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Fluctuations around Nash equilibria in game theory

Matteo Marsili, Yi-Cheng Zhang *

Institut de Physique Théorique, Université de Fribourg, CH-1700, Fribourg, Switzerland

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Abstract

We investigate the fluctuations induced by irrationality in simple games with a large number of competing players. We show that Nash equilibria in such games are "weakly" stable: irrationality propagates and amplifies through players' interactions so that huge fluctuations can result from a small amount of irrationality. In the presence of multiple Nash equilibria, our statistical approach allows to establish which is the globally stable equilibrium. However, characteristic times to reach this state can be very large.

Game theory [1] provides strategical thinking for modern economics and socio-political decisions. It has been an active research subject, in the economist's community, in the past half century. Extremely refined analysis is now being preformed, however, there is growing frustration recently [2] that such studies prove to be far too idealized as to deal with the real world in economics. One of the main pitfalls lies in the fact that the assumption of rationality makes game theory deterministic. On the other hand, we know that the economic world is characterized by large fluctuations. These fluctuations are, for obvious reasons, of great interest for people in economics. They are now inspiring great interest also among physicists; it has been realized [3] that economic systems share scaling and self-organized critical behaviors with more traditional subjects in statistical physics.

In real world irrationality is ubiquitous. This gives us reason to use physics tools to include it in game theory. Strikingly we find that irrationality propagates and amplifies through player's interactions and it can lead to huge fluctuations, growing with the number of players. This shows that irrationality is indeed a "relevant parameter", which should be included in game theory.

* Corresponding author. E-mail: yi-cheng.zhang@unifr.ch.

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A game is defined as a mathematical model for optimal strategies among competing players. Typically a player has a utility function depending on the strategies of all the players. In this paper we will limit ourselves to the so-called complete information games where every player is aware of all other players' strategies and benefits. Under the basic assumption of rationality of all players, solutions in game theory are given by *Nash* equilibria of all players' strategies. The nature of a Nash equilibrium differs qualitatively from that of an equilibrium state in statistical mechanics. Nash equilibria do not result from just the maximization or minimization of some global function (such as e.g. the free energy in statistical mechanics) but rather from the requirement that each player's strategy must simultaneously be a local maximum with respect to his own strategy. Loosely speaking, in game theory there is not a unique Hamiltonian, but rather each player has his own Hamiltonian to minimize. The interactions among players need not be symmetric and their goals may be in conflict with one another. Finally Nash equilibria gives an exact deterministic answer in the sense that it includes no fluctuations. In parallel with statistical mechanics, one could say that game theory is a "zero temperature" theory similar to ground states. The aim of this paper is to include "thermal" fluctuations in game theory through the Langevin approach. For any realistic game, this issue is of utmost importance since no player can have infinitely precise actions. The "zero temperature" nature of game theory, resulting from complete rationality, was indeed recently questioned [2]. We show that the effects of fluctuations, in standard games of a large number of competing players, can be quite dramatic and that they characterize the stability of *Nash* equilibria.

The simplest, economy motivated model of game theory was introduced by Cournot in 1838 [4]: 2 firms produce quantities x_1 and x_2, respectively, of a homogeneous product. The market-clearing price of the product depends, through the law of demand-and-offer, on the total quantity $X = x_1 + x_2$ produced: $P(X) = a - bX$. The larger X the smaller P is. The model assumes that the cost of producing a quantity x_i is cx_i and $c < a$. The firms choose their strategies (i.e. x_i) with the goal to maximize their profit (utility): $u_i = x_i[P(x_1 + x_2) - c]$. The problem is to find x_i assuming that both firms behave rationally. The best response $x_1^*(x_2)$ of firm 1 to any given strategy x_2 of firm 2 is obtained by maximizing $u_1(x_1, x_2)$ with respect to x_1 with fixed x_2. Firm 2, assuming that 1 behaves rationally (i.e. that it will play $x_1^*(x_2)$ whatever x_2 is) will choose x_2^* which maximizes $u_2(x_1^*(x_2), x_2)$. This leads to $x_1^* = x_2^* = (a - c)/3b$. This solution highlights the essential point of the concept of Nash equilibrium [5], which applies also to more general games.

In a situation with n players, we consider

$$u_i = x_i V(x_1 + \cdots + x_i + \cdots + x_n)\,. \tag{1}$$

In general, one requires that $V(X)$ be a decreasing function of X. This describes, apart from a demand-and-offer law, also situations where the gain of each player depends on a common resource. As $X = x_1 + \cdots + x_n$ grows, the resource is depleted ($V(X)$ decreases). $V(X)$ can eventually turn negative for $X > X_0$: the resource has been exhausted and production gives rise to negative benefit for all the players. Generally one has $V' \sim -$

$n^{-1}V$ (V' denotes derivative here and below) as a consequence of the fact that each x_i have an effect $1/n$ on a global quantity V. We shall consider $-\infty < x_i < \infty$. A negative x_i is a quantity that, instead of being produced and sold, is bought by player i. We shall also discuss briefly the effects of the constraints $x_i > 0$.

Technically, the Nash equilibrium is obtained by solving

$$\frac{\partial}{\partial x_i} u_i(x_1,\dots,x_n)\bigg|_{x_j = x_i\ \forall j} = 0 \quad \forall i\,. \tag{2}$$

This equation contains the maximization of the utility of *player* i and his expectation that all other players will do the same [6]. For the generalized Cournot model with n firms, $a - c = 1$ and $b = 1/n$ (i.e. $V = 1 - X/n$), Eq. (2) gives the Nash equilibrium

$$x_i = x_N \equiv \frac{n}{n+1} \quad \text{and} \quad u_i = \frac{n}{(n+1)^2}\,. \tag{3}$$

Note that u_i is of order $1/n$: the common resource is nearly exhausted $V \simeq 0$ due to the aggressive strategies $x_i \simeq 1$.

It is interesting to compare the above to the case where each player acts to maximize the total utility $U(x_1,\dots,x_n) = \sum_i u_i$. In this case x_i is given by $\partial U/\partial x_i = 0$ and the result is quite different: $x_i = \frac{1}{2}$ and $u_i = \frac{1}{4}$. Strikingly the profit of each player in this case is a factor n larger than in the previous case!

This is a typical lesson [7] of game theory: when each player acts to maximize his own utility u_i, the global utility is very small. The global utility is maximized when all players have a common goal. This is very similar to the dynamics in statistical mechanics where all degrees of freedom evolve to optimize an Hamiltonian. The maximal utility state, in spite of being "socially" better (everybody behaves less aggressively and receives a better payoff), is unfortunately never achieved since incentives to cheat are large. This fact will emerge clearly from the analysis of fluctuations.

In the Nash equilibrium instead, everybody is more aggressive (larger x_i) and *per* player benefit is much more meager. The crucial features which makes this state more relevant than the social one is its stability: The Nash equilibrium is stable because each player has no incentive to cheat since an over-aggressive move ($x_i > x_N$) would hurt the player himself.

It is important to note that the Nash equilibrium can be reached dynamically, like for example in a repeated game where the players adjust their strategies according to the gradient: $\partial_t x_i = \partial u_i/\partial x_i$. This observation suggests that a "finite temperature" can be included in the system, by considering the Langevin-like equation (in suitable units of time):

$$\partial_t x_i = \frac{\partial u_i}{\partial x_i} + \eta_i, \tag{4}$$

where $\eta_i(t)$ is gaussian noise with $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t)\eta_j(t') \rangle = D\delta_{i,j}\delta(t - t')$. D, in the statistical mechanics analogy, plays the role of a finite temperature.

If u_i is given by Eq. (1), it is possible to find the stationary state distribution $P(\boldsymbol{x})$. Indeed, since V depends only on $X = \sum_i x_i$, it is convenient to perform an *orthonormal* transformation in the space spanned by $\boldsymbol{x}=(x_1,\ldots,x_n)$ into $\boldsymbol{y}=(y_1,\ldots,y_n)$ in such a way that $y_n = X/\sqrt{n}$. The Gram-Schmidt method [8] then gives $y_k = (\sum_{i\leqslant k} x_i - kx_{k+1})/\sqrt{k(k+1)}$ for $k<n$.

In the new variables, the dynamics reads

$$\partial_t y_k = V' y_k + \tilde{\eta}_k, \quad 1 \leqslant k < n, \tag{5}$$

$$\partial_t y_n = \sqrt{n} V + V' y_n + \tilde{\eta}_n, \tag{6}$$

and, by orthonormality $\langle \tilde{\eta}_i(t)\tilde{\eta}_j(t')\rangle = D\delta_{i,j}\delta(t-t')$.

This transformation has the virtue of displaying the statistical dependence of the variables in a natural way. Since V and V' depends on y_n only, y_n has a dynamics which is independent of the y_k, whereas each y_k is coupled to y_n. Therefore the stationary distribution can, in general, be expressed as $P(\boldsymbol{y}) = P(y_n)\prod_{k<n} P(y_k|y_n)$, where $P(y_k|y_n)$ is the distribution of y_k conditional to y_n. Eq. (6) describes a "particle" in a potential with thermal fluctuations and can be solved using standard techniques [9]. The same holds for Eq. (5), where y_n appears as a parameter. We find $P(\boldsymbol{y}) \propto \exp[-H/D]$ and

$$H = -\frac{V'}{2}\sum_{k=1}^{n-1} y_k^2 - \frac{V y_n}{\sqrt{n}} - \frac{n-1}{\sqrt{n}}\int_0^{y_n} \mathrm{d}x\, V(\sqrt{n}x). \tag{7}$$

This form of the stationary distribution is reminiscent of an equilibrium system with Hamiltonian H. It would however be misleading to identify $-H$ with some measure of the utility U. This stationary state has a completely dynamic origin.

The equilibrium distribution, for D small, can be expanded around its maximum. The maximum of $H(\boldsymbol{y})$ is attained at $y_k^* = \delta_{k,n}\sqrt{n}x_N$, in agreement with Eq. (3). The gaussian fluctuations around the Nash equilibrium are found in the standard way: Expand $H(\boldsymbol{y})$ up to second order in $\delta y_k = y_k - y_k^*$. The inverse of the matrix of the quadratic form, yields the fluctuations $\langle \delta y_k \delta y_j\rangle$. In view of Eq. (7), one finds $\langle \delta y_k \delta y_j\rangle = \delta_{k,j}D/|V'|$ for $k<n$. Note that since $|V'| \sim n^{-1}$, these fluctuations are of order n. Because of these huge fluctuations, we shall call y_k "soft modes". The fluctuations of y_n instead turn out to be of order 1. One can infer the fluctuations of the x_i's by using the identity

$$\sum_{i=1}^{n} x_i^2 = \sum_{k=1}^{n} y_k^2 \tag{8}$$

and assuming $\langle \delta x_i \delta x_j\rangle = (A-C)\delta_{i,j} + C$. We discuss here only the case $V = 1 - X/n$ which allows more compact expressions. The same features discussed below apply to any $V(x)$ such that $V' \sim -n^{-1}$. Observing that $\langle y_n^2\rangle = A + (n-1)C$, and using Eq. (8) one finds

$$\langle \delta x_i^2\rangle = \frac{2n^2+n-2}{2(n+1)}D, \qquad \langle \delta x_i \delta x_j\rangle = -\frac{2n+1}{2(n+1)}D. \tag{9}$$

The main message of Eq. (9) is that fluctuations around the Nash equilibrium are very strong: The relative fluctuation of x_i, given that the average of x_i is close to one, is proportional to $\sqrt{nD}$. This depends on the fact that $|V'| \sim n^{-1}$, which is a very general feature in large games of the form (1). The variable x_i fluctuates the same order of magnitude as the sum of x_i over all $i = 1, \dots, n$. This is possible because of the negative correlation among the variables. A fluctuation of one of the variables is compensated by opposite fluctuations of the others.

Let us see what happens to the total utility. This is best seen in the variables y_k, because $U = \sqrt{n} y_n V(\sqrt{n} y_n)$ depends only on y_n. Therefore expanding U up to second order around $\mathbf{y}^*$ and taking the average, we find

$$\langle U \rangle = \frac{n^2}{(n+1)^2} - \frac{nD}{n+1}, \tag{10}$$

where, again we assumed $V(X) = 1 - X/n$. As can be easily seen, the fluctuations decrease the utility by a term $\delta U \simeq -D$ and they can have a dramatic effect: If $D > D_c \simeq 1$ the average utility becomes negative!

With respect to the dynamics, it is easy to check that the correlation function of the "soft" modes y_k, in the steady state for the linear $V(x)$, is

$$\langle y_k(t) y_k(t+\tau) \rangle \simeq nD \exp(-\tau/n), \tag{11}$$

which implies very long correlation times in the stationary state. This applies to the correlations of x_i as well.

The features discussed thus far hold the same if the constraint $x_i > 0$ is imposed. The fluctuation around the Nash equilibrium Eq. (3), at the level of the Gaussian approximation are still given by the above results. These characterize correctly the neighborhood of the Nash equilibrium. The corrections to the gaussian fluctuations are negligible when δx_i is much less than x_N, which occurs for $D \ll n^{-1}$. Numerical simulations show that, even for larger D, the same qualitative features (large fluctuations and eventually $U < 0$) hold also in the presence of the constraint $x_i > 0$.

It is instructive to study the "social" equilibrium in the same way. Now each player attempts to maximize the total utility U, and the Langevin equation is $\partial_t x_i = \partial U / \partial x_i + \eta_i$. In the variables $\mathbf{y}$ we find: $\partial y_n = \sqrt{n} V + n y_n V' + \tilde{\eta}_n$ and $\partial y_k = \tilde{\eta}_k$ for $k < n$. Note that y_k *now behave as random walks*. The distribution of y_k at long times is $P(\mathbf{y}, t) \propto \exp\left[-y_k^2/(2Dt)\right]$ and the correlations are $\langle y_k^2 \rangle = Dt$ for $k < n$ and $\langle \delta y_n^2 \rangle \sim D$. This implies unbounded fluctuations of x_i (i.e. $\langle \delta x_i^2 \rangle \simeq Dt$) and a negative correlation $\langle \delta x_i \delta x_j \rangle / \langle \delta x_i^2 \rangle \to -1/(n-1)$ such that the fluctuations of the sum X are finite. This implies that the average utility remains finite. The absence of a stationary distribution in the "social equilibrium" reflects its instability.

The results generalize with little qualitative changes when one considers a more general correlation among η_i or a mixed "social" egoistic model. These and other generalizations, as well as more detailed calculations, will be presented in a forthcoming publication.

It is generally recognized in economy that in realistic situations the relation between the utility and wealth (net profit) is not linear [11]. One source of non-linearity, for example, is inefficiency in capital management. This can be tolerated by the rich whereas it is very dangerous for the poor. Most studies [11] assume empirically a quadratic relation [12]. This, assuming $x_i V$ as a measure of the wealth of player i, leads to

$$u_i = x_i V(X)[1 - r x_i V(X)] . \tag{12}$$

Let us define $\bar{x} = X/n$ and assume that $V = 1 - \bar{x}$ (with little loss of generality since non-linearities in V do not change qualitatively the results). The interesting feature of this model is that if $r > 2$ a new Nash equilibrium appears. Indeed

$$\left.\frac{\partial u_i}{\partial x_i}\right|_{x_i=\bar{x}} = \left(1 - \frac{n+1}{n}\bar{x}\right)[1 - 2r\bar{x}(1-\bar{x})] = 0 , \tag{13}$$

for $r \geqslant 2$, has three solutions: $\bar{x} = x_N$, the usual Nash equilibrium Eq. (3), $\bar{x} = x_r = (r - \sqrt{r(r-2)})/2r$, which is a new Nash equilibrium, and $\bar{x} = x_+ = (r + \sqrt{r(r-2)})/2r$, which is an unstable equilibrium (i.e. a minimum of the utility). Provided $2 < r < n/2$, one has $x_r < x_+ < x_N$. The utility in the new equilibrium is $u_i(x_j = x_r,\ \forall j) = 1/(4r)$ which is positive and finite as compared to that at x_N which is $O(1/n)$. In the presence of two equilibria a player will choose one or the other according to what he judges other players will do.

Situations with more than one stable solution are frequent and of great interest in economy [2]. In particular one would like to know under what conditions a state is selected. The framework of Langevin dynamics (4) is particularly appealing. Indeed as we shall see it allows to understand which state is globally stable and how long a transition from the other state into it will take.

From Eq. (4) we can derive the equation for $\bar{x}$:

$$\partial_t \bar{x} = \left(1 - \frac{\bar{x}}{x_N}\right)[1 - 2r\bar{x}(1-\bar{x})] + \frac{2r(1-\bar{x})}{n^2}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) + \bar{\eta} . \tag{14}$$

Here $\bar{\eta}$ is the average of η_i, i.e. it is a white noise with equal time correlation D/n. In spite of the fact that all x_i appear in Eq. (14), it is still useful to use the variables y_k. Indeed one can use the identity (8) and average over the degrees of freedom y_k in Eq. (14). This amounts to replacing the term in brackets by $(n-1)\langle y_k^2|\bar{x}\rangle$ [13] which is the average of y_k^2 conditional to a fixed $\bar{x}$. In order to close the equations, we assume that in the Langevin equation for y_k,

$$\partial_t y_k = -\left[2r(1-\bar{x})^2 + \frac{1}{n}\right] y_k + \frac{2r(1-\bar{x})}{n\sqrt{k(k+1)}}\left[\sum_{i=1}^{k} x_i^2 - k x_{k+1}^2\right] + \tilde{\eta}_k .$$

We can neglect the second term in the right-hand side. This can be justified by the expectation that this term is negligible for $n \gg 1$ if $x_i^2 \approx x_j^2$. The equation for y_k then simplifies considerably and one finds that, in the steady state,

$$\langle y_k^2|\bar{x}\rangle = \frac{Dn}{4nr(1-\bar{x})^2+2}\,. \tag{15}$$

This, in Eq. (14), gives (to leading order in n)

$$\partial_t\bar{x} = \frac{x_N-\bar{x}}{x_N}[1-2r\bar{x}(1-\bar{x})] + \frac{rD(1-\bar{x})}{4nr(1-\bar{x})^2+2} + \bar{\eta}\,, \tag{16}$$

which can be cast in the form $\partial_t\bar{x} = -\mathrm{d}H/\mathrm{d}\bar{x}+\bar{\eta}$, where $-H(\bar{x})$ is the integral of the deterministic part of Eq. (16). Since $\langle\bar{\eta}(t)\bar{\eta}(t')\rangle = (D/n)\delta(t-t')$, the stationary solution is $P(\bar{x}) \propto \exp\,[-nH(\bar{x})/D]$. Here H plays the role of a free energy. Indeed it has the form $H=E-TS$ where $T=D/n$ is the analogous of the temperature and S is the entropy. The entropy enters from the fluctuations of the degrees of freedom y_k which have been self-consistently retained in the equation for $\bar{x}$. Let us discuss, from this point of view, the statistics of $\bar{x}$ in the steady state: Fixing $E(x_+)=0$, the energy in the two equilibrium states are, to leading order in n,

$$E(x_N) = \frac{2r-3-4r(r-2)x_r}{24r} + O(n^{-1})\,,$$
$$E(x_r) = \tfrac{1}{6}(r-2)(1-2x_r) + O(n^{-1})\,.$$

Therefore $E(x_N)<E(x_r)$ in the interval $2\leqslant r<\frac{9}{4}$ whereas $E(x_N)>E(x_r)$ for $r>\frac{9}{4}$. It is important to stress that this energy cannot be interpreted as $-U$. Indeed note that the minimum energy is at x_N for $2\leqslant r<\frac{9}{4}$, whereas the maximum utility U is always at x_r.

Energy alone suggests therefore that the system will fall in the minimum energy minimum for $t\to\infty$, and this, in the limit $n\to\infty$, is x_N for $r<\frac{9}{4}$ and x_r for $r>\frac{9}{4}$. This conclusion holds to leading order in n even if one considers also the entropy. The reason is that, for $n\to\infty$, one is considering very small temperatures $T=D/n$. Direct calculation shows that $S(x_N)=(\log n)/4+O(n^{-2})$ while $S(x_r)=[\log(2rx_r-1)]/4+O(n^{-1})$. The entropy in x_N is considerably larger than that in x_r. Indeed the fluctuations of y_k are of order $\sqrt{n}$ in x_N, whereas in x_r they are finite [see Eq. (15)]. In other words, since $\langle\delta x_i^2\rangle \simeq \langle y_k^2|\bar{x}\rangle$, the set $\{x_i\}$ is much more widely spread in the x_N minimum than in the x_r one. And this is an effect which is correctly accounted by the entropy above.

Even though we can identify a globally stable equilibrium (x_N for $r<\frac{9}{4}$ and x_r otherwise) which will ultimately attract the system under the Langevin dynamics, it is important to stress that the other *metastable* equilibrium can be stable over times which are exponentially large in n. Indeed the energy barrier between the two minima is finite, but the temperature is very small $T=D/n$. If $r>\frac{9}{4}$ and initially the system is in the x_N equilibrium, it will not visit the state x_r before a time of the order of $\sim\exp\,[nE(x_N)/D]$. This time can be infinite for all practical purposes. In other words, the system is very sensible to initial conditions.

We have presented a general approach to extend game theory to include fluctuations. We used the Langevin formulation which provides a natural bridge between game theory and statistical mechanics. The essential difference is that individual utility functions replace a global Hamiltonian. Fluctuations describe in a natural way the stability nature of various equilibria. We find that the Nash equilibrium of simple games with competition is stable against thermal fluctuations, even though the amplitude of fluctuations is very large. On the contrary, the "socially ideal" state is marginally unstable due to the presence of "soft modes". The approach also allows to study situation with more than one Nash equilibria and identifies the globally stable one as well as the criteria under which a state is reached by the dynamics.

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