right]\hat{\mathbf{e}}_z-\varepsilon Y(z-z_c)\times\frac{B_0}{2L}(x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y), \tag{109}$$

where $L=B_0/dB_z(z)/dz$, Y is the Heaviside function, z_c is some distance, and $\varepsilon=\pm1$ (the field can increase or decrease when $z>z_c$). The magnetic field should satisfy

$$\nabla\cdot\mathbf{B}=\mathbf{0} \tag{110}$$

and

$$\nabla\times\mathbf{B}=\mathbf{0}. \tag{111}$$

The magnetic field given by Eq. (109) does not satisfy Eq. (111) at $z=z_c$. This is because of the discontinuity introduced by the Heaviside function. This discontinuity has no physical meaning, and is thought not to affect the existence of the acceleration mechanism.

In this case the vector potential is

$$\mathbf{A}=\left(-\frac{B_{0z}}{2}y+\frac{E_0}{\omega_0}\cos(\omega_0t-k_0z)\right)\hat{\mathbf{e}}_x+\left(\frac{B_{0z}}{2}x+\frac{E_0}{\omega_0}\sin(\omega_0t-k_0z)\right)\hat{\mathbf{e}}_y. \tag{112}$$

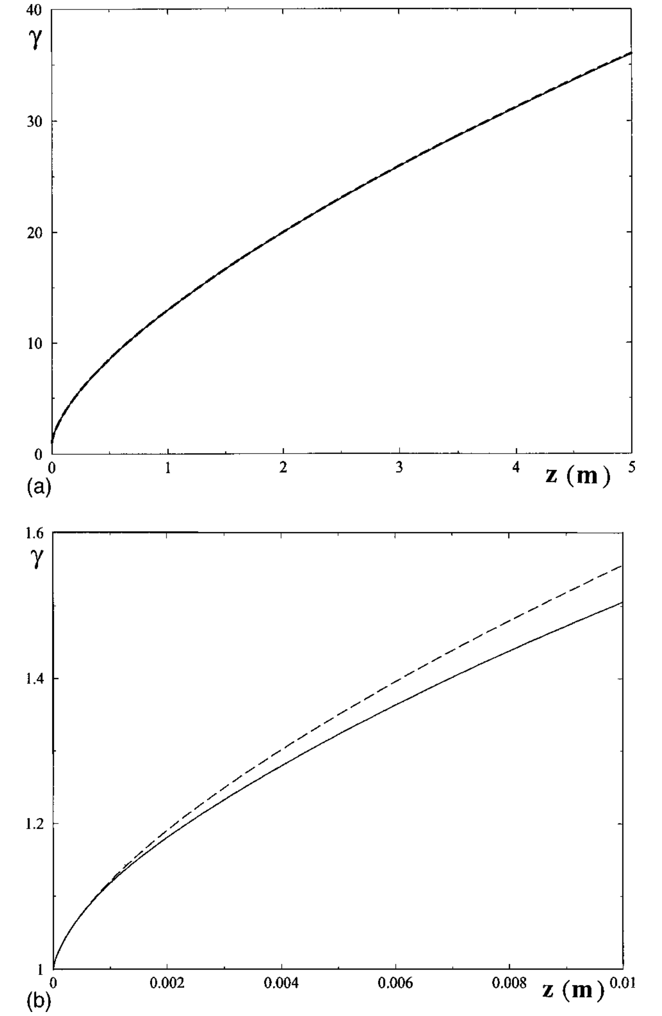

FIG. 25. (a) γ vs z when the particle is initially resonant and at rest. $E_0=10^7$ V/m and $f=2\times10^7$ Hz ($a\approx46.68$) when the magnetic field is the one of the Helmoltz coils with $R=0.5$ m and $d=0.25$ m (solid line), and when the magnetic field is homogeneous (long dashed line). (b) γ vs z when the particle is initially resonant and at rest. $E_0=10^7$ V/m and $f=10^9$ Hz ($a\approx0.933$) when the magnetic field is the one of the Helmoltz coils with $R=10^{-3}$ m and $d=0.5\times10^{-3}$ m (solid line) and when the magnetic field is homogeneous (long dashed line).

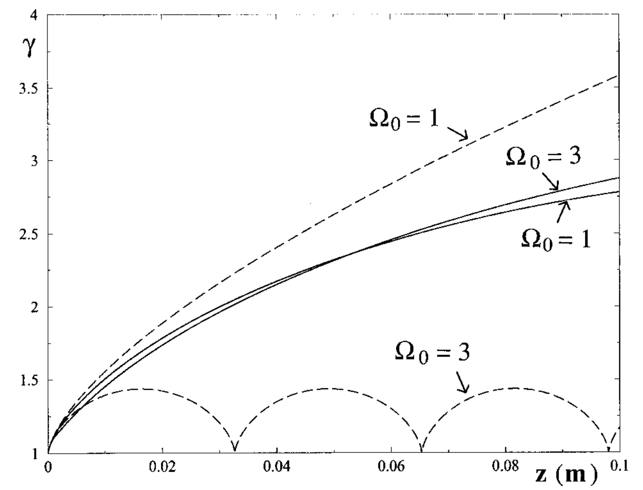

FIG. 26. γ vs z when the particle is initially at rest. $E_0=10^7$ V/m and $f=10^9$ Hz ($a\approx0.933$) when the magnetic field is the one of the Helmoltz coils with $R=10^{-3}$ m and $d=0.5\times10^{-3}$ m (solid line), and when the magnetic field is homogeneous (long dashed line). Cases when the particle is initially resonant ($\Omega_0=1$) and nonresonant ($\Omega_0=3$) are considered.

Using the same dimensionless variables and parameters as before, a normalized Hamiltonian can be built to describe the system

$$\hat{H}=\left[\left(\hat{P}_x+a\cos(\hat{t}-\hat{z})-\frac{\Omega_{0z}}{2}\hat{y}\right)^2+\left(\hat{P}_y+a\sin(\hat{t}-\hat{z})+\frac{\Omega_{0z}}{2}\hat{x}\right)^2+\hat{P}_z^2+1\right]^{1/2}, \quad (113)$$

where $\Omega_{0z}=eB_{0z}/m\omega_0$.

For a given increasing magnetic field gradient, Fig. 15 shows the normalized energy of the particle versus z compared to the one obtained by using the simple synchronous resonance only. It must be pointed out that when the particle is introduced to the nonhomogeneous magnetic field domain, it is kicked backward along the z axis by the $ev_\perp B_{0r}$ force ($v_\perp$ is the velocity in the plane perpendicular to the z axis, and B_{0r} is the radial constant magnetic field). Over the distance along the z axis considered in Figs. 15–21, it can be checked numerically that the effect of this force is overwhelmed by the $ev_\perp B$ force (B is the magnetic field of the wave). When these calculations are run for longer times, the charged particle goes backward along the z axis after some distance, as the average force in the z direction, acting on the particle, becomes more and more negative. It has been shown numerically that, when one has a positive linear magnetic field gradient, the additional acceleration is due mainly to the ev_zB_{0r} (v_z is the component of the velocity along the z axis) transverse force, which is much greater than that in the homogeneous magnetic field case. Figure 16 shows the transverse energy, and Fig. 17 the longitudinal one. In each case, the energy reached finally is higher when some magnetic field gradient is taken into account. When the gradient is strong enough, its effect can be positive up to a certain distance and become negative afterwards (Fig. 18). Figure 19 shows, for another of the electric field magnitude, the range of magnetic field gradients which leads to a positive effect. Figure 20 displays, for the same electric field, the very important gain obtained in transverse energy. The fact that an increasing linear gradient can be used to diminish the effect of inaccuracies in the initial conditions is shown in Fig. 21.

When the magnetic field decreases linearly after some distance $z_c(\varepsilon=-1)$, the ev_zB_{0r} force kicks the charged particle forward in the z direction, but this force is not sufficient to beget the additional acceleration shown in the growing magnetic field case. The other forces are not high enough compared to those one has in the homogeneous magnetic field to increase the acceleration. At this point, it must be mentioned that we have considered that $\Omega_0=0$ when $z>z_c+L$ in order not to let the magnetic field become negative. Still, even if the additional acceleration process did not exist in the domain that we have explored, E_0 is assumed to be in the range 10^6–10^9 V/m and the frequency f in the range

10^5-10^9 Hz, another very interesting phenomenon takes place. It has been found numerically that, when the magnitude of the electric field is high enough, the particle is accelerated just as if the synchronous interaction regime continued up to a distance of about five times the constant magnetic field gradient length (with an accuracy of about 5% in most cases). The rough inequality

$$E_0 \gtrsim 3\times 10^{-9} L f^2 \quad \text{(V/m)}, \tag{114a}$$

must be satisfied when $a>1$. Parameter z_c plays no part in this condition. When $0.4<a<1$, the inequality

$$E_0 \gtrsim 1.35\times 10^{-9} L f^2 \quad \text{(V/m)}, \tag{114b}$$

must be verified when $z_c \geqslant 2\times L$. When $a<0.4$ and $z_c \geqslant 2.5\times L$, one has to satisfy

$$E_0 \gtrsim 6.74\times 10^{-10} L f^2 \quad \text{(V/m)}. \tag{114c}$$

Figures 22(a) and 22(b) illustrate these results in two cases. Moreover, it has been verified that the good agreement between the two evolutions of γ is improved when inequalities (114) are amply verified.

Let us now consider the magnetic field produced by Helmoltz coils,

$$B_{0z}=\overline{B_0}R^3\{[(d-z)^2+R^2]^{-3/2}+[(d+z)^2+R^2]^{-3/2}\}, \tag{115}$$

where $2d$ is the distance between the two coils, R is the radius of each coil, $\overline{B_0}$ is the magnetic field created by one coil at its center, and z measures the position on the axis of symmetry of the two coils (Fig. 23). In order to have an almost uniform magnetic field over a short distance, the distance between the two coils is assumed to be equal to the radius of each coil. The charged particle is considered to be initially resonant and at rest at $z=0$. In such a case, Fig. 24 shows the normalized magnetic field versus z. E_0 and f are assumed to be in the same range as above, while R is supposed to be in the range $10^{-3}-1$ m. The additional acceleration process which exists when the magnetic field gradient increases linearly does not exist in the domain that we have explored, but another very interesting phenomenon takes place. The particle is accelerated exactly as if the magnetic field where homogeneous up to a distance of about ten times the radius of the coils with roughly the same accuracy as above. The condition

$$E_0 \gtrsim 2.25\times 10^{-10} R f^2 \quad \text{(V/m)}, \tag{116a}$$

when $a>1$, or

$$E_0 \gtrsim 2.25\times 10^{-9} R f^2 \quad \text{(V/m)}, \tag{116b}$$

when $a<1$ must be satisfied. This is displayed, in one case, in Figs. 25(a) and 25(b). The effect on the initial conditions is shown in Fig. 26.

IV. CONCLUSIONS

It has been shown that, in an almost circularly polarized traveling wave, a charged particle can have a constant average velocity along its propagation direction using the Hamiltonian formalism. In a cold electron plasma, the wave equations derived by Akhiezer and Polovin permit one to show that this effect is relevant only when the electron densities are low compared to the nonrelativistic critical density or when the intensity of the wave is relativistically very strong.

Still, with the help of the Hamiltonian formalism, the problem of relativistic motion of a charged particle in a constant homogeneous magnetic field and a transverse circularly polarized traveling wave has been studied. The integrability of this problem was shown in two different ways. Noether's theorem was applied to find a constant of motion of the system. Then two other integrals were derived; these three invariants are independent and in involution with each other. This is sufficient to prove integrability, since this problem has three degrees of freedom. Then, using canonical transformations, we reduced the system to a time-dependent one with two degrees of freedom. It has a constant of motion, which is the one found previously using Noether's theorem. Another one is the resonance condition, when the value of the constant is calculated with resonant initial conditions. As these two constants are independent and in involution, we have shown a second time that the problem is integrable. We have also proved that the system can be solved by quadratures in the resonant case. This consists of deriving and solving an equation for the energy.

The problem of an almost circularly polarized wave propagating along a constant homogeneous magnetic field has also been discussed. It was shown that the synchronous solution still exists. A condition to obtain the same particle energy evolution as in the pure circular polarization case was also given.

New acceleration mechanisms have been described when the charged particle is first accelerated for a short distance by using the synchronous resonance, and then introduced to a region where the magnetic field is no longer homogeneous and exhibits a very steep linearly growing magnetic field. Over a short distance the final energy reached can be much higher than if the magnetic field were homogeneous all the way. It has also been shown that when a particle is resonant and at rest and when the magnetic field is decreasing, if, on a distance which is roughly five times the length of the constant magnetic field gradient, the magnitude of the electric field is higher than some threshold, then the particle is accelerated just as if the magnetic field were homogeneous.

These acceleration mechanisms are also very important, as the synchronous interaction regime is very sensitive to inaccuracies in the initial conditions for long distance accelerations. Using magnetic field gradients allows one to diminish the strong influence of the inaccuracies in the initial conditions of the charged particle. Thus a spatial cyclotron accelerator using a magnetic field gradient is realistic.

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